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Research Article

Uniqueness of Successive Positive Solution for Nonlocal Singular Higher-Order Fractional Differential Equations Involving Arbitrary Derivatives

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In this article, by means of fixed point theorem on mixed monotone operator, we establish the uniqueness of positive solution for some nonlocal singular higher-order fractional differential equations involving arbitrary derivatives. We also give iterative schemes for approximating this unique positive solution.

1. Introduction

We are interested in investigating the existence and iterative schemes of the unique positive solution for the following fractional differential equation (FDE):

$$\begin{aligned} &D_{0+}^{\alpha}u(t)+f\left(t,u(t),D_{0+}^{\delta}u(t)\right)=0,\quad 0<t<1,\\ &D_{0+}^{\delta}u(0)=D_{0+}^{\delta+1}u(0)=\cdots=D_{0+}^{\delta+n-2}u(0)=0,\\ &D_{0+}^{\vartheta}u(1)=\lambda\int_0^{\eta}h(t)D_{0+}^{\iota}u(t)\,\mathrm{d}t, \end{aligned} \tag{1}$$

where D_{0+}^{α} is the standard Riemann-Liouville derivative, $n-1<\alpha\le n, n\ge 3, \vartheta-\delta\ge 1, \alpha-\vartheta-1>0, 0\le\delta<1, 0<\eta\le 1$, λ is a positive parameter with $0\le\lambda\Gamma(\alpha-\vartheta)\int_0^{\eta}h(t)t^{\alpha-\iota-1}\mathrm{d}t<\Gamma(\alpha-\iota)$, $f\in C(J\times R_+'\times R_+', R_+)$, $J=(0,1)$, $R_+'=(0,+\infty)$, $R_+=[0,+\infty)$, $0<\lambda\Gamma(\alpha-\vartheta)\int_0^{\eta}h(t)t^{\alpha-\iota-1}\mathrm{d}t<\Gamma(\alpha-\iota)$, $h\in L^1[0,1]$ is nonnegative, $f(t,x_1,x_2)$ permits singularities at $x_i=0$ $(i=1,2)$, and $t=0,1$.

In recent years, fractional calculus and fractional models play more and more significant role in describing a wide spectrum of nonlinear phenomena in natural sciences, engineering, economics, biology, and signal and image processing; see books and monographs [1–3] and references [4–32] to name a few. More and more attention has been paid to nonlocal problem of fractional differential equation because of its wide applications to applied mathematics and physics such as chemical engineering, underground water flow, heat conduction, thermoelasticity, and plasma physics. Under different conjugate type integral conditions such as no parameters, only one or two parameters involved in boundary conditions, [8–16, 33, 34] investigate the existence, uniqueness, and multiplicity of positive solutions for FDEs when f is either continuous or singular. Very recently, in [16], we give two uniqueness results of solution for the following FDE:

$$\begin{aligned} &D_{0+}^{\alpha}u(t)+f(t,u(t))=0,\quad 0\le t\le 1,\\ &u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0,\\ &D_{0+}^{\beta}u(1)=\lambda\int_0^{\eta}h(t)D_{0+}^{\gamma}u(t)\,\mathrm{d}t, \end{aligned} \tag{2}$$

where $f\in C(I\times R,R)$, $I=[0,1]$, $R=(-\infty,+\infty)$, and $h\in L^1[0,1]$ is nonnegative. The whole discussion is based

on the Banach contraction map principle and the theory of u_0-positive linear operator.

Motivated by the above papers, in this article we aim to obtain the uniqueness result of solution for BVP (1) by means of theory on mixed monotone operator. This article admits some new features. First, compared to [6–16] the problem considered in this paper performs more general form since another parameter δ is contained in boundary conditions. Second, the nonlinearity f is not only singular on space variable u but also relative with δ-order derivative of an unknown variable u. Finally, the method used in this paper is different from that in [16].

2. Preliminaries and Several Lemmas

Definition 1 (see [3]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $x : (0,\infty) \longrightarrow R$ is given by

$$I_{0+}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)\,\mathrm{d}s \tag{3}$$

provided the right-hand side is pointwise defined on $(0,\infty)$.

Definition 2 (see [3]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $x : (0,\infty) \longrightarrow R$ is given by

$$D_{0+}^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_0^t \frac{x(s)}{(t-s)^{\alpha-n+1}}\mathrm{d}s \tag{4}$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number α, provided that the right-hand side is pointwise defined on $(0,\infty)$.

Lemma 3 (see [23]). *(1) If $x \in L(0,1)$, $\nu > \sigma > 0$, then*

$$\begin{aligned} I_{0^+}^{\nu} I_{0^+}^{\sigma} x(t) &= I_{0^+}^{\nu+\sigma} x(t), \\ D_{0^+}^{\sigma} I_{0^+}^{\nu} x(t) &= I_{0^+}^{\nu-\sigma} x(t), \\ D_{0^+}^{\sigma} I_{0^+}^{\sigma} x(t) &= x(t). \end{aligned} \tag{5}$$

(2) If $\nu > 0, \sigma > 0$, then

$$D_{0^+}^{\nu} t^{\sigma-1} = \frac{\Gamma(\sigma)}{\Gamma(\sigma-\nu)} t^{\sigma-\nu-1}. \tag{6}$$

Let $u(t) = I_{0^+}^{\delta} x(t)$, $x(t) \in C[0,1]$. According to the definition of Riemann-Liouville derivative and Lemma 3, we have

$$\begin{aligned} D_{0^+}^{\alpha} u(t) &= \frac{\mathrm{d}^n}{\mathrm{d}t^n} I_{0^+}^{n-\alpha} u(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} I_{0^+}^{n-\alpha} I_{0^+}^{\delta} x(t) \\ &= \frac{\mathrm{d}^n}{\mathrm{d}t^n} I_{0^+}^{n-\alpha+\delta} x(t) = D_{0^+}^{\alpha-\delta} x(t); \\ D_{0^+}^{\delta} u(t) &= D_{0^+}^{\delta} I_{0^+}^{\delta} x(t) = x(t); \\ D_{0^+}^{\delta+1} u(t) &= x'(t), \cdots, D_{0^+}^{\delta+n-2} u(t) = x^{(n-2)}(t); \\ D_{0^+}^{\beta+\delta} u(t) &= D_{0^+}^{\beta} x(t), \\ D_{0^+}^{\gamma+\delta} u(t) &= D_{0^+}^{\gamma} x(t). \end{aligned} \tag{7}$$

Let $\beta = \vartheta - \delta, \gamma = \iota - \delta$. Then, by (7), BVP (1) can reduce to the following modified fractional boundary value problems (MFBVP):

$$\begin{aligned} &D_{0+}^{\alpha-\delta} x(t) + f\left(t, I_{0^+}^{\delta} x(t), x(t)\right) = 0, \quad 0 < t < 1, \\ &x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, \\ &D_{0+}^{\beta} x(1) = \lambda \int_0^{\eta} h(t) D_{0+}^{\gamma} x(t)\,\mathrm{d}t. \end{aligned} \tag{8}$$

In a similar way, we can transform (8) into the form (1). Thus, MFBVP (8) is equivalent to BVP (1).

Lemma 4. *Let $u(t) = I_{0^+}^{\delta} x(t)$, $x(t) \in C[0,1]$. Then BVP (1) can transform to (8). In addition, if $x \in C[0,1]$ is a positive solution for (8), then $I_{0^+}^{\delta} x$ is a positive solution for BVP (1).*

Proof. Substituting $u(t) = I_{0^+}^{\delta} x(t)$ into (1), we know from Lemma 3 and (7) that

$$\begin{aligned} &D_{0^+}^{\alpha} u(t) = D_{0^+}^{\alpha-\delta} x(t); \\ &D_{0^+}^{\delta} u(t) = x(t); \\ &D_{0^+}^{\delta+1} u(t) = x'(t), \cdots, D_{0^+}^{\delta+n-2} u(t) = x^{(n-2)}(t); \\ &D_{0^+}^{\vartheta} u(t) = D_{0^+}^{\beta+\delta} u(t) = D_{0^+}^{\beta} x(t), \\ &D_{0^+}^{\iota} u(t) = D_{0^+}^{\gamma+\delta} u(t) = D_{0^+}^{\gamma} x(t). \end{aligned} \tag{9}$$

Considering this together with the boundary value conditions, we have

$$\begin{aligned} &x(0) = D_{0^+}^{\delta} u(0) = 0, \\ &x^{(i)}(t) = D_{0^+}^{\delta+i} u(0) = 0, \\ &\qquad\qquad i = 1, 2, \cdots, n-2; \\ &D_{0+}^{\beta} x(1) = D_{0+}^{\vartheta} u(1) = \lambda \int_0^{\eta} h(t) D_{0+}^{\iota} u(t)\,\mathrm{d}t \\ &\qquad = \lambda \int_0^{\eta} h(t) D_{0+}^{\gamma} x(t)\,\mathrm{d}t, \end{aligned} \tag{10}$$

and

$$D_{0+}^{\alpha-\delta} x(t) = -f\left(t, I_{0^+}^{\delta} x(t), x(t)\right). \tag{11}$$

Thus, (1) is converted to (8).

Additionally, suppose that $x \in C[0,1]$ is a positive of (8). Let $u(t) = I_{0^+}^{\delta} x(t)$. Then, by Lemma 3, one gets

$$\begin{aligned} D_{0^+}^{\alpha} u(t) &= D_{0^+}^{\alpha} I_{0^+}^{\delta} x(t) = D_{0^+}^{\alpha-\delta} x(t) \\ &= -f\left(t, I_{0^+}^{\delta} x(t), x(t)\right) \\ &= -f\left(t, u(t), D_{0+}^{\delta} u(t)\right). \end{aligned} \tag{12}$$

The boundary condition $x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0$, $D_{0+}^{\beta}x(1) = \lambda \int_0^{\eta} h(t)D_{0+}^{\gamma}x(t)\mathrm{d}t$ together with (9) implies that

$$\begin{aligned} &D_{0+}^{\delta}u(0) = D_{0+}^{\delta+1}u(0) = \cdots = D_{0+}^{\delta+n-2}u(0) = 0, \\ &D_{0+}^{\vartheta}u(1) = \lambda \int_0^{\eta} h(t)\, D_{0+}^{\iota}u(t)\,\mathrm{d}t. \end{aligned} \tag{13}$$

That is to say, $I_{0^+}^{\delta}x(t)$ is a positive solution of BVP (1). □

Remark 5. Direct computation implies that

$$\begin{aligned} I_{0^+}^{\delta}t^{\alpha-\delta-1} &= \frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}s^{\alpha-\delta-1}\mathrm{d}s \\ &= \frac{B(\delta,\alpha-\delta)}{\Gamma(\delta)}t^{\alpha-1} = \frac{\Gamma(\alpha-\delta)}{\Gamma(\alpha)}t^{\alpha-1}. \end{aligned} \tag{14}$$

The following two lemmas are isomorphic forms of those in [16].

Lemma 6 (see [16]). *Assume that* $\lambda\Gamma(\alpha-\delta-\beta)\int_0^{\eta} h(t)t^{\alpha-\delta-\gamma-1}\mathrm{d}t \neq \Gamma(\alpha-\delta-\gamma)$. *Then for any* $y \in L(0,1)$, *the unique solution of the boundary value problems*

$$\begin{aligned} &D_{0+}^{\alpha-\delta}x(t) + y(t) = 0, \quad 0 < t < 1, \\ &x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, \\ &D_{0+}^{\beta}x(1) = \lambda\int_0^{\eta} h(t)\, D_{0+}^{\gamma}x(t)\,\mathrm{d}t, \end{aligned} \tag{15}$$

solves

$$x(t) = \int_0^1 G(t,s)\,y(s)\,\mathrm{d}s, \quad t \in [0,1], \tag{16}$$

where

$$G(t,s) = G_1(t,s) + G_2(t,s), \tag{17}$$

$$G_1(t,s) = \begin{cases} \dfrac{t^{\alpha-\delta-1}(1-s)^{\alpha-\delta-\beta-1} - (t-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)}, & 0 \le s \le t \le 1, \\ \dfrac{t^{\alpha-\delta-1}(1-s)^{\alpha-\delta-\beta-1}}{\Gamma(\alpha-\delta)}, & 0 \le t \le s \le 1, \end{cases} \tag{18}$$

$$\begin{aligned} G_2(t,s) = &\frac{\lambda\Gamma(\alpha-\delta-\beta)\,t^{\alpha-\delta-1}}{\Gamma(\alpha-\delta-\gamma) - \lambda\Gamma(\alpha-\delta-\beta)\int_0^{\eta} h(t)\,t^{\alpha-\delta-\gamma-1}\mathrm{d}t} \\ &\cdot \int_0^{\eta} h(t)\,H(t,s)\,\mathrm{d}t, \end{aligned} \tag{19}$$

$$H(t,s) = \begin{cases} \dfrac{t^{\alpha-\delta-\gamma-1}(1-s)^{\alpha-\delta-\beta-1} - (t-s)^{\alpha-\delta-\gamma-1}}{\Gamma(\alpha-\delta)}, & 0 \le s \le t \le 1, \\ \dfrac{t^{\alpha-\delta-\gamma-1}(1-s)^{\alpha-\delta-\beta-1}}{\Gamma(\alpha-\delta)}, & 0 \le t \le s \le 1, \end{cases} \tag{20}$$

Here, $G(t,s)$ is called the Green function of BVP (15). Obviously, $G(t,s)$ is continuous on $[0,1]\times[0,1]$.

Clearly, x is a positive solution of BVP (8) if and only if $x \in C[0,1]$ is a solution of the following nonlinear integral equation:

$$x(t) = \int_0^1 G(t,s)\,f\left(s, I_{0^+}^{\delta}x(s), x(s)\right)\mathrm{d}s. \tag{21}$$

Lemma 7 (see [16]). *The functions $G_1(t,s)$ and $G(t,s)$ given by (18) and (17), respectively, admit the following properties:*

(a_1) $G_1(t,s) \ge (1/\Gamma(\alpha-\delta))t^{\alpha-\delta-1}s(1-s)^{\alpha-\delta-\beta-1}$, $\forall\, t,s \in [0,1]$;

(a_2) $G_1(t,s) \le (1/\Gamma(\alpha-\delta))(\alpha-\delta-1)s(1-s)^{\alpha-\delta-\beta-1}$, $\forall\, t,s \in [0,1]$;

(a_3) $G(t,s) \le J(s)$, $J(s) = (1/\Gamma(\alpha-\delta))(\alpha-\delta-1)s(1-s)^{\alpha-\delta-\beta-1} + \lambda\Gamma(\alpha-\delta-\beta)/(\Gamma(\alpha-\delta-\gamma) - \lambda\Gamma(\alpha-\delta-\beta)\int_0^{\eta} h(t)t^{\alpha-\delta-\gamma-1}\mathrm{d}t)\cdot\int_0^{\eta} h(t)H(t,s)\mathrm{d}t$, $\forall\, t,s \in [0,1]$;

(a_4) $(1/(\alpha-\delta-1))t^{\alpha-\delta-1}J(s) \le G(t,s) \le (1/\Gamma(\alpha-\delta))\Lambda t^{\alpha-\delta-1}(1-s)^{\alpha-\delta-\beta-1}$, *here* $\Lambda = (\alpha-\delta-1) + (\lambda\Gamma(\alpha-\delta-\beta)/(\Gamma(\alpha-\delta-\gamma) - \lambda\Gamma(\alpha-\delta-\beta)\int_0^{\eta} h(t)t^{\alpha-\delta-\gamma-1}\mathrm{d}t))\cdot\int_0^{\eta} h(t)t^{\alpha-\delta-\gamma-1}\mathrm{d}t$, $\forall\, t,s \in [0,1]$.

Let P be a normal cone in a Banach space E and nonzero element $e \in P$ with $\|e\| \le 1$. A subset of cone P is given as follows:

$$\begin{aligned} Q_e = \{x \in P \mid &\text{there exist constants } m, M \\ &> 0 \text{ such that } me \le x \le Me\}. \end{aligned} \tag{22}$$

Definition 8 (see [35]). Assume that $A : Q_e \times Q_e \longrightarrow Q_e$. A is said to be mixed monotone if $A(x,y)$ is nondecreasing in x and nonincreasing in y, i.e., if $x_1 \le x_2$ $(x_1, x_2 \in Q_e)$ implies $A(x_1,y) \le A(x_2,y)$ for any $y \in Q_e$, and $y_1 \le y_2$ $(y_1, y_2 \in Q_e)$ implies $A(x,y_1) \le A(x,y_2)$ for any $x \in Q_e$. The element $x^* \in Q_e$ is said to be a fixed point of A if $A(x^*,x^*) = x^*$.

Lemma 9 (see [17]). *Suppose that $A : Q_e \times Q_e \longrightarrow Q_e$ is a mixed monotone operator and there exists a constant σ, $0 \le \sigma < 1$, such that*

$$A\left(tx, \frac{1}{t}y\right) \ge t^{\sigma}A(x,y), \quad \forall x,y \in Q_e,\ 0 < t < 1. \tag{23}$$

Then A has a unique fixed point $x^* \in Q_e$. Moreover, for any $(x_0,y_0) \in Q_e \times Q_e$,

$$\begin{aligned} x_k &= A(x_{k-1}, y_{k-1}), \\ y_k &= A(y_{k-1}, x_{k-1}), \\ &\qquad k = 1, 2, \cdots \end{aligned} \tag{24}$$

satisfy

$$\begin{aligned} x_k &\longrightarrow x^*, \\ y_k &\longrightarrow x^*, \end{aligned} \tag{25}$$

where

$$\begin{aligned} \|x_k - x^*\| &= o\left(1 - r^{\sigma^k}\right), \\ \|y_k - x^*\| &= o\left(1 - r^{\sigma^k}\right), \end{aligned} \tag{26}$$

where, r is a constant, $0 < r < 1$, and dependent on x_0, y_0.

3. Main Result

Throughout this paper, we adopt the following assumptions:

(H_1) $f(t,x_1,x_2)=\phi(t,x_1,x_2)+\psi(t,x_1,x_2)$, where, $\phi,\psi: J\times R_+^I\times R_+^I \longrightarrow R_+$ is continuous, $\phi(t,x_1,x_2)$ is nondecreasing on x_i, and $\psi(t,x_1,x_2)$ is nonincreasing on x_i $(i=1,2)$.

(H_2) There exists $\sigma\in J$ such that for $x_i>0$ $(i=1,2)$, $t,c\in J$

$$\begin{aligned}\phi(t,cx_1,cx_2)&\geq c^{\sigma}\phi(t,x_1,x_2);\\ \psi\left(t,c^{-1}x_1,c^{-1}x_2\right)&\geq c^{\sigma}\psi(t,x_1,x_2).\end{aligned}\tag{27}$$

(H_3)

$$\begin{aligned}0&<\int_0^1(1-s)^{\alpha-\vartheta-1}s^{-\sigma(\alpha-1)}\psi(s,1,1)\,ds<+\infty;\\ 0&<\int_0^1(1-s)^{\alpha-\vartheta-1}\phi(s,1,1)\,ds<+\infty.\end{aligned}\tag{28}$$

Remark 10. According to (H_2), for any $t\in J, c\geq 1, x_i>0$ $(i=1,2)$, one has

$$\begin{aligned}\phi(t,cx_1,cx_2)&\leq c^{\sigma}\phi(t,x_1,x_2);\\ \psi\left(t,c^{-1}x_1,c^{-1}x_2\right)&\leq c^{\sigma}\psi(t,x_1,x_2).\end{aligned}\tag{29}$$

Theorem 11. *Assume that (H_1)–(H_3) hold. Then, the BVP (1) has a unique solution u^*, and there exists a constant $D>1$ such that*

$$\frac{\Gamma(\alpha-\delta)}{D\Gamma(\alpha)}t^{\alpha-1}\leq u^*(t)\leq\frac{D\Gamma(\alpha-\delta)}{\Gamma(\alpha)}t^{\alpha-1},\quad \forall t\in[0,1].\tag{30}$$

Moreover, for any u_0, we construct a successive sequence

$$\begin{aligned}u_{k+1}(t)=I_{0^+}^{\delta}\Bigg\{\int_0^1 G(t,s)\Big[&\phi\left(s,u_k(s),D_{0^+}^{\delta}u_k(s)\right)\\ &+\psi\left(s,u_k(s),D_{0^+}^{\delta}u_k(s)\right)\Big]\,ds\Bigg\},\quad k=1,2,\cdots,\end{aligned}\tag{31}$$

and we have $\|u_k-u^*\|\longrightarrow 0$ as $k\longrightarrow\infty$; the convergence rate is

$$\left\|u_k-u^*\right\|=\left\|I_{0^+}^{\delta}x_k-I_{0^+}^{\delta}x^*\right\|=o\left(1-r^{\sigma^k}\right),\tag{32}$$

where r is a constant with $0<r<1$ and is dependent on u_0.

Proof. Let $e(t)=t^{\alpha-\delta-1}$, and we define

$$Q_e=\left\{x\in C[0,1]\mid\frac{1}{D}e(t)\leq x(t)\leq De(t)\right\},\tag{33}$$

where

$$\begin{aligned}D>\max\Bigg\{&\left[\frac{1}{\Gamma(\alpha-\delta)}\int_0^1 s(1-s)^{\alpha-\vartheta-1}\left(\rho^{\sigma}s^{\sigma(\alpha-1)}\phi(s,1,1)+2^{-\sigma}b^{-\sigma}\psi(s,1,1)\right)ds\right]^{1/(\sigma-1)},1,2\rho,\\ &\left[\frac{\Lambda}{\Gamma(\alpha-\delta)}\int_0^1(1-s)^{\alpha-\vartheta-1}\left(2^{\sigma}b^{\sigma}\phi(s,1,1)+\rho^{-\sigma}s^{-\sigma(\alpha-1)}\psi(s,1,1)\right)ds\right]^{1/(1-\sigma)}\Bigg\},\end{aligned}\tag{34}$$

where $b=\max\{\Gamma(\alpha-\delta)/\Gamma(\alpha),1\}$, $\rho=\min\{1,\Gamma(\alpha-\delta)/\Gamma(\alpha)\}$. We consider the existence of positive solution for (8). For any $x,y\in Q_e$, define an operator T as follows:

$$\begin{aligned}T(x,y)(t)=\int_0^1 G(t,s)\Big[&\phi\left(s,I_{0^+}^{\delta}x(s),x(s)\right)\\ &+\psi\left(s,I_{0^+}^{\delta}y(s),y(s)\right)\Big]\,ds.\end{aligned}\tag{35}$$

It is clear that x is a positive solution of BVP (8) if and only if $x\in C[0,1]$ is a fixed point of the operator T.

First, we are in position to show that $T: Q_e\times Q_e\longrightarrow Q_e$ is well defined. By (H_2) and Remark 5, for any $x,y\in Q_e$ we have

$$\begin{aligned}&\phi\left(s,I_{0^+}^{\delta}x(s),x(s)\right)\\ &\quad\leq\phi\left(s,\frac{D\Gamma(\alpha-\delta)}{\Gamma(\alpha)}s^{\alpha-1},Ds^{\alpha-\delta-1}\right)\\ &\quad\leq\phi(s,Db+1,Db+1)\leq(Db+1)^{\sigma}\phi(s,1,1)\\ &\quad\leq 2^{\sigma}b^{\sigma}D^{\sigma}\phi(s,1,1),\quad s\in(0,1),\end{aligned}\tag{36}$$

and

$$\begin{aligned}\psi\left(s,I_{0^+}^{\delta}y(s),y(s)\right)&\leq\psi\left(s,\frac{\Gamma(\alpha-\delta)}{D\Gamma(\alpha)}s^{\alpha-1},\frac{1}{D}s^{\alpha-\delta-1}\right)\\ &\leq\psi\left(s,\frac{\rho}{D}s^{\alpha-1},\frac{\rho}{D}s^{\alpha-1}\right)\\ &\leq\left(\frac{\rho}{D}s^{\alpha-1}\right)^{-\sigma}\psi(s,1,1)\\ &=\rho^{-\sigma}D^{\sigma}s^{-\sigma(\alpha-1)}\psi(s,1,1),\\ &\qquad s\in(0,1).\end{aligned}\tag{37}$$

Considering the fact that $(\rho/D)s^{\alpha-1} < 1$, we have from (34), (H_2), and Remark 5 that

$$\begin{aligned}\phi\left(s, I_{0^+}^{\delta}x(s), x(s)\right) &\geq \phi\left(s, \frac{\Gamma(\alpha-\delta)}{D\Gamma(\alpha)}s^{\alpha-1}, \frac{1}{D}s^{\alpha-\delta-1}\right)\\ &\geq \phi\left(s, \frac{\rho}{D}s^{\alpha-1}, \frac{\rho}{D}s^{\alpha-\delta-1}\right)\\ &\geq \phi\left(s, \frac{\rho}{D}s^{\alpha-1}, \frac{\rho}{D}s^{\alpha-1}\right)\\ &\geq \left(\frac{\rho}{D}s^{\alpha-1}\right)^{\sigma}\phi(s,1,1)\\ &\geq \rho^{\sigma}D^{-\sigma}s^{\sigma(\alpha-1)}\phi(s,1,1),\\ &\qquad s\in(0,1),\end{aligned} \tag{38}$$

and

$$\begin{aligned}&\psi\left(s, I_{0^+}^{\delta}y(s), y(s)\right)\\ &\quad\geq \psi\left(s, \frac{D\Gamma(\alpha-\delta)}{\Gamma(\alpha)}s^{\alpha-1}, Ds^{\alpha-\delta-1}\right)\\ &\quad\geq \psi\left(s, Dbs^{\alpha-1}, Dbs^{\alpha-\delta-1}\right) \geq \psi(s, Db+1, Db+1)\\ &\quad\geq (Db+1)^{-\sigma}\psi(s,1,1) \geq 2^{-\sigma}b^{-\sigma}D^{-\sigma}\psi(s,1,1),\\ &\qquad s\in(0,1).\end{aligned} \tag{39}$$

Thus, it follows from (36), (37), Lemma 7, and (H_3) that

$$\begin{aligned}&\int_0^1 G(t,s)\left[\phi\left(s, I_{0^+}^{\delta}x(s), x(s)\right)\right.\\ &\quad\left.+\psi\left(s, I_{0^+}^{\delta}y(s), y(s)\right)\right]ds \leq \frac{1}{\Gamma(\alpha-\delta)}\\ &\quad\cdot \Lambda t^{\alpha-\delta-1}\int_0^1 (1-s)^{\alpha-\delta-\beta-1}\left[2^{\sigma}b^{\sigma}D^{\sigma}\phi(s,1,1)\right.\\ &\quad\left.+\rho^{-\sigma}D^{\sigma}s^{-\sigma(\alpha-1)}\psi(s,1,1)\right]ds = \frac{1}{\Gamma(\alpha-\delta)}\\ &\quad\cdot \Lambda t^{\alpha-\delta-1}\int_0^1 (1-s)^{\alpha-\vartheta-1}\left[2^{\sigma}b^{\sigma}D^{\sigma}\phi(s,1,1)\right.\\ &\quad\left.+\rho^{-\sigma}D^{\sigma}s^{-\sigma(\alpha-1)}\psi(s,1,1)\right]ds < +\infty,\\ &\qquad \forall t\in[0,1].\end{aligned} \tag{40}$$

This means that $T: Q_e \times Q_e \longrightarrow P$ is well defined.

On the other hand, we can easily see from (34) and (40) that

$$T(x,y)(t) \leq Dt^{\alpha-\delta-1} = De(t), \quad \forall t\in[0,1]. \tag{41}$$

At the same time, by (38), (39), and Lemma 7, we know that

$$\begin{aligned}T(x,y)(t) &= \int_0^1 G(t,s)\left[\phi\left(s, I_{0^+}^{\delta}x(s), x(s)\right)\right.\\ &\quad\left.+\psi\left(s, I_{0^+}^{\delta}y(s), y(s)\right)\right]ds \geq \frac{1}{\Gamma(\alpha-\delta)}\\ &\quad\cdot t^{\alpha-\delta-1}\int_0^1 s(1-s)^{\alpha-\delta-\beta-1}\left[\rho^{\sigma}D^{-\sigma}s^{\sigma(\alpha-1)}\phi(s,1,1)\right.\\ &\quad\left.+2^{-\sigma}b^{-\sigma}D^{-\sigma}\psi(s,1,1)\right]ds = \frac{1}{\Gamma(\alpha-\delta)}\\ &\quad\cdot t^{\alpha-\delta-1}\int_0^1 s(1-s)^{\alpha-\vartheta-1}\left[\rho^{\sigma}D^{-\sigma}s^{\sigma(\alpha-1)}\phi(s,1,1)\right.\\ &\quad\left.+2^{-\sigma}b^{-\sigma}D^{-\sigma}\psi(s,1,1)\right]ds \geq \frac{1}{D}t^{\alpha-\delta-1} = \frac{1}{D}\\ &\quad\cdot e(t), \quad \forall t\in[0,1].\end{aligned} \tag{42}$$

It follows from (40)-(42) that $T: Q_e \times Q_e \longrightarrow Q_e$ is well defined.

Next, we shall prove that $T: Q_e \times Q_e \longrightarrow Q_e$ is a mixed monotone operator. To this end, let $x_1, x_2 \in Q_e$ with $x_1 \leq x_2$. For any $y \in Q_e$, it follows from (H_1) together with the monotonicity of the operator $I_{0^+}^{\delta}$ that

$$\begin{aligned}T(x_1,y)(t) &= \int_0^1 G(t,s)\left[\phi\left(s, I_{0^+}^{\delta}x_1(s), x_1(s)\right)\right.\\ &\quad\left.+\psi\left(s, I_{0^+}^{\delta}y(s), y(s)\right)\right]ds \leq \int_0^1 G(t,s)\\ &\quad\cdot\left[\phi\left(s, I_{0^+}^{\delta}x_2(s), x_2(s)\right)\right.\\ &\quad\left.+\psi\left(s, I_{0^+}^{\delta}y(s), y(s)\right)\right]ds = T(x_2,y)(t),\end{aligned} \tag{43}$$

which implies that $T(x,y)$ is nondecreasing in x for any $y \in Q_e$. In a similar manner, for any $x, y_1, y_2 \in Q_e$ with $y_1 \leq y_2$, we have

$$\begin{aligned}T(x,y_1)(t) &= \int_0^1 G(t,s)\left[\phi\left(s, I_{0^+}^{\delta}x(s), x(s)\right)\right.\\ &\quad\left.+\psi\left(s, I_{0^+}^{\delta}y_1(s), y_1(s)\right)\right]ds \geq \int_0^1 G(t,s)\\ &\quad\cdot\left[\phi\left(s, I_{0^+}^{\delta}x(s), x(s)\right)\right.\\ &\quad\left.+\psi\left(s, I_{0^+}^{\delta}y_2(s), y_2(s)\right)\right]ds = T(x,y_2)(t).\end{aligned} \tag{44}$$

This is to say, $T(x,y)$ is nonincreasing in y for any $x \in Q_e$. Thus, $T: Q_e \times Q_e \longrightarrow Q_e$ is a mixed monotone operator.

Finally, by (H_2), one has

$$\begin{aligned}T\left(cx, c^{-1}y\right)(t) &= \int_0^1 G(t,s)\left[\phi\left(s, cI_{0^+}^{\delta}x(s), cx(s)\right)\right.\\ &\quad\left.+\psi\left(s, c^{-1}I_{0^+}^{\delta}y(s), c^{-1}y(s)\right)\right]ds \geq \int_0^1 G(t,s)\end{aligned}$$

$$\cdot c^{\sigma}\left[\phi\left(s, I_{0^+}^{\delta}x(s), x(s)\right) + \psi\left(s, I_{0^+}^{\delta}y(s), y(s)\right)\right]\mathrm{d}s = c^{\sigma}T(x,t)(t), \quad \forall t \in [0,1],\ x, y \in Q_e, \tag{45}$$

which means that (23) in Lemma 9 is satisfied. Hence, Lemma 9 guarantees that there exists a unique positive solution x^* for BVP (8). Let $u^*(t) = I_{0^+}^{\delta}x^*(t)$; then u^* is the unique positive solution for BVP (1).

In addition, for any $u_0(t) = I_{0^+}^{\delta}x_0(t) \in Q_e$, by Lemma 9, constructing a successive sequence

$$x_{k+1}(t) = \int_0^1 \mathrm{G}(t,s)\left[\phi\left(s, I_{0^+}^{\delta}x_k(s), x_k(s)\right) + \psi\left(s, I_{0^+}^{\delta}x_k(s), x_k(s)\right)\right]\mathrm{d}s, \quad k = 1, 2, \cdots, \tag{46}$$

and $u_{k+1}(t) = I_{0^+}^{\delta}x_{k+1}(t)$, then

$$u_{k+1}(t) = I_{0^+}^{\delta}\left\{\int_0^1 G(t,s)\left[\phi\left(s, u_k(s), D_{0^+}^{\delta}u_k(s)\right) + \psi\left(s, u_k(s), D_{0^+}^{\delta}u_k(s)\right)\right]\mathrm{d}s\right\}, \quad k = 1, 2, \cdots, \tag{47}$$

and we have $\|u_k - u^*\| = \|I_{0^+}^{\delta}x_k - I_{0^+}^{\delta}x^*\| \longrightarrow 0$ as $k \longrightarrow \infty$; the convergence rate is

$$\|u_k - u^*\| = \left\|I_{0^+}^{\delta}x_k - I_{0^+}^{\delta}x^*\right\| = o\left(1 - r^{\sigma^k}\right), \tag{48}$$

where r is a constant with $0 < r < 1$ and is dependent on u_0. Moreover, by Remark 5, we get

$$\frac{\Gamma(\alpha-\delta)}{D\Gamma(\alpha)}t^{\alpha-1} \le u^*(t) = I_{0^+}^{\delta}x^*(t) \le \frac{D\Gamma(\alpha-\delta)}{\Gamma(\alpha)}t^{\alpha-1}, \quad \forall t \in [0,1]. \tag{49}$$

□

4. An Example

Example 1. Consider the following fractional differential equation integral boundary value problems:

$$\begin{aligned} &D_{0+}^{9/2}u(t) + f\left(t, u(t), D_{0^+}^{1/4}u(t)\right) = 0, \quad 0 < t < 1, \\ &D_{0^+}^{1/4}u(0) = D_{0^+}^{5/4}u(0) = D_{0^+}^{9/4}u(0) = D_{0^+}^{13/4}u(0) = 0, \\ &D_{0+}^{7/4}u(1) = \frac{2}{3}\int_0^{4/5} t^{-3/4}D_{0+}^{3/2}u(t)\,\mathrm{d}t, \end{aligned} \tag{50}$$

where $\alpha = 9/2, n = 5, \delta = 1/4, \vartheta = 7/4, \iota = 3/2, \lambda = 2/3, h(t) = t^{-3/4}$,

$$\begin{aligned} f\left(t, u(t), D_{0^+}^{1/4}u(t)\right) = {} & \left(t^{-2/3} + \cos t\right)u^{1/8}(t) \\ & + t^{-1/4}u^{-1/9}(t) \\ & + 6t^{4/5}\left(D_{0^+}^{1/4}u(t)\right)^{1/7} \\ & + (3-2t)\left(D_{0^+}^{1/4}u(t)\right)^{-1/7}. \end{aligned} \tag{51}$$

By simple computation, we have $\lambda\Gamma(\alpha-\vartheta)\int_0^{\eta} h(t)t^{\alpha-\iota-1}\mathrm{d}t = 2/3 \times \Gamma(11/4) \times \int_0^{4/5} t^{-3/4}\cdot t^2\mathrm{d}t = 2/3 \times 1.6084 \times 0.2690 = 0.2884 < 2 = \Gamma(\alpha-\iota)$. It is easy to know that (H_1) holds for

$$\begin{aligned} \phi(t, x_1, x_2) &= \left(t^{-2/3} + \cos t\right){x_1}^{1/8} + 6t^{4/5}{x_2}^{1/7}, \\ \psi(t, x_1, x_2) &= t^{-1/4}{x_1}^{-1/9} + (3-2t){x_2}^{-1/7}. \end{aligned} \tag{52}$$

At the same time, for any $(t, x_1, x_2) \in J \times R_+' \times R_+'$ and $c \in J$, one has

$$\begin{aligned} \phi(t, cx_1, cx_2) &= c^{1/8}\left(t^{-2/3} + \cos t\right){x_1}^{1/8} + c^{1/7}6t^{4/5}{x_2}^{1/7} \\ &\ge c^{1/7}\phi(t, x_1, x_2), \\ \psi\left(t, c^{-1}x_1, c^{-1}x_2\right) &= c^{1/9}t^{-1/4}{x_1}^{-1/9} + c^{1/7}(3-2t){x_2}^{-1/7} \\ &\ge c^{1/7}\psi(t, x_1, x_2). \end{aligned} \tag{53}$$

Thus, (H_2) is valid for $\sigma = 1/7$. Notice that $\phi(s,1,1) = s^{-2/3} + \cos s + 6s^{4/5}$, $\psi(s,1,1) = s^{-1/4} + (3-2s)$, and one gets

$$\begin{aligned} &\int_0^1 (1-s)^{\alpha-\vartheta-1}s^{-\sigma(\alpha-1)}\psi(s,1,1)\,\mathrm{d}s \\ &\quad = \int_0^1 (1-s)^{7/4}s^{-1/2}\left(s^{-1/4} + 3 - 2s\right)\mathrm{d}s \approx 5.9264 \\ &\quad < +\infty, \end{aligned} \tag{54}$$

and

$$\begin{aligned} &\int_0^1 (1-s)^{\alpha-\vartheta-1}\phi(s,1,1)\,\mathrm{d}s \\ &\quad = \int_0^1 (1-s)^{7/4}\left(s^{-2/3} + \cos s + 6s^{4/5}\right)\mathrm{d}s \\ &\quad < \int_0^1 (1-s)^{7/4}\left(s^{-2/3} + 1 + 6s^{4/5}\right)\mathrm{d}s \approx 3.0765 \\ &\quad < +\infty, \end{aligned} \tag{55}$$

which implies that (H_3) is also satisfied. Thus, by Theorem 11 we know that BVP (50) has a unique positive solution.

In addition, for any initial $u_0 = I_{0^+}^{1/4} x_0 \in Q_e$, we construct a successive sequence

$$x_{k+1}(t) = \int_0^1 G(t,s)\left[\phi\left(s, I_{0^+}^{1/4} x_k(s), x_k(s)\right) + \psi\left(s, I_{0^+}^{1/4} x_k(s), x_k(s)\right)\right] \mathrm{d}s, \quad k = 1, 2, \cdots, \tag{56}$$

and $u_{k+1}(t) = I_{0^+}^{1/4} x_{k+1}(t)$; then

$$u_{k+1}(t) = I_{0^+}^{1/4}\left\{\int_0^1 G(t,s)\left[\phi\left(s, u_k(s), D_{0^+}^{1/4} u_k(s)\right) + \psi\left(s, u_k(s), D_{0^+}^{1/4} u_k(s)\right)\right] \mathrm{d}s\right\}, \quad k = 1, 2, \cdots, \tag{57}$$

and we have $\|u_k - u^*\| = \|I_{0^+}^{1/4} x_k - I_{0^+}^{1/4} x^*\| \longrightarrow 0$ as $k \longrightarrow \infty$; the convergence rate is

$$\left\|u_k - u^*\right\| = \left\|I_{0^+}^{\delta} x_k - I_{0^+}^{1/4} x^*\right\| = o\left(1 - r^{(1/7)^k}\right), \tag{58}$$

where r is a constant with $0 < r < 1$ and is dependent on u_0.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All the authors read and approved the final manuscript.

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