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Research Article

Uniform Boundedness Principle for Nonlinear Operators on Cones of Functions

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We prove a uniform boundedness principle for the Lipschitz seminorm of continuous, monotone, positively homogeneous, and subadditive mappings on suitable cones of functions. The result is applicable to several classes of classically nonlinear operators.

1. Introduction and Preliminaries

Uniform boundedness principle for bounded linear operators (Banach-Steinhaus theorem) is one of the cornerstones of classical functional analysis (see, e.g., [1–3] and the references cited therein). In this article we prove a new uniform boundedness principle for monotone, positively homogeneous, subadditive, and Lipschitz mappings defined on a suitable cone of functions (Theorem 2). This result is applicable to several classes of classically nonlinear operators (Examples 4 and 5 and Remarks 7 and 8).

Let Ω be a nonempty set. Throughout the article let X denote a vector space of all functions $\varphi : \Omega \to \mathbb{R}$ or a vector space of all equivalence classes of (almost equal everywhere) real measurable functions on Ω, if $(\Omega, \mathscr{M}, \mu)$ is a measure space. As usual, $|\varphi|$ denotes the absolute value of $\varphi \in X$.

Let $Y \subset X$ be a vector space and let Y_+ denote the positive cone of Y, that is, the set of all $\varphi \in Y$ such that $\varphi(\omega) \geq 0$ for all (almost all) $\omega \in \Omega$. The space Y is called an ordered vector space with the partial ordering induced by the cone Y_+. If, in addition, Y is a normed space it is called an ordered normed space. The vector space $Y \subset X$ is called a vector lattice (or a Riesz space) if for every $\varphi, \psi \in Y$ we have a supremum and infimum (greatest lower bound) in Y. If, in addition, Y is a normed space and if $|\varphi| \leq |\psi|$ implies $\|\varphi\| \leq \|\psi\|$, then Y is a called normed vector lattice (or a normed Riesz space). Note that in a normed vector lattice Y we have $\||\varphi|\| = \|\varphi\|$ for all $\varphi \in Y$. A complete normed vector lattice is called a Banach lattice. Observe that X itself is a vector lattice.

Let $Y \subset X$ be a normed space. The cone Y_+ is called normal if and only if there exists a constant $C > 0$ such that $\|\varphi\| \leq C\|\psi\|$ whenever $\varphi \leq \psi$, $\varphi, \psi \in Y_+$. A cone Y_+ is normal if and only if there exists an equivalent monotone norm $|||\cdot|||$ on Y; that is, $|||\varphi||| \leq |||\psi|||$ whenever $0 \leq \varphi \leq \psi$ (see, e.g., [4, Theorem 2.38]). A positive cone of a normed vector lattice is closed and normal. Every closed cone in a finite dimensional Banach space is necessarily normal.

Let $Z \subset Y$ be a cone (not necessarily equal to Y_+). A cone Z is said to be complete if it is a complete metric space in the topology induced by Y. In the case when Y is a Banach space this is equivalent to Z being closed in Y.

A mapping $A : Z \to Z$ is called positively homogeneous (of degree 1) if $A(t\varphi) = tA(\varphi)$ for all $t \geq 0$ and $\varphi \in Z$. A mapping $A : Z \to Z$ is called Lipschitz if there exists $L > 0$ such that $\|A\varphi - A\psi\| \leq L\|\varphi - \psi\|$ for all $\varphi, \psi \in Z$ and we denote

$$\|A\|_{\mathrm{LIP}} = \sup_{\varphi,\psi\in Z,\varphi\neq\psi} \frac{\|A\varphi - A\psi\|}{\|\varphi - \psi\|}. \tag{1}$$

If A is Lipschitz and positively homogeneous, then

$$\|A\|_{\mathrm{LIP}} = \sup_{\varphi,\psi,\in Z,\|\varphi-\psi\|=1} \|A\varphi - A\psi\|. \tag{2}$$

Note also that a Lipschitz and positively homogeneous mapping A on Z is always bounded on Z; that is,

$$\|A\| = \sup_{\varphi\in Z,\varphi\neq 0} \frac{\|A\varphi\|}{\|\varphi\|} = \sup_{\varphi\in Z,\|\varphi\|=1} \|A\varphi\| \tag{3}$$

is finite and it holds that $\|A\| \le \|A\|_{\mathrm{LIP}}$. Moreover, a positively homogeneous mapping $A : Z \to Z$, which is continuous at 0, is bounded on Z.

A set $K \subset Y$ is called a wedge if $K + K \subset K$ and if $tK \subset K$ for all $t \ge 0$. A wedge K induces on Y a vector preordering $\le_K$ ($\varphi \le_K \psi$ if and only if $\psi - \varphi \in K$), which is reflexive and transitive, but not necessarily antisymmetric.

If $K \subset Y$ is a wedge, then $A : K \to Y$ is called subadditive if $A(\varphi + \psi) \le A\varphi + A\psi$ for $\varphi, \psi \in K$ and is called monotone (order preserving) if $A\varphi \le A\psi$ whenever $\varphi \le \psi$, $\varphi, \psi \in K$. Note that in this definition of subadditivity and monotonicity we consider on Y (and on K) a partial ordering $\le_{Y_+}$ induced by Y_+ (not a preordering $\le_K$). One of the reasons for this choice is that, for example, it may happen that a nonlinear map is monotone with respect to the ordering $\le_{Y_+}$, but it is not monotone with respect to the preordering $\le_K$ (see, e.g., [5, Section 5] and max-type operators, or [6] and the "renormalization operators" which occur in discussing diffusion on fractals). Moreover, for similar reasons wherever in our article we consider a subcone $Z \subset Y_+$ we consider on Z a partial ordering $\le_{Y_+}$ induced by Y_+ (not a partial ordering $\le_Z$). Observe that in this setting the set $Z - Z$ is a vector subspace in Y and thus a wedge.

In our main result (Theorem 2) we will consider a normed space $Y \subset X$ with a normal cone Y_+ and a complete subcone $Z \subset Y_+$ that satisfies $|\varphi - \psi| \in Z$ for all $\varphi, \psi \in Z$ and such that $\|\varphi\| = \||\varphi|\|$ for all $\varphi = \varphi_1 - \varphi_2$, where $\varphi_1, \varphi_2 \in Z$. Since X itself is a vector lattice the above assumptions make sense. Note also that a positive cone $Z = Y_+$ of each Banach lattice Y or, in particular, of each Banach function space (see, e.g., [2, 7–11] and the references cited therein) satisfies these properties. For the theory of cones, wedges, linear and nonlinear operators on cones and wedges, Banach ordered spaces, Banach function spaces, vector and Banach lattices, and applications, for example, in financial mathematics, we refer the reader to [2, 4, 5, 7, 8, 12–19] and the references cited therein.

2. Results

We will need the following lemma.

Lemma 1. *Let $Y \subset X$ be a vector space and let $Z \subset Y_+$ be a subcone such that $|\varphi - \psi| \in Z$ for all $\varphi, \psi \in Z$. If $A : Z - Z \to Y$ is a subadditive and monotone mapping, then*

$$|A\varphi - A\psi| \le A|\varphi - \psi|, \tag{4}$$

for all $\varphi, \psi \in Z$.

If, in addition, Y is a normed space such that Y_+ is normal and $Z \subset Y_+$ is a subcone such that $\|\varphi\| = \||\varphi|\|$ for all $\varphi = \varphi_1 - \varphi_2$, where $\varphi_1, \varphi_2 \in Z$, and if $AZ \subset Z$ and A is bounded on Z, then A is Lipschitz on Z.

Proof. Let $\varphi, \psi \in Z$. Since $A : Z - Z \to Y$ is a subadditive, we have

$$A\varphi = A(\varphi - \psi + \psi) \le A(\varphi - \psi) + A\psi. \tag{5}$$

It follows that $A\varphi - A\psi \le A(\varphi - \psi) \le A|\varphi - \psi|$, since A is monotone and $\varphi - \psi \le |\varphi - \psi|$. Similarly one obtains that $A\psi - A\varphi \le A|\varphi - \psi|$, which proves (4).

Assume that, in addition, Y is a normed space such that Y_+ is normal (with a normality constant C) and $Z \subset Y_+$ a subcone such that $\|\varphi_1 - \varphi_2\| = \||\varphi_1 - \varphi_2|\|$ for all $\varphi_1, \varphi_2 \in Z$ and that $AZ \subset Z$ and A is bounded on Z. It follows from (4) that

$$\begin{aligned} \|A\varphi - A\psi\| &= \||A\varphi - A\psi|\| \le C\|A|\varphi - \psi|\| \\ &\le C\|A\|\|\varphi - \psi\|, \end{aligned} \tag{6}$$

and thus A is Lipschitz on Z (and $\|A\|_{\mathrm{LIP}} \le C\|A\|$), which completes the proof. □

The following uniform boundedness principle is the central result of this article.

Theorem 2. *Let $Y \subset X$ be a normed space such that Y_+ is normal and let $Z \subset Y_+$ be a complete subcone, such that $|\varphi - \psi| \in Z$ for all $\varphi, \psi \in Z$ and such that $\|\varphi\| = \||\varphi|\|$ for all $\varphi \in Z - Z$. Assume that $\mathscr{A}$ is a set of subadditive and monotone mappings $A : Z - Z \to Y$ such that $AZ \subset Z$ and that each $A \in \mathscr{A}$ is positively homogeneous and continuous on Z.*

If the set $\{A\varphi\colon A \in \mathscr{A}\}$ is bounded for each $\varphi \in Z$ (i.e., for each $\varphi \in Z$ there exists $M_\varphi > 0$ such that $\|A\varphi\| \le M_\varphi$ for all $A \in \mathscr{A}$), then there exists $M > 0$ such that $\|A\|_{LIP} \le M$ for all $A \in \mathscr{A}$.

Proof. Since Z is closed and each $A \in \mathscr{A}$ is continuous on Z the set

$$A_n = \{\varphi \in Z\colon \|A\varphi\| \le n \ \forall A \in \mathscr{A}\} \tag{7}$$

is closed in Y for each $n \in \mathbb{N}$. Moreover, Z is a complete metric space and $Z = \bigcup_{n=1}^{\infty} A_n$. By Baire's theorem there exist $n_0 \in \mathbb{N}$, $\varphi_0 \in Z$, and $\varepsilon > 0$ such that an open ball $\mathcal{O}(\varphi_0, 3\varepsilon C) = \{\varphi \in Z\colon \|\varphi - \varphi_0\| < 3\varepsilon C\} \subset A_{n_0}$, where C is the normality constant of Y_+.

Let $\varphi, \psi \in Z$ such that $\|\varphi - \psi\| = 1$ and $A \in \mathscr{A}$. Since Z is a normal cone and A is positively homogeneous on Z, we have by (4)

$$\begin{aligned} \|A\varphi - A\psi\| &= \||A\varphi - A\psi|\| \le C\|A|\varphi - \psi|\| \\ &= \frac{C}{\varepsilon}\|A(\varepsilon|\varphi - \psi|)\|. \end{aligned} \tag{8}$$

Since A is subadditive and monotone on $Z - Z$ we have

$$\begin{aligned} A(\varepsilon|\varphi - \psi|) &= A(\varphi_0 + \varepsilon|\varphi - \psi| - \varphi_0) \\ &\le A\varphi_0 + A(\varepsilon|\varphi - \psi| - \varphi_0) \\ &\le A\varphi_0 + A(|\varepsilon|\varphi - \psi| - \varphi_0|), \end{aligned} \tag{9}$$

which together with (8) implies

$$\begin{aligned}\|A\varphi - A\psi\| &\le \frac{C^2}{\varepsilon}\left(\|A\varphi_0\| + \|A\,|\varepsilon|\,\varphi-\psi\,|-\varphi_0|\|\right)\\ &\le \frac{C^2}{\varepsilon}\left(n_0 + \|A\,|\varepsilon|\,\varphi-\psi\,|-\varphi_0|\|\right).\end{aligned} \tag{10}$$

We also have

$$\left||\varepsilon|\,\varphi-\psi\right| - \varphi_0\,|-\varphi_0| < 3\varepsilon\,|\varphi-\psi|. \tag{11}$$

Indeed, if $\varepsilon|\varphi(\omega)-\psi(\omega)|-\varphi_0(\omega)\le 0$, then

$$\begin{aligned}&\left||\varepsilon|\,\varphi(\omega)-\psi(\omega)\right| - \varphi_0(\omega)\,|-\varphi_0(\omega)|\\ &\quad= \varepsilon\left|\varphi(\omega)-\psi(\omega)\right|,\end{aligned} \tag{12}$$

and if $\varepsilon|\varphi(\omega)-\psi(\omega)|-\varphi_0(\omega)>0$, then

$$\begin{aligned}&\left||\varepsilon|\,\varphi(\omega)-\psi(\omega)\right| - \varphi_0(\omega)\,|-\varphi_0(\omega)|\\ &\quad= |\varepsilon|\,\varphi(\omega)-\psi(\omega)\,|-2\varphi_0(\omega)|\\ &\quad\le \varepsilon\left|\varphi(\omega)-\psi(\omega)\right| + 2\varphi_0(\omega) < 3\varepsilon\left|\varphi(\omega)-\psi(\omega)\right|,\end{aligned} \tag{13}$$

which proves (11).

It follows from (11) that $\||\varepsilon|\varphi-\psi|-\varphi_0|-\varphi_0\| \le 3C\varepsilon\|\varphi-\psi\| = 3C\varepsilon$ and thus $|\varepsilon|\varphi-\psi|-\varphi_0| \in A_{n_0}$ and so $\|A|\varepsilon|\varphi-\psi|-\varphi_0|\| \le n_0$. Therefore

$$\|A\varphi - A\psi\| \le \frac{2C^2 n_0}{\varepsilon}, \tag{14}$$

and so $\|A\|_{\text{LIP}} \le 2C^2 n_0/\varepsilon$. □

Remark 3. (i) Each $A\in\mathscr{A}$ satisfies $\|A\|\le\|A\|_{\text{LIP}}\le C\|A\|$ (see the proof of Lemma 1). Therefore we could alternatively set $\psi=0$ in the proof above and prove a uniform upper bound $\|A\|\le 2Cn_0/\varepsilon$ for all $A\in\mathscr{A}$, which gives the same conclusion.

(ii) In the proofs of Lemma 1 and Theorem 2 we did not need the assumption $Z\cap(-Z)=\{0\}$, so it suffices to assume that Z is a wedge in these two results (not necessarily a cone).

(iii) Also the assumption on normality of Y_+ can be slightly weakened in Lemma 1 and Theorem 2. Instead of normality of Y_+ it suffices to assume that there exists a constant $C>0$ such that $\|\varphi\|\le C\|\psi\|$ whenever $\varphi\le\psi$, $\varphi,\psi\in Z$ (where again $\varphi\le\psi$ means $\varphi\le_{Y_+}\psi$).

Our results can be applied to various classes of nonlinear operators. In particular, they apply to various max-kernel operators (and their isomorphic versions) appearing in the literature (see, e.g., [5, 19–21] and the references cited therein). We point out the following two related examples from [5, 17–19].

Example 4. Given $a>0$, let $Y=C[0,a]$ be Banach lattice of continuous functions on $[0,a]$ equipped with $\|\cdot\|_\infty$ norm. Consider the following max-type kernel operators $A: C[0,a]\to C[0,a]$ of the form

$$(A(\varphi))(s) = \max_{t\in[\alpha(s),\beta(s)]} k(s,t)\,\varphi(t), \tag{15}$$

where $\varphi\in C[0,a]$ and $\alpha,\beta:[0,a]\to[0,a]$ are given continuous functions satisfying $\alpha\le\beta$. The kernel $k:S\to[0,\infty)$ is a given nonnegative continuous function, where S denotes the compact set

$$S=\{(s,t)\in[0,a]\times[0,a]: t\in[\alpha(s),\beta(s)]\}. \tag{16}$$

It is clear that for $Z=C_+[0,a]$ it holds that $AZ\subset Z$. The eigenproblem of these operators arises in the study of periodic solutions of a class of differential-delay equations

$$\varepsilon y'(t) = g(y(t), y(t-\tau)),\quad \tau=\tau(y(t)), \tag{17}$$

with state-dependent delay (see, e.g., [19]).

The mapping $A:Y\to Y$ is subadditive and monotone and is positively homogeneous and Lipschitz on Z. Moreover, $\|A\|_{\text{LIP}} = \|A\| = \max_{(s_0,s_1)\in\mathscr{S}_1} k(s_0,s_1)$, where $\mathscr{S}_1 = \{(s_0,s_1): s_0\in[0,a],\ s_1\in[\alpha(s_0),\beta(s_0)]\}$. Clearly, Theorem 2 applies to sets of such mappings.

Consequently, Theorem 2 applies also to isomorphic max-plus mappings (see, e.g., [19] and the references cited therein) and a Lipschitz seminorm with respect to a suitably induced metric. Note that a related result for uniform boundedness (in fact contractivity) result for a Lipschitz seminorm of semigroups of max-plus mappings was stated in [22, 23]. However, observe that the Lipschitz seminorm there is defined with respect to a different metric than that in our case.

We also point out the following related example from [17, 18].

Example 5. Let M be a nonempty set and let Y be the set of all bounded real functions on M. With the norm $\|f\|_\infty = \sup\{|f(t)|: t\in M\}$ and natural operations, Y is a Banach lattice. Let $Z=Y_+$ and let $k: M\times M\to[0,\infty)$ satisfy $\sup\{k(t,s): t,s\in M\}<\infty$. Let $A:Y\to Y$ be defined by $(Af)(s)=\sup\{k(s,t)f(t): t\in M\}$. Then $A:Y\to Y$ is subadditive and monotone mapping that satisfies $AZ\subset Z$ and is positively homogeneous and Lipschitz on Z; therefore Theorem 2 applies to sets of such mappings. It also holds that $\|A\|_{\text{LIP}}=\|A\|=\sup\{k(t,s): t,s\in M\}$. In particular, if M is the set of all natural numbers $\mathbb{N}$, our results apply to infinite bounded nonnegative matrices $k=[k(i,j)]$ (i.e., $k(i,j)\ge 0$ for all $i,j\in\mathbb{N}$ and $\|k\|_\infty = \sup_{i,j\in\mathbb{N}} k(i,j)<\infty$). In this case, $Y=l^\infty$ and $Z=l^\infty_+$ and $\|A\|_{\text{LIP}}=\|A\|=\|k\|_\infty$.

Remark 6. The special case of Example 5 when $M=\{1,\dots,n\}$ for some $n\in\mathbb{N}$ is well known and studied under the name max-algebra (an analogue of linear algebra). Together with its isomorphic versions (max-plus algebra and min-plus algebra also known as tropical algebra) it provides an attractive way of describing a class of nonlinear problems appearing, for instance, in manufacturing and transportation scheduling, information technology, discrete event-dynamic systems, combinatorial optimization, mathematical physics, and DNA analysis (see, e.g., [24–29] and the references cited therein).

Remark 7. Our results apply also to more general max-type operators studied in [5, Section 5]. The authors considered there finite sums of more general operators than in Example 4 defined on a Banach space of continuous functions and their restrictions to suitable closed cones. The assumptions of our results are satisfied also for these mappings and therefore also for a special case of cone-linear Perron-Frobenius operators studied there.

Remark 8. Theorem 2 applies to several classes of nonlinear integral operators (under suitable assumptions on the kernels and on the defining nonlinearities) including Hammerstein type operators (see, e.g., [12, Chapter 12, p. 338] and [30, 31]), Uryson type operators (see, e.g., [12, Chapter 12, p. 339] and [32]), and Hardy-Littlewood type operators (see, e.g., [33–36]).

Conflicts of Interest

The author declares that they have no conflicts of interest.

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