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Research Article

# Boundedness and Continuity of Several Integral Operators with Rough Kernels in $W \mathscr{F} \beta\left(S^{n-1}\right)$ on Triebel-Lizorkin Spaces 

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#### Abstract

A systematic treatment is given of singular integrals and Marcinkiewicz integrals associated with surfaces generated by polynomial compound mappings as well as related maximal functions with rough kernels in $W \mathscr{F} \beta\left(S^{n-1}\right)$, which relates to the Grafakos-Stefanov function class. Certain boundedness and continuity for these operators on Triebel-Lizorkin spaces and Besov spaces are proved by applying some criterions of bounds and continuity for several operators on the above function spaces.


## 1. Introduction

Let $\mathbb{R}^{n}(n \geq 2)$ be the $n$-dimensional Euclidean space and $S^{n-1}$ denote the unit sphere in $\mathbb{R}^{n}$ equipped with the induced Lebesgue measure $d \sigma$. Assume that $\Omega \in L^{1}\left(S^{n-1}\right)$ is a homogeneous function of degree zero and satisfies

$$
\begin{equation*}
\int_{S^{n-1}} \Omega(u) d \sigma(u)=0 \tag{1}
\end{equation*}
$$

For a suitable function $h$ defined on $\mathbb{R}_{+}:=(0, \infty)$, a complex number $\rho=\varsigma+i \tau(\varsigma, \tau \in \mathbb{R}$ with $\varsigma>0)$, and a suitable mapping $\Gamma: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$, we consider the singular integral operators $T_{h, \Omega, \Gamma}$ and parametric Marcinkiewicz integral operators $\mathscr{M}_{h, \Omega, \Gamma, \rho}$ in $\mathbb{R}^{n}$ by

$$
\begin{gather*}
T_{h, \Omega, \Gamma} f(x)=\text { p.v. } \int_{\mathbb{R}^{n}} f(x-\Gamma(y)) \frac{h(|y|) \Omega(y)}{|y|^{n}} d y  \tag{2}\\
\mathscr{M}_{h, \Omega, \Gamma, \rho} f(x)=\left(\int_{0}^{\infty} \left\lvert\, \frac{1}{t^{\rho}}\right.\right. \\
\left.\left.\cdot \int_{|y| \leq t} f(x-\Gamma(y)) \frac{h(|y|) \Omega(y)}{|y|^{n-\rho}} d y\right|^{2} \frac{d t}{t}\right)^{1 / 2} \tag{3}
\end{gather*}
$$

Define the related maximal operators $\mathcal{S}_{\Omega, \Gamma}$ and $\mathscr{M}_{\Omega, \Gamma, \rho}$ by

$$
\begin{align*}
\mathcal{S}_{\Omega, \Gamma} f(x) & =\sup _{h \in \mathscr{K}_{2}}\left|T_{h, \Omega, \Gamma} f(x)\right|,  \tag{4}\\
\mathscr{M}_{\Omega, \Gamma, \rho} f(x) & =\sup _{h \in \mathscr{K}_{2}}\left|\mathscr{M}_{h, \Omega, \Gamma, \rho} f(x)\right|, \tag{5}
\end{align*}
$$

where $\mathscr{K}_{2}$ is the set of all measurable functions $h: \mathbb{R}_{+} \rightarrow \mathbb{R}$ with $\|h\|_{L^{2}\left(\mathbb{R}_{+}, r^{-1} d r\right)} \leq 1$.

The primary purpose of this paper is to study the bounds and continuity of the singular integral operators and Marcinkiewicz integral operators associated with surfaces generated by polynomial compound mappings as well as related maximal functions with rough kernels in $W \mathscr{F}_{\beta}\left(\mathrm{S}^{n-1}\right)$ on the Triebel-Lizorkin spaces and Besov spaces. Before stating our main results, let us recall some pertinent definitions, notations, and backgrounds.

Definition 1 (function class $W \mathscr{F}_{\beta}\left(S^{n-1}\right)$ ). For $\beta>0$, the function class $W \mathscr{F}_{\beta}\left(\mathrm{S}^{n-1}\right)$ is the set of all $L^{1}\left(\mathrm{~S}^{n-1}\right)$ functions $\Omega$ which satisfy

$$
\begin{align*}
& \sup _{\xi \in S^{n-1}} \iint_{S^{n-1} \times S^{n-1}}\left|\Omega(\theta) \Omega\left(u^{\prime}\right)\right| \\
& \quad \cdot \log ^{\beta} \frac{2 e}{\left|\left(\theta-u^{\prime}\right) \cdot \xi\right|} d \sigma(\theta) d \sigma\left(u^{\prime}\right)  \tag{6}\\
& \quad<\infty .
\end{align*}
$$

The function class $W \mathscr{F} \beta\left(S^{n-1}\right)$ was originally introduced by Fan and Sato [1]. It is closely related to the GrafakosStefanov function class $\mathscr{F}_{\beta}\left(\mathrm{S}^{n-1}\right)$, which was first introduced in [2] and is given by

$$
\begin{align*}
& \mathscr{F}_{\beta}\left(\mathrm{S}^{n-1}\right):=\{\Omega \\
& \quad \in L^{1}\left(\mathrm{~S}^{n-1}\right): \sup _{\xi \in \mathrm{S}^{n-1}} \int_{S^{n-1}}\left|\Omega\left(y^{\prime}\right)\right| \log ^{\beta} \frac{2}{\left|\xi \cdot y^{\prime}\right|} d \sigma\left(y^{\prime}\right)  \tag{7}\\
& \quad<\infty\}, \quad \beta>0
\end{align*}
$$

It was shown in $[1,3]$ that

$$
\begin{aligned}
& \mathscr{F}_{\beta}\left(\mathrm{S}^{1}\right) \subset W \mathscr{F}_{\beta}\left(\mathrm{S}^{1}\right), \\
& W \mathscr{F}_{2 \beta}\left(\mathrm{~S}^{n-1}\right) \backslash \mathscr{F}_{\beta}\left(\mathrm{S}^{n-1}\right) \neq \emptyset, \\
& \begin{array}{l}
\bigcup_{r>1} L^{r}\left(\mathrm{~S}^{n-1}\right) \subset \mathscr{F}_{\beta_{2}}\left(\mathrm{~S}^{n-1}\right) \subset \mathscr{F}_{\beta_{1}}\left(\mathrm{~S}^{n-1}\right), \\
0<\beta_{1}<\beta_{2}<\infty ; \\
\\
\bigcup_{r>1} L^{r}\left(\mathrm{~S}^{n-1}\right) \subset W \mathscr{F}_{\beta_{2}}\left(\mathrm{~S}^{n-1}\right) \subset W \mathscr{F}_{\beta_{1}}\left(\mathrm{~S}^{n-1}\right), \\
0<\beta_{1}<\beta_{2}<\infty .
\end{array}
\end{aligned}
$$

To introduce some known results, we need to recall one more function space $\Delta_{\gamma}\left(\mathbb{R}_{+}\right)$.

Definition 2 (function class $\Delta_{\gamma}\left(\mathbb{R}_{+}\right)$). For $1 \leq \gamma \leq \infty$, the function class $\Delta_{\gamma}\left(\mathbb{R}_{+}\right)$is the set of all measurable functions $h: \mathbb{R}_{+} \rightarrow \mathbb{R}$ satisfying

$$
\begin{equation*}
\|h\|_{\Delta_{\gamma}\left(\mathbb{R}_{+}\right)}:=\sup _{R>0}\left(R^{-1} \int_{0}^{R}|h(t)|^{\gamma} d t\right)^{1 / \gamma}<\infty . \tag{9}
\end{equation*}
$$

It is clear that $\Delta_{\gamma_{2}}\left(\mathbb{R}_{+}\right) \subsetneq \Delta_{\gamma_{1}}\left(\mathbb{R}_{+}\right)$for $1 \leq \gamma_{1}<\gamma_{2} \leq \infty$ and $\Delta_{\infty}\left(\mathbb{R}_{+}\right)=L^{\infty}\left(\mathbb{R}_{+}\right)$.

When $\Gamma(y)=y$, the operators defined in (2) reduce to the classical Calderón-Zygmund operator

$$
\begin{equation*}
T_{h, \Omega} f(x)=\text { p.v. } \int_{\mathbb{R}^{n}} f(x-y) \frac{h(|y|) \Omega(y)}{|y|^{n}} d y \tag{10}
\end{equation*}
$$

which was originally studied by Calderón and Zygmund [4] and later investigated by many authors (see [1, 2, 5, 6], etc.). In 2009, Fan and Sato [1] first studied the $L^{p}$ bounds for $T_{h, \Omega}$ with $\Omega$ which belongs to $W \mathscr{F}_{\beta}\left(\mathrm{S}^{n-1}\right)$. More precisely, the above authors established the $L^{p}$ bounds for $T_{h, \Omega}$ with $|1 / p-1 / 2|<1 / \max \left\{2, \gamma^{\prime}\right\}-1 / \beta$ if $h \in \Delta_{\gamma}\left(\mathbb{R}_{+}\right)$for $\gamma>1$ and $\Omega \in W \mathscr{F} \beta\left(S^{n-1}\right)$ for some $\beta>\max \left\{\gamma^{\prime}, 2\right\}$. Recently, Liu and Wu [7] extended the result of [1] to the singular integrals along polynomial compound curves in the mixed homogeneity setting.

Let us recall the definitions of Triebel-Lizorkin spaces and Besov spaces.

Definition 3 (Triebel-Lizorkin spaces and Besov spaces). Let $\delta^{\prime}\left(\mathbb{R}^{n}\right)$ be the tempered distribution class on $\mathbb{R}^{n}$. For $\alpha \in \mathbb{R}$ and $0<p, q \leq \infty(p \neq \infty)$, the homogeneous TriebelLizorkin spaces $\dot{F}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)$ and Besov spaces $\dot{B}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)$ are defined by

$$
\begin{align*}
& \dot{F}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right):=\left\{f \in \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right):\|f\|_{\dot{F}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)}\right.  \tag{11}\\
& \left.\quad=\left\|\left(\sum_{i \in \mathbb{Z}} 2^{-i \alpha q}\left|\Psi_{i} * f\right|^{q}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)}<\infty\right\} ; \\
& \left.\quad=\left(\sum_{i \in \mathbb{Z}} 2^{-i \alpha q}\left\|\Psi_{i} * f\right\|_{L^{p}\left(\mathbb{R}^{n}\right)}^{q}\right)^{1 / q}<\infty\right\},
\end{align*}
$$

where $\widehat{\Psi_{i}}(\xi)=\phi\left(2^{i} \xi\right)$ for $i \in \mathbb{Z}$ and $\phi \in \mathscr{C}_{c}^{\infty}\left(\mathbb{R}^{n}\right)$ satisfies the conditions $0 \leq \phi(x) \leq 1 ; \operatorname{supp}(\phi) \subset\{x: 1 / 2 \leq$ $|x| \leq 2\} ; \phi(x)>c>0$ if $3 / 5 \leq|x| \leq 5 / 3$. The inhomogeneous versions of Triebel-Lizorkin spaces and Besov spaces, which are denoted by $F_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)$ and $B_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)$, respectively, are obtained by adding the term $\|\Theta * f\|_{L^{p}\left(\mathbb{R}^{n}\right)}$ to the right hand side of (11) or (12) with $\sum_{i \in \mathbb{Z}}$ replaced by $\sum_{i \geq 1}$, where $\Theta \in \mathcal{S}\left(\mathbb{R}^{n}\right)$ (the Schwartz class), $\operatorname{supp}(\widehat{\Theta}) \subset\{\xi:|\xi| \leq$ $2\}, \widehat{\Theta}(x)>c>0$ if $|x| \leq 5 / 3$.

The following properties are well-known (see [8,9] for more details):

$$
\begin{align*}
& \dot{F}_{0}^{p, 2}\left(\mathbb{R}^{n}\right)=L^{p}\left(\mathbb{R}^{n}\right), \quad 1<p<\infty ;  \tag{13}\\
& \dot{F}_{\alpha}^{p, p}\left(\mathbb{R}^{n}\right)=\dot{B}_{\alpha}^{p, p}\left(\mathbb{R}^{n}\right), \quad \alpha \in \mathbb{R}, 1<p<\infty ;  \tag{14}\\
& F_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right) \sim \dot{F}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right) \cap L^{p}\left(\mathbb{R}^{n}\right), \\
& \|f\|_{F_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)} \sim\|f\|_{\dot{F}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)}+\|f\|_{L^{p}\left(\mathbb{R}^{n}\right)}, \tag{15}
\end{align*}
$$

$$
\alpha>0
$$

$$
\begin{align*}
& B_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right) \sim \dot{B}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right) \cap L^{p}\left(\mathbb{R}^{n}\right) \\
& \|f\|_{B_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)} \sim\|f\|_{\dot{B}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)}+\|f\|_{L^{p}\left(\mathbb{R}^{n}\right)} \tag{16}
\end{align*}
$$

$$
\alpha>0 .
$$

Recently, the investigation of the bounds for singular integrals with rough kernels in $W \mathscr{F}_{\beta}\left(\mathrm{S}^{n-1}\right)$ on TriebelLizorkin spaces and Besov spaces has received some attention of many authors (see [3, 10, 11]). Particularly, Liu et al. [10] obtained the following result.

Theorem A (see [10]). Let $\Gamma(y)=P_{N}(|y|) y^{\prime}$, where $P_{N}$ is a real-valued polynomial with $P_{N}(0)=0$ and $\operatorname{deg}\left(P_{N}\right)=N$.

Suppose that $h \in \Delta_{\gamma}\left(\mathbb{R}_{+}\right)$for some $\gamma>1$ and $\Omega \in W \mathscr{F}_{\beta}\left(S^{n-1}\right)$ for some $\beta>\max \left\{2, \gamma^{\prime}\right\}$ satisfying (1). Then
(i) $T_{h, \Omega, \Gamma}$ is bounded on $\dot{F}_{\alpha}^{P, q}\left(\mathbb{R}^{n}\right)$ for $\alpha \in \mathbb{R}$ and $\max \{\mid 1 / p-$ $1 / 2|,|1 / q-1 / 2|\}<1 / \max \left\{2, \gamma^{\prime}\right\}-1 / \beta$;
(ii) $T_{h, \Omega, \Gamma}$ is bounded on $\dot{B}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)$ for $\alpha \in \mathbb{R}, q \in(1, \infty)$, and $|1 / p-1 / 2|<1 / \max \left\{2, \gamma^{\prime}\right\}-1 / \beta$.

It should be pointed out that there is a gap in the proof of part (i) in Theorem A. To the best of my knowledge, it is unknown whether Theorem A(i) holds. However, we can obtain the following result.

Theorem 4. Let $\Gamma(y)=\left(P_{1}(|y|) y_{1}^{\prime}, \ldots, P_{n}(|y|) y_{n}^{\prime}\right)$ with each $P_{j}$ being a real-valued polynomial on $\mathbb{R}$ satisfying $P_{j}(0)=0$. Suppose that $h \in \Delta_{\gamma}\left(\mathbb{R}_{+}\right)$for some $\gamma>1$ and $\Omega \in W \mathscr{F}_{\beta}\left(S^{n-1}\right)$ for some $\beta>\max \left\{2, \gamma^{\prime}\right\}$ satisfying (1).
(i) Then, for $\alpha \in \mathbb{R}$ and $(1 / p, 1 / q) \in \mathscr{R}_{\gamma, \beta}$, there exists a constant $C>0$ such that

$$
\begin{equation*}
\left\|T_{h, \Omega, \Gamma} f\right\|_{\dot{F}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)} \leq C\|f\|_{\dot{F}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)} \tag{17}
\end{equation*}
$$

for all $f \in \dot{F}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)$, where $C=C_{n, \alpha, p, q, \gamma, \beta}$ is independent of the coefficients of $\left\{P_{j}\right\}_{j=1}^{n}$. Here $\mathscr{R}_{\gamma, \beta}$ is the set of all interiors of the convex hull of three squares $\left(1 / 2,1 / 2+1 / \max \left\{2, \gamma^{\prime}\right\}-\right.$ $1 / \beta)^{2},\left(1 / 2-1 / \max \left\{2, \gamma^{\prime}\right\}+1 / \beta, 1 / 2\right)^{2}$ and $\left(\max \left\{2, \gamma^{\prime}\right\} / 2 \beta+\right.$ $(1 / 2 \gamma)\left(1-\max \left\{2, \gamma^{\prime}\right\} / \beta\right), \max \left\{2, \gamma^{\prime}\right\} / 2 \beta+(1-1 / 2 \gamma)(1-$ $\left.\left.\max \left\{2, \gamma^{\prime}\right\} / \beta\right)\right)^{2}$.
(ii) Then, for $\alpha \in \mathbb{R}, q \in(1, \infty)$, and $|1 / p-1 / 2|<$ $1 / \max \left\{2, \gamma^{\prime}\right\}-1 / \beta$, there exists a constant $C>0$ such that

$$
\begin{equation*}
\left\|T_{h, \Omega, \Gamma} f\right\|_{\dot{B}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)} \leq C\|f\|_{\dot{B}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)} \tag{18}
\end{equation*}
$$

for all $f \in \dot{B}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)$, where $C=C_{n, \alpha, p, q, \gamma, \beta}$ is independent of the coefficients of $\left\{P_{j}\right\}_{j=1}^{n}$.

Applying a switched method following from [12], Theorem 4 yields the following more general result.

Theorem 5. Let $\Gamma(y)=\left(P_{1}(\varphi(|y|)) y_{1}^{\prime}, \ldots, P_{n}(\varphi(|y|)) y_{n}^{\prime}\right)$ with each $P_{j}$ being a real-valued polynomial on $\mathbb{R}$ with $P_{j}(0)=0$ and $\varphi \in \mathscr{G}$. Here $\mathscr{G}$ is the set of all nonnegative (or nonpositive) and monotonic $\mathscr{C}^{1}\left(\mathbb{R}_{+}\right)$functions $\varphi$ satisfying $\Upsilon(t):=$ $\varphi(t) / t \varphi^{\prime}(t)$ with $|\Upsilon(t)| \leq C$, where $C>0$ depends only on $\varphi$. Suppose that $h \in \Delta_{\gamma}\left(\mathbb{R}_{+}\right)$for some $\gamma>1$ and $\Omega \in W \mathscr{F}_{\beta}\left(S^{n-1}\right)$ for some $\beta>\max \left\{2, \gamma^{\prime}\right\}$ satisfying (1). Let $\mathscr{R}_{\gamma, \beta}$ be given as in Theorem 4.
(i) Then, for $\alpha \in \mathbb{R}$ and $(1 / p, 1 / q) \in \mathscr{R}_{\gamma, \beta}$, there exists a constant $C>0$ such that

$$
\begin{equation*}
\left\|T_{h, \Omega, \Gamma} f\right\|_{\dot{F}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)} \leq C\|f\|_{\dot{F}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)} \tag{19}
\end{equation*}
$$

for all $f \in \dot{F}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)$, where $C=C_{n, \alpha, p, q, \gamma, \beta, \varphi}$ is independent of the coefficients of $\left\{P_{j}\right\}_{j=1}^{n}$.
(ii) Then, for $\alpha \in \mathbb{R}, q \in(1, \infty)$, and $|1 / p-1 / 2|<$ $1 / \max \left\{2, \gamma^{\prime}\right\}-1 / \beta$, there exists a constant $C>0$ such that

$$
\begin{equation*}
\left\|T_{h, \Omega, \Gamma} f\right\|_{\dot{B}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)} \leq C\|f\|_{\dot{B}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)} \tag{20}
\end{equation*}
$$

for all $f \in \dot{B}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)$, where $C=C_{n, \alpha, p, q, \gamma, \beta, \varphi}$ is independent of the coefficients of $\left\{P_{j}\right\}_{j=1}^{n}$.

Remark 6. (i) It is not difficult to see that $\mathscr{R}_{\gamma_{1}, \beta} \subsetneq \mathscr{R}_{\gamma_{2}, \beta}$ when $1<\gamma_{1}<\gamma_{2} \leq \infty$ and

$$
\begin{align*}
& \mathscr{R}_{\gamma, \beta} \subsetneq\left\{(p, q) ; \max \left\{\left|\frac{1}{p}-\frac{1}{2}\right|,\left|\frac{1}{q}-\frac{1}{2}\right|\right\}\right. \\
& \left.\quad<\frac{1}{\max \left\{2, \gamma^{\prime}\right\}}-\frac{1}{\beta}\right\} . \tag{21}
\end{align*}
$$

(ii) If $\varphi \in \mathscr{G}$, then $\lim _{t \rightarrow 0} \varphi(t)=0$ and $\lim _{t \rightarrow \infty}|\varphi(t)|=$ $\infty$ if $\varphi$ is nonnegative and increasing, or nonpositive and decreasing; $\lim _{t \rightarrow 0}|\varphi(t)|=\infty$ and $\lim _{t \rightarrow \infty} \varphi(t)=0$ if $\varphi$ is nonnegative and decreasing or nonpositive and increasing (see [12]).
(iii) It follows from Theorem 5 that $T_{h, \Omega, \Gamma}$ is bounded on $L^{p}\left(\mathbb{R}^{n}\right)$ for $\beta>\max \left\{2, \gamma^{\prime}\right\}$ and $|1 / p-1 / 2|<1 / \max \left\{2, \gamma^{\prime}\right\}-$ $1 / \beta$ under the same conditions of Theorem 5.

It is well known that the operators defined in (3) have their roots in the classical Marcinkiewicz integral operators $\mathscr{M}_{\Omega}$, corresponding to $\rho=1, h(t) \equiv 1$, and $\Gamma(y)=y$. The $L^{p}$ bounds for parametric Marcinkiewicz integrals have been extensively studied by many authors (see [13-15], etc.). In recent years, the investigation of boundedness for parametric Marcinkiewicz integral operators on the Triebel-Lizorkin spaces has also attracted the attention of many authors (see [16-19] for examples). Particularly, Yabuta [18] proved the following result.

Theorem B (see [18]). Let $\rho>0$ and $\Gamma(y)=\varphi(|y|) y^{\prime}$ with $\varphi \in \mathfrak{F}$, where $\mathfrak{F}$ is the set of all functions $\phi$ which satisfy the following conditions:
(a) $\phi$ is a positive increasing function on $\mathbb{R}_{+}$such that $t^{\delta} \phi^{\prime}(t)$ is monotonic on $\mathbb{R}_{+}$for some $\delta \in \mathbb{R}$;
(b) there exist positive constants $C_{\phi}$ and $c_{\phi}$ such that $t \phi^{\prime}(t) \geq C_{\phi} \phi(t)$ and $\phi(2 t) \leq c_{\phi} \phi(t)$ for all $t>0$.

Suppose that $h \in \Delta_{\gamma}\left(\mathbb{R}_{+}\right)$for some $\gamma>1$ and $\Omega \in$ $W \mathscr{F}_{\beta}\left(S^{n-1}\right)$ for some $\beta>\max \left\{2, \gamma^{\prime}\right\}$ satisfying (1). Let $\mathscr{R}_{\gamma, \beta}$ be given as in Theorem 4 . Then
(i) $\mathscr{M}_{h, \Omega, \Gamma, \rho}$ is bounded on $\dot{F}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)$ for $\alpha \in(0,1)$ and $(1 / p, 1 / q) \in \mathscr{R}_{\gamma, \beta} ;$
(ii) $\mathscr{M}_{h, \Omega, \Gamma, \rho}$ is bounded on $\dot{B}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)$ for $\alpha \in(0,1), q \in$ $(1, \infty)$, and $|1 / p-1 / 2|<1 / \max \left\{2, \gamma^{\prime}\right\}-1 / \beta$.

Remark 7. We notice that $\mathfrak{F} \subsetneq \mathscr{G}$. There are some examples for the class $\mathfrak{F}$, such as $t^{\alpha}(\alpha>0), t^{\beta} \ln (1+t)(\beta \geq 1)$, $t \ln \ln (e+t)$, and real-valued polynomials $P$ on $\mathbb{R}$ with positive coefficients and $P(0)=0$. It should be pointed out that there exists $B_{\varphi}>1$ such that $\varphi(2 t) \geq B_{\varphi} \varphi(t)$ for any $\varphi \in \mathfrak{F}$ (see [16]).

The second one of main results is listed as follows.
Theorem 8. Let $\Gamma(y)=\left(P_{1}(\varphi(|y|)) y_{1}^{\prime}, \ldots, P_{n}(\varphi(|y|)) y_{n}^{\prime}\right)$ with each $P_{j}$ being a real-valued polynomial on $\mathbb{R}$ satisfying $P_{j}(0)=$ 0 and $\varphi \in \mathfrak{F}$. Suppose that $h \in \Delta_{\gamma}\left(\mathbb{R}_{+}\right)$for some $\gamma>1$ and
$\Omega \in W \mathscr{F} \beta\left(S^{n-1}\right)$ for some $\beta>\max \left\{2, \gamma^{\prime}\right\}$ satisfying (1). Let $\mathscr{R}_{\gamma, \beta}$ be given as in Theorem 4.
(i) Then for $\alpha \in(0,1)$ and $(1 / p, 1 / q) \in \mathscr{R}_{\gamma, \beta}$, there exists a constant $C>0$ such that

$$
\begin{equation*}
\left\|\mathscr{M}_{h, \Omega, \Gamma, \rho} f\right\|_{\dot{F_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)}} \leq C\|f\|_{\dot{F}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)} \tag{22}
\end{equation*}
$$

for all $f \in \dot{F}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)$, where $C=C_{n, \zeta, \alpha, p, q, \gamma, \beta, \varphi}$ is independent of the coefficients of $\left\{P_{j}\right\}_{j=1}^{n}$.
(ii) $\mathscr{M}_{h, \Omega, \Gamma, \rho}$ is continuous from $F_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)$ to $\dot{F}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)$ for $\alpha \in(0,1)$ and $(1 / p, 1 / q) \in \mathscr{R}_{\gamma, \beta}$.
(iii) Then, for $\alpha \in(0,1), q \in(1, \infty)$, and $|1 / p-1 / 2|<$ $1 / \max \left\{2, \gamma^{\prime}\right\}-1 / \beta$, there exists a constant $C>0$ such that

$$
\begin{equation*}
\left\|\mathscr{M}_{h, \Omega, \Gamma, \rho} f\right\|_{\dot{B}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)} \leq C\|f\|_{\dot{B}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)} \tag{23}
\end{equation*}
$$

for all $f \in \dot{B}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)$, where $C=C_{n, \varsigma, \alpha, p, q, \gamma, \beta, \varphi, \varphi}$ is independent of the coefficients of $\left\{P_{j}\right\}_{j=1}^{n}$.
(iv) $\mathscr{M}_{h, \Omega, \Gamma, \rho}$ is continuous from $B_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)$ to $\dot{B}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)$ for $\alpha \in(0,1), q \in(1, \infty)$, and $|1 / p-1 / 2|<1 / \max \left\{2, \gamma^{\prime}\right\}-1 / \beta$.

Remark 9. Parts (i) and (iii) in Theorem 8 extend Theorem B , which corresponds to the case $P_{1}(t)=P_{2}(t)=\cdots=$ $P_{n}(t)=t$. Comparing with the singular integral operators, the continuity of the singular integral operators on the Triebel-Lizorkin spaces and Besov spaces can be obtained automatically by the corresponding boundedness since the singular integral operators are linear. However, the continuity of the Marcinkiewicz integral operators on the above function spaces is nontrivial. The reason for this is twofold. First, the Marcinkiewicz integral operators are not linear. Second, $f \leq$ $g$ can not imply $\|f\|_{\dot{F}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)} \leq\|g\|_{\dot{F}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)}$ and $\|f\|_{\dot{B}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)} \leq$ $\|g\|_{\dot{B}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)}$.

Remark 10. By employing the method in the proof of [20, Theorem 1.4] and applying some estimates about Fourier transforms of measures appearing in the proof of Theorem 8, one can obtain that $\mathscr{M}_{h, \Omega, \Gamma, \rho}$ is bounded on $L^{p}\left(\mathbb{R}^{n}\right)$ for $|1 / p-1 / 2|<1 / \max \left\{2, \gamma^{\prime}\right\}-1 / \beta$ if $h, \Omega, \Gamma, \beta$ are given as in Theorem 8.

The study of integral operators in form (4) is motivated by the early work of Fefferman on singular integral operators with rough kernels multiplied by bounded radials functions [6] and was introduced by Chen and Lin [21]. Recently, the Triebel-Lizorkin space and Besov space bounds for maximal operators have also been investigated by many authors. For example, see $[22,23]$ for the Hardy-Littlewood maximal operator and $[24,25]$ for the maximal functions related to rough singular integrals and Marcinkiewicz integrals. Motivated by the above works, we shall establish the following theorem.

Theorem 11. Let $\Gamma(y)=\left(P_{1}(\varphi(|y|)) y_{1}^{\prime}, \ldots, P_{n}(\varphi(|y|)) y_{n}^{\prime}\right)$ with each $P_{j}$ being a real-valued polynomial on $\mathbb{R}$ satisfying $P_{j}(0)=$ 0 and $\varphi \in \mathfrak{F}$. Suppose that $\Omega \in W \mathscr{F}_{\beta}\left(S^{n-1}\right)$ for some $\beta>2$ satisfying (1).
(i) Then, for $\alpha \in(0,1)$ and $(1 / p, 1 / q) \in \mathscr{G}_{\beta}$, there exists a constant $C>0$ such that

$$
\begin{equation*}
\left\|\mathcal{S}_{\Omega, \Gamma} f\right\|_{\dot{F}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)}+\left\|M_{\Omega, \Gamma, \rho} f\right\|_{\dot{F}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)} \leq C\|f\|_{\dot{F}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)} \tag{24}
\end{equation*}
$$

for all $f \in \dot{F}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)$, where $C=C_{n, \alpha, p, q, \beta, \varphi}$ is independent of the coefficients of $\left\{P_{j}\right\}_{j=1}^{n}$. Here $\mathscr{G}_{\beta}$ is the set of all interiors of the convex hull of two squares $(1 / \beta, 1 / 2)^{2}$ and $(1 / 2,1-1 / \beta)^{2}$.
(ii) $\mathcal{S}_{\Omega, \Gamma}$ and $M_{\Omega, \Gamma, \rho}$ are continuous from $F_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)$ to $\dot{F}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)$ for $\alpha \in(0,1), p \in[2, \beta)$, and $1 / q \in(1 / \beta, 1 / p+$ $1 / 2-1 / \beta)$.
(iii) Then, for $\alpha \in(0,1), p \in[2, \beta)$, and $q \in(1, \infty)$, there exists a constant $C>0$ such that

$$
\begin{equation*}
\left\|\mathcal{S}_{\Omega, \Gamma} f\right\|_{\dot{B}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)}+\left\|M_{\Omega, \Gamma, \rho} f\right\|_{\dot{B}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)} \leq C\|f\|_{\dot{B}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)} \tag{25}
\end{equation*}
$$

for all $f \in \dot{B}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)$, where $C=C_{n, \alpha, p, q, \beta, \varphi}$ is independent of the coefficients of $\left\{P_{j}\right\}_{j=1}^{n}$.
(iv) $\mathcal{S}_{\Omega, \Gamma}$ and $\mathscr{M}_{\Omega, \Gamma, \rho}$ are continuous from $B_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)$ to $\dot{B}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)$ for $\alpha \in(0,1), p \in[2, \beta)$, and $q \in(1, \infty)$.

Remark 12. Note that $\mathscr{G}_{\beta}=\mathscr{R}_{2, \beta}$. By using the estimates of measures appearing in the proof of Theorem 11 and the arguments similar to those used in deriving [7, Theorem 1.9], we can obtain that $S_{\Omega, \Gamma}$ is bounded on $L^{p}$ for $p \in[2, \beta)$ under the conditions of Theorem 11.

Applying (15)-(16), Theorems 4, 5, 8, and 11, and Remarks $6,9,10$, and 12 , we can obtain the following result immediately.

Theorem 13. Under the same conditions of Theorems 4, 5, 8, and 11 and Remarks 6, 9, 10, and 12 with $\alpha>0$, these operators are bounded and continuous on $F_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)$ and $B_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)$, respectively.

Due to the fact that $\mathscr{F}_{\beta}\left(\mathrm{S}^{1}\right) \subset W \mathscr{F}_{\beta}\left(\mathrm{S}^{1}\right)$, Theorem 13 may yield directly the following conclusion.

Theorem 14. Let $n=2$ and $\Gamma(y)=\left(P_{1}(\varphi(|y|)) y_{1}^{\prime}, \ldots\right.$, $\left.P_{2}(\varphi(|y|)) y_{n}^{\prime}\right)$ with each $P_{j}$ being a real-valued polynomial on $\mathbb{R}$ satisfying $P_{j}(0)=0$ and $\varphi \in \mathfrak{F}$.
(i) Ifh $\in \Delta_{\gamma}\left(\mathbb{R}_{+}\right)$for some $\gamma>1$ and $\Omega \in \mathscr{F}_{\beta}\left(\mathrm{S}^{1}\right)$ for some $\beta>\max \left\{2, \gamma^{\prime}\right\}$ satisfying (1), let $\mathscr{R}_{\gamma, \beta}$ be given as in Theorem 4. Then
(a) $T_{h, \Omega, \Gamma}$ is bounded and continuous on $F_{\alpha}^{p, q}\left(\mathbb{R}^{2}\right)$ for $\alpha \in$ $(0, \infty)$ and $(1 / p, 1 / q) \in \mathscr{R}_{\gamma, \beta} . T_{h, \Omega, \Gamma}$ is also bounded and continuous on $B_{\alpha}^{p, q}\left(\mathbb{R}^{2}\right)$ for $\alpha \in \mathbb{R}, q \in(1, \infty)$, and $\mid 1 / p-$ $1 / 2 \mid<1 / \max \left\{2, \gamma^{\prime}\right\}-1 / \beta$;
(b) $\mathscr{M}_{h, \Omega, \Gamma, \rho}$ is bounded and continuous on $F_{\alpha}^{p, q}\left(\mathbb{R}^{2}\right)$ for $\alpha \in(0,1)$ and $(1 / p, 1 / q) \in \mathscr{R}_{\gamma, \beta} . \mu_{h, \Omega, \Gamma, \rho}$ is also bounded and continuous on $B_{\alpha}^{p, q}\left(\mathbb{R}^{2}\right)$ for $\alpha \in(0,1), q \in(1, \infty)$, and $\mid 1 / p-$ $1 / 2 \mid<1 / \max \left\{2, \gamma^{\prime}\right\}-1 / \beta$.
(ii) If $\Omega \in \mathscr{F}_{\beta}\left(\mathrm{S}^{1}\right)$ for some $\beta>2$ satisfying (1), then $\mathcal{S}_{\Omega, \mathrm{\Gamma}}$ and $\mathscr{M}_{\Omega, \Gamma, \rho}$ are bounded and continuous on $F_{\alpha}^{p, q}\left(\mathbb{R}^{2}\right)$ for $\alpha \in$ $(0,1), p \in[2, \beta)$, and $1 / q \in(1 / \beta, 1 / p+1 / 2-1 / \beta) . \mathcal{S}_{\Omega, \Gamma}$ and $M_{\Omega, \Gamma, \rho}$ are also bounded and continuous on $B_{\alpha}^{p, q}\left(\mathbb{R}^{2}\right)$ for $\alpha \in(0,1), p \in[2, \beta)$, and $q \in(1, \infty)$.

The paper is organized as follows. Section 2 contains some known results, which play key roles in the proofs of main results. In Section 3, we will present some criterions on the boundedness and continuity of several operators on Triebel-Lizorkin spaces and Besov spaces, which are the main ingredients of our proofs. The proofs of main results will be given in Section 4. We remark that the methods employed in this paper follow from a combination of ideas and arguments in [10, 12, 17, 18, 23, 24, 26, 27], among others. It should be also point out that our methods can be used to deal with other integral operators, such as singular integrals, Marcinkiewicz integrals, and related maximal functions associated with other surfaces with other rough kernels.

Throughout the paper, we denote $p^{\prime}$ by the conjugate index of $p$, which satisfies $1 / p+1 / p^{\prime}=1$. The letter $C$ or $c$, sometimes with certain parameters, will stand for positive constants not necessarily the same one at each occurrence but are independent of the essential variables. In what follows, we set $\Re_{n}=\left\{\xi \in \mathbb{R}^{n}: 1 / 2<|\xi| \leq 1\right\}$. We denote by $\Delta_{\zeta}$ the difference of $f$ for an arbitrary function $f$ defined on $\mathbb{R}^{n}$ and $\zeta \in \mathbb{R}^{n}$; that is, $\Delta_{\zeta} f(x)=f(x+\zeta)-f(x)$. We also set $\sum_{j \in \emptyset} a_{j}=0$ and $\prod_{j \in \emptyset} a_{j}=1$.

## 2. Preliminary Lemmas

This section is devoted to recalling some known lemmas, which plays key roles in the proofs of main theorems. Let us begin with the following lemma of van der Corput type.

Lemma 15 (see [28]). Let $\Phi(t)=t^{\alpha_{1}}+\mu_{2} t^{\alpha_{2}}+\cdots+\mu_{n} t^{\alpha_{n}}$ and $\Psi \in C^{1}([0,1])$, where $\mu_{2}, \ldots, \mu_{n}$ are real parameters,
and $\alpha_{1}, \ldots, \alpha_{n}$ are distinct positive (not necessarily integer) exponents. Then, for $\lambda \neq 0$, the following holds:

$$
\begin{align*}
& \left|\int_{a}^{b} \exp (i \lambda \Phi(t)) \Psi(t) d t\right| \\
& \quad \leq C|\lambda|^{-\epsilon}\left\{\sup _{a \leq t \leq b}|\Psi(t)|+\int_{a}^{b}\left|\Psi^{\prime}(t)\right| d t\right\} \tag{26}
\end{align*}
$$

where $\epsilon=\min \left\{1 / \alpha_{1}, 1 / n\right\}$ and $C$ does not depend on $\mu_{2}, \ldots, \mu_{n}$ as long as $0 \leq a<b \leq 1$.

Applying Lemma 15 and the arguments similar to those used in the proof of [16, Lemma 2.2], we can obtain the following result.

Lemma 16. Let $\Phi(t)=t^{\alpha_{1}}+\mu_{2} t^{\alpha_{2}}+\cdots+\mu_{n} t^{\alpha_{n}}$, where $\mu_{2}, \ldots, \mu_{n}$ are real parameters, and $\alpha_{1}, \ldots, \alpha_{n}$ are distinct positive (not necessarily integer) exponents. Suppose that $\varphi \in \mathfrak{F}$ satisfying $t^{\delta} \varphi^{\prime}(t)$ is monotonic on $\mathbb{R}_{+}$for some $\delta \in \mathbb{R}$. Then, for any $r>0$ and $\lambda \neq 0$, the following holds:

$$
\begin{equation*}
\left|\int_{r / 2}^{r} \exp (i \lambda \Phi(\varphi(t))) \frac{d t}{t}\right| \leq C\left|\lambda \varphi(r)^{\alpha_{1}}\right|^{-\epsilon} \tag{27}
\end{equation*}
$$

with $\epsilon=\min \left\{1 / \alpha_{1}, 1 / n\right\}$, where $C$ is independent of $\mu_{2}, \ldots, \mu_{n}$, but may depend on $\varrho, \varphi$, and $\delta$.

Proof. By the change of variables, we have

$$
\begin{align*}
& \int_{r / 2}^{r} \exp (i \lambda \Phi(\varphi(t))) \frac{d t}{t}=\int_{\varphi(r / 2)}^{\varphi(r)} \exp (i \lambda \Phi(t)) \frac{d t}{\varphi^{\prime}\left(\varphi^{-1}(t)\right) \varphi^{-1}(t)} \\
& \quad=\int_{\varphi(r / 2)}^{\varphi(r)} \exp (i \lambda \Phi(t))\left(\varphi^{-1}(t)\right)^{\delta-1} \frac{d t}{\left(\varphi^{-1}(t)\right)^{\delta} \varphi^{\prime}\left(\varphi^{-1}(t)\right)}=\varphi(r)  \tag{28}\\
& \quad \cdot \int_{\varphi(r / 2) / \varphi(r)}^{1} \exp (i \lambda \Phi(\varphi(r) t))\left(\varphi^{-1}(\varphi(r) t)\right)^{\delta-1} \frac{d t}{\left(\varphi^{-1}(\varphi(r) t)\right)^{\delta} \varphi^{\prime}\left(\varphi^{-1}(\varphi(r) t)\right)} \\
& \quad=\varphi(r) \int_{\varsigma}^{1} \exp (i \lambda \Phi(\varphi(r) t)) g_{r, \varphi}(t) \psi(t) d t
\end{align*}
$$

where $\varsigma=\varphi(r / 2) / \varphi(r), \quad g_{r, \varphi}(t)=1 /$ $\left(\varphi^{-1}(\varphi(r) t)\right)^{\delta} \varphi^{\prime}\left(\varphi^{-1}(\varphi(r) t)\right)$, and $\psi(t)=\left(\varphi^{-1}(\varphi(r) t)\right)^{\delta-1}$. We can also write

$$
\begin{equation*}
\int_{r / 2}^{r} \exp (i \lambda \Phi(\varphi(t))) \frac{d t}{t}=\varphi(r) \int_{\varsigma}^{1} \psi(t) d J(t) \tag{29}
\end{equation*}
$$

where

$$
\begin{equation*}
J(t)=\int_{\varsigma}^{t} \exp (i \lambda \Phi(\varphi(r) s)) g_{r, \varphi}(s) d s, \quad \varsigma \leq t \leq 1 \tag{30}
\end{equation*}
$$

Since $g_{r, \varphi} \in C^{1}([0,1])$ is monotonic, applying Lemma 15, we obtain

$$
\begin{align*}
|J(t)| \leq & C\left|\lambda \varphi(r)^{\alpha_{1}}\right|^{-\epsilon} \\
& \cdot\left(\left(r^{\delta} \varphi^{\prime}(r)\right)^{-1}+\left(\left(\frac{r}{2}\right)^{\delta} \varphi^{\prime}\left(\frac{r}{2}\right)\right)^{-1}\right) \tag{31}
\end{align*}
$$

for $\varsigma \leq t \leq 1$, where $\epsilon=\min \left\{1 / \alpha_{1}, 1 / n\right\}$. By (29) and the integration by parts, we have

$$
\left|\int_{r / 2}^{r} \exp (i \lambda \Phi(\varphi(t))) \frac{d t}{t}\right| \leq \varphi(r)
$$

$$
\begin{align*}
& \left(|J(1) \psi(1)|+\int_{\varsigma}^{1}|J(t)|\left|\psi^{\prime}(t)\right| d t\right) \leq C \varphi(r) \\
& \cdot\left|\lambda \varphi(r)^{\alpha_{1}}\right|^{-\epsilon}\left(\left(r^{\delta} \varphi^{\prime}(r)\right)^{-1}+\left(\left(\frac{r}{2}\right)^{\delta} \varphi^{\prime}\left(\frac{r}{2}\right)\right)^{-1}\right) \\
& \cdot\left(r^{\delta-1}+\left(\frac{r}{2}\right)^{\delta-1}\right) \leq C(\varphi, \delta)\left|\lambda \varphi(r)^{\alpha_{1}}\right|^{-\epsilon} \tag{32}
\end{align*}
$$

This proves Lemma 16.
Lemmas 17-19 are known and will play key roles in the proofs of Theorems 4, 8 , and 11 , respectively.

Lemma 17 (see [26]). Let $\Gamma(y)=\left(P_{1}(|y|) a_{1}(y /|y|), \ldots\right.$, $\left.P_{n}(|y|) a_{n}(y| | y \mid)\right)$, where $P_{1}, \ldots, P_{n}$ are real-valued polynomials defined on $\mathbb{R}$ and $a_{1}, \ldots, a_{n}$ are arbitrary functions defined on $\mathrm{S}^{n-1}$. Suppose that $\Omega \in L^{1}\left(\mathrm{~S}^{n-1}\right)$ is a homogeneous function of degree zero and $h \in \Delta_{\gamma}\left(\mathbb{R}_{+}\right)$for some $\gamma>1$. Define the measures $\left\{\sigma_{k, \Gamma, \Omega}\right\}_{k \in \mathbb{Z}}$ by

$$
\begin{align*}
& \int_{\mathbb{R}^{n}} f(x) d \sigma_{k, \Gamma, \Omega}(x) \\
& \quad=\int_{2^{k} \leq|x|<2^{k+1}} f(\Gamma(x)) \frac{h(|x|) \Omega(x)}{|x|^{n}} d x \tag{33}
\end{align*}
$$

If $(1 / p, 1 / q)$ belongs to the interior of the convex hull of three squares $\left(1 / 2,1 / 2+1 / \max \left\{2, \gamma^{\prime}\right\}\right)^{2},\left(1 / 2-1 / \max \left\{2, \gamma^{\prime}\right\}, 1 / 2\right)^{2}$, and $(1 / 2 \gamma, 1-1 / 2 \gamma)^{2}$, then, for arbitrary functions $\left\{g_{k, j}\right\}_{k, j \in \mathbb{Z}} \in$ $L^{p}\left(\ell^{q}\left(\ell^{2}\right), \mathbb{R}^{n}\right)$, there exists $C>0$ independent of $\gamma$ such that

$$
\begin{align*}
& \left\|\left(\sum_{j \in \mathbb{Z}}\left(\sum_{k \in \mathbb{Z}}\left|\sigma_{k, \Gamma, \Omega} * g_{k, j}\right|^{2}\right)^{q / 2}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)} \\
& \leq C\|\Omega\|_{L^{1}\left(S^{n-1}\right)}\|h\|_{\Delta_{\gamma}\left(\mathbb{R}_{+}\right)}  \tag{34}\\
& \quad \cdot\left\|\left(\sum_{j \in \mathbb{Z}}\left(\sum_{k \in \mathbb{Z}}\left|g_{k, j}\right|^{2}\right)^{q / 2}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)}
\end{align*}
$$

The constant $C$ is independent of $\Omega$ and the coefficients of $\left\{P_{j}\right\}_{j=1}^{n}$.

Lemma 18 (see [17]). Let $\Gamma(y)=\left(P_{1}(\varphi(|y|)) a_{1}(y), \ldots\right.$, $\left.P_{n}(\varphi(|y|)) a_{n}(y)\right)$, where $\varphi \in \mathfrak{F}, P_{1}, \ldots, P_{n}$ are real-valued polynomials on $\mathbb{R}$ and $a_{1}(y), \ldots, a_{n}(y)$ are arbitrary functions independent of $|y|$. Define the family of measures $\left\{\left|\sigma_{t, \Gamma}\right|\right\}_{t \in \mathbb{R}_{+}}$on $\mathbb{R}^{n}$ by

$$
\begin{align*}
& \int_{\mathbb{R}^{n}} f(x) d\left|\sigma_{t, \Gamma}\right|(x) \\
& \quad=\frac{1}{t^{\rho}} \int_{t / 2<|x| \leq t} f(\Gamma(x)) \frac{|h(|x|) \Omega(x)|}{|x|^{n-\rho}} d x . \tag{35}
\end{align*}
$$

Suppose that $\in \Delta_{\gamma}\left(\mathbb{R}_{+}\right)$for some $\gamma>1$ and $\Omega \in L^{1}\left(S^{n-1}\right)$. If $(1 / p, 1 / q, 1 / r)$ belongs to the interior of the convex hull
of three cubes $\left(1 / 2,1 / 2+1 / \max \left\{2, \gamma^{\prime}\right\}\right)^{3},(1 / 2-1 / \max \{2$, $\left.\left.\gamma^{\prime}\right\}, 1 / 2\right)^{3}$, and $(1 / 2 \gamma, 1-1 / 2 \gamma)^{3}$, then, for arbitrary functions $\left\{g_{j \zeta, \zeta, k}\right\}_{j, \zeta, k} \in L^{p}\left(\ell^{q}\left(L^{r}\left(\ell^{2}\right)\right), \mathbb{R}^{n}\right)$, there exists $C>0$ such that

$$
\begin{align*}
& \left\|\left(\sum_{j \in \mathbb{Z}}\left(\int_{\Re_{n}}\left(\sum_{k \in \mathbb{Z}} \int_{2^{k}}^{2^{k+1}} \| \sigma_{t, \Gamma}\left|* g_{j, \zeta, k}\right|^{2} \frac{d t}{t}\right)^{1 / 2} d \zeta\right)^{q}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)} \\
& \left.\leq C\|\Omega\|_{L^{1}\left(S^{n-1}\right)}\left\|\left(\sum_{j \in \mathbb{Z}} \|\left(\sum_{k \in \mathbb{Z}} \mid g_{j, \zeta,,\left.\right|^{2}}\right)^{2}\right)^{1 / 2}\right\|_{L^{r}\left(\Re_{n}\right)}^{q}\right)^{1 / q} \|_{L^{p}\left(\mathbb{R}^{n}\right)} \tag{36}
\end{align*}
$$

The constant $C>0$ is independent of $\Omega$ and the coefficients of $\left\{P_{j}\right\}_{j=1}^{n}$.

Lemma 19 (see [24]). Let $\Gamma(y)=\left(P_{1}(\varphi(|y|)) a_{1}(y /|y|), \ldots\right.$, $\left.P_{n}(\varphi(|y|)) a_{n}(y /|y|)\right)$, where $\varphi \in \mathfrak{F}$ and $P_{1}, P_{2}, \ldots, P_{n}$ are real-valued polynomials on $\mathbb{R}_{+}$, and $a_{1}, a_{2}, \ldots, a_{n}$ are arbitrary functions defined on $\mathrm{S}^{n-1}$. Suppose that $\Omega \in L^{1}\left(\mathrm{~S}^{n-1}\right)$. Define the measures $\left\{\left|\sigma_{t, \Gamma}\right|\right\}_{t \in \mathbb{R}_{+}}$by

$$
\begin{equation*}
\widehat{\left|\sigma_{t, \Gamma}\right|}(x)=\int_{S^{n-1}} e^{-2 \pi i \Gamma\left(t y^{\prime}\right) \cdot x}\left|\Omega\left(y^{\prime}\right)\right| d \sigma\left(y^{\prime}\right) \tag{37}
\end{equation*}
$$

If $(1 / p, 1 / q, 1 / r)$ belongs to the interior of the convex hull of two cubes $(0,1 / 2)^{3}$ and $(1 / 2,1)^{3}$, then, for arbitrary functions $\left\{g_{j, \zeta, k}\right\}_{j, \zeta, k} \in L^{p}\left(\ell^{q}\left(L^{r}\left(\ell^{2}\right)\right), \mathbb{R}^{n}\right)$, there exists $C>0$ independent of $\Omega$ and the coefficients of $\left\{P_{j}\right\}_{j=1}^{n}$ such that

$$
\begin{align*}
& \left\|\left(\sum_{j \in \mathbb{Z}}\left(\int_{\mathfrak{R}_{n}}\left(\sum_{k \in \mathbb{Z}} \int_{2^{k}}^{2^{k+1}} \| \sigma_{t, \Gamma}\left|* g_{j, \zeta, k}\right|^{2} \frac{d t}{t}\right)^{1 / 2} d \zeta\right)^{q}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)}  \tag{38}\\
& \leq C\|\Omega\|_{L^{1}\left(S^{n-1}\right)}\left\|\left(\sum_{j \in \mathbb{Z}}\left\|\left(\sum_{k \in \mathbb{Z}}\left|g_{j, \zeta, k}\right|^{2}\right)^{1 / 2}\right\|_{L^{r}\left(\Re_{n}\right)}^{q}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)}
\end{align*}
$$

Below is the vector-valued inequality of the Hardy-Littlewood maximal functions, which is one of the main ingredients of our proofs.

Lemma 20 (see [16]). Let $M_{(n)}$ be the Hardy-Littlewood maximal operator on $\mathbb{R}^{n}$. Then

$$
\begin{align*}
& \left\|\left(\sum_{j \in \mathbb{Z}}\left\|\left(\sum_{k \in \mathbb{Z}}\left|M_{(n)} g_{j, \zeta, k}\right|^{2}\right)^{1 / 2}\right\|_{L^{r}\left(\Re_{n}\right)}^{q}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)}  \tag{39}\\
& \quad \leq C\left\|\left(\sum_{j \in \mathbb{Z}}\left\|\left(\sum_{k \in \mathbb{Z}}\left|g_{j, \zeta, k}\right|^{2}\right)^{1 / 2}\right\|_{L^{r}\left(\Re_{n}\right)}^{q}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)}
\end{align*}
$$

for any $1<p, q, r<\infty$.
In order to deal with Marcinkiewicz integrals and maximal functions, we need a useful characterization of TriebelLizorkin spaces and Besov spaces.

Lemma 21 (see [18]). Let $0<\alpha<1$.
(i) If $1<p<\infty, 1<q \leq \infty$, and $1 \leq r<\min \{p, q\}$, then
$\|f\|_{\dot{F}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)}$

$$
\begin{equation*}
\sim\left\|\left(\sum_{k \in \mathbb{Z}} 2^{k q \alpha}\left(\int_{\mathfrak{R}_{n}}\left|\Delta_{2^{-k} \zeta} f\right|^{r} d \zeta\right)^{q / r}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)} \tag{40}
\end{equation*}
$$

(ii) If $1 \leq p<\infty, 1 \leq q \leq \infty$, and $1 \leq r \leq p$, then
$\|f\|_{\dot{B}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)}$

$$
\begin{equation*}
\sim\left(\sum_{k \in \mathbb{Z}} 2^{k q \alpha}\left\|\left(\int_{\mathfrak{R}_{n}}\left|\triangle_{2^{-k} \zeta} f\right|^{r} d \zeta\right)^{1 / r}\right\|_{L^{p}\left(\mathbb{R}^{d}\right)}^{q}\right)^{1 / q} \tag{41}
\end{equation*}
$$

To prove Theorem 5, we need the following results.
Lemma 22 (see [12]). Let $\Upsilon, \varphi$ be given as in Theorem 5. Suppose that $h \in \Delta_{\gamma}\left(\mathbb{R}_{+}\right)$for some $\gamma>1$, then $h\left(\varphi^{-1}\right) \Upsilon\left(\varphi^{-1}\right) \in$ $\Delta_{\gamma}\left(\mathbb{R}_{+}\right)$.

The following lemma is a key switched result about singular integrals associated with compound surfaces.

Lemma 23 (see [29]). Let $\varphi \in \mathscr{G}$ and $\Upsilon$ be given as in Theorem 5. Let $T_{h, \Omega, \Gamma}$ be defined as in (2) and $\widetilde{\Omega}(y)=\Omega(-y)$. Define the operator $T_{h, \Omega, \Gamma, \varphi}$ by

$$
\begin{align*}
& T_{h, \Omega, \Gamma, \varphi} f(x) \\
& \quad=p \cdot v \cdot \int_{\mathbb{R}^{n}} f\left(x-\Gamma\left(\varphi(|y|) y^{\prime}\right)\right) \frac{h(|y|) \Omega(y)}{|y|^{n}} d y . \tag{42}
\end{align*}
$$

(i) If $\varphi$ is nonnegative and increasing, then $T_{h, \Omega, \Gamma, \varphi} f=$ $T_{h\left(\varphi^{-1}\right) \Upsilon\left(\varphi^{-1}\right), \Omega, \Gamma} f$.
(ii) If $\varphi$ is nonnegative and decreasing, then $T_{h, \Omega, \varphi} f=$ $-T_{h\left(\varphi^{-1}\right) \Upsilon\left(\varphi^{-1}\right), \Omega, \Gamma} f$.
(iii) If $\varphi$ is nonpositive and decreasing, then $T_{h, \Omega, \Gamma, \varphi} f=$ $T_{h\left(\varphi^{-1}\right) \curlyvee\left(\varphi^{-1}\right), \widetilde{\Omega}, \Gamma} f$.
(iv) If $\varphi$ is nonpositive and increasing, then $T_{h, \Omega, \Gamma, \varphi} f=$ $-T_{h\left(\varphi^{-1}\right) \Upsilon\left(\varphi^{-1}\right), \widetilde{\Omega}, \Gamma} f$.

## 3. Some Criterions

To prove Theorem 4, we need the following criterion on the boundedness of the convolution operators on TriebelLizorkin spaces, which is a variant of [15, Theorem 1.10]. This can be proved by making some minor modifications in the proof of [15, Theorem 1.10]. We omit the details.

Proposition 24. Let $l \in \mathbb{N} \backslash\{0\}$ and $\left\{\sigma_{s, k}: 0 \leq s \leq l\right.$ and $k \in$ $\mathbb{Z}\}$ be a family of measures on $\mathbb{R}^{n}$. For $1 \leq s \leq l$, let $\left\{a_{k, s}\right\}_{k \in \mathbb{Z}}$ be some sequences of positive real numbers with satisfying

$$
\begin{equation*}
\inf _{k \in \mathbb{Z}} \frac{a_{k+1, s}}{a_{k, s}} \geq \eta^{s} \tag{43}
\end{equation*}
$$

for some $\eta>1$. For $1 \leq s \leq l$, let $\ell_{s} \in \mathbb{N} \backslash\{0\}$ and $L_{s}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{\ell_{s}}$ be linear transformations. Suppose that there exists constants $C>0$ and $\beta>1$ such that
(i) $\sigma_{0, k}=0$ for every $k \in \mathbb{Z}$;
(ii) $\left\|\sigma_{s, k}\right\| \leq C$ for every $k \in \mathbb{Z}$ and $1 \leq s \leq l$;
(iii) $\left|\widehat{\sigma_{s, k}}(\xi)\right| \leq c \log \left(\left|a_{k, s} L_{s}(\xi)\right|\right)^{-\beta}$ if $\left|a_{k, s} L_{s}(\xi)\right|>1$ for $\xi \in \mathbb{R}^{n}, k \in \mathbb{Z}$, and $1 \leq s \leq l$;
(iv) $\left|\widehat{\sigma_{s, k}}(\xi)-\widehat{\sigma_{s-1, k}}(\xi)\right| \leq c\left|a_{k, s} L_{s}(\xi)\right|$ for $\xi \in \mathbb{R}^{n}, k \in \mathbb{Z}$, and $1 \leq s \leq l$;
(v) for any $1 \leq s \leq l$ and arbitrary functions $\left\{g_{k, j}\right\}_{k, j} \in$ $L^{p}\left(\ell^{q}\left(\ell^{2}\right), \mathbb{R}^{n}\right)$, there exists a positive constant $C$ which is independent of $\left\{L_{s}\right\}_{s=1}^{l}$ such that

$$
\begin{align*}
& \left\|\left(\sum_{j \in \mathbb{Z}}\left(\sum_{k \in \mathbb{Z}}\left|\sigma_{s, k} * g_{k, j}\right|^{2}\right)^{q / 2}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)}  \tag{44}\\
& \quad \leq C\left\|\left(\sum_{j \in \mathbb{Z}}\left(\sum_{k \in \mathbb{Z}}\left|g_{k, j}\right|^{2}\right)^{q / 2}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)}
\end{align*}
$$

for some $p_{0}, q_{0} \in(1, \infty)$ with $p_{0} \neq 2$ and $q_{0} \neq 2$.
Let $P_{1} P_{2}$ be the line segment from $P_{1}$ to $P_{2}$ with $P_{1}=$ $(1 / 2,1 / 2)$ and $P_{2}=\left(1 / 2 \beta+\left(1 / p_{0}\right)(1-1 / \beta), 1 / 2 \beta+\left(1 / q_{0}\right)(1-\right.$ $1 / \beta)$. Then there exists a positive constant $C$ such that

$$
\begin{equation*}
\left\|\sum_{k \in \mathbb{Z}} \sigma_{l, k} * f\right\|_{\dot{F}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)} \leq C\|f\|_{\dot{F}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)} \tag{45}
\end{equation*}
$$

holds for any $\alpha \in \mathbb{R}$ and $(1 / p, 1 / q) \in P_{1} P_{2} \backslash\left\{P_{2}\right\}$.
To establish the Triebel-Lizorkin space boundedness parts in Theorems 8 and 11, we will give the following lemma, which is the heart of the proofs of Theorems 8 and 11 .

Proposition 25. Let $\Lambda \in \mathbb{N} \backslash\{0\}$ and $\left\{\sigma_{s, t}: t>0,1 \leq s \leq\right.$ $\Lambda\}$ be a family of Borel measures on $\mathbb{R}^{n}$. Let $\left|\sigma_{s, t}\right|$ be the total variation of $\sigma_{s, t}$. For $1 \leq s \leq \Lambda$, let $\left\{a_{k, s}\right\}_{k \in \mathbb{Z}}$ be some sequences of positive real numbers with satisfying

$$
\begin{equation*}
\delta_{s} \geq \inf _{k \in \mathbb{Z}} \frac{a_{k+1, s}}{a_{k, s}} \geq \eta_{s}>1 \tag{46}
\end{equation*}
$$

for some $\eta_{s}, \delta_{s}>1$. For $1 \leq s \leq \Lambda$, let $\ell_{s} \in \mathbb{N} \backslash\{0\}$ and $L_{s}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{M_{s}}$ be linear transformations. Suppose that there exist $p_{0}, q_{0}>1,1<r_{0}<\min \left\{p_{0}, q_{0}\right\}, \beta>1$, and $C>0$ such that the following conditions hold for $1 \leq s \leq \Lambda, t>0, \xi \in \mathbb{R}^{n}$, and $\left\{g_{l, \zeta, k}\right\} \in L^{p_{0}}\left(\ell^{q_{0}}\left(L^{r_{0}}\left(\ell^{2}, \mathfrak{R}_{n}\right)\right), \mathbb{R}^{n}\right)$ :
(i) $\sigma_{0, t}=0$;
(ii) $\left(\int_{2^{k}}^{2^{k+1}}\left|\widehat{\sigma_{s, t}}(\xi)-\widehat{\sigma_{s-1, t}}(\xi)\right|^{2}(d t / t)\right)^{1 / 2} \leq C \min \{1$, $\left.\left|a_{k+1, s} L_{s}(\xi)\right|\right\} ;$
(iii) $\left(\int_{2^{k}}^{2^{k+1}}\left|\widehat{\sigma_{s, t}}(\xi)\right|(d t / t)\right)^{1 / 2} \leq C\left(\log \left|a_{k, s} L_{s}(\xi)\right|\right)^{-\beta}$ if $\left|a_{k, s} L_{s}(\xi)\right|>1 ;$
(iv)

$$
\begin{equation*}
\left\|\left(\sum_{l \in \mathbb{Z}}\left(\int_{\mathfrak{R}_{n}}\left(\sum_{k \in \mathbb{Z}} \int_{2^{k}}^{2^{k+1}} \| \sigma_{s, t}\left|* g_{l \zeta \zeta, k}\right|^{2} \frac{d t}{t}\right)^{1 / 2} d \zeta\right)^{q_{0}}\right)^{1 / q_{0}}\right\|_{L^{p_{0}\left(\mathbb{R}^{n}\right)}} \leq C\left\|\left(\sum_{l \in \mathbb{Z}}\left\|\left(\sum_{k \in \mathbb{Z}}\left|g_{l \zeta, k}\right|^{2}\right)^{1 / 2}\right\|_{L^{0}\left(\Re_{n}\right)}^{q_{0}}\right)^{1 / q_{0}}\right\|_{L^{p_{0}\left(\mathbb{R}^{n}\right)}} \tag{47}
\end{equation*}
$$

Let $P_{1} P_{2}$ be the line segment from $P_{1}$ to $P_{2}$ with $P_{1}=(1 / 2,1 / 2)$ and $P_{2}=\left(1 / 2 \beta+\left(1 / p_{0}\right)(1-1 / \beta), 1 / 2 \beta+\left(1 / q_{0}\right)(1-1 / \beta)\right)$. Then there exists a positive constant $C$ such that

$$
\begin{align*}
& \left\|\left(\sum_{l \in \mathbb{Z}} 2^{l q \alpha}\left(\int_{\mathfrak{R}_{n}}\left(\int_{0}^{\infty}\left|\sigma_{\Lambda, t} * \Delta_{2^{-l} \zeta} f\right|^{2} \frac{d t}{t}\right)^{1 / 2} d \zeta\right)^{q}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)}  \tag{48}\\
& \quad \leq C\|f\|_{\tilde{F}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)}
\end{align*}
$$

holds for any $\alpha \in(0,1)$ and $(1 / p, 1 / q) \in P_{1} P_{2} \backslash\left\{P_{2}\right\}$.
Proof. For any $1 \leq s \leq \Lambda$, let $l_{s}=\operatorname{rank}\left(L_{s}\right) \leq \min \left\{n, \ell_{s}\right\}$. By [5, Lemma 6.1], there are two nonsingular linear transformations $\mathscr{H}_{s}: \mathbb{R}^{l_{s}} \rightarrow \mathbb{R}^{l_{s}}$ and $\mathscr{G}_{s}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ such that

$$
\begin{equation*}
\left|\mathscr{H}_{s} \pi_{l_{s}}^{n} \mathscr{G}_{s} \xi\right| \leq\left|L_{s}(\xi)\right| \leq M_{s}\left|\mathscr{H}_{s} \pi_{l_{s}}^{n} \mathscr{G}_{s} \xi\right|, \quad \xi \in \mathbb{R}^{n} . \tag{49}
\end{equation*}
$$

For $t>0$ and $1 \leq s \leq \Lambda$, we define the family of measures $\left\{\tau_{s, t}\right\}_{t>0}$ by

$$
\begin{align*}
\widehat{\tau_{s, t}}(\xi)= & \widehat{\sigma_{s, t}}(\xi) \prod_{j=s+1}^{\Lambda} \psi\left(\left|a_{k, j} \mathscr{H}_{j} \pi_{l_{j}}^{n} \mathscr{G}_{j} \xi\right|\right) \\
& -\widehat{\sigma_{s-1, t}}(\xi) \prod_{j=s}^{\Lambda} \psi\left(\left|a_{k, j} \mathscr{H}_{j} \pi_{l_{j}}^{n} \mathscr{G}_{j} \xi\right|\right), \tag{50}
\end{align*}
$$

where $\psi \in \mathscr{C}_{0}^{\infty}(\mathbb{R})$ such that $\psi(t) \equiv 1$ for $|t| \leq 1 / 2$ and $\psi(t) \equiv$ 0 for $|t|>1$. Then (50) together with assumption (i) implies that

$$
\begin{equation*}
\sigma_{\Lambda, t}=\sum_{s=1}^{\Lambda} \tau_{s, t} . \tag{51}
\end{equation*}
$$

It follows that

$$
\begin{align*}
& \left\|\left(\sum_{l \in \mathbb{Z}} 2^{l q \alpha}\left(\int_{\mathfrak{R}_{n}}\left(\int_{0}^{\infty}\left|\sigma_{\Lambda, t} * \Delta_{2^{-l} \zeta} f\right|^{2} \frac{d t}{t}\right)^{1 / 2} d \zeta\right)^{q}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)}  \tag{52}\\
& \quad \leq \sum_{s=1}^{\Lambda}\left\|\left(\sum_{l \in \mathbb{Z}} 2^{l q \alpha}\left(\int_{\Re_{n}}\left(\int_{0}^{\infty}\left|\tau_{s, t} * \Delta_{2^{-l}} f\right|^{2} \frac{d t}{t}\right)^{1 / 2} d \zeta\right)^{q}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)} .
\end{align*}
$$

Therefore, to prove (48), it suffices to show that there exists $C>0$ such that

$$
\begin{align*}
& \left\|\left(\sum_{l \in \mathbb{Z}} 2^{l q \alpha}\left(\int_{\mathfrak{R}_{n}}\left(\int_{0}^{\infty}\left|\tau_{s, t} * \Delta_{2^{-l}} f\right|^{2} \frac{d t}{t}\right)^{1 / 2} d \zeta\right)^{q}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)}  \tag{53}\\
& \quad \leq C\|f\|_{F_{\alpha}^{p,}\left(\mathbb{R}^{n}\right)}
\end{align*}
$$

for any $1 \leq s \leq \Lambda, \alpha \in(0,1)$, and $(1 / p, 1 / q) \in P_{1} P_{2} \backslash\left\{P_{2}\right\}$.
We now prove (53). By our assumptions (ii)-(iii), we have

$$
\begin{aligned}
& \left(\int_{2^{k}}^{2^{k+1}}\left|\widehat{\tau_{s, t}}(\xi)\right| \frac{d t}{t}\right)^{1 / 2} \leq C \min \left\{1,\left|a_{k+1, s} L_{s}(\xi)\right|\right\} \\
& \left(\int_{2^{k}}^{2^{k+1}}\left|\widehat{\tau_{s, t}}(\xi)\right| \frac{d t}{t}\right)^{1 / 2} \leq C\left(\log \left|a_{k, s} L_{s}(\xi)\right|\right)^{-\beta} \\
& \text { if }\left|a_{k, s} L_{s}(\xi)\right|>M_{s}
\end{aligned}
$$

Let $\left\{v_{k, s}\right\}_{k \in \mathbb{Z}}$ be a collection of $\mathscr{C}^{\infty}$ functions on $(0, \infty)$ with the following properties:

$$
\begin{align*}
& \operatorname{supp}\left(v_{k, s}\right) \subset\left[a_{k+1, s}^{-1}, a_{k-1, s}^{-1}\right] \\
& \qquad 0 \leq v_{k, s}(t) \leq 1 ; \sum_{k \in \mathbb{Z}} v_{k, s}(t)=1 \tag{55}
\end{align*}
$$

Define the multiplier operator $S_{k, s}$ on $\mathbb{R}^{n}$ by

$$
\begin{equation*}
\widehat{S_{k, s} f}(\xi)=v_{k, s}\left(\left|\mathscr{H}_{s} \pi_{l_{s}}^{n} \mathscr{Q}_{s} \xi\right|\right) \widehat{f}(\xi) \tag{56}
\end{equation*}
$$

Note that $\delta_{s} \geq a_{k+1, s} / a_{k, s} \geq \eta_{s}>1$. By [16, Lemma 2.5] we obtain

$$
\begin{align*}
& \left\|\left(\sum_{l \in \mathbb{Z}}\left\|\left(\sum_{k \in \mathbb{Z}}\left|S_{k, s} f_{\zeta, l}\right|^{2}\right)^{1 / 2}\right\|_{L^{r}\left(\Re_{n}\right)}^{q}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)}  \tag{57}\\
& \quad \leq C\left\|\left(\sum_{l \in \mathbb{Z}}\left\|f_{\zeta, l}\right\|_{L^{r}\left(\Re_{n}\right)}^{q}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)}
\end{align*}
$$

By Minkowski's inequality we have

$$
\begin{align*}
& \left\|\left(\sum_{l \in \mathbb{Z}} 2^{l q \alpha}\left(\int_{\mathfrak{R}_{n}}\left(\int_{0}^{\infty}\left|\tau_{s, t} * \Delta_{2^{-l} \zeta} f\right|^{2} \frac{d t}{t}\right)^{1 / 2} d \zeta\right)^{q}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)} \\
& =\left\|\left(\sum_{l \in \mathbb{Z}} 2^{l q \alpha}\left(\int_{\mathfrak{R}_{n}}\left(\sum_{k \in \mathbb{Z}} \int_{2^{k}}^{2^{k+1}}\left|\tau_{s, t} * \sum_{j \in \mathbb{Z}} S_{j-k, s} \Delta_{2^{-l \zeta}} f\right|^{2} \frac{d t}{t}\right)^{1 / 2} d \zeta\right)^{q}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)}  \tag{58}\\
& \quad \leq \sum_{j \in \mathbb{Z}}\left\|\left(\sum_{l \in \mathbb{Z}} 2^{l q \alpha}\left(\int_{\mathfrak{R}_{n}}\left(\sum_{k \in \mathbb{Z}} \int_{2^{k}}^{2^{k+1}}\left|\tau_{s, t} * S_{j-k, s} \triangle_{2^{-l} \zeta} f\right|^{2} \frac{d t}{t}\right)^{1 / 2} d \zeta\right)^{q}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)}
\end{align*}
$$

Define the mixed norm $\|\cdot\|_{E_{\alpha}^{p, q}}$ for measurable functions on $\mathbb{R}^{n} \times \mathfrak{R}_{n} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{R}_{+}$by

$$
\begin{equation*}
\|g\|_{E_{\alpha}^{p, q}}:\left\|\left(\sum_{l \in \mathbb{Z}} 2^{l q \alpha}\left(\int_{\mathfrak{R}_{n}}\left(\sum_{k \in \mathbb{Z}} \int_{0}^{\infty}|g(x, \zeta, l, k, t)|^{2} \frac{d t}{t}\right)^{1 / 2} d \zeta\right)^{q}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)} \tag{59}
\end{equation*}
$$

For any $j \in \mathbb{Z}$, let Then we have

$$
\begin{align*}
& V_{j, s}(f)(x, \zeta, l, k, t)  \tag{60}\\
& \quad:=\tau_{s, t} * S_{j-k, s} \triangle_{2^{-l} \zeta} f(x) \chi_{\left[2^{k}, 2^{k+1}\right]}(t)
\end{align*}
$$

$$
\begin{equation*}
\left\|\left(\sum_{l \in \mathbb{Z}} 2^{l q \alpha}\left(\int_{\mathfrak{R}_{n}}\left(\sum_{k \in \mathbb{Z}} \int_{2^{k}}^{2^{k+1}}\left|\tau_{s, t} * \Delta_{2^{-} \zeta} f\right|^{2} \frac{d t}{t}\right)^{1 / 2} d \zeta\right)^{q}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)} \leq \sum_{j \in \mathbb{Z}}\left\|V_{j, s}(f)\right\|_{E_{\alpha}^{p, q}} \tag{61}
\end{equation*}
$$

By (54), Hölder's inequality, Minkowski's inequality, Fubini's theorem, Plancherel's theorem, and Lemma 21(ii), we have

$$
\begin{align*}
\left\|V_{j, s}(f)\right\|_{E_{\alpha}^{22}}^{2} & =\left\|\left(\sum_{l \in \mathbb{Z}} 2^{2 l \alpha}\left(\int_{\mathfrak{R}_{n}}\left(\sum_{k \in \mathbb{Z}} \int_{2^{k}}^{2^{k+1}}\left|\tau_{s, t} * S_{j-k, s} \Delta_{2^{-l} \zeta} f\right|^{2} \frac{d t}{t}\right)^{1 / 2} d \zeta\right)^{2}\right)^{1 / 2}\right\|^{2} \\
& =\int_{\mathbb{R}^{n}} \sum_{l \in \mathbb{Z}} 2^{2 l \alpha}\left(\int_{\mathfrak{R}_{n}\left(\mathbb{R}^{n}\right)}\left(\sum_{k \in \mathbb{Z}} \int_{2^{k}}^{2^{k+1}}\left|\tau_{s, t} * S_{j-k, s} \triangle_{2^{-l} \zeta} f(x)\right|^{2} \frac{d t}{t}\right)^{1 / 2} d \zeta\right)^{2} d x  \tag{62}\\
& \leq C \sum_{l \in \mathbb{Z}} 2^{2 l \alpha} \int_{\Re_{n}} \sum_{k \in \mathbb{Z}} \int_{2^{k}}^{2^{k+1}} \int_{\mathbb{R}^{n}}\left|\tau_{s, t} * S_{j-k, s} \triangle_{2^{-l} \zeta} f(x)\right|^{2} d x \frac{d t}{t} d \zeta \\
& \leq C \sum_{l \in \mathbb{Z}} 2^{2 l \alpha} \int_{\mathfrak{R}_{n}} \sum_{k \in \mathbb{Z}} \int_{E_{j-k, s}} \int_{2^{k}}^{2^{k+1}}\left|\widehat{\tau_{s, t}}(x)\right|^{2} \frac{d t}{t}\left|\widehat{\triangle_{2^{-l} \zeta} f}(x)\right|^{2} d x d \zeta \leq C B_{j, s}^{2}\|f\|_{\dot{B}_{\alpha}^{2,2}\left(\mathbb{R}^{n}\right)}^{2}
\end{align*}
$$

where $B_{j, s}=M_{s} \eta_{s}^{-j+1} \chi_{\left\{j \geq k_{s}\right\}}(j)+|j|^{-\beta} \chi_{\left\{j<k_{s}\right\}}(j)$ with $k_{s}=$ $\max \left\{k \in \mathbb{Z}: k<-1-\log _{\eta_{s}} M_{s}\right\}$ and

$$
\begin{equation*}
E_{j+k, s}=\left\{x \in \mathbb{R}^{n}: a_{j+k+1, s}^{-1} \leq\left|L_{s}(\xi)\right| \leq M_{s} a_{j+k-1, s}^{-1}\right\} . \tag{63}
\end{equation*}
$$

It follows from (62) and (14) that

$$
\begin{equation*}
\left\|V_{j, s}(f)\right\|_{E_{\alpha}^{2,2}} \leq C B_{j, s}\|f\|_{\dot{F}_{\alpha}^{2,2}\left(\mathbb{R}^{n}\right)} \tag{64}
\end{equation*}
$$

We now prove

$$
\begin{equation*}
\left\|V_{j, s}(f)\right\|_{E_{\alpha}^{p_{0}, q_{0}}} \leq C\|f\|_{\dot{F}_{\alpha}^{p_{0}, q_{0}}\left(\mathbb{R}^{n}\right)} \tag{65}
\end{equation*}
$$

For $1 \leq s \leq \Lambda$, let $\Phi^{s}$ be a radial function in $\mathcal{S}\left(\mathbb{R}^{l_{s}}\right)$ defined by $\widehat{\Phi^{s}}(x)=\psi(|x|)$, where $x \in \mathbb{R}^{l_{s}}$ and $\psi$ is given as in (50). Define $J_{s}$ and $X_{s}$ by

$$
\begin{align*}
J_{s} f(x) & :=f\left(\mathscr{G}_{s}^{t}\left(\mathscr{H}_{s}^{t} \otimes i d_{\mathbb{R}^{n-l_{s}}}\right) x\right), \\
X_{s} f(x) & =\sup _{k \in \mathbb{Z}} \sup _{t \in\left[2^{k}, 2^{k+1}\right]}\left|X_{k, t ; s} f(x)\right|, \tag{66}
\end{align*}
$$

where $X_{k, t ; s} f(x)=J_{s}^{-1}\left(\left(\Phi_{k, t ; s} \otimes \delta_{\mathbb{R}^{n-l_{s}}}\right) * J_{s} f\right)(x)$ and $\Phi_{k, t ; s}\left(x^{0}\right)$ $=\left(\varphi(t)^{\gamma_{s}}\right)^{-l_{s}} \Phi^{s}\left(\varphi(t)^{-\gamma_{s}} x^{0}\right)$ with $x^{0} \in \mathbb{R}^{l_{s}}$. It is easy to check that

$$
\begin{equation*}
\left|X_{s} f(x)\right| \leq C_{s}\left[J_{s}^{-1} \circ\left(M_{\left(l_{s}\right)} \otimes i d_{\mathbb{R}^{n-l_{s}}}\right) \circ J_{s}\right](f)(x), \tag{67}
\end{equation*}
$$

where $x=\left(x^{0}, x^{1}\right) \in \mathbb{R}^{l_{s}} \times \mathbb{R}^{n-l_{s}}$. (67) together with Lemma 20 yields that

$$
\begin{align*}
& \left\|\left(\sum_{l \in \mathbb{Z}}\left\|\left(\sum_{k \in \mathbb{Z}}\left|X_{s} g_{l, \zeta, k}\right|^{2}\right)^{1 / 2}\right\|_{L^{r}\left(\Re_{n}\right)}^{q}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)}^{p} \\
& \quad \leq C\left\|\left(\sum_{l \in \mathbb{Z}}\left\|\left(\sum_{k \in \mathbb{Z}}\left|\left[J_{s}^{-1} \circ\left(M_{\left(l_{s}\right)} \otimes i d_{\mathbb{R}^{n-l_{s}}}\right) \circ J_{s}\right]\left(g_{l, \zeta, k}\right)\right|^{2}\right)^{1 / 2}\right\|_{L^{r}\left(\Re_{n}\right)}^{q}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)}^{p} \leq C\left|J_{s}\right|  \tag{68}\\
& \quad \cdot \int_{\mathbb{R}^{n-l_{s}}} \int_{\mathbb{R}^{l_{s}}}\left(\sum_{l \in \mathbb{Z}}\left\|\left(\sum_{k \in \mathbb{Z}} M_{\left(l_{s}\right)}\left[\left(J_{s} g_{l, \zeta, k}\left(\cdot, x^{1}\right)\right)\right]\left(x^{0}\right)^{2}\right)^{1 / 2}\right\|_{L^{r}\left(\Re_{n}\right)}^{q}\right)^{p / q} d x^{0} d x^{1} \\
& \quad \leq C\left\|\left(\sum_{l \in \mathbb{Z}}\left\|\left(\sum_{k \in \mathbb{Z}}\left|g_{l, \zeta, k}\right|^{2}\right)^{1 / 2}\right\|_{L^{r}\left(\Re_{n}\right)}^{q}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)}^{p}
\end{align*}
$$

for any $1 \leq s \leq \Lambda$ and $1<p, q, r<\infty$. Define $X^{s} f=$ $X_{s} \circ X_{s+1} \circ \cdots \circ X_{\Lambda} f$ for $1 \leq s \leq \Lambda$. We get from (68) that, for any $1 \leq s \leq \Lambda$ and $1<p, q, r<\infty$,

$$
\begin{align*}
& \left\|\left(\sum_{l \in \mathbb{Z}}\left\|\left(\sum_{k \in \mathbb{Z}}\left|X^{s} g_{l, \zeta, k}\right|^{2}\right)^{1 / 2}\right\|_{L^{r}\left(\Re_{n}\right)}^{q}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)}^{p}  \tag{69}\\
& \quad \leq C\left\|\left(\sum_{l \in \mathbb{Z}}\left\|\left(\sum_{k \in \mathbb{Z}}\left|g_{l, \zeta, k}\right|^{2}\right)^{1 / 2}\right\|_{L^{r}\left(\Re_{n}\right)}^{q}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)}^{p}
\end{align*}
$$

On the other hand, by the definition of $X_{k, t ; s}$ we have

$$
\begin{aligned}
\tau_{s, t} * f= & \sigma_{s, t} *\left(X_{k, t ; s+1} \circ X_{k, t ; s+2} \circ \cdots \circ X_{k, t ; \Lambda} f\right) \\
& -\sigma_{s-1, t} *\left(X_{k, t ; s} \circ X_{k, t ; s+1} \circ \cdots \circ X_{k, t ; \Lambda} f\right) .
\end{aligned}
$$

It follows that

$$
\begin{align*}
& \int_{2^{k}}^{2^{k+1}}\left|\tau_{s, t} * f\right|^{2} \frac{d t}{t} \leq 2\left(\int_{2^{k}}^{2^{k+1}}| | \sigma_{s, t}\left|* X^{s+1} f\right|^{2} \frac{d t}{t}\right. \\
& \left.\quad+\int_{2^{k}}^{2^{k+1}}| | \sigma_{s-1, t}\left|* X^{s} f\right|^{2} \frac{d t}{t}\right) . \tag{71}
\end{align*}
$$

By (69), (71), and assumption (iv), we have

$$
\begin{align*}
& \left\|\left(\sum_{l \in \mathbb{Z}}\left(\int_{\Re_{n}}\left(\sum_{k \in \mathbb{Z}} \int_{2^{k}}^{2^{k+1}}\left|\tau_{s, t} * g_{l \zeta, k}\right|^{2} \frac{d t}{t}\right)^{1 / 2} d \zeta\right)^{q_{0}}\right)^{1 / q_{0}}\right\|_{L_{\rho_{0}\left(\mathbb{R}^{n}\right)}}  \tag{72}\\
& \leq C\left\|\left(\sum_{l \in \mathbb{Z}}\left\|\left(\sum_{k \in \mathbb{Z}}\left|g_{l \zeta, \zeta, k}\right|^{2}\right)^{1 / 2}\right\|_{L^{0}\left(\Re_{n}\right)}^{q_{0}}\right)^{1 / q_{0}}\right\|_{L_{0}\left(\mathbb{R}^{n}\right)}
\end{align*}
$$

for arbitrary functions $\left\{g_{l \zeta, \zeta, k}\right\} \in L^{p_{0}}\left(\ell^{q_{0}}\left(L^{r_{0}}\left(\ell^{2}, \Re_{d}\right)\right), \mathbb{R}^{n}\right)$ and $1 \leq s \leq \Lambda$. Then (72) together with Lemma 21(i) and (57) leads to

$$
\begin{aligned}
& \left\|V_{j, s}(f)\right\|_{E_{\alpha}^{p_{0} q_{0}}}=\|(\sum_{l \in \mathbb{Z}} 2^{l_{0} \alpha}(\int_{\Re_{n}}(\sum_{k \in \mathbb{Z}} \int_{2^{k}}^{2^{k+1}}|\tau_{s, t} * S_{j-k, s s^{-l} \underbrace{-l}} f|^{2} \frac{d t}{t})^{1 / 2} d \zeta)^{q_{0}})^{1 / q_{0}}\|_{L^{p_{0}\left(\mathbb{R}^{n}\right)}}
\end{aligned}
$$

$$
\begin{aligned}
& \leq C\left(\frac{B_{\varphi}^{\delta_{s}}}{B_{\varphi}^{\delta_{s}}-1}\right)^{n+2}\|f\|_{\dot{F}_{\alpha}^{p_{\alpha}, 0_{0}\left(\mathbb{R}^{n}\right)}} .
\end{aligned}
$$

This proves (65).
By the interpolation between (64) and (65), we obtain that, for $\alpha \in(0,1)$ and $(1 / p, 1 / q) \in P_{1} P_{2} \backslash\left\{P_{2}\right\}$, there exists $\theta \in(1 / \beta, 1]$ such that

$$
\begin{equation*}
\left\|V_{j, s}(f)\right\|_{E_{\alpha}^{p, q}} \leq C B_{j, s}^{\theta}\|f\|_{\dot{F}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)} \tag{74}
\end{equation*}
$$

Then, (74) together with (61) yields (48) and completes the proof of Proposition 25.

The following result is a criterion on the boundedness and continuity of several operators on Besov spaces, which can be used to prove the boundedness and continuity result on Besov spaces in Theorems 8 and 11.

Proposition 26 (see [23]). Assume that $T: L^{p}\left(\mathbb{R}^{n}\right) \rightarrow$ $L^{p}\left(\mathbb{R}^{n}\right)$ for some $p \in(1, \infty)$. If

$$
\begin{equation*}
\left|\Delta_{\zeta}(T f)(x)\right| \leq\left|T\left(\Delta_{\zeta}(f)\right)(x)\right| \tag{75}
\end{equation*}
$$

for any $x, \zeta \in \mathbb{R}^{n}$. Then $T$ is bounded on $\dot{B}_{s}^{p, q}\left(\mathbb{R}^{n}\right)$ for any $s \in$ $(0,1)$ and $q \in(1, \infty)$. Particularly, if $T$ satisfies

$$
\begin{equation*}
|T f-T g| \leq|T(f-g)| \tag{76}
\end{equation*}
$$

for arbitrary functions $f, g$ defined on $\mathbb{R}^{n}$, then $T$ is continuous from $B_{s}^{p, q}\left(\mathbb{R}^{n}\right)$ to $\dot{B}_{s}^{p, q}\left(\mathbb{R}^{n}\right)$ for any $s \in(0,1)$ and $q \in(1, \infty)$.

To establish the Triebel-Lizorkin space continuity parts in Theorems 8 and 11, we will give the following criterion of continuity for several sublinear operators on Triebel-Lizorkin spaces.

Proposition 27. Assume that $T$ is a sublinear operator and the following conditions hold.
(i) $T: L^{p}\left(\mathbb{R}^{d}\right) \rightarrow L^{p}\left(\mathbb{R}^{d}\right)$ for some $p \in(1, \infty)$.
(ii) For all $x, \zeta \in \mathbb{R}^{n}$, the following holds:

$$
\begin{equation*}
\left|\triangle_{\zeta}(T f)(x)\right| \leq\left|T\left(\triangle_{\zeta}(f)\right)(x)\right| . \tag{77}
\end{equation*}
$$

(iii) For arbitrary functions $f, g$ defined on $\mathbb{R}^{n}$, the following holds:

$$
\begin{equation*}
|T f-T g| \leq|T(f-g)| \tag{78}
\end{equation*}
$$

(iv) There exist $\alpha \in(0,1)$ and $q \in(1, \infty)$ such that

$$
\begin{align*}
& \left\|\left(\sum_{l \in \mathbb{Z}} 2^{l q \alpha}\left(\int_{\mathfrak{R}_{n}}\left|T\left(\Delta_{2^{-l} \zeta} f\right)\right| d \zeta\right)^{q}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)}  \tag{79}\\
& \quad \leq C\|f\|_{\dot{F}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)} .
\end{align*}
$$

Then $T$ is continuous from $F_{s}^{p, q}\left(\mathbb{R}^{n}\right)$ to $\dot{F}_{s}^{p, q}\left(\mathbb{R}^{n}\right)$.
Proof. Let $f_{j} \rightarrow f$ in $F_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)$ as $j \rightarrow \infty$. By (15) we see that $f_{j} \rightarrow f$ in $\dot{F}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)$ and in $L^{p}\left(\mathbb{R}^{n}\right)$ as $j \rightarrow \infty$. Since $\left\|f_{j}-f\right\|_{L^{p}\left(\mathbb{R}^{n}\right)} \rightarrow 0$ as $j \rightarrow \infty$, by assumptions (i) and (iii) we obtain that $T f_{j} \rightarrow T f$ in $L^{p}\left(\mathbb{R}^{n}\right)$ as $j \rightarrow \infty$. Therefore, it suffices to show that $T f_{j} \rightarrow T f$ in $\dot{F}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)$ as $j \rightarrow \infty$.

We shall prove this claim by contradiction. Without loss of generality we may assume that there exists $c>0$ such that

$$
\begin{equation*}
\left\|T f_{j}-T f\right\|_{\dot{F}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)}>c \tag{80}
\end{equation*}
$$

for every $j$. Since $T f_{j} \rightarrow T f$ in $L^{p}\left(\mathbb{R}^{n}\right)$ as $j \rightarrow \infty$, by extracting a subsequence we may assume that $\left|T f_{j}(x)-T f(x)\right| \rightarrow 0$ as $j \rightarrow \infty$ for almost every $x \in \mathbb{R}^{n}$. It follows that $\Delta_{2^{-l \zeta}}\left(T f_{j}\right.$ $-T f)(x) \rightarrow 0$ as $j \rightarrow \infty$ for every $(l, \zeta) \in \mathbb{Z} \times \mathfrak{R}_{n}$ and almost every $x \in \mathbb{R}^{n}$. We get from assumption (ii) and the sublinearity of $T$ that

$$
\begin{align*}
& \left|\triangle_{2^{-l} \zeta}\left(T f_{j}-T f\right)(x)\right|  \tag{81}\\
& \quad \leq 2 T\left(\triangle_{2^{-l} \zeta} f\right)(x)+T\left(\triangle_{2^{-l} \zeta}\left(f_{j}-f\right)\right)(x)
\end{align*}
$$

for $(x, l, \zeta) \in \mathbb{R}^{n} \times \mathbb{Z} \times \mathfrak{R}_{n}$. For convenience, we set

$$
\begin{equation*}
\|g\|_{p, q, \alpha}:=\left\|\left(\sum_{l \in \mathbb{Z}} 2^{l q \alpha}\left(\int_{\mathfrak{R}_{n}}|g(x, l, \zeta)| d \zeta\right)^{q}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)} \tag{82}
\end{equation*}
$$

for $\alpha \in \mathbb{R}$ and $(p, q) \in(1, \infty)^{2}$. It follows from Lemma 21(i) that $\|f\|_{\dot{F}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)} \sim\left\|\Delta_{2^{-l} \zeta} f\right\|_{p, q, \alpha}$ for $\alpha \in(0,1)$ and $(p, q) \in$ $(1, \infty)^{2}$. By assumption (iv) we obtain

$$
\left\|T\left(\triangle_{2^{-l \zeta}} f\right)\right\|_{p, q, \alpha}
$$

$$
\begin{align*}
& \leq C\left\|\left(\sum_{l \in \mathbb{Z}} 2^{l q_{\alpha}}\left(\int_{\mathfrak{R}_{n}}\left|T\left(\Delta_{2^{-}-\zeta} f\right)\right| d \zeta\right)^{q}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)} \\
& \leq C\|f\|_{\dot{F}_{\alpha}^{p q}\left(\mathbb{R}^{n}\right)} \tag{83}
\end{align*}
$$

It follows that $\left\|T\left(\triangle_{2^{-l \zeta}}\left(f_{j}-f\right)\right)\right\|_{p, q, \alpha} \leqslant\left\|f_{j}-f\right\|_{\dot{F}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)} \rightarrow$ 0 as $j \rightarrow \infty$. One can extract a subsequence such that $\sum_{j=1}^{\infty}\left\|T\left(\triangle_{2^{-l} \zeta}\left(f_{j}-f\right)\right)\right\|_{p, q, \alpha}<\infty$. Define a function $G$ : $\mathbb{R}^{n} \times \mathbb{Z} \times \mathfrak{R}_{n} \rightarrow \mathbb{R}$ by

$$
\begin{align*}
G(x, l, \zeta)= & \sum_{j=1}^{\infty} T\left(\triangle_{2^{-l} \zeta}\left(f_{j}-f\right)\right)(x)  \tag{84}\\
& +2 T\left(\triangle_{2^{-l} \zeta} f\right)(x) .
\end{align*}
$$

One can easily check that $\|G\|_{p, q, \alpha}<\infty$ and

$$
\begin{align*}
& \left|\triangle_{2^{-l} \zeta}\left(T f_{j}-T f\right)(x)\right| \leq G(x, l, \zeta)  \tag{85}\\
& \text { for almost every }(x, l, \zeta) \in \mathbb{R}^{n} \times \mathbb{Z} \times \mathfrak{R}_{n}
\end{align*}
$$

Since $\|G\|_{p, q, \alpha}<\infty$, we have that $\int_{\mathfrak{R}_{d}} G(x, k, \zeta) d \zeta<\infty$ for every $k \in \mathbb{Z}$ and almost every $x \in \mathbb{R}^{n}$. (85) together with the dominated convergence theorem leads to

$$
\begin{equation*}
\int_{\mathfrak{R}_{n}}\left|\triangle_{2^{-l}}\left(T f_{j}-T f\right)(x)\right| d \zeta \longrightarrow 0 \quad \text { as } j \longrightarrow \infty \tag{86}
\end{equation*}
$$

for every $l \in \mathbb{Z}$ and almost every $x \in \mathbb{R}^{n}$. By the fact $\|G\|_{p, q, \alpha}<$ $\infty$ again, we have

$$
\begin{equation*}
\left(\sum_{l \in \mathbb{Z}} 2^{l q \alpha}\left(\int_{\mathfrak{R}_{n}} G(x, l, \zeta) d \zeta\right)^{q}\right)^{1 / q}<\infty \tag{87}
\end{equation*}
$$

for almost every $x \in \mathbb{R}^{n}$. Using (85) we obtain

$$
\begin{equation*}
\int_{\mathfrak{R}_{n}}\left|\triangle_{2^{-l \zeta}}\left(T f_{j}-T f\right)(x)\right| d \zeta \leq \int_{\Re_{n}} G(x, l, \zeta) d \zeta \tag{88}
\end{equation*}
$$

for almost every $x \in \mathbb{R}^{d}$ and $l \in \mathbb{Z}$. It follows from (86)-(88) and the dominated convergence theorem that

$$
\begin{align*}
&\left(\sum_{l \in \mathbb{Z}} 2^{l q \alpha}\left(\int_{\mathfrak{R}_{n}}\left|\triangle_{2^{-l} \zeta}\left(T f_{j}-T f\right)(x)\right| d \zeta\right)^{q}\right)^{1 / q} \longrightarrow 0  \tag{89}\\
& \text { as } j \longrightarrow \infty
\end{align*}
$$

for almost every $x \in \mathbb{R}^{n}$. By (85) again, it holds that

$$
\begin{align*}
& \left(\sum_{l \in \mathbb{Z}} 2^{l q \alpha}\left(\int_{\mathfrak{R}_{n}}\left|\triangle_{2^{-l} \zeta}\left(T f_{j}-T f\right)(x)\right| d \zeta\right)^{q}\right)^{1 / q} \\
& \quad \leq\left(\sum_{l \in \mathbb{Z}} 2^{l q \alpha}\left(\int_{\Re_{n}}|G(x, l, \zeta)| d \zeta\right)^{q}\right)^{1 / q} \tag{90}
\end{align*}
$$

for almost every $x \in \mathbb{R}^{n}$. By (89)-(90), the fact $\|G\|_{p, q, \alpha}<\infty$, and the dominated convergence theorem, we obtain

$$
\begin{equation*}
\left\|\Delta_{2^{-l} \zeta}\left(T f_{j}-T f\right)\right\|_{p, q, \alpha} \longrightarrow 0 \quad \text { as } j \longrightarrow \infty . \tag{91}
\end{equation*}
$$

This yields that $\left\|T f_{j}-T f\right\|_{\dot{F}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)} \rightarrow 0$ as $j \rightarrow \infty$ and gives a contradiction.

## 4. Proofs of Theorems 4, 5, and 11

In this section we shall prove Theorems 4-11. In what follows, let $\operatorname{deg}(P)=\max _{1 \leq j \leq n} \operatorname{deg}\left(P_{j}\right)$. For $1 \leq j \leq n$, we set $P_{l}(t)=$ $\sum_{i=1}^{\operatorname{deg}(P)} a_{i, j} t^{i}$. Then there are integers $0<l_{1}<l_{2}<\cdots<l_{\Lambda} \leq$ $\operatorname{deg}(P)$ such that $P_{j}(t)=\sum_{i=1}^{\Lambda} a_{l_{i}, j} t^{l_{i}}$ for any $1 \leq j \leq n$ and $\left(a_{l_{i}, 1}, a_{l_{i}, 2}, \ldots, a_{l_{i}, n}\right) \neq(0,0, \ldots, 0) \in \mathbb{R}^{n}$ for all $1 \leq i \leq \Lambda$. For $1 \leq j \leq n$ and $0 \leq s \leq \Lambda$, we set $P_{j}^{(s)}(t)=\sum_{i=1}^{s} a_{l_{i}, j} t^{l_{i}}$ for $1 \leq s \leq \Lambda$ and $P_{j}^{(0)}(t)=(0, \ldots, 0)$. For $1 \leq s \leq \Lambda$, we define the linear transformation $L_{i}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ by

$$
\begin{equation*}
L_{i}(\xi)=\left(a_{l_{i}, 1} \xi_{1}, a_{l_{i}, 2} \xi_{2}, \ldots, a_{l_{i}, n} \xi_{n}\right) \tag{92}
\end{equation*}
$$

We now turn to prove Theorems 4, 5, and 11
Proof of Theorem 4. Define $\left\{\Phi_{s}\right\}_{s=0}^{\Lambda}$ by

$$
\begin{equation*}
\Phi_{s}(y)=\left(P_{1}^{(s)}(|y|) y_{1}^{\prime}, \ldots, P_{n}^{(s)}(|y|) y_{n}^{\prime}\right), \tag{93}
\end{equation*}
$$

$$
0 \leq s \leq \Lambda
$$

It is clear to see that

$$
\begin{equation*}
\Phi_{s}(x) \cdot \xi=\sum_{j=1}^{n} P_{j}^{(s)}(|x|) x^{\prime} \cdot \xi_{l}=\sum_{i=1}^{s}|x|^{l_{i}}\left(L_{i}(\xi) \cdot x^{\prime}\right) \tag{94}
\end{equation*}
$$

for any $x, \xi \in \mathbb{R}^{n}$ and $1 \leq s \leq \Lambda$. For $0 \leq s \leq \Lambda$ and $\xi \in \mathbb{R}^{n}$, we define the measures $\left\{\sigma_{k, s}\right\}_{k \in \mathbb{Z}}$ by

$$
\begin{equation*}
\widehat{\sigma_{k, s}}(\xi)=\int_{2^{k}<|y| \leq 2^{k+1}} e^{-2 \pi i \Phi_{s}(|y|) y^{\prime} \cdot \xi} \frac{\Omega(y) h(|y|)}{|y|^{n}} d y . \tag{95}
\end{equation*}
$$

It is clear that

$$
\begin{equation*}
T_{h, \Omega, \Gamma} f=\sum_{k \in \mathbb{Z}} \sigma_{k, \Lambda} * f \tag{96}
\end{equation*}
$$

By the change of the variables, we have

$$
\begin{align*}
& \left|\widehat{\sigma_{k, s}}(\xi)-\widehat{\sigma_{k, s-1}}(\xi)\right|=\mid \int_{2^{k}}^{2^{k+1}} \int_{S^{n-1}} \Omega\left(y^{\prime}\right) \\
& \quad \cdot\left(e^{-2 \pi i \Phi_{s}(t) y^{\prime} \cdot \xi}-e^{-2 \pi i \Phi_{s-1}(t) y^{\prime} \cdot \xi}\right) d \sigma\left(y^{\prime}\right)  \tag{97}\\
& \left.\quad \cdot h(t) \frac{d t}{t} \right\rvert\, \leq C\|\Omega\|_{L^{1}\left(S^{n-1}\right)}\|h\|_{\Delta_{\gamma}\left(\mathbb{R}_{+}\right)} \\
& \quad \cdot\left|2^{(k+1) s} L_{s}(\xi)\right|
\end{align*}
$$

On the other hand, it is easy to check that

$$
\left\|\sigma_{k, s}\right\| \leq C\|\Omega\|_{L^{1}\left(S^{n-1}\right)}\|h\|_{\Delta_{\gamma}\left(\mathbb{R}_{+}\right)}
$$

$$
\begin{equation*}
\sigma_{k, 0}=0 \tag{98}
\end{equation*}
$$

By the change of the variables and Hölder's inequality, we have

$$
\begin{align*}
& \left|\widehat{\sigma_{k, s}}(\xi)\right|=\left|\int_{2^{k}}^{2^{k+1}} \int_{S^{n-1}} \Omega\left(y^{\prime}\right) e^{-2 \pi i \Phi_{s}(t) y^{\prime} \cdot \xi} d \sigma\left(y^{\prime}\right) h(t) \frac{d t}{t}\right| \leq C\|h\|_{\Delta_{y}\left(\mathbb{R}_{+}\right)} \\
& \quad \cdot\left(\int_{2^{k}}^{2^{k+1}}\left|\int_{S^{n-1}} \Omega\left(y^{\prime}\right) e^{-2 \pi i \Phi_{s}(t) y^{\prime} \cdot \xi} d \sigma\left(y^{\prime}\right)\right|^{y^{\prime}} \frac{d t}{t}\right)^{1 / \gamma^{\prime}} \leq C\left(\int_{2^{k}}^{2^{k+1}}\left|\int_{S^{n-1}} \Omega\left(y^{\prime}\right) e^{-2 \pi i \Phi_{s}(t)^{\prime} \xi} d \sigma\left(y^{\prime}\right)\right|^{2} \frac{d t}{t}\right)^{1 / \max \left\{2, y^{\prime}\right\}}  \tag{99}\\
& \quad \leq C\left(\int_{2^{k}}^{k^{k+1}} \iint_{S^{n-1} \times S^{n-1}} \Omega\left(y^{\prime}\right) \overline{\Omega(\theta)} e^{-2 \pi i \Phi_{s}(t)\left(y^{\prime}-\theta\right) \cdot \xi} d \sigma\left(y^{\prime}\right) d \sigma(\theta) \frac{d t}{t}\right)^{1 / \max \left\{2, \gamma^{\prime}\right\}} \leq C\left(\iint_{S^{n-1} \times s^{n-1}} H_{k}\left(\xi, y^{\prime}, \theta\right) \Omega\left(y^{\prime}\right)\right. \\
& \left.\quad \cdot \overline{\Omega(\theta)} d \sigma\left(y^{\prime}\right) d \sigma(\theta)\right)^{1 / \max \left\{2, \gamma^{\prime}\right\}},
\end{align*}
$$

where

$$
\begin{equation*}
H_{k}\left(\xi, y^{\prime}, \theta\right)=\int_{2^{k}}^{2^{k+1}} e^{-2 \pi i \Phi_{s}(t)\left(y^{\prime}-\theta\right) \cdot \xi} \frac{d t}{t} \tag{100}
\end{equation*}
$$

By the Van der Corput lemma, there exists a constant $C>0$, such that

$$
\begin{align*}
& \left|H_{k}\left(\xi, y^{\prime}, \theta\right)\right| \\
& \quad \leq C \min \left\{1,\left|2^{(k+1) s} L_{s}(\xi) \cdot\left(y^{\prime}-\theta\right)\right|^{-1 / s}\right\} \tag{101}
\end{align*}
$$

When $\left|2^{(k+1) s} L_{s} \xi\right|>1$, since $t /(\log t)^{\beta}$ is increasing in $\left(e^{\beta}\right.$, $\infty$ ), we have

$$
\begin{equation*}
\left|H_{k}\left(\xi, y^{\prime}, \theta\right)\right| \leq C \frac{\left(\log 2 e^{\beta \lambda}\left|\eta \cdot\left(y^{\prime}-\theta\right)\right|^{-1}\right)^{\beta}}{\left(\log \left|2^{(k+1) s} L_{s}(\xi)\right|\right)^{\beta}} \tag{102}
\end{equation*}
$$

where $\eta=L_{s}(\xi) /\left|L_{s}(\xi)\right|$. Combining (99), (102) with the fact that $\Omega \in W \mathscr{F}_{\beta}\left(S^{n-1}\right)$ yields that

$$
\begin{align*}
\left|\widehat{\sigma_{k, s}}(\xi)\right| \leq C\left(\log \left|2^{(k+1) s} L_{s}(\xi)\right|\right)^{-\beta / \max \left\{2, \gamma^{\prime}\right\}}  \tag{103}\\
\quad \text { if }\left|2^{(k+1) s} L_{s}(\xi)\right|>1
\end{align*}
$$

On the other hand, Lemma 17 yields that

$$
\begin{align*}
& \left\|\left(\sum_{j \in \mathbb{Z}}\left(\sum_{k \in \mathbb{Z}}\left|\sigma_{k, s} * g_{k, j}\right|^{2}\right)^{q / 2}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)} \\
& \quad \leq C\left\|\left(\sum_{j \in \mathbb{Z}}\left(\sum_{k \in \mathbb{Z}}\left|g_{k, j}\right|^{2}\right)^{q / 2}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)} \tag{104}
\end{align*}
$$

for $(1 / p, 1 / q)$ belonging to the interior of the convex hull of three squares $\left(1 / 2,1 / 2+1 / \max \left\{2, \gamma^{\prime}\right\}\right)^{2},(1 / 2-1 / \max \{2$, $\left.\left.\gamma^{\prime}\right\}, 1 / 2\right)^{2}$, and $(1 / 2 \gamma, 1-1 / 2 \gamma)^{2}$. Here $C>0$ is independent of the coefficients of $\left\{P_{j}\right\}_{j=1}^{n}$.

Take $a_{k, s}=2^{k s}$. By (96)-(98), (103)-(104) and Proposition 24, we obtain

$$
\begin{equation*}
\left\|T_{h, \Omega, \Gamma} f\right\|_{\dot{F}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)} \leq C\|f\|_{\dot{F}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)} \tag{105}
\end{equation*}
$$

for $\beta>\max \left\{2, \gamma^{\prime}\right\}, \alpha \in \mathbb{R}$, and all $(1 / p, 1 / q) \in \mathscr{R}_{\gamma, \beta}$, where $\mathscr{R}_{\gamma, \beta}$ is given as in Theorem 4. This proves Theorem 4(i).

On the other hand, it follows from Theorem 4(i) and (13) that $T_{h, \Omega, \Gamma}$ is bounded on $L^{p}\left(\mathbb{R}^{n}\right)$ for $\beta>\max \left\{2, \gamma^{\prime}\right\}$ and $|1 / p-1 / 2|<1 / \max \left\{2, \gamma^{\prime}\right\}-1 / \beta$. This together with the arguments similar to those used in deriving [30, Theorem 1.2] yields Theorem 4(ii).
Proof of Theorem 5. Theorem 5 follows from Theorem 4 and Lemmas 22 and 23.

Proof of Theorem 8. Define $\Phi_{0}, \Phi_{1}, \ldots, \Phi_{\Lambda}$ by

$$
\begin{align*}
& \Phi_{s}(y)=\left(P_{1}^{(s)}(\varphi(|y|)) y_{1}^{\prime}, \ldots, P_{n}^{(s)}(\varphi(|y|)) y_{n}^{\prime}\right)  \tag{106}\\
& 0 \leq s \leq \Lambda .
\end{align*}
$$

## Clearly,

$$
\begin{align*}
\Phi_{s}(x) \cdot \xi & =\sum_{j=1}^{n} P_{j}^{(s)}(\varphi(|x|)) x^{\prime} \cdot \xi_{l}  \tag{107}\\
& =\sum_{i=1}^{s} \varphi(|x|)^{l_{i}}\left(L_{i}(\xi) \cdot x^{\prime}\right)
\end{align*}
$$

for any $x, \xi \in \mathbb{R}^{n}$ and $1 \leq s \leq \Lambda$. For $0 \leq s \leq \Lambda$, define the family of measures $\left\{\sigma_{t, s}\right\}_{t \in \mathbb{R}_{+}}$by

$$
\begin{align*}
& \int_{\mathbb{R}^{n}} f(x) d \sigma_{t, s}(x) \\
& \quad=\frac{1}{t^{\rho}} \int_{t / 2<|x| \leq t} f\left(\Phi_{s}(x)\right) \frac{h(|x|) \Omega(x)}{|x|^{n-\rho}} d x \tag{108}
\end{align*}
$$

where $\left|\sigma_{t, s}\right|$ is defined in the same way as $\sigma_{t, s}$, but with $h$ and $\Omega$ replaced by $h$ and $|\Omega|$, respectively. By the change of variables and Minkowski's inequality, we have

$$
\begin{aligned}
& \mathscr{M}_{h, \Omega, \Gamma, \rho} f(x) \\
& \quad=\left(\int_{0}^{\infty}\left|\sum_{k=-\infty}^{0} 2^{k \rho} \sigma_{2^{k} t, \Lambda} * f(x)\right|^{2} \frac{d t}{t}\right)^{1 / 2}
\end{aligned}
$$

$$
\begin{align*}
& \left\|\left(\sum_{j \in \mathbb{Z}}\left(\int_{\Re_{n}}\left(\sum_{k \in \mathbb{Z}} \int_{2^{k}}^{2^{k+1}} \| \sigma_{t, s}\left|* g_{j, \zeta, k}\right|^{2} \frac{d t}{t}\right)^{1 / 2} d \zeta\right)^{q}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)}  \tag{110}\\
& \left.\quad \leq C\|\Omega\|_{L^{1}\left(S^{n-1}\right)} \|\left(\sum_{j \in \mathbb{Z}}\left\|\left(\sum_{k \in \mathbb{Z}}\left|g_{j, \zeta, k}\right|^{2}\right)^{1 / 2}\right\|^{q}\right)_{L^{r}\left(\Re_{n}\right)}^{1 / q}\right) \|_{L^{p}\left(\mathbb{R}^{n}\right)}
\end{align*}
$$

for $(1 / p, 1 / q, 1 / r)$ belonging to the interior of the convex hull of three cubes $\left(1 / 2,1 / 2+1 / \max \left\{2, \gamma^{\prime}\right\}\right)^{3},(1 / 2-$ $\left.1 / \max \left\{2, \gamma^{\prime}\right\}, 1 / 2\right)^{3}$, and $(1 / 2 \gamma, 1-1 / 2 \gamma)^{3}$. Here $C>0$ is independent of the coefficients of $\left\{P_{j}\right\}_{j=1}^{n}$.

One can easily check that

$$
\begin{align*}
\sigma_{t, 0} & =0  \tag{111}\\
\left|\widehat{\sigma_{t, s}}(\xi)-\widehat{\sigma_{t, s-1}}(\xi)\right| & \leq C \min \left\{1,\left|\varphi(t)^{l_{s}} L_{s}(\xi)\right|\right\} \tag{112}
\end{align*}
$$

$$
\begin{align*}
& \leq \sum_{k=-\infty}^{0} 2^{k \varsigma}\left(\int_{0}^{\infty}\left|\sigma_{2^{k} t, \Lambda} * f(x)\right|^{2} \frac{d t}{t}\right)^{1 / 2} \\
& \leq \frac{1}{1-2^{-\varsigma}}\left(\int_{0}^{\infty}\left|\sigma_{t, \Lambda} * f(x)\right|^{2} \frac{d t}{t}\right)^{1 / 2} \tag{109}
\end{align*}
$$

By Lemma 18, we obtain

It follows from (112) that

$$
\begin{align*}
& \left(\int_{2^{k}}^{2^{k+1}}\left|\widehat{\sigma_{t, s}}(\xi)-\widehat{\sigma_{t, s-1}}(\xi)\right|^{2} \frac{d t}{t}\right)^{1 / 2}  \tag{113}\\
& \quad \leq C \min \left\{1,\left|\varphi\left(2^{k+1}\right)^{l_{s}} L_{s}(\xi)\right|\right\}
\end{align*}
$$

$$
\begin{align*}
\left|\widehat{\sigma_{t, s}}(\xi)\right|= & \left|\frac{1}{t^{\rho}} \int_{t / 2}^{t} \int_{S^{n-1}} \exp \left(-2 \pi i \sum_{j=1}^{s}\left(L_{j}(\xi) \cdot y^{\prime}\right) \varphi(r)^{l_{j}}\right) \Omega\left(y^{\prime}\right) d \sigma\left(y^{\prime}\right) h(r) \frac{d r}{r^{1-\rho}}\right| \\
\leq & \int_{t / 2}^{t}\left|\int_{S^{n-1}} \exp \left(-2 \pi i \sum_{j=1}^{s}\left(L_{j}(\xi) \cdot y^{\prime}\right) \varphi(r)^{l_{j}}\right) \Omega\left(y^{\prime}\right) d \sigma\left(y^{\prime}\right)\right||h(r)| \frac{d r}{r}  \tag{114}\\
\leq & C\|h\|_{\Delta_{\gamma}\left(\mathbb{R}_{+}\right)}\|\Omega\|_{L^{1}\left(S^{n-1}\right)}^{\max \left\{0,1-2 / \gamma^{\prime}\right\}} \\
& \times\left(\int_{t / 2}^{t}\left|\int_{S^{n-1}} \exp \left(-2 \pi i \sum_{j=1}^{s}\left(L_{j}(\xi) \cdot y^{\prime}\right) \varphi(r)^{l_{j}}\right) \Omega\left(y^{\prime}\right) d \sigma\left(y^{\prime}\right)\right|^{\frac{d r}{r}}\right)^{1 / \max \left\{2, \gamma^{\prime}\right\}}
\end{align*}
$$

Since

$$
\begin{aligned}
& \int_{t / 2}^{t}\left|\int_{S^{n-1}} \exp \left(-2 \pi i \sum_{j=1}^{s}\left(L_{j}(\xi) \cdot y^{\prime}\right) \varphi(r)^{l_{j}}\right) \Omega\left(y^{\prime}\right) d \sigma\left(y^{\prime}\right)\right|^{2} \frac{d r}{r} \\
& \quad \leq \int_{t / 2}^{t} \iint_{S^{n-1} \times S^{n-1}} \exp \left(-2 \pi i \sum_{j=1}^{s}\left(L_{j}(\xi) \cdot\left(y^{\prime}-u^{\prime}\right)\right) \varphi(r)^{l_{j}}\right) \Omega\left(y^{\prime}\right) \overline{\Omega\left(u^{\prime}\right)} d \sigma\left(y^{\prime}\right) d \sigma\left(u^{\prime}\right) \frac{d r}{r}
\end{aligned}
$$

$$
\begin{equation*}
\leq \iint_{S^{n-1} \times \Upsilon^{n-1}}\left|\Omega\left(y^{\prime}\right) \overline{\Omega\left(u^{\prime}\right)}\right|\left|\int_{t / 2}^{t} \exp \left(-2 \pi i \sum_{j=1}^{s}\left(L_{j}(\xi) \cdot\left(y^{\prime}-u^{\prime}\right)\right) \varphi(r)^{l_{j}}\right) \frac{d r}{r}\right| d \sigma\left(y^{\prime}\right) d \sigma\left(u^{\prime}\right) \tag{115}
\end{equation*}
$$

by Lemma 16, we have

$$
\begin{align*}
& \left|\int_{t / 2}^{t} \exp \left(-2 \pi i \sum_{j=1}^{s}\left(L_{j}(\xi) \cdot\left(y^{\prime}-u^{\prime}\right)\right) \varphi(r)^{l_{j}}\right) \frac{d r}{r}\right|  \tag{116}\\
& \quad \leq C \min \left\{1,\left|\varphi(t)^{l_{s}} L_{s}(\xi) \cdot\left(y^{\prime}-u^{\prime}\right)\right|^{-1 / s}\right\} .
\end{align*}
$$

For $\left|\varphi(t)^{l_{s}} L_{s}(\xi)\right|>1$, since $r /(\log r)^{\beta}$ is increasing in $\left(e^{\beta}, \infty\right)$, we have

$$
\begin{align*}
& \left|\int_{t / 2}^{t} \exp \left(-2 \pi i \sum_{j=1}^{s}\left(L_{j}(\xi) \cdot\left(y^{\prime}-u^{\prime}\right)\right) \varphi(r)^{l_{j}}\right) \frac{d r}{r}\right| \\
& \quad \leq C \frac{\left(\log 2 e^{\beta s}\left|\eta \cdot\left(y^{\prime}-\theta\right)\right|^{-1}\right)^{\beta}}{\left(\log \left|\varphi(t)^{l_{s}} L_{s}(\xi)\right|\right)^{\beta}} \tag{117}
\end{align*}
$$

where $\eta=L_{s}(\xi) /\left|L_{s}(\xi)\right|$. Combining (114), (115), and (117) with the fact that $\Omega \in W \mathscr{F}_{\beta}\left(\mathrm{S}^{n-1}\right)$ yields that

$$
\begin{equation*}
\left|\widehat{\sigma_{t, s}}(\xi)\right| \leq C\left(\log \left|\varphi(t)^{l_{s}} L_{s}(\xi)\right|\right)^{-\beta / \max \left\{2, \gamma^{\prime}\right\}} \tag{118}
\end{equation*}
$$

when $\left|\varphi(t)^{l_{s}} L_{s}(\xi)\right|>1$. It follows from (118) that

$$
\begin{align*}
& \left(\int_{2^{k}}^{2^{k+1}}\left|\widehat{\sigma_{t, s}}(\xi)\right|^{2} \frac{d t}{t}\right)^{1 / 2}  \tag{119}\\
& \quad \leq C\left(\log \left|\varphi\left(2^{k}\right)^{l_{s}} L_{s}(\xi)\right|\right)^{-\beta / \max \left\{2, \gamma^{\prime}\right\}}
\end{align*}
$$

when $\left|\varphi\left(2^{k}\right)^{l_{s}} L_{s}(\xi)\right|>1$.
Take $a_{k, s}=\varphi\left(2^{k}\right)^{l_{s}}$. By Remark 7 we have that $c_{\varphi}^{l_{s}} \geq$ $a_{k+1, s} / a_{k, s} \geq B_{\varphi}^{l_{s}}>1$. It follows from (110)-(111), (113), (119), and Proposition 25 that

$$
\begin{align*}
& \left\|\left(\sum_{l \in \mathbb{Z}} 2^{l q \alpha}\left(\int_{\mathfrak{R}_{n}}\left(\int_{0}^{\infty}\left|\sigma_{t, \Lambda} * \Delta_{2^{-l} \zeta} f\right|^{2} \frac{d t}{t}\right)^{1 / 2} d \zeta\right)^{q}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)}  \tag{120}\\
& \quad \leq C\|f\|_{\dot{F}_{\alpha}^{p q}\left(\mathbb{R}^{n}\right)}
\end{align*}
$$

holds for $\beta>\max \left\{2, \gamma^{\prime}\right\}$, any $\alpha \in(0,1)$, and all $(1 / p, 1 / q) \in$ $\mathscr{R}_{\gamma, \beta}$. Thus (120) together with (109) yields that

$$
\begin{align*}
& \left\|\left(\sum_{l \in \mathbb{Z}} 2^{l q \alpha}\left(\int_{\mathfrak{R}_{n}}\left|\mathscr{M}_{h, \Omega, \Gamma, \rho}\left(\Delta_{2^{-l}} f\right)\right| d \zeta\right)^{q}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)}  \tag{121}\\
& \quad \leq C\|f\|_{\hat{F}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)}
\end{align*}
$$

holds for $\beta>\max \left\{2, \gamma^{\prime}\right\}$, any $\alpha \in(0,1)$, and all $(1 / p, 1 / q) \in$ $\mathscr{R}_{\gamma, \beta}$. Here $C=C_{n, \varsigma, \alpha, p, q, \gamma, \beta, \varphi}$ is independent of the coefficients of $\left\{P_{j}\right\}_{j=1}^{n}$. On the other hand, one can easily check that

$$
\begin{equation*}
\left|\Delta_{\zeta}\left(\mathscr{M}_{h, \Omega, \Gamma, \rho} f\right)(x)\right| \leq\left|\mathscr{M}_{h, \Omega, \Gamma, \rho}\left(\Delta_{\zeta}(f)\right)(x)\right| \tag{122}
\end{equation*}
$$

for any $x, \zeta \in \mathbb{R}^{n}$ and

$$
\begin{equation*}
\left|\mathscr{M}_{h, \Omega, \Gamma, \rho} f-\mathscr{M}_{h, \Omega, \Gamma, \rho} g\right| \leq\left|\mathscr{M}_{h, \Omega, \Gamma, \rho}(f-g)\right| \tag{123}
\end{equation*}
$$

for arbitrary functions $f$ and $g$ defined on $\mathbb{R}^{n}$. By Lemma 21(i), we have

$$
\begin{align*}
& \left\|\mathscr{M}_{h, \Omega, \Gamma, \rho} f\right\|_{\dot{F}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)} \\
& \leq C\left\|\left(\sum_{l \in \mathbb{Z}} 2^{l q \alpha}\left(\int_{\mathfrak{R}_{n}}\left|\Delta_{2^{-l} \zeta}\left(\mathscr{M}_{h, \Omega, \Gamma, \rho} f\right)\right| d \zeta\right)^{q}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)} \tag{124}
\end{align*}
$$

for all $\alpha \in(0,1)$ and $(p, q) \in(1, \infty)^{2}$. Here $C=$ $C_{n, \zeta, \alpha, p, p, q, \gamma, \beta, \varphi}$ is independent of the coefficients of $\left\{P_{j}\right\}_{j=1}^{n}$. And (122) and (124) yield Theorem 8(i). We get from Remark 10 that $\mathscr{M}_{h, \Omega, \Gamma, \rho}$ is bounded on $L^{p}\left(\mathbb{R}^{n}\right)$ for $\beta>\max \left\{2, \gamma^{\prime}\right\}$ and $|1 / p-1 / 2|<1 / \max \left\{2, \gamma^{\prime}\right\}-1 / \beta$. Note that $\mathscr{M}_{h, \Omega, \Gamma, \rho}$ is a sublinear operator. These facts together with (121)-(123) yield Theorem 8(ii). Theorem 8(iii) and (iv) follow from the $L^{p}$ bounds for $\mu_{h, \Omega, \Gamma, \rho}$ and (122)-(123).

Proof of Theorem 11. We first consider the operator $\mathcal{S}_{\Omega, \Gamma}$. One can easily check that

$$
\left|\Delta_{\zeta}\left(\mathcal{S}_{\Omega, \Gamma} f\right)(x)\right| \leq\left|\mathcal{S}_{\Omega, \Gamma}\left(\Delta_{\zeta}(f)\right)(x)\right|, ~\left(~ \forall x, \zeta \in \mathbb{R}^{d} .\right.
$$

By Lemma 21(i) and (125) we have

$$
\begin{align*}
& \left\|\mathcal{S}_{\Omega, \Gamma} f\right\|_{\dot{F}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)} \\
& \leq C\left\|\left(\sum_{l \in \mathbb{Z}} 2^{l q \alpha}\left(\int_{\mathfrak{R}_{n}}\left|\Delta_{2^{-l} \zeta}\left(\delta_{\Omega, \Gamma} f\right)\right| d \zeta\right)^{q}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)}  \tag{126}\\
& \leq C\left\|\left(\sum_{l \in \mathbb{Z}} 2^{l q \alpha}\left(\int_{\Re_{n}}\left|\mathcal{S}_{\Omega, \Gamma}\left(\Delta_{2^{-l} \zeta} f\right)\right| d \zeta\right)^{q}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)}
\end{align*}
$$

for $\alpha \in(0,1)$ and $(p, q) \in(1, \infty)^{2}$. Therefore, to prove Theorem 11(i) for $\delta_{\Omega, \Gamma}$, it suffices to show that

$$
\begin{align*}
& \left\|\left(\sum_{l \in \mathbb{Z}} 2^{l q \alpha}\left(\int_{\Re_{n}}\left|\mathcal{S}_{\Omega, \Gamma}\left(\triangle_{2^{-l} \zeta} f\right)\right| d \zeta\right)^{q}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)}  \tag{127}\\
& \quad \leq C\|f\|_{\dot{F}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)}
\end{align*}
$$

for $\alpha \in(0,1)$ and $(1 / p, 1 / q)$ belonging to the set of all interiors of the convex hull of two squares $(1 / \beta, 1 / 2)^{2}$ and $(1 / 2,1-$ $1 / \beta)^{2}$. Here $C=C_{n, \alpha, p, q, \beta, \varphi}$ is independent of the coefficients of $\left\{P_{j}\right\}_{j=1}^{n}$.

Let $\Phi_{s}, L_{s}$, and $\Lambda$ be given as in the proof of Theorem 8. Define the family of measures $\left\{\sigma_{t, s}\right\}_{t \in \mathbb{R}_{+}}$and $\left\{\left|\sigma_{t, s}\right|\right\}_{t \in \mathbb{R}_{+}}$on $\mathbb{R}^{n}$ by

$$
\begin{align*}
\widehat{\sigma_{t, s}}(x) & =\int_{S^{n-1}} e^{-2 \pi i \Phi_{s}\left(t y^{\prime}\right) \cdot x} \Omega\left(y^{\prime}\right) d \sigma\left(y^{\prime}\right)  \tag{128}\\
\widehat{\mid \sigma_{t, s}}(x) & =\int_{S^{n-1}} e^{-2 \pi i \Phi_{s}\left(t y^{\prime}\right) \cdot x}\left|\Omega\left(y^{\prime}\right)\right| d \sigma\left(y^{\prime}\right)
\end{align*}
$$

By duality we have

$$
\begin{equation*}
\mathcal{S}_{\Omega, \Gamma} f(x)=\left(\int_{0}^{\infty}\left|\sigma_{t, \Lambda} * f(x)\right|^{2} \frac{d t}{t}\right)^{1 / 2} \tag{129}
\end{equation*}
$$

One can easily check that

$$
\begin{aligned}
& \sigma_{t, 0}=0 \\
& \left(\int_{2^{k}}^{2^{k+1}}\left|\widehat{\sigma_{t, s}}(\xi)-\widehat{\sigma_{t, s-1}}(\xi)\right|^{2} \frac{d t}{t}\right)^{1 / 2} \\
& \quad \leq\left(\int_{2^{k}}^{2^{k+1}} \max \left\{1,\left|\varphi(t)^{l_{s}} L_{s}(\xi)\right|\right\}^{2} \frac{d t}{t}\right)^{1 / 2} \\
& \quad \leq C \min \left\{1,\left|\varphi\left(2^{k+1}\right)^{l_{s}} L_{s}(\xi)\right|\right\}
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
& \left|\widehat{\sigma_{t, s}}(\xi)\right|^{2}=\left|\int_{S^{n-1}} e^{-2 \pi i \Phi_{s}\left(t y^{\prime}\right) \cdot \xi} \Omega\left(y^{\prime}\right) d \sigma\left(y^{\prime}\right)\right|^{2} \\
& \quad=\mid \int_{S^{n-1}} \exp \left(-2 \pi i \sum_{i=1}^{s} \varphi(t)^{l_{i}}\left(L_{i}(\xi) \cdot y^{\prime}\right)\right)
\end{aligned}
$$

$$
\begin{equation*}
\left.\cdot \Omega\left(y^{\prime}\right) d \sigma\left(y^{\prime}\right)\right|^{2}\left\|\left(\sum_{j \in \mathbb{Z}}\left(\int_{\mathfrak{R}_{n}}\left(\sum_{k \in \mathbb{Z}} \int_{2^{k}}^{2^{k+1}}| | \sigma_{t, s}\left|* g_{j, \zeta, k}\right|^{2} \frac{d t}{t}\right)^{1 / 2} d \zeta\right)^{q}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)} \tag{136}
\end{equation*}
$$

$$
\leq C\|\Omega\|_{L^{1}\left(S^{n-1}\right)}\left\|\left(\sum_{j \in \mathbb{Z}}\left\|\left(\sum_{k \in \mathbb{Z}}\left|g_{j ; \zeta, k}\right|^{2}\right)^{1 / 2}\right\|_{L^{r}\left(\Re_{n}\right)}^{q}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)}
$$

for $(1 / p, 1 / q, 1 / r)$ belonging to the interior of the convex hull of two cubes $(0,1 / 2)^{3}$ and $(1 / 2,1)^{3}$. Here $C>0$ is independent of the coefficients of $\left\{P_{j}\right\}_{j=1}^{n}$.

$$
\begin{align*}
& =\iint_{S^{n-1} \times S^{n-1}} \exp \left(-2 \pi i \sum_{i=1}^{s} \varphi(t)^{l_{i}}\left(L_{i}(\xi) \cdot\left(y^{\prime}-\theta\right)\right)\right) \\
& \cdot \Omega\left(y^{\prime}\right) \overline{\Omega(\theta)} d \sigma\left(y^{\prime}\right) d \sigma(\theta) \tag{131}
\end{align*}
$$

It follows that

$$
\begin{align*}
& \int_{2^{k}}^{2^{k+1}}\left|\widehat{\sigma_{t, s}}(\xi)\right|^{2} \frac{d t}{t} \\
& \quad \leq \iint_{S^{n-1} \times S^{n-1}}\left|\int_{2^{k}}^{2^{k+1}} \exp \left(-2 \pi i \sum_{i=1}^{s} \varphi(t)^{l_{i}}\left(L_{i}(\xi) \cdot\left(y^{\prime}-\theta\right)\right)\right) \frac{d t}{t}\right|(  \tag{132}\\
& \quad \cdot\left|\Omega\left(y^{\prime}\right) \overline{\Omega(\theta)}\right| d \sigma\left(y^{\prime}\right) d \sigma(\theta) .
\end{align*}
$$

Invoking Lemma 16 we obtain

$$
\begin{align*}
& \left|\int_{2^{k}}^{2^{k+1}} \exp \left(-2 \pi i \sum_{i=1}^{s} \varphi(t)^{l_{i}}\left(L_{i}(\xi) \cdot\left(y^{\prime}-\theta\right)\right)\right) \frac{d t}{t}\right|  \tag{133}\\
& \quad \leq \min \left\{1,\left|\varphi\left(2^{k+1}\right)^{l_{s}} L_{s}(\xi) \cdot\left(y^{\prime}-\theta\right)\right|^{-1 / s}\right\}
\end{align*}
$$

When $\left|\varphi\left(2^{k+1}\right)^{l_{s}} L_{s}(\xi)\right|>1$, since $t /(\log t)^{\beta}$ is increasing in $\left(e^{\beta}, \infty\right)$, we have

$$
\begin{align*}
& \left|\int_{2^{k}}^{2^{k+1}} \exp \left(-2 \pi i \sum_{i=1}^{s} \varphi(t)^{l_{i}}\left(L_{i}(\xi) \cdot\left(y^{\prime}-\theta\right)\right)\right) \frac{d t}{t}\right| \\
& \quad \leq C \frac{\left(\log 2 e^{\beta \lambda}\left|\eta \cdot\left(y^{\prime}-\theta\right)\right|^{-1}\right)^{\beta}}{\left(\log \left|\varphi\left(2^{k+1}\right)^{l_{s}} L_{s}(\xi)\right|\right)^{\beta}} \tag{134}
\end{align*}
$$

where $\eta=L_{s}(\xi) /\left|L_{s}(\xi)\right|$. Combining (132), (134) with the fact that $\Omega \in W \mathscr{F}_{\beta}\left(\mathrm{S}^{n-1}\right)$ yields that

$$
\begin{align*}
& \left(\int_{2^{k}}^{2^{k+1}}\left|\widehat{\sigma_{t, s}}(\xi)\right|^{2} \frac{d t}{t}\right)^{1 / 2}  \tag{135}\\
& \quad \leq C\left(\log \left|\varphi\left(2^{k+1}\right)^{l_{s}} L_{s}(\xi)\right|\right)^{-\beta / 2}
\end{align*}
$$

if $\left|\varphi\left(2^{k+1}\right)^{l_{s}} L_{s}(\xi)\right|>1$. By Lemma 19 we have

$$
\begin{align*}
& \left\|\left(\sum_{l \in \mathbb{Z}} 2^{l q \alpha}\left(\int_{\mathfrak{R}_{n}}\left(\int_{0}^{\infty}\left|\sigma_{t, \Lambda} * \Delta_{2^{-l \zeta}} f\right|^{2} \frac{d t}{t}\right)^{1 / 2} d \zeta\right)^{q}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)}  \tag{137}\\
& \quad \leq C\|f\|_{\dot{F}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)}
\end{align*}
$$

for $\beta>2, \alpha \in(0,1)$ and $(1 / p, 1 / q)$ belonging to the interior of the convex hull of two squares $(1 / \beta, 1 / 2)^{2}$ and $(1 / 2,1-$ $1 / \beta)^{2}$. Here $C=C_{n, \alpha, p, q, \beta, \varphi}$ is independent of the coefficients of $\left\{P_{j}\right\}_{j=1}^{n}$. Equation (137) together with (129) yields (127). By arguments similar to those used in deriving (3.14) and (3.16) in [31], one can obtain

$$
\begin{equation*}
\mathscr{M}_{\Omega, \Gamma, \rho} f(x) \leq C_{\varsigma} \mathcal{S}_{\Omega, \Gamma} f(x) \quad \forall x \in \mathbb{R}^{n} \tag{138}
\end{equation*}
$$

Thus (138) together with (127) yields that

$$
\begin{align*}
& \left\|\left(\sum_{l \in \mathbb{Z}} 2^{l q \alpha}\left(\int_{\mathfrak{R}_{n}}\left|M_{\Omega, \Gamma, \rho}\left(\Delta_{2^{-l} \zeta} f\right)\right| d \zeta\right)^{q}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)}  \tag{139}\\
& \quad \leq C\|f\|_{\dot{F}_{\alpha}^{p, q}\left(\mathbb{R}^{n}\right)}
\end{align*}
$$

for $\alpha \in(0,1)$ and $(1 / p, 1 / q)$ belonging to the set of all interiors of the convex hull of two squares $(1 / \beta, 1 / 2)^{2}$ and $(1 / 2,1-1 / \beta)^{2}$. Here $C=C_{n, \varsigma, \alpha, p, q, \beta, \varphi}$ is independent of the coefficients of $\left\{P_{j}\right\}_{j=1}^{n}$. One can easily check that

$$
\begin{align*}
&\left|\Delta_{\zeta}\left(\mathscr{M}_{\Omega, \Gamma, \rho} f\right)(x)\right| \leq\left|\mathscr{M}_{\Omega, \Gamma, \rho}\left(\Delta_{\zeta} f\right)(x)\right|  \tag{140}\\
& \forall x, \zeta \in \mathbb{R}^{n} .
\end{align*}
$$

By Lemma 21(i) and (140) we have

$$
\begin{align*}
& \left\|\mathscr{M}_{\Omega, \Gamma, \rho} f\right\|_{\dot{F}_{\alpha}^{p q}\left(\mathbb{R}^{n}\right)} \\
& \leq C\left\|\left(\sum_{l \in \mathbb{Z}} 2^{l q \alpha}\left(\int_{\mathfrak{R}_{n}}\left|\Delta_{2^{-l} \zeta}\left(\mathscr{M}_{\Omega_{,, \rho}, \rho} f\right)\right| d \zeta\right)^{q}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)}  \tag{141}\\
& \leq C\left\|\left(\sum_{l \in \mathbb{Z}} \sum^{l q \alpha}\left(\int_{\Re_{n}}\left|M_{\Omega_{, \Gamma, \rho}}\left(\Delta_{2^{-l} \zeta} f\right)\right| d \zeta\right)^{q}\right)^{1 / q}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)}
\end{align*}
$$

for $\alpha \in(0,1)$ and $(p, q) \in(1, \infty)^{2}$. Then Theorem 11(i) follows from (126)-(127), (139), and (141).

It is known that both $\mathcal{S}_{\Omega, \Gamma}$ and $\mathscr{M}_{\Omega, \Gamma, \rho}$ are sublinear operators. Moreover, one can easily check that

$$
\begin{align*}
\left|\mathcal{S}_{\Omega, \Gamma} f-\mathcal{S}_{\Omega, \Gamma} g\right| & \leq\left|\mathcal{S}_{\Omega, \Gamma}(f-g)\right| \\
\left|\mathscr{M}_{\Omega, \Gamma, \rho} f-\mathscr{M}_{\Omega, \Gamma, \rho} g\right| & \leq\left|\mathscr{M}_{\Omega, \Gamma, \rho}(f-g)\right| \tag{142}
\end{align*}
$$

for arbitrary functions $f, g$ defined on $\mathbb{R}^{n}$. It follows from Remark 12 that

$$
\begin{align*}
& \max \left\{\left\|\mathcal{S}_{\Omega, \Gamma} f\right\|_{L^{p}\left(\mathbb{R}^{n}\right)},\left\|M_{\Omega, \Gamma, \rho} f\right\|_{L^{p}\left(\mathbb{R}^{n}\right)}\right\}  \tag{143}\\
& \left.\quad \leq C\|f\|_{L^{p}\left(\mathbb{R}^{n}\right)}\right\}
\end{align*}
$$

for $p \in[2, \beta)$. It follows from (142)-(143), (127), (139), and Proposition 27 that Theorem 11(ii) holds. Theorem 11(iii)-(iv) follows from (125), (140), (142)-(143), and Proposition 26.

## Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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