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Research Article

Reducing Subspaces of the Dual Truncated Toeplitz Operator

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We define the dual truncated Toeplitz operators and give some basic properties of them. In particular, spectrum and reducing subspaces of some special dual truncated Toeplitz operator are characterized.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$ and $\mathbb{T}$ denote the unit circle. As usual, L^2 denotes the Hilbert space of Lebesgue square integral functions on $\mathbb{T}$ with the inner product:

$$\langle u, v\rangle = \int_{\mathbb{T}} u\bar{v}\, dm, \tag{1}$$

where $u, v \in L^2$, m is the normalized Lebesgue measure. As usual, H^2 will be identified with the subspace of L^2 consisting of the functions whose Fourier coefficients with negative indices vanish. Let P denote the projection from L^2 to H^2, which is given explicitly by the Cauchy integral:

$$(Pf)(\lambda) = \int_{\mathbb{T}} \frac{f(\xi)}{1-\bar{\lambda}\xi}\, dm(\xi), \quad \lambda \in \mathbb{D}. \tag{2}$$

For $\varphi \in L^\infty$, the Toeplitz operator T_φ on H^2, with symbol φ, is defined by

$$T_\varphi f = P(\varphi f), \tag{3}$$

where $f \in H^2$. Let S denote the unilateral shift operator on H^2. Its adjoint, the backward shift, is given by

$$(S^* f)(z) = \frac{f(z) - f(0)}{z}. \tag{4}$$

For the remainder of the paper, u will denote a nonconstant inner function. The subspace $K_u^2 = H^2 \ominus uH^2$ is a proper nontrivial invariant subspace of S^*, the most general one by the well-known theorem of A. Beurling [1]. Truncated Toeplitz operators (TTO) are compressions of the standard Toeplitz operators on the Hardy space H^2 to its coinvariant, the so-called model space K_u^2. Let P_u denote the orthogonal projection from L^2 onto the subspace K_u^2. For $\varphi \in L^2$, the truncated Toeplitz operator A_φ with symbol φ is defined by

$$A_\varphi f = P_u(\varphi f), \tag{5}$$

on the dense subset $K_u^2 \cap L^\infty$ of the space K_u^2. The symbol is not unique [2].

Although the truncated Toeplitz operators share many fundamental properties of classical Toeplitz operators on the Hardy space, they differ in many crucial ways. For example, compact Toeplitz operators on Hardy space are zero, but there are many nonzero compact truncated Toeplitz operators. In part motivated by several of the problems posed in the aforementioned article [2], the area has undergone vigorous development during the past several years [3–22]. While several of the initial questions raised by Sarason have now been resolved, the study of truncated Toeplitz operators has nevertheless proven to be fertile ground, spawning both new questions and unexpected results.

Recently, Gu [23] defined truncated Hankel operator (THO) as the compression of Hankel operator to invariant subspaces for the backward shift and proved a number of algebraic properties of them. Some of the properties in his paper reveal the relation between the THOs and TTOs. Later, Kang and Kim [24] characterized the pairs of truncated Hankel operator on the model spaces K_u^2 whose products result in truncated Toeplitz operators when the inner function u has a certain symmetric property.

In this paper, we will define dual truncated Toeplitz operators and introduce some algebraic properties of them. As is well known, the orthogonal complement of K_u^2, $K_u^{2\perp} = uH^2 \oplus \overline{H}_0^2$, which H_0^2 is the subspace of H^2 with $f(0) = 0$. Let $\varphi \in L^\infty$, and the dual truncated Toeplitz operator B_φ on $K_u^{2\perp}$ with symbol φ is defined by

$$B_\varphi f = (I - P_u)(f\varphi), \tag{6}$$

where $f \in K_u^{2\perp}$. For $\lambda \in \mathbb{D}$, the kernel function in $K_u^{2\perp}$ for the functional of evaluation at λ will be denoted by K_λ^u, which is

$$K_\lambda^u(z) = \frac{\lambda \overline{z}}{1 - \lambda \overline{z}} + \frac{\overline{u(\lambda)} u(z)}{1 - \overline{\lambda} z}. \tag{7}$$

The paper is organized as follows. In Section 2, we give some basic properties of dual truncated Toeplitz operator. In Section 3, we characterize the spectrum and reducing subspaces of some special dual truncated Toeplitz operator.

2. Basic Properties

An operator A on $K_u^{2\perp}$ is called C-symmetric if $CAC = A^*$, where C is an antiunitary involution on L^2 defined by $(Cf)(\xi) = u(\xi)\overline{\xi f(\xi)}$ ($|\xi| = 1$). In fact, it is easily seen that C maps uH^2 to $\overline{H}_0^2$ and $\overline{H}_0^2$ to uH^2.

Lemma 1. *Dual truncated Toeplitz operators are C-symmetric.*

Proof. Let f be in L^∞ with B_f bounded. For g and g in $K_u^{2\perp}$, we have

$$\begin{aligned}\left\langle CB_fCg, h\right\rangle &= \left\langle Ch, B_fCg\right\rangle \\ &= \int u(\xi)\,\overline{\xi}\,\overline{h(\xi) f(\xi) u(\xi)} \xi g(\xi)\, dm(\xi) \\ &= \int \overline{h(\xi)} g(\xi)\, \overline{f(\xi)} dm(\xi) = \left\langle B_{\overline{f}} g, h\right\rangle \\ &= \left\langle B_f^* g, h\right\rangle.\end{aligned} \tag{8}$$

This completes the proof. □

In the following, we will discuss the boundedness and compactness of dual Truncated Toeplitz operators.

Theorem 2. *For $\phi \in L^\infty$, we have $\|B_\phi\| = \|\phi\|_\infty = r(B_\phi)$.*

Proof. Let $\sigma_{ap}(T)$ denote the approximate point spectrum of the bounded linear operator T. It is well known that multiplication operators M_ϕ on the Hardy space H^2, $\|M_\phi\| = \|\phi\|_\infty = r(M_\phi)$, and $\sigma(M_\phi) = \sigma_{ap}(M_\phi)$.

Assuming that $\lambda \in \sigma_{ap}(M_\phi)$, then there exists a sequence $\{f_n\}$ of functions in L^2 with $\|f_n\| = 1$ such that

$$\left\|\left(M_\phi - \lambda\right) f_n\right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{9}$$

By removing a "tail" of small norm and renormalizing, for each n there is g_n of norm 1 that has only a finite number of nonzero Fourier coefficients corresponding to positive indices and satisfy $\|f_n - g_n\| \le 1/n$. Then

$$\begin{aligned}\left\|\left(M_\phi - \lambda\right) g_n\right\| &\le \left\|\left(M_\phi - \lambda\right) f_n\right\| \\ &\quad + \left\|\left(M_\phi - \lambda\right)(f_n - g_n)\right\| \longrightarrow 0 \\ &\quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{10}$$

Let $W^* = M_{e^{-i\theta}}$, and, for each n, there exists a positive integer M_n such that $W^{*M_n} g_n$ is in $\overline{H}_0^2$. Since W^* is unitary and commute with M_ϕ, it follows that

$$\begin{aligned}\left\|\left(M_\phi - \lambda\right) W^{*M_n} g_n\right\| &= \left\|W^{*M_n}\left(M_\phi - \lambda\right) g_n\right\| \\ &= \left\|\left(M_\phi - \lambda\right) g_n\right\|\end{aligned} \tag{11}$$

and

$$\left\|W^{*M_n} g_n\right\| = \|g_n\| = 1. \tag{12}$$

For each n, define $h_n = W^{*M_n} g_n$. Then each h_n is in $\overline{H}_0^2$,

$$\begin{gathered}\|h_n\| = 1 \\ \text{and } \left\|\left(M_\phi - \lambda\right) h_n\right\| \longrightarrow 0.\end{gathered} \tag{13}$$

Since

$$\begin{aligned}\left\|\left(B_\phi - \lambda\right) h_n\right\| &= \left\|\left(I - P_u\right)\left(M_\phi - \lambda\right) h_n\right\| \\ &\le \left\|\left(M_\phi - \lambda\right) h_n\right\|,\end{aligned} \tag{14}$$

we have $\|(B_\phi - \lambda)h_n\| \to 0$, which implies that $\lambda \in \sigma_{ap}(B_\phi)$. Thus we get that $r(M_\phi) \le r(B_\phi)$. Hence

$$\left\|M_\phi\right\| = r\left(M_\phi\right) \le r\left(B_\phi\right) \le \left\|B_\phi\right\|. \tag{15}$$

On the other hand,

$$\left\|B_\phi\right\| \le \left\|(I - P_u)\right\| \left\|M_\phi\right\| \le \left\|M_\phi\right\|, \tag{16}$$

which completes the proof. □

In this paper, let e_k denote the orthonormal basis. For $k \ge 0$, we have $e_k = u(z)z^k$. For $k < 0$, we get $e_k = \overline{z}^{-k}$.

Theorem 3. *The only compact dual truncated Toeplitz operator is zero.*

Proof. Let B_f denote a compact dual Truncated Toeplitz operator. For nonnegative integers s and t, since $\{e_{s+n}\}$ converges weakly to zero as $n \to +\infty$, and it is obtained that

$$\left|\left\langle B_f e_{s+n}, e_{t+n}\right\rangle\right| \longrightarrow 0 \quad \text{as } n \longrightarrow +\infty. \tag{17}$$

Let $f = \sum_{k=-\infty}^{+\infty} a_k e^{ik\theta}$, where a_k is the $k-th$ Fourier coefficient of f, and it follows that

$$\begin{aligned} \lim_{n\to+\infty} \left|\left\langle B_f e_{s+n}, e_{t+n}\right\rangle\right| &= \lim_{n\to+\infty} \left|\left\langle f e_{s+n}, e_{t+n}\right\rangle\right| \\ &= \lim_{n\to+\infty} \left|\left\langle \sum_{k=-\infty}^{+\infty} a_k e^{ik\theta} u(z) z^{s+n}, u(z) z^{t+n}\right\rangle\right| \\ &= \left|a_{t-s}\right|. \end{aligned} \tag{18}$$

Hence $a_{t-s} = 0$ for all nonnegative integers s and t. So $a_k = 0$ for all integer k which implies that $f = 0$. Thus $B_f = 0$. □

With the above theorem, we get the following corollary.

Corollary 4. *A dual truncated Toeplitz operator is self-adjoint if and only if the symbol is real-valued almost everywhere.*

3. Spectrum and Reducing Subspace

In this section, we discuss the spectrum and reducing subspace of some special dual truncated Toeplitz operator. Let $\sigma(T)$ denote the spectrum of the linear operator T and $\sigma_e(T)$ denote the essential spectrum and $\sigma_p(T)$ denote the point spectrum.

Theorem 5. *For dual Truncated Toeplitz operator B_z, if $u(0) \neq 0$, then*

$$\begin{aligned} \sigma(B_z) &= \sigma_e(B_z) = \mathbb{T}, \\ \sigma_p(B_z) &= \emptyset. \end{aligned} \tag{19}$$

Proof. It is clear that the spectrum of B_z is contained in $\overline{\mathbb{D}}$. First, we will show that how the dual Truncated Toeplitz operator B_z act on the orthonormal basis $\{e_n\}$. A direct computation gives

$$\begin{aligned} B_z \overline{z}^n &= \overline{z}^{n-1}, \quad n \geq 2, \\ B_z \overline{z} &= \overline{u(0)} u, \\ B_z\left(u(z) z^n\right) &= u(z) z^{n+1}, \quad n \geq 0. \end{aligned} \tag{20}$$

From the above, if $u(0) = 1$, B_z is the bilateral shift. By corollary 24.4 in [25], we have the spectrum of B_z that is the unit circle. In the following, assume that $u(0) \neq 1$. Then B_z is a special weight shift with the sequence of scalar $\{\alpha_n, n \in \mathbb{Z}\}$, which $\alpha_n = 1$ for $n \neq -1$ and $\alpha_n = \overline{u(0)}$ for $n = -1$.

By proposition 27.7(c) in [22], we have $\sigma_e(B_z) = \mathbb{T}$. Then we will see the point spectrum of B_z. Suppose that $f \in uH^2 \oplus \overline{H_0^2}$, $f = \sum_{n=0}^{+\infty} u(z) a_n z^n + \sum_{m=1}^{+\infty} b_m \overline{z}^m$. Let $\lambda \in \overline{\mathbb{D}}$, and it is obtained that

$$B_z f = \sum_{n=0}^{+\infty} u(z) a_n z^{n+1} + \sum_{m=2}^{+\infty} b_m \overline{z}^{m-1} + b_1 \overline{u(0)} u(z) \tag{21}$$

and

$$\lambda f = \lambda \sum_{n=0}^{+\infty} u(z) a_n z^n + \lambda \sum_{m=1}^{+\infty} b_m \overline{z}^m. \tag{22}$$

From above, we get that $B_z f = 0$ implies $f = 0$ almost everywhere. Suppose that $B_z f = \lambda f$, we have

$$\begin{aligned} b_m &= \lambda b_{m-1}, \quad m \geq 2, \\ b_1 \overline{u(0)} &= a_0 \lambda, \quad m = 1, \\ a_n &= \lambda a_{n+1}, \quad n \geq 0. \end{aligned} \tag{23}$$

Since $|\lambda| \leq 1$ and $f \in H^2$, we have that $f = 0$ almost everywhere. Hence we have that the point spectrum of B_z is $\emptyset$.

It suffices for us to show that any point in $\mathbb{D}$ is not the spectrum of the dual truncated Toeplitz operator B_z. Since it is known that $\sigma_e(B_z) = \mathbb{T}$, for any $\lambda \in \mathbb{D}$, it follows that $\lambda I - B_z$ is Fredholm operator. Let $\widetilde{U}$ be an operator defined on $uH^2 \oplus \overline{H_0^2}$ as follows

$$\widetilde{U} f = \begin{cases} M_z f, & f \in uH^2 \oplus \overline{z}\overline{H_0^2}, \\ u, & f = \overline{z}, \end{cases} \tag{24}$$

where f is in $uH^2 \oplus \overline{H_0^2}$. It is clear that $\widetilde{\widetilde{U}}$ is an unitary operator and $\sigma(\widetilde{U}) = \mathbb{T}$. Hence $\lambda I - \widetilde{U}$ is invertible. It is obtained that $ind(\lambda I - \widetilde{U}) = 0$. A direct computation gives that

$$\lambda I - B_z = \lambda I - \widetilde{U} + \left(1 - \overline{u(0)}\right) u \otimes \overline{z}, \tag{25}$$

hence we have $ind(\lambda I - B_z) = 0$. As $\sigma_p(B_z) = \emptyset$, it follows that

$$\dim \ker\left(\lambda I - B_z\right) = \dim \ker\left(\lambda I - B_z\right)^* = 0. \tag{26}$$

So we get that $\lambda I - B_z$ is invertible and $\lambda \notin \sigma(B_z)$. This completes the proof. □

In the rest of this section, we characterize the structure of lattice of reducing subspaces for B_{z^2} and B_z. To do so, we need to give some notations.

Let S be a bounded linear operator on a Hilbert space $\mathscr{H}$. A closed subspace $\mathscr{M}$ is said to be a reducing subspace for S, if $S\mathscr{M} \subseteq \mathscr{M}$ and $S\mathscr{M}^\perp \subseteq \mathscr{M}^\perp$. Or equivalently, $\mathscr{M}$ is a reducing subspace for S if and only if $SP_\mathscr{M} = P_\mathscr{M} S$ and $S^* P_\mathscr{M} = P_\mathscr{M} S^*$, where $P_\mathscr{M}$ is the orthogonal projection from $\mathscr{H}$ onto $\mathscr{M}$.

In addition, $\mathscr{M}$ is called minimal if there is no nonzero reducing subspace $\mathscr{N}$ contained in $\mathscr{M}$ properly. The operator S is said to be completely reducible if its lattice of reducing subspaces has no nonzero minimal elements [26].

By some easy computations, we have the following equations:

$$B_{z^2}\overline{z}^j = (I - P_u)\overline{z}^{j-2} = \begin{cases} \overline{z}^{j-2}, & j > 2, \\ \overline{a_1}u + \overline{a_0}uz, & j = 1, \\ \overline{a_0}u, & j = 2. \end{cases} \quad (27)$$

$$B_{z^2}uz^j = uz^{j+2}, \quad j \geq 0. \quad (28)$$

$$B^*_{z^2}\overline{z}^j = \overline{z}^{j+2}, \quad j \geq 1. \quad (29)$$

$$B^*_{z^2}uz^j = \begin{cases} uz^{j-2}, & j \geq 2, \\ a_1\overline{z} + a_0\overline{z}^2, & j = 0, \\ a_0\overline{z}, & j = 1. \end{cases} \quad (30)$$

Moreover, we have

$$B^*_{z^2}B_{z^2}\overline{z}^j = \begin{cases} \left(|a_1|^2 + |a_0|^2\right)\overline{z} + a_0\overline{a_1}\,\overline{z}^2, & j = 1, \\ \overline{a_0}a_1\overline{z} + |a_0|^2\overline{z}^2, & j = 2, \\ \overline{z}^j, & j \geq 3. \end{cases} \quad (31)$$

$$B^*_{z^2}B_{z^2}uz^i = uz^i, \quad i \geq 0. \quad (32)$$

$$B_{z^2}B^*_{z^2}\overline{z}^j = \overline{z}^j, \quad j \geq 1. \quad (33)$$

$$B_{z^2}B^*_{z^2}uz^i = \begin{cases} \left(|a_1|^2 + |a_0|^2\right)u + \overline{a_0}a_1uz, & i = 0, \\ a_0\overline{a_1}u + |a_0|^2 uz, & i = 1, \\ uz^i, & i \geq 2. \end{cases} \quad (34)$$

Let $u(z) = \sum_{n=0}^{\infty} a_n z^n$. Let $[f]$ denote the reducing subspace generated by f, that is, the minimal reducing subspace containing f. Denote by $\mathbb{Z}_+$, $\mathbb{N}$ all the nonnegative integers and positive integers, respectively. Write $T \in \{S, S^*\}'$ if $TS = ST$ and $TS^* = S^*T$.

Theorem 6. *If $a_0a_1 \neq 0$ and $|a_1|^2 \neq (1 - |a_0|^2)^2$, then B_{z^2} has no nontrivial reducing subspaces on $(K_u^2)^\perp$.*

Proof. Suppose there exists a reducing subspace $\mathscr{M} \subseteq (K_u^2)^\perp$ for B_{z^2} and $P_{\mathscr{M}}$ denote the orthogonal projection from $(K_u^2)^\perp$ onto $\mathscr{M}$. Then $P_{\mathscr{M}} \in \{B_{z^2}, B^*_{z^2}\}'$. Moreover, from (32) and (34), we have

$$\begin{aligned} a_0\overline{a_1}P_{\mathscr{M}}u &= P_{\mathscr{M}}\left[B_{z^2}B^*_{z^2} - |a_0|^2 B^*_{z^2}B_{z^2}\right](uz) \\ &= \left[B_{z^2}B^*_{z^2} - |a_0|^2 B^*_{z^2}B_{z^2}\right]P_{\mathscr{M}}(uz), \end{aligned} \quad (35)$$

$$\begin{aligned} \overline{a_0}a_1P_{\mathscr{M}}(uz) &= P_{\mathscr{M}}\left[B_{z^2}B^*_{z^2} - \left(|a_0|^2 + |a_1|^2\right)B^*_{z^2}B_{z^2}\right]u \\ &= \left[B_{z^2}B^*_{z^2} - \left(|a_0|^2 + |a_1|^2\right)B^*_{z^2}B_{z^2}\right]P_{\mathscr{M}}u. \end{aligned} \quad (36)$$

In the following, we shall prove that $\mathscr{M} = (K_u^2)^\perp$ or $\mathscr{M} = \{0\}$.

Let

$$\begin{aligned} P_{\mathscr{M}}u &= \sum_{k=1}^{\infty} e_k\overline{z}^k + \sum_{n=0}^{\infty} c_n uz^n, \\ P_{\mathscr{M}}(uz) &= \sum_{k=1}^{\infty} b_k\overline{z}^k + \sum_{n=0}^{\infty} d_n uz^n. \end{aligned} \quad (37)$$

Taking the two equalities above into (35) and (36), along with (31), (32), (33) and (34), we deduce

$$\begin{aligned} &a_0\overline{a_1}\left(\sum_{k=1}^{\infty} e_k\overline{z}^k + \sum_{n=0}^{\infty} c_n uz^n\right) \\ &= \left(1 - |a_0|^2\right)\sum_{k=3}^{\infty} b_k\overline{z}^k + \left(1 - |a_0|^2\right)\sum_{n=2}^{\infty} d_n uz^n \\ &+ \left(|a_1|^2 d_0 + a_0\overline{a_1}d_1\right)u + \overline{a_0}a_1d_0uz \\ &+ \left[b_1 - |a_0|^2\left(|a_0|^2 + |a_1|^2\right)b_1 - |a_0|^2\overline{a_0}a_1b_2\right]\overline{z} \\ &+ \left[\left(1 - |a_0|^4\right)b_2 - |a_0|^2 a_0\overline{a_1}b_1\right]\overline{z}^2, \end{aligned} \quad (38)$$

and

$$\begin{aligned} &\overline{a_0}a_1\left(\sum_{k=1}^{\infty} b_k\overline{z}^k + \sum_{n=0}^{\infty} d_n uz^n\right) = \left[1 - \left(|a_0|^2 + |a_1|^2\right)\right] \\ &\cdot \sum_{k=3}^{\infty} e_k\overline{z}^k + \left[1 - \left(|a_0|^2 + |a_1|^2\right)\right]\sum_{n=2}^{\infty} c_n uz^n + a_0\overline{a_1}c_1u \\ &+ \left(\overline{a_0}a_1c_0 - |a_1|^2 c_1\right)uz + \Big[e_1 - \left(|a_0|^2 + |a_1|^2\right)^2 e_1 \\ &- \overline{a_0}a_1\left(|a_0|^2 + |a_1|^2\right)e_2\Big]\overline{z} + \Big[e_2 \\ &- |a_0|^2\left(|a_0|^2 + |a_1|^2\right)e_2 - a_0\overline{a_1}\left(|a_0|^2 + |a_1|^2\right)e_1\Big] \\ &\cdot \overline{z}^2. \end{aligned} \quad (39)$$

Comparing the coefficients of $\overline{z}^k$ $(k \geq 3)$ in (38) and (39), we get

$$\begin{aligned} a_0\overline{a_1}e_k &= \left(1 - |a_0|^2\right)b_k, \\ \overline{a_0}a_1b_k &= \left(1 - |a_0|^2 - |a_1|^2\right)e_k. \end{aligned} \quad (40)$$

Likewise, comparing the coefficients of uz^n $(n \geq 2)$, we obtain

$$\begin{aligned} a_0\overline{a_1}c_n &= \left(1 - |a_0|^2\right)d_n, \\ \overline{a_0}a_1d_n &= \left(1 - |a_0|^2 - |a_1|^2\right)c_n. \end{aligned} \quad (41)$$

If $(1 - |a_0|^2)(1 - |a_0|^2 - |a_1|^2) = 0$, then immediately we have $e_k = b_k = 0$ for $k \geq 3$ and $c_n = d_n = 0$ for $n \geq 2$; or else, if $(1 - |a_0|^2)(1 - |a_0|^2 - |a_1|^2) \neq 0$, it follows from (40) and (41) that $e_k = b_k = 0$ for $k \geq 3$ and $c_n = d_n = 0$ for $n \geq 2$ since $|a_1|^2 \neq (1-|a_0|^2)^2$.

Considering the coefficients of u, uz, $\overline{z}$, $\overline{z}^2$, we have

$$c_1 = \frac{\overline{a_0}a_1}{a_0\overline{a_1}}d_0, \quad c_0 = d_0, \quad d_1 = \left(1 - \frac{|a_1|^2}{a_0\overline{a_1}}\right)d_0, \tag{42}$$

and

$$\begin{aligned} a_0\overline{a_1}e_1 &= \left[1-|a_0|^2\left(|a_0|^2+|a_1|^2\right)\right]b_1 - |a_0|^2\,\overline{a_0}a_1b_2, \\ \overline{a_0}a_1b_1 &= \left[1-\left(|a_0|^2+|a_1|^2\right)^2\right]e_1 \\ &\quad - \left(|a_0|^2+|a_1|^2\right)\overline{a_0}a_1e_2, \\ a_0\overline{a_1}e_2 &= \left(1-|a_0|^4\right)b_2 - a_0\overline{a_1}\,|a_0|^2\,b_1, \\ \overline{a_0}a_1b_2 &= \left[1-|a_0|^2\left(|a_0|^2+|a_1|^2\right)\right]e_2 \\ &\quad - a_0\overline{a_1}\left(|a_0|^2+|a_1|^2\right)e_1, \end{aligned} \tag{43}$$

where (42) holds since $a_0\overline{a_1} \neq 0$. By some computations, from (43), we obtain

$$\begin{aligned} &a_0\overline{a_1}\left(2|a_0|^2+|a_1|^2\right)b_1 \\ &\quad = \left[1-|a_0|^2\left(|a_0|^2+|a_1|^2\right)\right]b_2, \end{aligned} \tag{44}$$

and

$$\begin{aligned} &\left[1-\left(|a_0|^2+|a_1|^2\right)^2 - |a_0|^2|a_1|^2\right]b_1 \\ &\quad = \overline{a_0}a_1\left(2|a_0|^2+|a_1|^2\right)b_2. \end{aligned} \tag{45}$$

If $[1-|a_0|^2(|a_0|^2+|a_1|^2)][1-(|a_0|^2+|a_1|^2)^2-|a_0|^2|a_1|^2] = 0$, obviously we have $b_1 = b_2 = e_1 = e_2 = 0$; if not, since $|a_0|^2|a_1|^2(2|a_0|^2+|a_1|^2)^2 = [1-|a_0|^2(|a_0|^2+|a_1|^2)][1-(|a_0|^2+|a_1|^2)^2-|a_0|^2|a_1|^2]$ if and only if $|a_1|^2 = (1-|a_0|^2)^2$, along with the assumption $|a_1|^2 \neq (1-|a_0|^2)^2$, we also have $b_1 = b_2 = e_1 = e_2 = 0$.

Associated with (42), we deduce

$$\begin{aligned} P_{\mathscr{M}}u &= d_0u + \frac{\overline{a_0}a_1}{a_0\overline{a_1}}d_0(uz), \\ P_{\mathscr{M}}(uz) &= d_0u + \left(1-\frac{|a_1|^2}{a_0\overline{a_1}}\right)d_0(uz). \end{aligned} \tag{46}$$

If $d_0 = 0$, then $u, uz \in \mathscr{M}^{\perp}$; otherwise, we have

$$P_{\mathscr{M}}u - P_{\mathscr{M}}(uz) = \left(\frac{\overline{a_0}a_1}{a_0\overline{a_1}} - 1 + \frac{|a_1|^2}{a_0\overline{a_1}}\right)d_0(uz) \in \mathscr{M}, \tag{47}$$

which proves that $u, uz \in \mathscr{M}$.

Suppose $u, uz \in \mathscr{M}$. Then the following statements hold:

(i) $uz^j \in \mathscr{M}$, $j \in \mathbb{Z}_+$ since $B_{z^2}uz^j = uz^{j+2}$, $j \in \mathbb{Z}_+$.

(ii) $B^*_{z^2}(uz) = a_0\overline{z}$ and $B^*_{z^2}u = a_1\overline{z} + a_0\overline{z}^2$ show that $\overline{z}, \overline{z}^2 \in \mathscr{M}$.

(iii) On the basis of (ii), we have $\overline{z}^j \in \mathscr{M}$, $j \in \mathbb{N}$ since $B^*_{z^2}\overline{z}^j = \overline{z}^{j+2}$, $j \in \mathbb{N}$.

These give that $(K_u^2)^{\perp} \subseteq \mathscr{M}$, which shows that $(K_u^2)^{\perp} = \mathscr{M}$. Similarly, we can demonstrate $(K_u^2)^{\perp} = \mathscr{M}^{\perp}$ if we assume $u, uz \in \mathscr{M}^{\perp}$, and thus $\mathscr{M} = \{0\}$.

The proof is complete. □

Theorem 7. *If $a_0a_1 \neq 0$ and $|a_1|^2 = (1-|a_0|^2)^2$, then*

(i) if $\mathscr{M}$ is a reducing subspace for B_{z^2} and $\langle P_{\mathscr{M}}u, u\rangle = 0$, then $\mathscr{M} = \{0\}$;

(ii) if $1-|a_0|^2 = a_0\overline{a_1}$, then B_{z^2} has no nontrivial reducing subspaces;

(iii) if $1-|a_0|^2 \neq a_0\overline{a_1}$, then

$$\begin{aligned} \mathscr{N} = \overline{span}\Bigg\{ &uz^{2n} \\ &- \frac{1-|a_0|^2}{a_0\overline{a_1}}uz^{2n+1}, \frac{-\left(1-|a_0|^2\right)|a_0|^2}{\overline{a_1}}\overline{z}^{2n+1} \\ &+ a_0\overline{z}^{2n+2} : n \in \mathbb{Z}_+ \Bigg\} \end{aligned} \tag{48}$$

is a minimal reducing subspace for B_{z^2}.

Proof. Suppose there exists a reducing subspace $\mathscr{M} \subseteq (K_u^2)^{\perp}$ for B_{z^2} and $P_{\mathscr{M}}$ denote the orthogonal projection from $(K_u^2)^{\perp}$ onto $\mathscr{M}$. Then $P_{\mathscr{M}} \in \{B_{z^2}, B^*_{z^2}\}'$.

The assumption in (i) indicates that $u \in \mathscr{M}^{\perp}$. $B_{z^2}B^*_{z^2}u = (|a_1|^2 + |a_0|^2)u + \overline{a_0}a_1uz$ implies $uz \in \mathscr{M}^{\perp}$. By the analysis in the proof of Theorem 6, we have $\mathscr{M} = \{0\}$.

Suppose $\mathscr{M}$ is a reducing subspace for B_{z^2} such that $\langle P_{\mathscr{M}}u, u\rangle \neq 0$. Let

$$\begin{aligned} P_{\mathscr{M}}u &= \sum_{k=1}^{\infty} e_k\overline{z}^k + \sum_{n=0}^{\infty} c_nuz^n, \\ P_{\mathscr{M}}(uz) &= \sum_{k=1}^{\infty} b_k\overline{z}^k + \sum_{n=0}^{\infty} d_nuz^n, \end{aligned} \tag{49}$$

where $c_0 \neq 0$. By (40), (41), (43) and the assumption $|a_1|^2 = (1-|a_0|^2)^2$, we obtain

$$\begin{aligned} e_k &= \frac{1-|a_0|^2}{a_0\overline{a_1}}b_k \quad \text{for } k \geq 1, \\ c_n &= \frac{1-|a_0|^2}{a_0\overline{a_1}}d_n \quad \text{for } n \geq 2. \end{aligned} \tag{50}$$

Therefore,

$$P_{\mathscr{M}}u = \frac{1-|a_0|^2}{a_0\overline{a_1}}\sum_{k=1}^{\infty}b_k\overline{z}^k + c_0u + c_1uz + \frac{1-|a_0|^2}{a_0\overline{a_1}}\sum_{n=2}^{\infty}d_nuz^n, \quad (51)$$

$$P_{\mathscr{M}}(uz) = \sum_{k=1}^{\infty}b_k\overline{z}^k + \sum_{n=0}^{\infty}d_nuz^n.$$

If $1 - |a_0|^2 = a_0\overline{a_1}$, then combining (42) with the assumption $|a_1|^2 = (1-|a_0|^2)^2$, we obtain

$$P_{\mathscr{M}}u - P_{\mathscr{M}}(uz) = (c_1 - d_1)uz = \frac{|a_1|^2}{a_0\overline{a_1}}d_0uz = \left(1-|a_0|^2\right)d_0uz \in \mathscr{M}. \quad (52)$$

Since $a_0a_1 \neq 0$ and $|a_1|^2 = (1-|a_0|^2)^2$, we have $1-|a_0|^2 \neq 0$. Because $c_0 = 0$ if and only if $d_0 = 0$, surely we have $uz \in \mathscr{M}$. $B_{z^2}^*B_{z^2}(uz) = a_0\overline{a_1}u + |a_0|^2uz$ shows that $u \in \mathscr{M}$. By the analysis in the proof of Theorem 7, we get $\mathscr{M} = (K_u^2)^{\perp}$. Associated with (i), (ii) is established.

If $1-|a_0|^2 \neq a_0\overline{a_1}$, from (31), one can easily get that

$$B_{z^2}B_{z^2}^*\left(u - \frac{1-|a_0|^2}{a_0\overline{a_1}}uz\right) = |a_0|^4\left(u - \frac{1-|a_0|^2}{a_0\overline{a_1}}uz\right). \quad (53)$$

Suppose $\mathscr{M}_0$ is a reducing subspace for B_{z^2} contained in $\mathscr{M}$. Set

$$P_{\mathscr{M}_0}\left(u - \frac{1-|a_0|^2}{a_0\overline{a_1}}uz\right) = \sum_{n=0}^{\infty}\alpha_n\left(\frac{-\left(1-|a_0|^2\right)|a_0|^2}{\overline{a_1}}\overline{z}^{2n+1} + a_0\overline{z}^{2n+2}\right) + \sum_{m=0}^{\infty}\beta_m\left(uz^{2m} - \frac{1-|a_0|^2}{a_0\overline{a_1}}uz^{2m+1}\right). \quad (54)$$

$P_{\mathscr{M}_0}B_{z^2}B_{z^2}^* = B_{z^2}B_{z^2}^*P_{\mathscr{M}_0}$ indicates that

$$|a_0|^4P_{\mathscr{M}_0}\left(u - \frac{1-|a_0|^2}{a_0\overline{a_1}}uz\right) = \sum_{n=0}^{\infty}\alpha_n\left(\frac{-\left(1-|a_0|^2\right)|a_0|^2}{\overline{a_1}}\overline{z}^{2n+1} + a_0\overline{z}^{2n+2}\right) + \sum_{m=1}^{\infty}\beta_m\left(uz^{2m} - \frac{1-|a_0|^2}{a_0\overline{a_1}}uz^{2m+1}\right) + |a_0|^4\left(u - \frac{1-|a_0|^2}{a_0\overline{a_1}}uz\right). \quad (55)$$

$|a_0| \neq 1$ shows $\alpha_n = 0$ for $n \in \mathbb{Z}_+$ and $\beta_m = 0$ for $m \in \mathbb{N}$. Thus, $P_{\mathscr{M}_0}(u - ((1-|a_0|^2)/a_0\overline{a_1})uz) = \beta_0(u - ((1-|a_0|^2)/a_0\overline{a_1})uz)$. Therefore, if $\beta_0 \neq 0$, then $[u-((1-|a_0|^2)/a_0\overline{a_1})uz] = \mathscr{M} \subseteq \mathscr{M}_0$, forcing $\mathscr{M}_0 = \mathscr{M}$; otherwise, if $\beta_0 = 0$, similarly we have $\mathscr{M}_0 = \{0\}$. Thus, (iii) holds. □

Theorem 8. *If u is an inner function, then the following statements hold:*

(i) *If $a_0 \neq 0$, and $a_1 = 0$, then $\mathscr{M}$ is a minimal reducing subspace for B_{z^2} if and only if $\mathscr{M}$ has the following forms:*

(1) $\mathscr{M} = \overline{span}\{\overline{z}^{2n}, uz^{2(n-1)} : n \in \mathbb{N}\}$;

(2) $\mathscr{M} = \overline{span}\{\overline{z}^{2n+1}, uz^{2n+1} : n \in \mathbb{Z}_+\}$;

(3) $\mathscr{M} = \overline{span}\{c_1\overline{z}^{2k-1} + c_2\overline{z}^{2k}, c_1uz^{2k-2} + c_2uz^{2k-1} : k \in \mathbb{N}\}$, *where $c_1, c_2 \in \mathbb{C}$ satisfying $c_1 - c_1^2 = |c_2|^2$, $c_1c_2 \neq 0$.*

(ii) *If $a_0 = 0$, $a_1 \neq 0$ and $|a_1| \neq 1$, then B_{z^2} has 3 minimal reducing subspaces.*

(iii) *If $a_0 = a_1 = 0$ and if $\mathscr{M}$ is a reducing subspace for B_{z^2}, then $\mathscr{M}$ is minimal if and only if*

$$\mathscr{M} = \overline{span}\left\{c_1\overline{z}^{2k-1} + c_2\overline{z}^{2k} : k \in \mathbb{N}\right\}, \quad (56)$$

or

$$\mathscr{M} = \overline{span}\left\{c_3uz^{2k} + c_4uz^{2k-1} : k \in \mathbb{N}\right\}, \quad (57)$$

where $c_1, c_2, c_3, c_4 \in \mathbb{C}$ satisfying $c_1 - c_1^2 = |c_2|^2$ and $c_3 - c_3^2 = |c_4|^2$.

Proof. Suppose $\mathscr{M} \subseteq (K_u^2)^{\perp}$ is a reducing subspace for B_{z^2} and let $P_{\mathscr{M}}$ denote the orthogonal projection from $(K_u^2)^{\perp}$ onto $\mathscr{M}$. Then $P_{\mathscr{M}} \in \{B_{z^2}, B_{z^2}^*\}'$.

To show (i), firstly we show that

$$\mathscr{M}_1 = \overline{\text{span}}\left\{\overline{z}^{2n}, uz^{2(n-1)} : n \in \mathbb{N}\right\}, \quad \mathscr{M}_2 = \overline{\text{span}}\left\{\overline{z}^{2n+1}, uz^{2n+1} : n \in \mathbb{Z}_+\right\} \quad (58)$$

are two minimal reducing subspaces for B_{z^2}. Obviously,

$$\left(K_u^2\right)^{\perp} = \mathscr{M}_1 \bigoplus \mathscr{M}_2, \quad (59)$$

and $\mathscr{M}_1$, $\mathscr{M}_2$ are reducing subspaces generated by u, uz, respectively. Moreover, they are both minimal for B_{z^2}. In fact, suppose there exists a reducing subspace $\mathscr{M}_0 \subseteq \mathscr{M}_1$ for B_{z^2}. Then $P_{\mathscr{M}_0}|_{\mathscr{M}_1} : \mathscr{M}_1 \to \mathscr{M}_0$ and $P_{\mathscr{M}_0} \in \{B_{z^2}, B_{z^2}^*\}'$. Write

$$P_{\mathscr{M}_0}u = \sum_{n=1}^{\infty}b_n\overline{z}^{2n} + \sum_{k=1}^{\infty}c_kuz^{2(k-1)}. \quad (60)$$

By (33), (34), along with $B_{z^2}B_{z^2}^*P_{\mathscr{M}_0}u = P_{\mathscr{M}_0}B_{z^2}B_{z^2}^*u$, we get

$$\sum_{n=1}^{\infty}b_n\overline{z}^{2n} + c_1|a_0|^2u + \sum_{k=2}^{\infty}c_kuz^{2(k-1)} = |a_0|^2P_{\mathscr{M}_0}u. \quad (61)$$

It follows that

$$b_n = |a_0|^2 b_n, \quad n \in \mathbb{N},$$
$$\text{and } c_k = |a_0|^2 c_k, \quad k \in \mathbb{N} \text{ and } k \neq 1, \tag{62}$$

which forces that $b_n = 0$ for $n \in \mathbb{N}$, and $c_k = 0$ for $k \in \mathbb{N}$ and $k \neq 1$ since $|a_0| \neq 1$. Therefore, we have $P_{\mathscr{M}_0} u = c_1 u$. If $c_1 = 0$, then $u \in \mathscr{M}_0^{\perp}$, showing $\mathscr{M}_0^{\perp} = [u] = \mathscr{M}_1$, that is, $\mathscr{M}_0 = \{0\}$; if not, $c_1 \neq 0$ shows that $u \in \mathscr{M}_0$, implying $\mathscr{M}_0 = [u] = \mathscr{M}_1$. In conclusion, $\mathscr{M}_1$ is minimal for B_{z^2}. Using a similar method, we can also prove $\mathscr{M}_2$ is minimal for B_{z^2}.

Next, suppose

$$P_{\mathscr{M}} \overline{z} = \sum_{k=1}^{\infty} e_k \overline{z}^k + \sum_{k=0}^{\infty} f_k u z^k, \tag{63}$$

$$P_{\mathscr{M}} \overline{z}^2 = \sum_{k=1}^{\infty} e_k' \overline{z}^k + \sum_{k=0}^{\infty} f_k' u z^k. \tag{64}$$

By the assumption, we have

$$|a_0|^2 P_{\mathscr{M}} \overline{z} = P_{\mathscr{M}} B_{z^2}^* B_{z^2} \overline{z} = B_{z^2}^* B_{z^2} P_{\mathscr{M}} \overline{z}. \tag{65}$$

Along with (63) and the fact that $|a_0| = |u(0)| < 1$, we have $e_k = 0$ for $k \geq 3$ and $f_k = 0$ for $k \geq 0$. Thus, we obtain that

$$P_{\mathscr{M}} \overline{z} = e_1 \overline{z} + e_2 \overline{z}^2. \tag{66}$$

Similarly, we get

$$P_{\mathscr{M}} \overline{z}^2 = e_1' \overline{z} + e_2' \overline{z}^2. \tag{67}$$

Case 1 ($\det \begin{pmatrix} e_1 & e_2 \\ e_1' & e_2' \end{pmatrix} \neq 0$). Then, from (66) and (67), we have $\overline{z}, \overline{z}^2 \in \mathscr{M}$. By the analysis above, we obtain that $\mathscr{M}_1$ and $\mathscr{M}_2$ are the only two minimal reducing subspaces for B_{z^2}.

Case 2 ($\det \begin{pmatrix} e_1 & e_2 \\ e_1' & e_2' \end{pmatrix} = 0$). Again from (66) and (67), we have

$$\left\langle P_{\mathscr{M}} \overline{z}^2, \overline{z} \right\rangle = e_1' = \left\langle \overline{z}^2, P_{\mathscr{M}} \overline{z} \right\rangle = \overline{e_2}, \tag{68}$$

that is, $e_1' = \overline{e_2}$. Moreover, $\langle P_{\mathscr{M}} \overline{z}, P_{\mathscr{M}} \overline{z}^2 \rangle = \langle P_{\mathscr{M}} \overline{z}, \overline{z}^2 \rangle$ shows that $e_2' = 1 - \overline{e}_1$. By the assumption, we have e_1 is real and $e_1 - e_1^2 = |e_2|^2$.

If $e_1' = e_2 = 0$, then $e_1 - e_1^2 = 0$. That is, $e_1 = 0$ or 1. Suppose $e_1 = 0$, then $e_2' = 1$. By (66) and (67), we have $\overline{z} \in \mathscr{M}^{\perp}$ and $\overline{z}^2 \in \mathscr{M}$. Otherwise, $e_1 = 1$, $e_2' = 0$ imply that $\overline{z} \in \mathscr{M}$ and $\overline{z}^2 \in \mathscr{M}^{\perp}$. In this way, we still obtain that $\mathscr{M}_1$ and $\mathscr{M}_2$ are the only two minimal reducing subspaces for B_{z^2}.

On the other hand, if $e_1' e_2 \neq 0$, it is clear that $e_1 e_2' \neq 0$. Immediately, we have $P_{\mathscr{M}} \overline{z}^2 = e_1 \overline{z} + e_2 \overline{z}^2 = (\overline{e_2}/e_1) P_{\mathscr{M}} \overline{z}$, where $e_1 - e_1^2 = |e_2|^2$, $e_1 e_2 \neq 0$. Then, $\mathscr{M}$ contains a minimal reducing subspace like

$$\mathscr{M} = \overline{\operatorname{span}} \left\{ e_1 \overline{z}^{2k-1} + e_2 \overline{z}^{2k}, e_1 u z^{2k-2} + e_2 u z^{2k-1} : k \in \mathbb{N} \right\}. \tag{69}$$

Furthermore, $\mathscr{M}$ is a minimal reducing subspace for B_{z^2} if and only if

$$\mathscr{M} = \overline{\operatorname{span}} \left\{ e_1 \overline{z}^{2k-1} + e_2 \overline{z}^{2k}, e_1 u z^{2k-2} + e_2 u z^{2k-1} : k \in \mathbb{N} \right\}, \tag{70}$$

where $e_1 - e_1^2 = |e_2|^2$, $e_1 e_2 \neq 0$.

The assumption in Condition (ii) and a similar method indicate that

$$P_{\mathscr{N}} \overline{z} = e_1 \overline{z} + e_2 \overline{z}^2,$$
$$P_{\mathscr{N}} \overline{z}^2 = e_2' \overline{z}^2, \tag{71}$$

where $\mathscr{N}$ is supposed to be a reducing subspace for B_{z^2}.

Note that there is always a reducing subspace $\mathscr{N}$ satisfying $\langle P_{\mathscr{N}} \overline{z}^2, \overline{z}^2 \rangle \neq 0$. Thus, $P_{\mathscr{N}} \overline{z}^2 = e_2' \overline{z}^2$ shows that $\overline{z}^2 \in \mathscr{N}$. Therefore, $\mathscr{N}_1 = [\overline{z}^2] = \overline{\operatorname{span}}\{\overline{z}^{2n} : n \in \mathbb{N}\}$ is definitely a minimal reducing subspace for B_{z^2}. Hence, $\overline{z} \in \mathscr{N}_1^{\perp}$ shows that if $\mathscr{N}$ is a reducing subspace such that $\langle P_{\mathscr{N}} \overline{z}, \overline{z} \rangle \neq 0$, then $P_{\mathscr{N}} \overline{z} = e_1 \overline{z}$, $e_1 \neq 0$. Similarly, $\mathscr{N}_2 = [\overline{z}] = \overline{\operatorname{span}}\{\overline{z}^{2n+1}, u z^{2n} : n \in \mathbb{Z}_+\}$ is a minimal reducing subspace for B_{z^2}. Denote by $\mathscr{N}_3 = (K_u^2)^{\perp} \ominus (\mathscr{N}_1 \oplus \mathscr{N}_2) = \overline{\operatorname{span}}\{u z^{2n+1} : n \in \mathbb{Z}_+\}$. We shall prove $\mathscr{N}_3$ is also minimal. Assume $\mathscr{N}_0 \subseteq \mathscr{N}_3$ is a reducing subspace for B_{z^2} and

$$P_{\mathscr{N}_0}(uz) = \sum_{k=0}^{\infty} f_k u z^{2k+1}. \tag{72}$$

From (34), we see that $B_{z^2} B_{z^2}^*(uz) = 0$. Therefore,

$$0 = P_{\mathscr{N}_0} B_{z^2} B_{z^2}^*(uz) = B_{z^2} B_{z^2}^* P_{\mathscr{N}_0}(uz)$$
$$= f_0 |a_1|^2 u + \sum_{k=2}^{\infty} f_k u z^k, \tag{73}$$

which forces that $f_0 = f_2 = f_3 = \cdots = 0$ and thus $P_{\mathscr{N}_0}(uz) = f_1 uz$. Hence, we have $\mathscr{N}_0 = \{0\}$ or $\mathscr{N}_0 = [uz] = \mathscr{N}_3$.

By the similar way in (ii), the statement in (iii) shows that

$$\mathscr{K}_1 = \overline{\operatorname{span}} \left\{ \overline{z}^{2n-1} : n \in \mathbb{N} \right\},$$
$$\mathscr{K}_2 = \overline{\operatorname{span}} \left\{ \overline{z}^{2n} : n \in \mathbb{N} \right\},$$
$$\mathscr{K}_3 = \overline{\operatorname{span}} \left\{ u z^{2n} : n \in \mathbb{Z}_+ \right\},$$
$$\mathscr{K}_4 = \overline{\operatorname{span}} \left\{ u z^{2n+1} : n \in \mathbb{Z}_+ \right\} \tag{74}$$

are four minimal reducing subspaces for B_{z^2}. Using a similar method as in (i), along with (27)-(34), we obtain that

$$P_{\mathscr{M}} \overline{z} = e_1 \overline{z} + e_2 \overline{z}^2,$$
$$P_{\mathscr{M}} \overline{z}^2 = e_1' \overline{z} + e_2' \overline{z}^2,$$
$$P_{\mathscr{M}} u = f_1 u + f_2 uz,$$
$$P_{\mathscr{M}}(uz) = f_1' u + f_2' uz, \tag{75}$$

where $e_1, e_2, e_1', e_2', f_1, f_2, f_1', f_2' \in \mathbb{C}$. A similar discussion as in (i) leads to the desired conclusion. □

Theorem 9. *If $a_0 \neq 0$, $a_1 = 0$ and $|a_0| = 1$, then B_{z^2} is completely reducible.*

Proof. Under the assumptions, along with (31)-(34), B_{z^2} is a normal operator on $(K_u^2)^\perp$. By the Spectral Theorem of the normal operators, the range of $E(\{\lambda\})$ is the eigenspace $\ker(B_{z^2} - \lambda)$, where E is the spectral measure for B_{z^2}. Thus, for every $\lambda \in \mathbb{C}$, $\ker(B_{z^2} - \lambda) = \{0\}$ if and only if B_{z^2} has no minimal reducing subspaces, that is, B_{z^2} is completely reducible.

Given $\lambda \in \mathbb{C}$, suppose there exists $f \in (K_u^2)^\perp$ such that

$$B_{z^2} f = \lambda f. \tag{76}$$

Assume

$$f = \sum_{n=1}^{\infty} u_n \overline{z}^n + \sum_{k=0}^{\infty} v_n u z^n. \tag{77}$$

Taking this into (76), we get

$$\begin{aligned} &\sum_{n=3}^{\infty} u_n \overline{z}^{n-2} + \overline{a_0}\left(u_2 u + u_1 u z\right) + \sum_{k=0}^{\infty} v_n u z^{n+2} \\ &\quad = \lambda\left(\sum_{n=1}^{\infty} u_n \overline{z}^n + \sum_{k=0}^{\infty} v_n u z^n\right), \end{aligned} \tag{78}$$

from which we can obtain

$$\begin{aligned} u_{2n-1} &= \lambda^n u_1, \\ u_{2n} &= \lambda^n u_2 \\ &\text{for } n \in \mathbb{N}, \end{aligned} \tag{79}$$

and

$$\begin{aligned} v_{2n+1} &= \frac{1}{\lambda^{n+1}} u_1 \overline{a_0}, \\ v_{2n} &= \frac{1}{\lambda^{n+1}} u_2 \overline{a_0} \\ &\text{for } n \in \mathbb{Z}_+. \end{aligned} \tag{80}$$

Then, (77) turns to be

$$\begin{aligned} f = {} & u_1 \sum_{n=1}^{\infty} \lambda^n \overline{z}^{2n-1} + u_2 \sum_{n=1}^{\infty} \lambda^n \overline{z}^{2n} + u_1 \overline{a_0} \sum_{k=0}^{\infty} \frac{1}{\lambda^{n+1}} u z^{2n+1} \\ & + u_2 \overline{a_0} \sum_{k=0}^{\infty} \frac{1}{\lambda^{n+1}} u z^{2n}. \end{aligned} \tag{81}$$

Thus, we must have $u_1 = u_2 = 0$, or $\|f\|_{(K_u^2)^\perp}$ is infinite. Therefore, we conclude $f \equiv 0$. Then, the desired result follows. □

Remark 10. Notice that the assumption in Theorem 9 implies that u is a constant function with module 1, and hence the conclusion in this theorem is trivial.

Remark 11. If the condition in Theorem 8 (ii) is changed to $a_0 = 0$ and $|a_1| = 1$, then the result is different. In fact, $\mathcal{N}_2 = \overline{\operatorname{span}}\{uz^{2n+1} : n \in \mathbb{Z}_+\}$, $\mathcal{N}_3 = \overline{\operatorname{span}}\{\overline{z}^{2n} : n \in \mathbb{N}\}$ are still both minimal for B_{z^2}. As given in the proof of Theorem 8, $\mathcal{N}_1 = \overline{\operatorname{span}}\{\overline{z}^{2n+1}, uz^{2n} : n \in \mathbb{Z}_+\}$. Since $a_0 = 0$, $|a_1| = 1$, $B_{z^2}|_{\mathcal{N}_1}$ is normal on $\mathcal{N}_1$. By the same method in Theorem 9, we obtain that B_{z^2} is completely reducible when restricted to $\mathcal{N}_1$.

The proofs in Theorems 6–9 apply to characterize the structure of the reducing subspaces for B_z. By some computations, we have

$$\begin{aligned} B_z \overline{z}^j &= \begin{cases} \overline{z}^{j-1}, & j > 1 \\ \overline{a_0} u, & j = 1, \end{cases} \\ B_z\left(uz^j\right) &= uz^{j+1}, \quad j \geq 0, \\ B_z^* \overline{z}^j &= \overline{z}^{j+1}, \quad j \geq 1, \\ B_z^* u &= \begin{cases} a_0 \overline{z}, & j = 0 \\ uz^{j-1}, & j \geq 1. \end{cases} \end{aligned} \tag{82}$$

Moreover,

$$\begin{aligned} B_z B_z^* \overline{z}^j &= \overline{z}^j \quad \text{for } j \geq 1, \\ B_z B_z^* u z^j &= \begin{cases} \left|a_0\right|^2 u, & j = 0 \\ uz^j, & j \geq 1. \end{cases} \end{aligned} \tag{83}$$

It is easy to verify that if $|a_0| \neq 1$, then B_z has no nontrivial reducing subspaces.

If $|a_0| = 1$, by (83), B_z is a normal operator on $(K_u^2)^\perp$. Theorem 5 gives $\sigma_p(B_z) = \emptyset$. Associated with the proof in Theorem 9, B_z is completely reducible on $(K_u^2)^\perp$

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Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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