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Research Article

Higher-Order Commutators of Parametric Marcinkiewicz Integrals on Herz Spaces with Variable Exponent

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Let $\Omega \in L^s(S^{n-1})$ for $s \geqslant 1$ be a homogeneous function of degree zero and b be BMO functions. In this paper, we obtain some boundedness of the parametric Marcinkiewicz integral operator μ_Ω^ρ and its higher-order commutator $[b^m, \mu_\Omega^\rho]$ on Herz spaces with variable exponent.

1. Introduction

Function spaces with variable exponent are being concerned with strong interest not only in harmonic analysis but also in applied mathematics. In the past 27 years, the theory of function spaces with variable exponent has made great progress since some elementary properties were given by Kováčik and Rákosník [1] in 1991. In [2–6], the authors proved the boundedness of some integral operators on variable L^p spaces, respectively. Lebesgue and Sobolev spaces with integrability exponent have been widely studied; see [3, 5] and the references therein. Many applications of these spaces were given, for example, in the modeling of electrorheological fluids, in the study of image processing, and in differential equations with nonstandard growth.

On the other hand, a class of function spaces called Herz-type spaces on $\mathbb{R}^n$ has attracted considerable attention in recent years because the interesting norm includes explicitly both local and global information of the function. In 2011, Izuki [7] studied the Herz spaces with variable exponent and proved the boundedness of some sublinear operators on the spaces. In addition, Wang and Liu [8] introduced a certain Herz-type Hardy spaces with variable exponent in 2012.

Suppose that S^{n-1} denotes the unit sphere in $\mathbb{R}^n$ ($n \geqslant 2$) equipped with normalized Lebesgue measure. Let $\Omega \in \mathrm{Lip}_\beta(\mathbb{R}^n)$ for $0 < \beta \leqslant 1$ be a homogeneous function of degree zero and

$$\int_{S^{n-1}} \Omega(x')\, d\sigma(x') = 0, \tag{1}$$

where $x' = x/|x|$ for any $x \neq 0$. In 1958, Stein [9] introduced the Marcinkiewicz integral related to the Littlewood-Paley g function on $\mathbb{R}^n$ as follows:

$$\mu_\Omega(f)(x) = \left(\int_0^\infty \left|F_{\Omega,t}(f)(x)\right|^2 \frac{dt}{t^3}\right)^{1/2}, \tag{2}$$

where

$$F_{\Omega,t}(f)(x) = \int_{|x-y|\leqslant t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)\, dy. \tag{3}$$

It was shown that μ_Ω is of type (p,p) for $1 < p \leqslant 2$ and of weak type $(1,1)$.

The parametric Marcinkiewicz integral is defined by

$$\mu_\Omega^\rho(f)(x) = \left(\int_0^\infty \left|F_{\Omega,t}^\rho(f)(x)\right|^2 \frac{dt}{t^{2\rho+1}}\right)^{1/2}, \tag{4}$$

where

$$F^{\rho}_{\Omega,t}(f)(x) = \int_{|x-y|\leqslant t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} f(y)\,dy, \quad \rho > 0,\ t > 0. \tag{5}$$

Let $m \in \mathbb{N}$ and $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the higher-order commutator generated by the parametric Marcinkiewicz integral μ^{ρ}_{Ω} and b is defined by

$$[b^m, \mu^{\rho}_{\Omega}](f)(x) = \left(\int_0^{\infty}\left|\int_{|x-y|\leqslant t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}}[b(x)-b(y)]^m f(y)\,dy\right|^2 \frac{dt}{t^{2\rho+1}}\right)^{1/2}. \tag{6}$$

Motivated by [10, 11], we will study the boundedness for the parametric Marcinkiewicz integral operator μ^{ρ}_{Ω} and its commutator $[b^m, \mu^{\rho}_{\Omega}]$ on the Herz spaces with variable exponent, where $\Omega \in L^s(S^{n-1})$ for $s \geqslant 1$.

Throughout this paper, we denote the Lebesgue measure and the characteristic function of a measurable set $A \subset \mathbb{R}^n$ by $|A|$ and χ_A, respectively. The notation $f \approx g$ means that there exist constants $C_1, C_2 > 0$ such that $C_1 g \leqslant f \leqslant C_2 g$. Let $B_k = \{x \in \mathbb{R}^n : |x| \leqslant 2^k\}$ and $A_k = B_k \setminus B_{k-1}$ for $k \in \mathbb{Z}$. Denote $\mathbb{Z}_+$ and $\mathbb{N}$ as the sets of all positive and nonnegative integers, $\chi_k = \chi_{A_k}$ for $k \in \mathbb{Z}$, $\widetilde{\chi}_k = \chi_k$ if $k \in \mathbb{Z}_+$ and $\widetilde{\chi}_0 = \chi_{B_0}$. In addition, δ_1 and δ_2 are the same as in Lemma 4.

2. Preliminaries

Firstly we give some notation and basic definitions on variable Lebesgue spaces. Given an open set $E \subset \mathbb{R}^n$, and a measurable function $p(\cdot) : E \longrightarrow [1, \infty)$. $p'(\cdot)$ is the conjugate exponent defined by $p'(\cdot) = p(\cdot)/(p(\cdot) - 1)$.

Define $\mathscr{P}^0(E)$ to be set of $p(\cdot) : E \longrightarrow (0, \infty)$ such that

$$p^- = \operatorname{ess\,inf}\{p(x) : x \in E\} > 0, \qquad p^+ = \operatorname{ess\,sup}\{p(x) : x \in E\} < \infty. \tag{7}$$

The set $\mathscr{P}(E)$ consists of all $p(\cdot) : E \longrightarrow [1, \infty)$ satisfying

$$p^- = \operatorname{ess\,inf}\{p(x) : x \in E\} > 1, \qquad p^+ = \operatorname{ess\,sup}\{p(x) : x \in E\} < \infty. \tag{8}$$

By $L^{p(\cdot)}(E)$ we denote the space of all measurable functions f on E such that for some $\lambda > 0$,

$$\int_E \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx < \infty. \tag{9}$$

This is a Banach function space with respect to the Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}(E)} = \inf\left\{\lambda > 0 : \int_E \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \leqslant 1\right\}. \tag{10}$$

The space $L^{p(\cdot)}_{\mathrm{loc}}(\Omega)$ is defined by

$$L^{p(\cdot)}_{\mathrm{loc}}(\Omega) := \left\{f : f \in L^{p(\cdot)}(E) \text{ for all compact subsets } E \subset \Omega\right\}. \tag{11}$$

Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$; the Hardy-Littlewood maximal operator is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)|\,dy, \tag{12}$$

where $B_r(x) = \{y \in \mathbb{R}^n : |x-y| < r\}$. The set $\mathscr{B}(\mathbb{R}^n)$ consists of $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ satisfying the condition that M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

In variable L^p spaces there are some important lemmas as follows.

Lemma 1 (see [2]). *If $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ and satisfies*

$$|p(x) - p(y)| \leqslant \frac{C}{-\log(|x-y|)}, \quad |x-y| \leqslant \frac{1}{2} \tag{13}$$

and

$$|p(x) - p(y)| \leqslant \frac{C}{\log(|x|+e)}, \quad |y| \geqslant |x|, \tag{14}$$

then $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$; that is, the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

Lemma 2 (see [1] generalized Hölder inequality). *Let $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$. If $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$, then fg is integrable on $\mathbb{R}^n$ and*

$$\int_{\mathbb{R}^n} |f(x)\,g(x)|\,dx \leqslant r_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}, \tag{15}$$

where

$$r_p = 1 + \frac{1}{p^-} - \frac{1}{p^+}. \tag{16}$$

Lemma 3 (see [7]). *Suppose $q(\cdot) \in \mathscr{B}(\mathbb{R}^n)$. Then there exists a constant $C > 0$ such that for all balls B in $\mathbb{R}^n$,*

$$\frac{1}{|B|} \|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \leqslant C. \tag{17}$$

Lemma 4 (see [7]). *Let $q(\cdot) \in \mathscr{B}(\mathbb{R}^n)$. Then there exists a positive constant C such that for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$,*

$$\frac{\|\chi_S\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \leqslant C\left(\frac{|S|}{|B|}\right)^{\delta_1} \quad \text{and} \quad \frac{\|\chi_S\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q'(\cdot)}(\mathbb{R}^n)}} \leqslant C\left(\frac{|S|}{|B|}\right)^{\delta_2}, \tag{18}$$

where δ_1, δ_2 are constants with $0 < \delta_1, \delta_2 < 1$.

Next we recall the definition of the Herz spaces with variable exponent.

Definition 5 (see [7]). Let $\alpha \in \mathbb{R}, 0 < p \leqslant \infty$, and $q(\cdot) \in \mathscr{P}(\mathbb{R}^n)$. The homogeneous Herz space $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ consists of all $f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ such that

$$\|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \right\}^{1/p} < \infty. \tag{19}$$

The nonhomogeneous Herz space $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined as the set of all $f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \left\{ \sum_{k=0}^{\infty} 2^{k\alpha p} \|f\widetilde{\chi}_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \right\}^{1/p} < \infty. \tag{20}$$

3. Boundedness of the Parametric Marcinkiewicz Integral Operator

In this section we will prove the boundedness of the parametric Marcinkiewicz integral operators μ^ρ_Ω on Herz spaces with variable exponent.

A nonnegative locally integrable function ω on $\mathbb{R}^n$ is said to belong to A_p $(1 < p < \infty)$, if

$$\sup_Q \left(\frac{1}{|Q|} \int_Q \omega(x)\,dx \right) \left(\frac{1}{|Q|} \int_Q \omega(x)^{1-p'}\,dx \right)^{p-1} < \infty, \tag{21}$$

where $p' = p/(p-1)$, Q denotes a cube in $\mathbb{R}^n$ with its sides parallel to the coordinate axes, and $|Q|$ denotes the Lebesgue measure of Q.

The weighted (L^p, L^p) boundedness of μ^ρ_Ω has been proved by Shi and Jiang [12].

Lemma 6 (see [12]). *Suppose that $\Omega \in L^s(S^{n-1})(s > 1)$ satisfying (1). If $\omega \in A_{p/s'}, s' < p < \infty$, then for any $f \in L^p(\omega)$, there is a constant C, independent of f, such that*

$$\int_{\mathbb{R}^n} |\mu^\rho_\Omega(f)(x)|^p \omega(x)\,dx \leqslant C \int_{\mathbb{R}^n} |f(x)|^p \omega(x)\,dx. \tag{22}$$

Lemma 7 (see [4]). *Given a family $\mathscr{F}$ and an open set $E \subset \mathbb{R}^n$, assume that for some p_0, $0 < p_0 < \infty$ and for every $\omega \in A_\infty$,*

$$\int_E f(x)^{p_0} \omega(x)\,dx \leqslant C_0 \int_E g(x)^{p_0} \omega(x)\,dx, \quad (f,g) \in \mathscr{F}. \tag{23}$$

Given $p(\cdot) \in \mathscr{P}^0(E)$ such that $p(\cdot)$ satisfies (13) and (14) in Lemma 1. Then for all $(f,g) \in \mathscr{F}$ such that $f \in L^{p(\cdot)}(E)$,

$$\|f\|_{L^{p(\cdot)}(E)} \leqslant C \|g\|_{L^{p(\cdot)}(E)}. \tag{24}$$

Since $A_{p/s'} \subset A_\infty$, by Lemmas 6 and 7 it is easy to get the $(L^{p(\cdot)}(\mathbb{R}^n), L^{p(\cdot)}(\mathbb{R}^n))$-boundedness of the parametric Marcinkiewicz integral operators μ^ρ_Ω.

To obtain the Theorem 11, we need the following lemmas.

Lemma 8 (see [13]). *If $a > 0, 1 \leqslant s \leqslant \infty, 0 < d \leqslant s$ and $-n + (n-1)d/s < \nu < \infty$, then*

$$\left(\int_{|y| \leqslant a|x|} |y|^\nu |\Omega(x-y)|^d\,dy \right)^{1/d} \leqslant C|x|^{(\nu+n)/d} \|\Omega\|_{L^s(S^{n-1})}. \tag{25}$$

Lemma 9 (see [14]). *Define a variable exponent $\widetilde{q}(\cdot)$ by $1/p(x) = 1/\widetilde{q}(x) + 1/q$ $(x \in \mathbb{R}^n)$. Then we have*

$$\|fg\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C \|f\|_{L^{\widetilde{q}(\cdot)}(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)} \tag{26}$$

for all measurable functions f and g.

Lemma 10 (see [5]). *Let $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ satisfy conditions (13) and (14) in Lemma 1. Then*

$$\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \approx \begin{cases} |Q|^{1/p(x)} & \text{if } |Q| \leqslant 2^n \text{ and } x \in Q, \\ |Q|^{1/p(\infty)} & \text{if } |Q| \geqslant 1 \end{cases} \tag{27}$$

for every cube (or ball) $Q \subset \mathbb{R}^n$, where $p(\infty) = \lim_{x \longrightarrow \infty} p(x)$.

Theorem 11. *Suppose that $0 < \nu \leqslant 1$, $0 < p \leqslant \infty$, $q(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ satisfies conditions (13) and (14) in Lemma 1; $\Omega \in L^s(S^{n-1})(s > q'^-)$ and $-n\delta_1 - \nu - n/s < \alpha < n\delta_2 - \nu - n/s$. Then μ^ρ_Ω is bounded on $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ and $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.*

Proof. We only prove homogeneous case. The nonhomogeneous case can be proved in the same way. We suppose $0 < p < \infty$, since the proof of the case $p = \infty$ is easier. Let $f \in \dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$. Denote $f_j = f\chi_j$ for each $j \in \mathbb{Z}$, then we have $f(x) = \sum_{j=-\infty}^{\infty} f_j(x)$. Then we have

$$\begin{aligned} \|\mu^\rho_\Omega(f)\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} &= \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|\mu^\rho_\Omega(f)\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \right\}^{1/p} \\ &\leqslant C \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \cdot \left(\sum_{j=-\infty}^{k-2} \|\mu^\rho_\Omega(f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \right\}^{1/p} \\ &\quad + C \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-1}^{k+1} \|\mu^\rho_\Omega(f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \right\}^{1/p} \\ &\quad + C \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k+2}^{\infty} \|\mu^\rho_\Omega(f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \right\}^{1/p} \\ &=: CI_1 + CI_2 + CI_3. \end{aligned} \tag{28}$$

We first estimate I_2, by the $(L^{q(\cdot)}(\mathbb{R}^n), L^{q(\cdot)}(\mathbb{R}^n))$-boundedness of the commutator μ^ρ_Ω; we have

$$I_2 \leqslant C \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \right\}^{1/p} = C \|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}. \tag{29}$$

Now we estimate I_1. We consider

$$\begin{aligned}
&\left|\mu_\Omega^\rho\left(f_j\right)(x)\right| \\
&\leqslant \left(\int_0^{|x|}\left|\int_{|x-y|\leqslant t}\frac{\Omega(x-y)}{|x-y|^{n-\rho}}f_j(y)\,dy\right|^2\frac{dt}{t^{2\rho+1}}\right)^{1/2} \\
&\quad+\left(\int_{|x|}^{\infty}\left|\int_{|x-y|\leqslant t}\frac{\Omega(x-y)}{|x-y|^{n-\rho}}f_j(y)\,dy\right|^2\frac{dt}{t^{2\rho+1}}\right)^{1/2} \\
&=: I_{11}+I_{12}.
\end{aligned} \tag{30}$$

Note that $x\in A_k$, $y\in A_j$, and $j\leqslant k-2$. So we know that $|x-y|\sim|x|$, and by mean value theorem we have

$$\left|\frac{1}{|x-y|^{2\rho}}-\frac{1}{|x|^{2\rho}}\right|\leqslant\frac{|y|}{|x-y|^{2\rho+1}}. \tag{31}$$

By (31), the Minkowski inequality, and the generalized Hölder inequality we have

$$\begin{aligned}
I_{11}&\leqslant C\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n-\rho}}\left|f_j(y)\right|\left(\int_{|x-y|}^{|x|}\frac{dt}{t^{2\rho+1}}\right)^{1/2}dy \\
&\leqslant C\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n-\rho}}\left|f_j(y)\right| \\
&\quad\cdot\left|\frac{1}{|x-y|^{2\rho}}-\frac{1}{|x|^{2\rho}}\right|^{1/2}dy \\
&\leqslant C\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n-\rho}}\left|f_j(y)\right|\frac{|y|^{1/2}}{|x-y|^{\rho+1/2}}dy\leqslant C \\
&\quad\cdot\frac{2^{j/2}}{|x|^{n+1/2}}\int_{A_j}|\Omega(x-y)|\,|f(y)|\,dy \\
&\leqslant C2^{(j-k)/2}2^{-kn}\left\|f_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\Omega(x-\cdot)\chi_j(\cdot)\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}.
\end{aligned} \tag{32}$$

Similarly, we consider I_{12}. Noting that $|x-y|\sim|x|$, by the Minkowski inequality and the generalized Hölder inequality, we have

$$\begin{aligned}
I_{12}&\leqslant C\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n-\rho}}\left|f_j(y)\right|\left(\int_{|x|}^{\infty}\frac{dt}{t^{2\rho+1}}\right)^{1/2}dy \\
&\leqslant C\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n}}\left|f_j(y)\right|dy \\
&\leqslant C2^{-kn}\left\|f_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\Omega(x-\cdot)\chi_j(\cdot)\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}.
\end{aligned} \tag{33}$$

So we have

$$\begin{aligned}
&\left|\mu_\Omega^\rho\left(f_j\right)(x)\right| \\
&\quad\leqslant C2^{-kn}\left\|f_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\Omega(x-\cdot)\chi_j(\cdot)\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}.
\end{aligned} \tag{34}$$

Noting $s>q'^{-}$, we denote $\widetilde{q'}(\cdot)>1$ and $1/q'(x)=1/\widetilde{q'}(x)+1/s$. By Lemmas 8 and 9 we have

$$\begin{aligned}
&\left\|\Omega(x-\cdot)\chi_j(\cdot)\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \\
&\quad\leqslant\left\|\Omega(x-\cdot)\chi_j(\cdot)\right\|_{L^{s}(\mathbb{R}^n)}\left\|\chi_j(\cdot)\right\|_{L^{\widetilde{q'}(\cdot)}(\mathbb{R}^n)} \\
&\quad\leqslant\left\|\Omega(x-\cdot)\chi_j(\cdot)\right\|_{L^{s}(\mathbb{R}^n)}\left\|\chi_{B_j}\right\|_{L^{\widetilde{q'}(\cdot)}(\mathbb{R}^n)} \\
&\quad\leqslant C2^{-j\nu}\left(\int_{A_j}|\Omega(x-y)|^s|y|^{s\nu}dy\right)^{1/s}\left\|\chi_{B_j}\right\|_{L^{\widetilde{q'}(\cdot)}(\mathbb{R}^n)} \\
&\quad\leqslant C2^{-j\nu}2^{k(\nu+n/s)}\|\Omega\|_{L^s(S^{n-1})}\left\|\chi_{B_j}\right\|_{L^{\widetilde{q'}(\cdot)}(\mathbb{R}^n)}.
\end{aligned} \tag{35}$$

When $|B_j|\leqslant 2^n$ and $x_j\in B_j$, by Lemma 10 we have

$$\left\|\chi_{B_j}\right\|_{L^{\widetilde{q'}(\cdot)}(\mathbb{R}^n)}\approx\left|B_j\right|^{1/\widetilde{q'}(x_j)}\approx\left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\left|B_j\right|^{-1/s}. \tag{36}$$

When $|B_j|\geqslant 1$ we have

$$\left\|\chi_{B_j}\right\|_{L^{\widetilde{q'}(\cdot)}(\mathbb{R}^n)}\approx\left|B_j\right|^{1/\widetilde{q'}(\infty)}\approx\left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\left|B_j\right|^{-1/s}. \tag{37}$$

So we obtain $\|\chi_{B_j}\|_{L^{\widetilde{q'}(\cdot)}(\mathbb{R}^n)}\approx\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}|B_j|^{-1/s}$.

By Lemmas 3 and 4 we have

$$\begin{aligned}
&\left\|\mu_\Omega^\rho\left(f_j\right)\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\leqslant C2^{-kn}\left\|f_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\quad\cdot\left\|\Omega(x-\cdot)\chi_j(\cdot)\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\left\|\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\quad\leqslant C2^{-kn}2^{-j\nu}2^{k(\nu+n/s)}\|\Omega\|_{L^s(S^{n-1})}\left\|f_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\quad\cdot\left\|\chi_{B_j}\right\|_{L^{\widetilde{q'}(\cdot)}(\mathbb{R}^n)}\left\|\chi_{B_k}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\quad\leqslant C2^{-kn}2^{-j\nu}2^{k(\nu+n/s)}\|\Omega\|_{L^s(S^{n-1})}\left\|f_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\quad\times\left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\left|B_j\right|^{-1/s}\left\|\chi_{B_k}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\quad\leqslant C2^{(k-j)(\nu+n/s)}\|\Omega\|_{L^s(S^{n-1})}\left\|f_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\frac{\left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\left\|\chi_{B_k}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}} \\
&\quad\leqslant C2^{(j-k)(n\delta_2-\nu-n/s)}\|\Omega\|_{L^s(S^{n-1})}\left\|f_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned} \tag{38}$$

Thus we obtain

$$\begin{aligned}
I_1&\leqslant C\|\Omega\|_{L^s(S^{n-1})}\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2}2^{(j-k)(n\delta_2-\nu-n/s)}\right.\right. \\
&\quad\left.\left.\cdot\left\|f_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p}=C\|\Omega\|_{L^s(S^{n-1})}
\end{aligned}$$

$$\cdot \left\{ \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} 2^{j\alpha} 2^{(j-k)(n\delta_2-\nu-n/s-\alpha)} \cdot \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \right\}^{1/p}. \tag{39}$$

If $1 < p < \infty$, take $1/p + 1/p' = 1$. Since $n\delta_2 - \nu - n/s - \alpha > 0$, by the Hölder inequality we have

$$\begin{aligned} I_1 &\leqslant C\|\Omega\|_{L^s(S^{n-1})} \left\{ \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} 2^{j\alpha p} \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \cdot 2^{(j-k)(n\delta_2-\nu-n/s-\alpha)p/2} \right) \times \left(\sum_{j=-\infty}^{k-2} 2^{(j-k)(n\delta_2-\nu-n/s-\alpha)p'/2} \right)^{p/p'} \right\}^{1/p} \\ &\leqslant C\|\Omega\|_{L^s(S^{n-1})} \left\{ \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} 2^{j\alpha p} \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \times 2^{(j-k)(n\delta_2-\nu-n/s-\alpha)p/2} \right) \right\}^{1/p} = C\|\Omega\|_{L^s(S^{n-1})} \\ &\quad \cdot \left\{ \sum_{j=-\infty}^{\infty} 2^{j\alpha p} \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \times \left(\sum_{k=j+2}^{\infty} 2^{(j-k)(n\delta_2-\nu-n/s-\alpha)p/2} \right) \right\}^{1/p} \\ &\leqslant C \left\{ \sum_{j=-\infty}^{\infty} 2^{j\alpha p} \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \right\}^{1/p} = C\|f\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)}. \end{aligned} \tag{40}$$

If $0 < p \leqslant 1$, then we have

$$\begin{aligned} I_1 &\leqslant C\|\Omega\|_{L^s(S^{n-1})} \left\{ \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{k-2} 2^{j\alpha p} \cdot 2^{(j-k)(n\delta_2-\nu-n/s-\alpha)p} \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \right\}^{1/p} \\ &= C\|\Omega\|_{L^s(S^{n-1})} \left\{ \sum_{j=-\infty}^{\infty} 2^{j\alpha p} \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \cdot \left(\sum_{k=j+2}^{\infty} 2^{(j-k)(n\delta_2-\nu-n/s-\alpha)p} \right) \right\}^{1/p} \leqslant C\|f\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)}. \end{aligned} \tag{41}$$

Let us now estimate I_3. Note that $x \in A_k$, $y \in A_j$, and $j \geqslant k+2$, so we have $|x-y| \sim |y|$. We consider

$$\begin{aligned} &\left|\mu_\Omega^\rho(f_j)(x)\right| \\ &\leqslant \left(\int_0^{|y|} \left| \int_{|x-y|\leqslant t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} f_j(y)\,dy \right|^2 \frac{dt}{t^{2\rho+1}} \right)^{1/2} \\ &\quad + \left(\int_{|y|}^{\infty} \left| \int_{|x-y|\leqslant t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} f_j(y)\,dy \right|^2 \frac{dt}{t^{2\rho+1}} \right)^{1/2} \\ &=: I_{31} + I_{32}. \end{aligned} \tag{42}$$

Similar to the estimate for I_{11}, we get

$$\begin{aligned} &I_{31} \\ &\leqslant C2^{(k-j)/2} 2^{-jn} \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^{q'(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{43}$$

Similar to the estimate for I_{12}, we get

$$I_{32} \leqslant C2^{-jn} \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^{q'(\cdot)}(\mathbb{R}^n)}. \tag{44}$$

So we have

$$\begin{aligned} &\left|\mu_\Omega^\rho(f_j)(x)\right| \\ &\leqslant C2^{-jn} \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^{q'(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{45}$$

By Lemmas 3 and 4 we have

$$\begin{aligned} \|\mu_\Omega^\rho(f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\leqslant C2^{-jn} \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \cdot \|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\leqslant C2^{-jn} 2^{-j\nu} 2^{k(\nu+n/s)} \|\Omega\|_{L^s(S^{n-1})} \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \cdot \|\chi_{B_j}\|_{L^{\tilde{q}'(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\leqslant C2^{-jn} 2^{-j\nu} 2^{k(\nu+n/s)} \|\Omega\|_{L^s(S^{n-1})} \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \times \|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)} |B_j|^{-1/s} \|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\leqslant C2^{(k-j)(\nu+n/s)} \|\Omega\|_{L^s(S^{n-1})} \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \frac{\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \\ &\leqslant C2^{(k-j)(n\delta_1+\nu+n/s)} \|\Omega\|_{L^s(S^{n-1})} \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{46}$$

Thus we obtain

$$I_3 \leqslant C\|\Omega\|_{L^s(S^{n-1})} \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k+2}^{\infty} 2^{(k-j)(n\delta_1+\nu+n/s)} \cdot \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \right\}^{1/p} = C\|\Omega\|_{L^s(S^{n-1})}$$

$$\cdot \left\{ \sum_{k=-\infty}^{\infty} \left(\sum_{j=k+2}^{\infty} 2^{j\alpha} \cdot 2^{(k-j)(n\delta_1+\nu+n/s+\alpha)} \left\| f_j \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \right\}^{1/p}. \tag{47}$$

If $1 < p < \infty$, take $1/p + 1/p' = 1$. Since $n\delta_1 + \nu + n/s + \alpha > 0$, by the Hölder inequality we have

$$\begin{aligned} I_3 &\leqslant C \|\Omega\|_{L^s(S^{n-1})} \left\{ \sum_{k=-\infty}^{\infty} \left(\sum_{j=k+2}^{\infty} 2^{j\alpha p} \left\| f_j \right\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \cdot 2^{(k-j)(n\delta_1+\nu+n/s+\alpha)p/2} \right) \right. \\ &\quad \left. \times \left(\sum_{j=k+2}^{\infty} 2^{(k-j)(n\delta_1+\nu+n/s+\alpha)p'/2} \right)^{p/p'} \right\}^{1/p} \\ &\leqslant C \|\Omega\|_{L^s(S^{n-1})} \left\{ \sum_{k=-\infty}^{\infty} \left(\sum_{j=k+2}^{\infty} 2^{j\alpha p} \left\| f_j \right\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \times 2^{(k-j)(n\delta_1+\nu+n/s+\alpha)p/2} \right) \right\}^{1/p} = C \|\Omega\|_{L^s(S^{n-1})} \\ &\quad \cdot \left\{ \sum_{j=-\infty}^{\infty} 2^{j\alpha p} \left\| f_j \right\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \times \left(\sum_{k=-\infty}^{j-2} 2^{(k-j)(n\delta_1+\nu+n/s+\alpha)p/2} \right) \right\}^{1/p} \\ &\leqslant C \left\{ \sum_{j=-\infty}^{\infty} 2^{j\alpha p} \left\| f_j \right\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \right\}^{1/p} = C \|f\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)}. \end{aligned} \tag{48}$$

If $0 < p \leqslant 1$, then we have

$$\begin{aligned} I_3 &\leqslant C \|\Omega\|_{L^s(S^{n-1})} \left\{ \sum_{k=-\infty}^{\infty} \sum_{j=k+2}^{\infty} 2^{j\alpha p} 2^{(k-j)(n\delta_1+\nu+n/s+\alpha)p} \cdot \left\| f_j \right\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \right\}^{1/p} = C \|\Omega\|_{L^s(S^{n-1})} \\ &\quad \cdot \left\{ \sum_{j=-\infty}^{\infty} 2^{j\alpha p} \left\| f_j \right\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \cdot \left(\sum_{k=-\infty}^{j-2} 2^{(k-j)(n\delta_1+\nu+n/s+\alpha)p} \right) \right\}^{1/p} \leqslant C \|f\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)}. \end{aligned} \tag{49}$$

Thus by (28), (29) and (40), (41), (48), and (49) we complete the proof of Theorem 11. □

4. BMO Estimate for the Higher-Order Commutator of Parametric Marcinkiewicz Integral Operator

Let us first recall that the space $\mathrm{BMO}(\mathbb{R}^n)$ consists of all locally integrable functions f such that

$$\|f\|_* = \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx < \infty, \tag{50}$$

where $f_Q = |Q|^{-1} \int_Q f(y) dy$, the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes, and $|Q|$ denotes the Lebesgue measure of Q.

Let $b \in \mathrm{BMO}(\mathbb{R}^n)$. The weighted (L^p, L^p) boundedness of $[b^m, \mu_\Omega^\rho]$ has been proved by Shi and Jiang [12].

Lemma 12 (see [12]). *Suppose that $\Omega \in L^s(S^{n-1})$ $(s > 1)$ satisfying (1). If $b(x) \in \mathrm{BMO}(\mathbb{R}^n)$ and $\omega \in A_{p/s'}$, $s' < p < \infty$, then for any $f \in L^p(\omega)$, there is a constant C, independent of f, such that*

$$\begin{aligned} &\int_{\mathbb{R}^n} \left| [b^m, \mu_\Omega^\rho](f)(x) \right|^p \omega(x) \, dx \\ &\quad \leqslant C \int_{\mathbb{R}^n} |f(x)|^p \omega(x) \, dx. \end{aligned} \tag{51}$$

By Lemmas 12 and 7 it is easy to get the $(L^{p(\cdot)}(\mathbb{R}^n), L^{p(\cdot)}(\mathbb{R}^n))$-boundedness of the commutator $[b^m, \mu_\Omega^\rho]$.

Next, we will give the corresponding result about the commutator $[b^m, \mu_\Omega^\rho]$ on Herz spaces with variable exponent.

Theorem 13. *Suppose that $b \in \mathrm{BMO}(\mathbb{R}^n)$, $0 < \nu \leqslant 1$, $0 < p \leqslant \infty$, $q(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ satisfies conditions (13) and (14) in Lemma 1; $\Omega \in L^s(S^{n-1})(s > q'^-)$ and $-n\delta_1 - \nu - n/s < \alpha < n\delta_2 - \nu - n/s$. Then $[b^m, \mu_\Omega^\rho]$ is bounded on $\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$ and $K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$.*

In the proof of Theorem 13, we also need the following lemma.

Lemma 14 (see [15]). *Let $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$, m be a positive integer, and B be a ball in $\mathbb{R}^n$. Then we have that, for all $b \in \mathrm{BMO}(\mathbb{R}^n)$ and all $j, i \in \mathbb{Z}$ with $j > i$,*

$$\begin{aligned} \frac{1}{C} \|b\|_*^m &\leqslant \sup_B \frac{1}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left\| (b - b_B)^m \chi_B \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\leqslant C \|b\|_*^m, \\ \left\| \left(b - b_{B_i}\right)^m \chi_{B_j} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\leqslant C (j-i)^m \|b\|_*^m \left\| \chi_{B_j} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}, \end{aligned} \tag{52}$$

where $B_i = \{x \in \mathbb{R}^n : |x| \leqslant 2^i\}$ and $B_j = \{x \in \mathbb{R}^n : |x| \leqslant 2^j\}$.

Proof of Theorem 13. Similar to Theorem 11, we only prove homogeneous case and still suppose $0 < p < \infty$. Let $f \in \dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$. Denote $f_j = f\chi_j$ for each $j \in \mathbb{Z}$, then we have $f(x) = \sum_{j=-\infty}^{\infty} f_j(x)$. Then

$$\begin{aligned}
\left\|[b^m,\mu_\Omega^\rho](f)\right\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} &= \left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \cdot \left\|[b^m,\mu_\Omega^\rho](f)\chi_k\right\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{1/p} \leqslant C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \cdot \left(\sum_{j=-\infty}^{k-2}\left\|[b^m,\mu_\Omega^\rho](f_j)\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p} \\
&\quad + C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \cdot \left(\sum_{j=k-1}^{k+1}\left\|[b^m,\mu_\Omega^\rho](f_j)\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p} \\
&\quad + C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \cdot \left(\sum_{j=k+2}^{\infty}\left\|[b^m,\mu_\Omega^\rho](f_j)\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p} =: CJ_1 \\
&\quad + CJ_2 + CJ_3.
\end{aligned} \tag{53}$$

Noting $[b^m,\mu_\Omega^\rho]$ is bounded on $L^{q(\cdot)}(\mathbb{R}^n)$, so we have

$$J_2 \leqslant C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left\|f_k\right\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{1/p} = C\left\|f\right\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}. \tag{54}$$

Now we estimate J_1. We consider

$$\begin{aligned}
\left|[b^m,\mu_\Omega^\rho](f_j)(x)\right| &\leqslant \left(\int_0^{|x|}\left|\int_{|x-y|\leqslant t}\frac{\Omega(x-y)}{|x-y|^{n-\rho}}[b(x)-b(y)]^m f_j(y)\,dy\right|^2\frac{dt}{t^{2\rho+1}}\right)^{1/2} \\
&\quad + \left(\int_{|x|}^{\infty}\left|\int_{|x-y|\leqslant t}\frac{\Omega(x-y)}{|x-y|^{n-\rho}}[b(x)-b(y)]^m f_j(y)\,dy\right|^2\frac{dt}{t^{2\rho+1}}\right)^{1/2} =: J_{11} + J_{12}.
\end{aligned} \tag{55}$$

Note that $x \in A_k$, $y \in A_j$, and $j \leqslant k-2$, and we know that $|x-y| \sim |x|$. By (31), the Minkowski inequality, and the generalized Hölder inequality we have

$$\begin{aligned}
J_{11} &\leqslant C\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n-\rho}}|b(x)-b(y)|^m|f_j(y)| \cdot \left(\int_{|x-y|}^{|x|}\frac{dt}{t^{2\rho+1}}\right)^{1/2}dy \\
&\leqslant C\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n-\rho}}|b(x)-b(y)|^m|f_j(y)| \cdot \left|\frac{1}{|x-y|^{2\rho}}-\frac{1}{|x|^{2\rho}}\right|^{1/2}dy \\
&\leqslant C\int_{\mathbb{R}^n}\frac{|\Omega(x-y)||b(x)-b(y)|^m|f_j(y)|}{|x-y|^{n-\rho}}\frac{|y|^{1/2}}{|x-y|^{\rho+1/2}}dy \\
&\leqslant C\frac{2^{j/2}}{|x|^{n+1/2}}\int_{A_j}|\Omega(x-y)||b(x)-b(y)|^m|f(y)|\,dy \\
&\leqslant C2^{(j-k)/2}2^{-kn}\left\{\left|b(x)-b_{B_j}\right|^m\int_{A_j}|\Omega(x-y)||f_j(y)|\,dy \right. \\
&\quad \left. + \int_{A_j}|\Omega(x-y)|\left|b_{B_j}-b(y)\right|^m|f_j(y)|\,dy\right\} \\
&\leqslant C2^{(j-k)/2}2^{-kn}\left\|f_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\{\left|b(x)-b_{B_j}\right|^m \cdot \left\|\Omega(x-\cdot)\chi_j(\cdot)\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \right. \\
&\quad \left. + \left\|\Omega(x-\cdot)\left(b_{B_j}-b(\cdot)\right)^m\chi_j(\cdot)\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\right\}.
\end{aligned} \tag{56}$$

Similarly, we consider J_{12}. Noting that $|x-y| \sim |x|$, by the Minkowski inequality and the generalized Hölder inequality, we have

$$\begin{aligned}
J_{12} &\leqslant C\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n-\rho}}|b(x)-b(y)|^m|f_j(y)| \cdot \left(\int_{|x|}^{\infty}\frac{dt}{t^{2\rho+1}}\right)^{1/2}dy \\
&\leqslant C\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n}}|b(x)-b(y)|^m|f_j(y)|\,dy
\end{aligned}$$

$$\leqslant C2^{-kn}\left\|f_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\{\left|b(x)-b_{B_j}\right|^m\right.$$
$$\cdot\left\|\Omega(x-\cdot)\chi_j(\cdot)\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}$$
$$\left.+\left\|\Omega(x-\cdot)\left(b_{B_j}-b(\cdot)\right)^m\chi_j(\cdot)\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\right\}. \tag{57}$$

So we have

$$\left|[b^m,\mu_\Omega^\rho](f_j)(x)\right|\leqslant C2^{-kn}\left\|f_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}$$
$$\cdot\left\{\left|b(x)-b_{B_j}\right|^m\left\|\Omega(x-\cdot)\chi_j(\cdot)\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\right.$$
$$\left.+\left\|\Omega(x-\cdot)\left(b_{B_j}-b(\cdot)\right)^m\chi_j(\cdot)\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\right\}. \tag{58}$$

Noting $s>q'^{-}$, we denote $\widetilde{q'}(\cdot)>1$ and $1/q'(x)=1/\widetilde{q'}(x)+1/s$. By Lemmas 8 and 9 we have

$$\begin{aligned}&\left\|\Omega(x-\cdot)\chi_j(\cdot)\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\\&\leqslant\left\|\Omega(x-\cdot)\chi_j(\cdot)\right\|_{L^s(\mathbb{R}^n)}\left\|\chi_j(\cdot)\right\|_{L^{\widetilde{q'}(\cdot)}(\mathbb{R}^n)}\\&\leqslant\left\|\Omega(x-\cdot)\chi_j(\cdot)\right\|_{L^s(\mathbb{R}^n)}\left\|\chi_{B_j}\right\|_{L^{\widetilde{q'}(\cdot)}(\mathbb{R}^n)}\\&\leqslant C2^{-j\nu}\left(\int_{A_j}|\Omega(x-y)|^s|y|^{s\nu}dy\right)^{1/s}\left\|\chi_{B_j}\right\|_{L^{\widetilde{q'}(\cdot)}(\mathbb{R}^n)}\\&\leqslant C2^{-j\nu}2^{k(\nu+n/s)}\|\Omega\|_{L^s(S^{n-1})}\left\|\chi_{B_j}\right\|_{L^{\widetilde{q'}(\cdot)}(\mathbb{R}^n)}.\end{aligned} \tag{59}$$

When $|B_j|\leqslant 2^n$ and $x_j\in B_j$, by Lemma 10 we have

$$\left\|\chi_{B_j}\right\|_{L^{\widetilde{q'}(\cdot)}(\mathbb{R}^n)}\approx\left|B_j\right|^{1/\widetilde{q'}(x_j)}\approx\left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\left|B_j\right|^{-1/s}. \tag{60}$$

When $|B_j|\geqslant 1$ we have

$$\left\|\chi_{B_j}\right\|_{L^{\widetilde{q'}(\cdot)}(\mathbb{R}^n)}\approx\left|B_j\right|^{1/\widetilde{q'}(\infty)}\approx\left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\left|B_j\right|^{-1/s}. \tag{61}$$

So we obtain $\|\chi_{B_j}\|_{L^{\widetilde{q'}(\cdot)}(\mathbb{R}^n)}\approx\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}|B_j|^{-1/s}$.

So we have

$$\begin{aligned}&\left\|\Omega(x-\cdot)\chi_j(\cdot)\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\\&\quad\leqslant C2^{(k-j)(\nu+n/s)}\|\Omega\|_{L^s(S^{n-1})}\left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}.\end{aligned} \tag{62}$$

Similarly, by Lemma 14 we have

$$\begin{aligned}&\left\|\Omega(x-\cdot)\left(b_{B_j}-b(\cdot)\right)^m\chi_j(\cdot)\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\\&\leqslant\left\|\Omega(x-\cdot)\chi_j(\cdot)\right\|_{L^s(\mathbb{R}^n)}\left\|\left(b_{B_j}-b(\cdot)\right)^m\chi_j(\cdot)\right\|_{L^{\widetilde{q'}(\cdot)}(\mathbb{R}^n)}\\&\leqslant C\|b\|_*^m\left\|\chi_{B_j}\right\|_{L^{\widetilde{q'}(\cdot)}(\mathbb{R}^n)}\left\|\Omega(x-\cdot)\chi_j(\cdot)\right\|_{L^s(\mathbb{R}^n)}\\&\leqslant C\|b\|_*^m2^{(k-j)(\nu+n/s)}\|\Omega\|_{L^s(S^{n-1})}\left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}.\end{aligned} \tag{63}$$

By (62), (63), Lemmas 3, 4, and 14 we have

$$\begin{aligned}&\left\|[b^m,\mu_\Omega^\rho](f_j)\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\leqslant C2^{-kn}\left\|f_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\&\cdot\left(2^{(k-j)(\nu+n/s)}\|\Omega\|_{L^s(S^{n-1})}\left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\right.\\&\times\left\|\left(b(\cdot)-b_{B_j}\right)^m\chi_{B_k}(\cdot)\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}+\|b\|_*^m\\&\left.\cdot2^{(k-j)(\nu+n/s)}\|\Omega\|_{L^s(S^{n-1})}\left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)\\&\leqslant C2^{-kn}\left\|f_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left((k-j)^m\|b\|_*^m\right.\\&\cdot2^{(k-j)(\nu+n/s)}\|\Omega\|_{L^s(S^{n-1})}\times\left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\left\|\chi_{B_k}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\&+\|b\|_*^m2^{(k-j)(\nu+n/s)}\|\Omega\|_{L^s(S^{n-1})}\left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\\&\left.\cdot\left\|\chi_{B_k}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)\leqslant C(k-j)^m\|b\|_*^m\left\|f_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\&\cdot2^{-kn}2^{(k-j)(\nu+n/s)}\|\Omega\|_{L^s(S^{n-1})}\times\left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\\&\cdot\left\|\chi_{B_k}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\leqslant C(k-j)^m\|b\|_*^m\left\|f_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\&\cdot2^{(k-j)(\nu+n/s)}\|\Omega\|_{L^s(S^{n-1})}\frac{\left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\left\|\chi_{B_k}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}\\&\leqslant C2^{(j-k)(n\delta_2-\nu-n/s)}(k-j)^m\|b\|_*^m\|\Omega\|_{L^s(S^{n-1})}\\&\cdot\left\|f_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}.\end{aligned} \tag{64}$$

Thus we obtain

$$\begin{aligned}J_1&\leqslant C\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2}2^{(j-k)(n\delta_2-\nu-n/s)}(k-j)^m\right.\right.\\&\left.\left.\times\|b\|_*^m\|\Omega\|_{L^s(S^{n-1})}\left\|f_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p}=C\|b\|_*^m\\&\cdot\|\Omega\|_{L^s(S^{n-1})}\\&\cdot\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2}2^{j\alpha}2^{(j-k)(n\delta_2-\nu-n/s-\alpha)}(k-j)^m\right.\right.\\&\left.\left.\times\left\|f_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p}.\end{aligned} \tag{65}$$

If $1<p<\infty$, take $1/p+1/p'=1$. Since $n\delta_2-\nu-n/s-\alpha>0$, by the Hölder inequality we have

$$\begin{aligned}J_1&\leqslant C\|b\|_*^m\|\Omega\|_{L^s(S^{n-1})}\\&\times\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2}2^{j\alpha p}\left\|f_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p2^{(j-k)(n\delta_2-\nu-n/s-\alpha)p/2}\right)\right.\end{aligned}$$

$$\times\left(\sum_{j=-\infty}^{k-2}2^{(j-k)(n\delta_2-\nu-n/s-\alpha)p'/2}(k-j)^{mp'}\right)^{p/p'}\Bigg\}^{1/p}$$
$$\leqslant C\|b\|_*^m\|\Omega\|_{L^s(S^{n-1})}$$
$$\times\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2}2^{j\alpha p}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p 2^{(j-k)(n\delta_2-\nu-n/s-\alpha)p/2}\right)\right\}^{1/p}$$
$$=C\|b\|_*^m\|\Omega\|_{L^s(S^{n-1})}\times\left\{\sum_{j=-\infty}^{\infty}2^{j\alpha p}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p\right.$$
$$\left.\cdot\left(\sum_{k=j+2}^{\infty}2^{(j-k)(n\delta_2-\nu-n/s-\alpha)p/2}\right)\right\}^{1/p}\leqslant C\|b\|_*^m$$
$$\cdot\left\{\sum_{j=-\infty}^{\infty}2^{j\alpha p}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p\right\}^{1/p}=C\|b\|_*^m\|f\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)}. \tag{66}$$

If $0<p\leqslant 1$, then we have

$$J_1\leqslant C\|b\|_*^m\|\Omega\|_{L^s(S^{n-1})}$$
$$\times\left\{\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k-2}2^{j\alpha p}2^{(j-k)(n\delta_2-\nu-n/s-\alpha)p}(k-j)^{mp}\right.$$
$$\left.\cdot\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p\right\}^{1/p}=C\|b\|_*^m\|\Omega\|_{L^s(S^{n-1})}$$
$$\times\left\{\sum_{j=-\infty}^{\infty}2^{j\alpha p}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p\right.$$
$$\left.\cdot\left(\sum_{k=j+2}^{\infty}2^{(j-k)(n\delta_2-\nu-n/s-\alpha)p}(k-j)^{mp}\right)\right\}^{1/p}$$
$$\leqslant C\|b\|_*^m\|f\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)}. \tag{67}$$

Let us now estimate J_3. Note that $x\in A_k$, $y\in A_j$, and $j\geqslant k+2$, so we have $|x-y|\sim|y|$. We consider

$$\left|[b^m,\mu_\Omega^\rho](f_j)(x)\right|\leqslant\left(\int_0^{|y|}\left|\int_{|x-y|\leqslant t}\frac{\Omega(x-y)}{|x-y|^{n-\rho}}[b(x)-b(y)]^m f_j(y)\,dy\right|^2\frac{dt}{t^{2\rho+1}}\right)^{1/2}$$
$$+\left(\int_{|y|}^{\infty}\left|\int_{|x-y|\leqslant t}\frac{\Omega(x-y)}{|x-y|^{n-\rho}}[b(x)-b(y)]^m f_j(y)\,dy\right|^2\frac{dt}{t^{2\rho+1}}\right)^{1/2}=:J_{31}+J_{32}. \tag{68}$$

Similar to the estimate for J_{11}, we get

$$J_{31}\leqslant C2^{(k-j)/2}2^{-jn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}$$
$$\cdot\left\{\left|b(x)-b_{B_k}\right|^m\|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\right.$$
$$\left.+\left\|\Omega(x-\cdot)\left(b_{B_k}-b(\cdot)\right)^m\chi_j(\cdot)\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\right\}. \tag{69}$$

Similar to the estimate for J_{12}, we get

$$J_{32}\leqslant C2^{-jn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}$$
$$\cdot\left\{\left|b(x)-b_{B_k}\right|^m\|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\right.$$
$$\left.+\left\|\Omega(x-\cdot)\left(b_{B_k}-b(\cdot)\right)^m\chi_j(\cdot)\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\right\}. \tag{70}$$

So we have

$$\left|[b^m,\mu_\Omega^\rho](f_j)(x)\right|\leqslant C2^{-jn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}$$
$$\cdot\left\{\left|b(x)-b_{B_k}\right|^m\|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\right.$$
$$\left.+\left\|\Omega(x-\cdot)\left(b_{B_k}-b(\cdot)\right)^m\chi_j(\cdot)\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\right\}. \tag{71}$$

By (62), (63), Lemmas 3, 4, and 14 we have

$$\|[b^m,\mu_\Omega^\rho](f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\leqslant C2^{-jn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left(\|b\|_*^m\right.$$
$$\cdot 2^{(k-j)(\nu+n/s)}\times\|\Omega\|_{L^s(S^{n-1})}\|\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}$$
$$+2^{(k-j)(\nu+n/s)}\|\Omega\|_{L^s(S^{n-1})}$$
$$\left.\cdot\left\|\left(b_{B_k}-b(\cdot)\right)^m\chi_{B_j}(\cdot)\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)$$
$$\leqslant C2^{-jn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left(\|b\|_*^m 2^{(k-j)(\nu+n/s)}\times\|\Omega\|_{L^s(S^{n-1})}\right.$$
$$\cdot\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}+(j-k)^m\|b\|_*^m$$
$$\left.\cdot 2^{(k-j)(\nu+n/s)}\|\Omega\|_{L^s(S^{n-1})}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)$$
$$\leqslant C(j-k)^m\|b\|_*^m\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}2^{(k-j)(\nu+n/s)}\|\Omega\|_{L^s(S^{n-1})}$$
$$\cdot\frac{\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}\leqslant C2^{(k-j)(n\delta_1+\nu+n/s)}(j-k)^m\|b\|_*^m$$
$$\cdot\|\Omega\|_{L^s(S^{n-1})}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}. \tag{72}$$

Thus we obtain

$$\begin{aligned} J_3 &\leqslant C\|b\|_*^m \|\Omega\|_{L^s(S^{n-1})} \times \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \right. \\ &\quad \left. \cdot \left(\sum_{j=k+2}^{\infty} 2^{(k-j)(n\delta_1+\nu+n/s)} (j-k)^m \left\| f_j \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \right\}^{1/p} \\ &= C\|b\|_*^m \|\Omega\|_{L^s(S^{n-1})} \times \left\{ \sum_{k=-\infty}^{\infty} \left(\sum_{j=k+2}^{\infty} 2^{j\alpha} \right. \right. \\ &\quad \left. \left. \cdot 2^{(k-j)(n\delta_1+\nu+n/s+\alpha)} (j-k)^m \left\| f_j \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \right\}^{1/p}. \end{aligned} \tag{73}$$

If $1 < p < \infty$, take $1/p + 1/p' = 1$. Since $n\delta_1 + \nu + n/s + \alpha > 0$, by the Hölder inequality we have

$$\begin{aligned} J_3 &\leqslant C\|b\|_*^m \|\Omega\|_{L^s(S^{n-1})} \times \left\{ \sum_{k=-\infty}^{\infty} \left(\sum_{j=k+2}^{\infty} 2^{j\alpha p} \right. \right. \\ &\quad \left. \cdot \left\| f_j \right\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p 2^{(k-j)(n\delta_1+\nu+n/s+\alpha)p/2} \right) \\ &\quad \left. \times \left(\sum_{j=k+2}^{\infty} 2^{(k-j)(n\delta_1+\nu+n/s+\alpha)p'/2} (j-k)^{mp'} \right)^{p/p'} \right\}^{1/p} \\ &\leqslant C\|b\|_*^m \|\Omega\|_{L^s(S^{n-1})} \times \left\{ \sum_{j=-\infty}^{\infty} 2^{j\alpha p} \left\| f_j \right\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \right. \\ &\quad \left. \cdot \left(\sum_{k=-\infty}^{j-2} 2^{(k-j)(n\delta_1+\nu+n/s+\alpha)p/2} \right) \right\}^{1/p} \leqslant C\|b\|_*^m \\ &\quad \cdot \left\{ \sum_{j=-\infty}^{\infty} 2^{j\alpha p} \left\| f_j \right\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \right\}^{1/p} = C\|b\|_*^m \|f\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)}. \end{aligned} \tag{74}$$

If $0 < p \leqslant 1$, then we have

$$\begin{aligned} J_3 &\leqslant C\|b\|_*^m \|\Omega\|_{L^s(S^{n-1})} \times \left\{ \sum_{j=-\infty}^{\infty} 2^{j\alpha p} \left\| f_j \right\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \right. \\ &\quad \left. \cdot \left(\sum_{k=-\infty}^{j-2} 2^{(k-j)(n\delta_1+\nu+n/s+\alpha)p} (j-k)^{mp} \right) \right\}^{1/p} \\ &\leqslant C\|b\|_*^m \|f\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)}. \end{aligned} \tag{75}$$

Thus by (53), (54) and (66), (67), (74), and (75) we complete the proof of Theorem 13. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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