Hindawi
Journal of Function Spaces
Volume 2018, Article ID 7351653, 10 pages
https://doi.org/10.1155/2018/7351653

Research Article

Positive Solutions for a System of Nonlinear Semipositone Boundary Value Problems with Riemann-Liouville Fractional Derivatives

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Received 18 January 2018; Accepted 28 March 2018; Published 6 May 2018

Academic Editor: Gennaro Infante

We study the existence of positive solutions for the system of nonlinear semipositone boundary value problems with Riemann-Liouville fractional derivatives $D_{0+}^{\alpha}D_{0+}^{\alpha}u = f_1(t,u,u',v,v')$, $0 < t < 1$, $D_{0+}^{\alpha}D_{0+}^{\alpha}v = f_2(t,u,u',v,v')$, $0 < t < 1$, $u(0) = u'(0) = u'(1) = D_{0+}^{\alpha}u(0) = D_{0+}^{\alpha+1}u(0) = D_{0+}^{\alpha+1}u(1) = 0$, and $v(0) = v'(0) = v'(1) = D_{0+}^{\alpha}v(0) = D_{0+}^{\alpha+1}v(0) = D_{0+}^{\alpha+1}v(1) = 0$, where $\alpha \in (2,3]$ is a real number and D_{0+}^{α} is the standard Riemann-Liouville fractional derivative of order α. Under some appropriate conditions for semipositone nonlinearities, we use the fixed point index to establish two existence theorems. Moreover, nonnegative concave and convex functions are used to depict the coupling behavior of our nonlinearities.

1. Introduction

In this paper, we investigate the existence of positive solutions for the system of nonlinear semipositone boundary value problems with Riemann-Liouville fractional derivatives

$$\begin{aligned} &D_{0+}^{\alpha}D_{0+}^{\alpha}u = f_1\left(t,u,u',v,v'\right), \quad 0<t<1, \\ &D_{0+}^{\alpha}D_{0+}^{\alpha}v = f_2\left(t,u,u',v,v'\right), \quad 0<t<1, \\ &u(0) = u'(0) = u'(1) = D_{0+}^{\alpha}u(0) = D_{0+}^{\alpha+1}u(0) \\ &\quad = D_{0+}^{\alpha+1}u(1) = 0, \\ &v(0) = v'(0) = v'(1) = D_{0+}^{\alpha}v(0) = D_{0+}^{\alpha+1}v(0) \\ &\quad = D_{0+}^{\alpha+1}v(1) = 0, \end{aligned} \tag{1}$$

where $\alpha \in (2,3]$ is a real number and D_{0+}^{α} is the standard Riemann-Liouville fractional derivative of order α. The nonlinear terms $f_i \in C([0,1]\times\mathbb{R}_+^4,\mathbb{R})$ ($\mathbb{R}_+ = [0,+\infty)$, $\mathbb{R} = (-\infty,+\infty)$) are bounded below; that is, f_i $(i=1,2)$ satisfy the following.

(H1) there exists a real number $M \geq 0$, such that $f_i(t,x_1,x_2,x_3,x_4) + M \geq 0$, $\forall t \in [0,1]$, $x_j \in \mathbb{R}_+$, $i = 1,2$, $j = 1,2,3,4$.

Existence and multiplicity of solutions for fractional differential equations are widely studied in the literature; see [1–14] and the references therein. For example, in [1], the authors used the Guo-Krasnosel'skii fixed point theorem to investigate the existence of positive solutions for the singular fractional differential system

$$\begin{aligned} &-D_{0+}^{\alpha}u(t) = \lambda f_1(t,u,v), \quad t\in(0,1), \\ &-D_{0+}^{\alpha}v(t) = \lambda f_2(t,u,v), \quad t\in(0,1), \\ &u(0) = u'(0) = 0, \\ &v(0) = v'(0) = 0, \\ &u(1) = av(\xi), \\ &v(1) = bu(\eta), \end{aligned} \tag{2}$$

where f_i $(i = 1, 2)$ satisfy

$$\frac{f_i(t,u,v)}{u+v} = 0, \text{ or } \infty, \quad \text{as } u+v \longrightarrow +\infty, \text{ uniformly for } t \in [0,1] \text{ or a subinterval.} \tag{3}$$

Condition (3) is used to study various types of fractional systems (see [1–12] and the references therein).

In this paper we use the fixed point index to study the existence of positive solutions for the system of nonlinear semipositone fractional boundary value problem (1). Under some appropriate conditions for f_i $(i = 1, 2)$, we use the fixed point index to obtain our results. Moreover, nonnegative concave and convex functions are used to depict the coupling behavior of our nonlinearities (see [13–15]), which depend on the unknown functions u, v and their derivatives u', v'.

2. Preliminary

Definition 1 (see [16, 17]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, +\infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s)\, ds, \tag{4}$$

where $n = [\alpha] + 1$ with $[\alpha]$ denoting the integer part of a number α, provided that the right hand side is pointwise defined on $(0, +\infty)$.

We first study the Green functions of problem (1). Let

$$G_1(t,s) := \frac{1}{\Gamma(\alpha)} \cdot \begin{cases} t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-1}(1-s)^{\alpha-2}, & 0 \le t \le s \le 1. \end{cases} \tag{5}$$

Then we have

$$G_2(t,s) := \frac{\partial}{\partial t} G_1(t,s) = \frac{\alpha-1}{\Gamma(\alpha)} \cdot \begin{cases} t^{\alpha-2}(1-s)^{\alpha-2} - (t-s)^{\alpha-2}, & 0 \le s \le t \le 1, \\ t^{\alpha-2}(1-s)^{\alpha-2}, & 0 \le t \le s \le 1. \end{cases} \tag{6}$$

Lemma 2. *Let f_i $(i = 1, 2)$ be as in (1). Then (1) is equivalent to*

$$\begin{aligned} D_{0+}^{\alpha} x &= -f_1\left(t, \int_0^1 G_1(t,s)x(s)\,ds, \int_0^1 G_2(t,s)x(s)\,ds, \int_0^1 G_1(t,s)y(s)\,ds, \int_0^1 G_2(t,s)y(s)\,ds\right), \\ D_{0+}^{\alpha} y &= -f_2\left(t, \int_0^1 G_1(t,s)x(s)\,ds, \int_0^1 G_2(t,s)x(s)\,ds, \int_0^1 G_1(t,s)y(s)\,ds, \int_0^1 G_2(t,s)y(s)\,ds\right), \end{aligned}$$

$$\begin{aligned} &x(0) = x'(0) = x'(1) = 0, \\ &y(0) = y'(0) = y'(1) = 0, \end{aligned} \tag{7}$$

which takes the form

$$\begin{aligned} x(t) &= \int_0^1 G_1(t,s) f_1\left(s, \int_0^1 G_1(s,\tau)x(\tau)\,d\tau, \int_0^1 G_2(s,\tau)x(\tau)\,d\tau, \int_0^1 G_1(s,\tau)y(\tau)\,d\tau, \int_0^1 G_2(s,\tau)y(\tau)\,d\tau\right) ds, \\ y(t) &= \int_0^1 G_1(t,s) f_2\left(s, \int_0^1 G_1(s,\tau)x(\tau)\,d\tau, \int_0^1 G_2(s,\tau)x(\tau)\,d\tau, \int_0^1 G_1(s,\tau)y(\tau)\,d\tau, \int_0^1 G_2(s,\tau)y(\tau)\,d\tau\right) ds. \end{aligned} \tag{8}$$

Let $D_{0+}^{\alpha} u = -x$, $D_{0+}^{\alpha} v = -y$. Then an argument similar to that in [18, Lemma 2.7] and [19, Lemma 3] establishes the result (we omit the standard details).

Lemma 3 ([19, Lemma 4]). *The functions $G_i(t,s) \in C([0,1] \times [0,1], \mathbb{R}_+)$ $(i = 1, 2)$. Moreover, the following inequalities are satisfied:*

$$t^{\alpha-1}s(1-s)^{\alpha-2} \le \Gamma(\alpha)G_1(t,s) \le s(1-s)^{\alpha-2} \quad \forall t,s \in [0,1], \tag{9}$$

$$(\alpha-1)(\alpha-2)t^{\alpha-2}(1-t)s(1-s)^{\alpha-2} \le \Gamma(\alpha)G_2(t,s) \le (\alpha-1)t^{\alpha-3}s(1-s)^{\alpha-2} \quad \forall t,s \in [0,1]. \tag{10}$$

Lemma 4 ([19, Lemma 5]). *Let $\varphi(t) = t(1-t)^{\alpha-2}$ for all $t \in [0,1]$. Let*

$$\begin{aligned} k_1 &:= \frac{\alpha\Gamma(\alpha-1)}{\Gamma(2\alpha)} \le k_2 := \frac{1}{\alpha(\alpha-1)\Gamma(\alpha)}, \\ k_3 &:= \frac{(\alpha-1)(\alpha-2)\Gamma(\alpha)}{\Gamma(2\alpha)} \le k_4 := \frac{\Gamma(\alpha-1)}{\Gamma(2\alpha-2)}. \end{aligned} \tag{11}$$

Then

$$k_{2i-1}\varphi(s) \le \int_0^1 G_i(t,s)\varphi(t)\,dt \le k_{2i}\varphi(s), \quad i = 1, 2, \ \forall s \in [0,1]. \tag{12}$$

Lemma 5. *(i) If $(x_*(t), y_*(t))$ is a positive solution of (7), then $(x_*(t)+w(t), y_*(t)+w(t))$ is a positive solution of the following differential equation:*

$$\begin{gathered} D_{0+}^{\alpha}x = -F_1\left(t, \int_0^1 G_1(t,s)(x(s)-w(s))\,ds, \int_0^1 G_2(t,s)(x(s)-w(s))\,ds, \int_0^1 G_1(t,s)(y(s)-w(s))\,ds, \int_0^1 G_2(t,s)(y(s)-w(s))\,ds\right), \\ D_{0+}^{\alpha}y = -F_2\left(t, \int_0^1 G_1(t,s)(x(s)-w(s))\,ds, \int_0^1 G_2(t,s)(x(s)-w(s))\,ds, \int_0^1 G_1(t,s)(y(s)-w(s))\,ds, \int_0^1 G_2(t,s)(y(s)-w(s))\,ds\right), \\ x(0)=x'(0)=x'(1)=0, \\ y(0)=y'(0)=y'(1)=0, \end{gathered} \tag{13}$$

where

$$F_i(t,x_1,x_2,x_3,x_4) := \begin{cases} \overline{f}_i(t,x_1,x_2,x_3,x_4), & t\in[0,1],\ x_1,x_2,x_3,x_4\ge 0, \\ \overline{f}_i(t,0,0,0,0), & t\in[0,1],\ x_1,x_2,x_3,x_4<0, \end{cases} \tag{14}$$

and $\overline{f}_i(t,x_1,x_2,x_3,x_4) = f_i(t,x_1,x_2,x_3,x_4)+M$, $\overline{f}_i : [0,1]\times\mathbb{R}_+^4 \to \mathbb{R}_+$ are continuous, and

$$w(t) := M\int_0^1 G_1(t,s)\,ds = \frac{M}{\Gamma(\alpha)}\left(\frac{t^{\alpha-1}}{\alpha-1}-\frac{t^{\alpha}}{\alpha}\right) \quad \forall t\in[0,1]. \tag{15}$$

(ii) If $(x(t), y(t))$ is a solution of (13) and $x(t)\ge w(t)$, $y(t)\ge w(t)$, $t\in[0,1]$, then $(x_*(t), y_*(t)) = (x(t)-w(t), y(t)-w(t))$ is a positive solution of (7).

Proof. If $(x_*(t), y_*(t))$ is a positive solution of (7) then (note $w(t) = M\int_0^1 G_1(t,s)\,ds$) we obtain $x_*(0)+w(0) = x_*'(0)+w'(0) = x_*'(1)+w'(1) = 0$ and

$$\begin{aligned} &D_{0+}^{\alpha}(x_*(t)+w(t)) + F_1\left(t, \int_0^1 G_1(t,s)x_*(s)\,ds, \int_0^1 G_2(t,s)x_*(s)\,ds, \int_0^1 G_1(t,s)y_*(s)\,ds, \int_0^1 G_2(t,s)y_*(s)\,ds\right) = D_{0+}^{\alpha}x_*(t) + D_{0+}^{\alpha}w(t) \\ &\quad + f_1\left(t, \int_0^1 G_1(t,s)x_*(s)\,ds, \int_0^1 G_2(t,s)x_*(s)\,ds, \int_0^1 G_1(t,s)y_*(s)\,ds, \int_0^1 G_2(t,s)\times y_*(s)\,ds\right)+M \\ &\quad = D_{0+}^{\alpha}w(t)+M = D_{0+}^{\alpha}M\int_0^1 G_1(t,s)\,ds+M \\ &\quad = -M+M=0. \end{aligned} \tag{16}$$

Similarly, we have

$$D_{0+}^{\alpha}(y_*(t)+w(t)) + F_2\left(t, \int_0^1 G_1(t,s)x_*(s)\,ds, \int_0^1 G_2(t,s)x_*(s)\,ds, \int_0^1 G_1(t,s)y_*(s)\,ds, \int_0^1 G_2(t,s)y_*(s)\,ds\right) = -M+M=0; \tag{17}$$

that is, $(x_*(t)+w(t), y_*(t)+w(t))$ satisfies (13). Therefore, (i) holds. Similarly, it is easy to prove (ii). This completes the proof. □

From Lemma 5, to obtain a positive solution of (7), we only need to find solutions $x(t)$, $y(t)$ of (13) satisfying $(t)\ge w(t)$, $y(t)\ge w(t)$, $t\in[0,1]$. If $x(t)$, $y(t)$ are solutions of (13), then $x(t)$, $y(t)$satisfy

$$\begin{aligned} x(t) = \int_0^1 G_1(t,s)F_1\Big(s, &\int_0^1 G_1(s,\tau)(x(\tau)-w(\tau))\,d\tau, \int_0^1 G_2(s,\tau)(x(\tau)-w(\tau))\,d\tau, \\ &\int_0^1 G_1(s,\tau)(y(\tau)-w(\tau))\,d\tau, \int_0^1 G_2(s,\tau)(y(\tau)-w(\tau))\,d\tau\Big)\,ds, \\ y(t) = \int_0^1 G_1(t,s)F_2\Big(s, &\int_0^1 G_1(s,\tau)(x(\tau)-w(\tau))\,d\tau, \int_0^1 G_2(s,\tau)(x(\tau)-w(\tau))\,d\tau, \end{aligned}$$

$$\int_0^1 G_1(s,\tau)(y(\tau)-w(\tau))\,d\tau,$$
$$\int_0^1 G_2(s,\tau)(y(\tau)-w(\tau))\,d\tau\Bigg)\,ds. \tag{18}$$

Let $E := C[0,1]$, $\|x\| := \max_{t\in[0,1]}|x(t)|$, $P := \{x \in E : x(t) \geq 0,\ t \in [0,1]\}$. Then $(E, \|\cdot\|)$ is a real Banach space, and P is a cone on E. We denote $B_\rho := \{x \in E : \|x\| < \rho\}$ for $\rho > 0$. Now, note that u, v solve (1) if and only if $x := -D_{0+}^{\alpha} u$, $y := -D_{0+}^{\alpha} v$ are fixed points of operator

$$\begin{aligned}
A_i(x,y)(t) &:= \int_0^1 G_1(t,s) F_i\Bigg(s, \\
&\int_0^1 G_1(s,\tau)(x(\tau)-w(\tau))\,d\tau, \\
&\int_0^1 G_2(s,\tau)(x(\tau)-w(\tau))\,d\tau, \\
&\int_0^1 G_1(s,\tau)\times(y(\tau)-w(\tau))\,d\tau, \\
&\int_0^1 G_2(s,\tau)(y(\tau)-w(\tau))\,d\tau\Bigg)\,ds, \\
A(x,y)(t) &= (A_1, A_2)(x,y)(t) \quad \text{for } x, y \in E.
\end{aligned} \tag{19}$$

Therefore, if (x,y) is a positive fixed for A with $x(t) \geq w(t)$, $y(t) \geq w(t)$ for $t \in [0,1]$, then $(x_*, y_*) = (x-w, y-w)$ is a positive solution for (1). Moreover, from the continuity of G_i and F_i $(i = 1,2)$, we know that $A_i : P \times P \to P$, $A : P \times P \to P \times P$ are continuous and completely continuous operators.

Lemma 6. *Let $P_0 := \{x \in P : x(t) \geq t^{\alpha-1}\|x\|,\ t \in [0,1]\}$. Then P_0 is a cone in E and $A(P \times P) \subset P_0^2$.*

Proof. From (9) for $t \in [0,1]$ we have

$$\begin{aligned}
A_1(x,y)(t) &= \int_0^1 G_1(t,s) F_1\Bigg(s, \\
&\int_0^1 G_1(s,\tau)(x(\tau)-w(\tau))\,d\tau, \\
&\int_0^1 G_2(s,\tau)(x(\tau)-w(\tau))\,d\tau, \\
&\int_0^1 G_1(s,\tau)\times(y(\tau)-w(\tau))\,d\tau, \\
&\int_0^1 G_2(s,\tau)(y(\tau)-w(\tau))\,d\tau\Bigg)\,ds \leq \frac{1}{\Gamma(\alpha)} \\
&\cdot \int_0^1 s(1-s)^{\alpha-2} F_1\Bigg(s, \\
&\int_0^1 G_1(s,\tau)(x(\tau)-w(\tau))\,d\tau, \\
&\int_0^1 G_2(s,\tau)(x(\tau)-w(\tau))\,d\tau, \\
&\int_0^1 G_1(s,\tau)\times(y(\tau)-w(\tau))\,d\tau, \\
&\int_0^1 G_2(s,\tau)(y(\tau)-w(\tau))\,d\tau\Bigg)\,ds.
\end{aligned} \tag{20}$$

Also from (9) and the above inequality, for every $(x,y) \in P \times P$, we obtain

$$\begin{aligned}
A_1(x,y)(t) &= \int_0^1 G_1(t,s) F_1\Bigg(s, \\
&\int_0^1 G_1(s,\tau)(x(\tau)-w(\tau))\,d\tau, \\
&\int_0^1 G_2(s,\tau)(x(\tau)-w(\tau))\,d\tau, \\
&\int_0^1 G_1(s,\tau)\times(y(\tau)-w(\tau))\,d\tau, \\
&\int_0^1 G_2(s,\tau)(y(\tau)-w(\tau))\,d\tau\Bigg)\,ds \geq \frac{t^{\alpha-1}}{\Gamma(\alpha)} \\
&\cdot \int_0^1 s(1-s)^{\alpha-2} F_1\Bigg(s, \\
&\int_0^1 G_1(s,\tau)(x(\tau)-w(\tau))\,d\tau, \\
&\int_0^1 G_2(s,\tau)(x(\tau)-w(\tau))\,d\tau, \\
&\int_0^1 G_1(s,\tau)\times(y(\tau)-w(\tau))\,d\tau, \\
&\int_0^1 G_2(s,\tau)(y(\tau)-w(\tau))\,d\tau\Bigg)\,ds \\
&\geq t^{\alpha-1}\|A_1(x,y)\|
\end{aligned} \tag{21}$$

for all $t \in [0,1]$. Similarly, $A_2(x,y)(t) \geq t^{\alpha-1}\|A_2(x,y)\|$. Therefore $A(P \times P) \subset P_0^2$. This completes the proof. □

To obtain a positive solution of (1), we seek a positive fixed point (x^*, y^*) of A with $x^* \geq w$, $y^* \geq w$ (note mean that

$x^*(t) = A_1(x^*, y^*)(t)$, $y^*(t) = A_2(x^*, y^*)(t)$ for $t \in [0,1]$). From Lemma 6, we have $x^*, y^* \in P_0$. For $x^* \in P_0$ we have

$$\begin{aligned} x^*(t) - w(t) &= x^*(t) - M \int_0^1 G_1(t,s)\,ds \\ &= x^*(t) - \frac{M}{\Gamma(\alpha)} \left(\frac{t^{\alpha-1}}{\alpha-1} - \frac{t^\alpha}{\alpha} \right) \\ &= x^*(t) - \frac{M t^{\alpha-1}}{\Gamma(\alpha)} \left(\frac{1}{\alpha-1} - \frac{t}{\alpha} \right) \\ &\geq x^*(t) - \frac{x^*(t)}{\|x^*\|} \frac{M}{(\alpha-1)\Gamma(\alpha)}. \end{aligned} \tag{22}$$

As a result, $x^*(t) \geq w(t)$ for $t \in [0,1]$ if $\|x^*\| \geq M/(\alpha-1)\Gamma(\alpha) := k_5$. Similarly, if $\|y^*\| \geq k_5$, we have $y^*(t) \geq w(t)$, for $t \in [0,1]$.

Lemma 7 (see [20]). *Let $\Omega \subset E$ be a bounded open set and $A : \overline{\Omega} \cap P \to P$ a continuous or completely continuous operator. If there exists $u_0 \in P \setminus \{0\}$ such that $u - Au \neq \mu u_0$ for all $\mu \geq 0$ and $u \in \partial\Omega \cap P$, then $i(A, \Omega \cap P, P) = 0$, where i denotes the fixed point index on P.*

Lemma 8 (see [20]). *Let $\Omega \subset E$ be a bounded open set with $0 \in \Omega$. Suppose $A : \overline{\Omega} \cap P \to P$ is a continuous or completely continuous operator. If $u \neq \mu Au$ for all $u \in \partial\Omega \cap P$ and $0 \leq \mu \leq 1$, then $i(A, \Omega \cap P, P) = 1$.*

3. Main Results

Let $K := \alpha/\Gamma(\alpha) \geq \max_{t,s\in[0,1]}(G_1(t,s) + G_2(t,s))$, In the sequel, we use $c_1, c_2, \ldots$ and $d_1, d_2, \ldots$ to stand for different positive constants. Now, we list our assumptions on F_i ($i = 1, 2$).

(H2) There exist $h, g \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that

(i) h is concave and strictly increasing on $\mathbb{R}_+$ (and $\lim_{x\to+\infty} h(x) = +\infty$);

(ii) there exist $c_1 > 0$, $d_1 > 1/k_1^2(k_1 + k_3)^2$, for all $(t, x_1, x_2, y_1, y_2) \in [0,1] \times \mathbb{R}_+^4$ such that

$$\begin{aligned} F_1(t, x_1, x_2, y_1, y_2) &\geq d_1 h(y_1 + y_2) - c_1, \\ F_2(t, x_1, x_2, y_1, y_2) &\geq g(x_1 + x_2) - c_1; \end{aligned} \tag{23}$$

(iii) $h(K^2 g(x)) \geq K^2 x - c_1$ for $x \in \mathbb{R}_+$.

(H3) For all $(t, x_1, x_2, y_1, y_2) \in [0,1] \times [0, k_5]^4$, there is a constant $M_1 \in (0, k_5 k_2^{-1})$ such that

$$F_i(t, x_1, x_2, y_1, y_2) \leq M_1, \quad i = 1, 2. \tag{24}$$

(H4) There exist $\beta, \gamma \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that

(i) β is convex and strictly increasing on $\mathbb{R}_+$ (and $\lim_{x\to+\infty} \beta(x) = +\infty$);

(ii) for all $(t, x_1, x_2, y_1, y_2) \in [0,1] \times \mathbb{R}_+^4$,

$$\begin{aligned} F_1(t, x_1, x_2, y_1, y_2) &\leq \beta(y_1 + y_2), \\ F_2(t, x_1, x_2, y_1, y_2) &\leq \gamma(x_1 + x_2); \end{aligned} \tag{25}$$

(iii) there exist $d_2 > 0$ such that $\beta(K^2\gamma(x)) \leq K^2 x + d_2$, for $x \in \mathbb{R}_+$.

(H5) There exist $Q : [0,1] \to \mathbb{R}$, $\theta \in (0,1]$, $t_0 \in [\theta, 1]$, for all $(t, x_1, x_2, y_1, y_2) \in [\theta, 1] \times [0, k_5]^4$, such that

$$f_i(t, x_1, x_2, y_1, y_2) + M \geq Q(t), \quad i = 1, 2, \tag{26}$$

where

$$\int_\theta^1 G_1(t_0, s) Q(s)\,ds > \frac{M}{(\alpha-1)\Gamma(\alpha)}. \tag{27}$$

Theorem 9. *Suppose that (H1)–(H3) hold. Then (1) has at least one positive solution.*

Proof. We first prove that there exists $R > k_5$ such that

$$\begin{aligned} &(x, y) \neq A(x, y) + \lambda(\phi, \phi), \\ &\qquad \forall (x, y) \in \partial B_R \cap (P \times P), \ \lambda \geq 0, \end{aligned} \tag{28}$$

where $\phi \in P_0$ is a given function. Suppose there exist $(x, y) \in \partial B_R \cap (P \times P)$, $\lambda \geq 0$ with $(x, y) = A(x, y) + \lambda(\phi, \phi)$, then $x(t) \geq A_1(x, y)(t)$, $y(t) \geq A_2(x, y)(t)$ for $t \in [0,1]$. From (i), (ii) of (H2) we have

$$\begin{aligned} x(t) \geq &\int_0^1 G_1(t,s) F_1 \bigg(s, \int_0^1 G_1(s,\tau)(x(\tau) - w(\tau))\,d\tau, \\ &\int_0^1 G_2(s,\tau)(x(\tau) - w(\tau))\,d\tau, \\ &\int_0^1 G_1(s,\tau)(y(\tau) - w(\tau))\,d\tau, \\ &\int_0^1 G_2(s,\tau)(y(\tau) - w(\tau))\,d\tau \bigg) ds \geq \int_0^1 G_1(t,s) \\ &\cdot \left[d_1 h \left(\int_0^1 [G_1(s,\tau) + G_2(s,\tau)][y(\tau) - w(\tau)]\,d\tau \right) \right. \\ &\left. - c_1 \right] ds \geq d_1 \int_0^1 G_1(t,s) \\ &\cdot h \left(\int_0^1 (G_1(s,\tau) + G_2(s,\tau))\, y(\tau)\,d\tau \right) ds \end{aligned}$$

$$\begin{aligned}
&- d_1 \int_0^1 G_1(t,s) \\
&\cdot h\left(\int_0^1 (G_1(s,\tau)+G_2(s,\tau))\, w(\tau)\, d\tau\right) ds \\
&- c_1 \int_0^1 G_1(t,s)\, ds \ge d_1 \int_0^1 G_1(t,s) \\
&\cdot h\left(\int_0^1 [G_1(s,\tau)+G_2(s,\tau)]\, y(\tau)\, d\tau\right) ds - c_2 \\
&= d_1 \int_0^1 G_1(t,s) \\
&\cdot h\left(\int_0^1 \frac{G_1(s,\tau)+G_2(s,\tau)}{K} K y(\tau)\, d\tau\right) ds - c_2 \\
&\ge d_1 \int_0^1 G_1(t,s) \\
&\cdot \int_0^1 \frac{G_1(s,\tau)+G_2(s,\tau)}{K} h(K y(\tau))\, d\tau\, ds - c_2.
\end{aligned} \tag{29}$$

From (ii) of (H2) we have

$$\begin{aligned}
K y(t) &\ge K \int_0^1 G_1(t,s)\, F_2\left(s, \int_0^1 G_1(s,\tau)\right. \\
&\cdot (x(\tau)-w(\tau))\, d\tau, \int_0^1 G_2(s,\tau) \\
&\cdot (x(\tau)-w(\tau))\, d\tau, \int_0^1 G_1(s,\tau) \\
&\times (y(\tau)-w(\tau))\, d\tau, \int_0^1 G_2(s,\tau) \\
&\left.\cdot (y(\tau)-w(\tau))\, d\tau\right) ds \ge K \int_0^1 G_1(t,s) \\
&\cdot g\left(\int_0^1 [G_1(s,\tau)+G_2(s,\tau)]\right. \\
&\left.\cdot [x(\tau)-w(\tau)]\, d\tau\right) ds - K c_1 \int_0^1 G_1(t, \\
&s)\, ds = K \int_0^1 G_1(t,s) \\
&\cdot g\left(\int_0^1 [G_1(s,\tau)+G_2(s,\tau)]\right. \\
&\left.\cdot [x(\tau)-w(\tau)]\, d\tau\right) ds - c_3.
\end{aligned} \tag{30}$$

From (30) and (i) of (H2) we obtain

$$\begin{aligned}
&h(K y(t) + c_3) \ge h\left(K \int_0^1 G_1(t,s)\right. \\
&\quad \left.\cdot g\left(\int_0^1 [G_1(s,\tau)+G_2(s,\tau)]\, [x(\tau)-w(\tau)]\, d\tau\right) ds\right).
\end{aligned} \tag{31}$$

This together with (iii) of (H2) yields

$$\begin{aligned}
h(K y(t)) &\ge h(K y(t) + c_3) - h(c_3) \ge h\left(K \int_0^1 G_1(t,s)\, g\left(\int_0^1 [G_1(s,\tau)+G_2(s,\tau)]\, [x(\tau)-w(\tau)]\, d\tau\right) ds\right) - h(c_3) \\
&\ge \int_0^1 h\left(\frac{G_1(t,s)}{K} K^2 g\left(\int_0^1 [G_1(s,\tau)+G_2(s,\tau)]\, [x(\tau)-w(\tau)]\, d\tau\right)\right) ds - h(c_3) \\
&\ge \int_0^1 \frac{G_1(t,s)}{K} h\left(K^2 g\left(\int_0^1 [G_1(s,\tau)+G_2(s,\tau)]\, [x(\tau)-w(\tau)]\, d\tau\right)\right) ds - h(c_3) \\
&\ge \int_0^1 K G_1(t,s) \int_0^1 [G_1(s,\tau)+G_2(s,\tau)]\, [x(\tau)-w(\tau)]\, d\tau\, ds - c_4.
\end{aligned} \tag{32}$$

Then (32) is substituted into (29) and we obtain

$$\begin{aligned}
x(t) &\ge d_1 \int_0^1 G_1(t,s) \int_0^1 \frac{G_1(s,\tau)+G_2(s,\tau)}{K} \left[\int_0^1 K G_1(\tau,r) \int_0^1 (G_1(r,l)+G_2(r,l)) \times (x(l)-w(l))\, dl\, dr - c_4\right] d\tau\, ds \\
&- c_2 \ge d_1 \int_0^1 G_1(t,s) \int_0^1 (G_1(s,\tau)+G_2(s,\tau)) \int_0^1 G_1(\tau,r) \int_0^1 (G_1(r,l)+G_2(r,l)) \times (x(l)-w(l))\, dl\, dr\, d\tau\, ds - c_5 \\
&\ge d_1 \int_0^1 G_1(t,s) \int_0^1 (G_1(s,\tau)+G_2(s,\tau)) \int_0^1 G_1(\tau,r) \int_0^1 (G_1(r,l)+G_2(r,l)) \times x(l)\, dl\, dr\, d\tau\, ds - c_6.
\end{aligned} \tag{33}$$

Multiplying by $\varphi(t)$ for (33) and integrating over $[0,1]$, we have

$$\int_0^1 \varphi(t)x(t)\,dt \geq d_1k_1^2(k_1+k_3)^2\int_0^1\varphi(t)x(t)\,dt - c_6k_2\Gamma(\alpha), \tag{34}$$

using the fact that

$$\begin{aligned} k_1\varphi(s) &\leq \int_0^1 G_1(t,s)\varphi(t)\,dt \leq k_2\varphi(s),\\ (k_1+k_3)\varphi(s) &\leq \int_0^1 (G_1(t,s)+G_2(t,s))\varphi(t)\,dt \\ &\leq (k_2+k_4)\varphi(s), \end{aligned} \tag{35}$$

which can be derived from (12) in Lemma 4. From (34) we obtain

$$\int_0^1 \varphi(t)x(t)\,dt \leq \frac{c_6k_2\Gamma(\alpha)}{d_1k_1^2(k_1+k_3)^2-1}. \tag{36}$$

Note that $x\in P_0$ (note that $x=A_1(x,y)+\lambda\phi$ and $A_1(x,y)\in P_0$ from Lemma 6 and $\phi\in P_0$) and we have

$$\begin{aligned} \int_0^1 \varphi(t)t^{\alpha-1}\|x\|\,dt &\leq \int_0^1\varphi(t)x(t)\,dt \\ &\leq \frac{c_6k_2\Gamma(\alpha)}{d_1k_1^2(k_1+k_3)^2-1}, \\ \|x\| &\leq \frac{c_6k_2}{d_1k_1^3(k_1+k_3)^2-k_1}. \end{aligned} \tag{37}$$

From (29) we have

$$\begin{aligned} &d_1\int_0^1 G_1(t,s)\int_0^1\frac{G_1(s,\tau)+G_2(s,\tau)}{K}h(Ky(\tau))\,d\tau\,ds \\ &\quad\leq x(t)+c_2\leq \|x(t)\|+c_2 \\ &\quad\leq \frac{c_6k_2}{d_1k_1^3(k_1+k_3)^2-k_1}+c_2. \end{aligned} \tag{38}$$

Multiplying by $\varphi(t)$ and integrating over $[0,1]$ we obtain

$$\begin{aligned} &\int_0^1 (x(t)+c_2)\varphi(t)\,dt \\ &\geq \int_0^1\varphi(t)d_1\int_0^1 G_1(t,s)\int\frac{G_1(s,\tau)+G_2(s,\tau)}{K}h(Ky(\tau))\,d\tau\,ds\,dt \\ &\geq d_1\int_0^1 k_1\varphi(s)\int_0^1\frac{G_1(s,\tau)+G_2(s,\tau)}{K}h(Ky(\tau))\,ds\,dt \\ &\geq \frac{d_1k_1(k_1+k_3)}{K}\int_0^1\varphi(t)h(Ky(t))\,dt. \end{aligned} \tag{39}$$

Consequently, we have

$$\begin{aligned} &\int_0^1 \varphi(t)h(Ky(t))\,dt \leq \frac{K}{d_1k_1(k_1+k_3)} \\ &\quad\cdot\int_0^1\left(\frac{c_6k_2}{d_1k_1^3(k_1+k_3)^2-k_1}+c_2\right)\varphi(t)\,dt \\ &\quad= \frac{K\Gamma(\alpha)k_2}{d_1k_1(k_1+k_3)}\left(\frac{c_6k_2}{d_1k_1^3(k_1+k_3)^2-k_1}+c_2\right) \\ &\quad:= N_1. \end{aligned} \tag{40}$$

Note that we may assume $y(t)\not\equiv 0$ for $t\in[0,1]$. Then $\|y\|>0$ and $h(K\|y\|)>0$. For $y\in P_0$, we have

$$\begin{aligned} k_1\Gamma(\alpha)K\|y\| &\leq \int_0^1 K\varphi(t)y(t)\,dt \\ &= \frac{K\|y\|}{h(K\|y\|)}\int_0^1\varphi(t)\frac{Ky(t)}{K\|y\|}h(K\|y\|)\,dt \\ &\leq \frac{K\|y\|}{h(K\|y\|)}\int_0^1\varphi(t)h(Ky(t))\,dt\leq\frac{K\|y\|}{h(K\|y\|)}N_1. \end{aligned} \tag{41}$$

Hence, $h(K\|y\|)\leq N_1/k_1\Gamma(\alpha)$. Note $\lim_{z\to+\infty}h(z)=+\infty$, and thus there exists $N_2\geq 0$ such that $\|Ky\|\leq N_2$. Therefore if $(x,y)\in\partial B_R\cap(P\times P)$, $\lambda\geq 0$ with $(x,y)=A(x,y)+\lambda(\phi,\phi)$ then $\|x\|\leq c_6k_2/(d_1k_1^3(k_1+k_3)^2-k_1)$ and $\|y\|\leq N_2/K$. Thus if we take $R>\max\{k_5, c_6k_2/(d_1k_1^3(k_1+k_3)^2-k_1), N_2/K\}$ then (28) is true. Lemma 7 implies

$$i(A, B_R\cap(P\times P), P\times P)=0. \tag{42}$$

Let $x,y\in\partial B_{k_5}\cap P$. From (H3) we have

$$\begin{aligned} A_1(x,y)(t) &= \int_0^1 G_1(t,s)F_1\Bigg(s, \\ &\int_0^1 G_1(s,\tau)(x(\tau)-w(\tau))\,d\tau, \\ &\int_0^1 G_2(s,\tau)(x(\tau)-w(\tau))\,d\tau, \\ &\int_0^1 G_1(s,\tau)\times(y(\tau)-w(\tau))\,d\tau, \\ &\int_0^1 G_2(s,\tau)(y(\tau)-w(\tau))\,d\tau\Bigg)ds\leq\int_0^1 G_1(t, \\ &s)M_1\,ds\leq\int_0^1\frac{s(1-s)^{\alpha-2}}{\Gamma(\alpha)}M_1\,ds<\frac{M}{(\alpha-1)\Gamma(\alpha)} \\ &= k_5=\|x\|, \end{aligned} \tag{43}$$

so $\|A_1(x,y)\|<\|x\|$. Similarly $\|A_2(x,y)\|<\|y\|$. Hence $\|A(x,y)\|<\|(x,y)\|$ for $x,y\in\partial B_{k_5}\cap P$. Thus

$$(x,y) \neq \lambda A(x,y), \quad \forall (x,y) \in \partial B_{k_5} \cap (P \times P), \; \lambda \in [0,1]. \tag{44}$$

It follows from Lemma 8 that

$$i\left(A, B_{k_5} \cap (P \times P), P \times P\right) = 1. \tag{45}$$

From (42) and (45) we have

$$i\left(A, \left(B_R \setminus \overline{B}_{k_5}\right) \cap (P \times P), (P \times P)\right) = 0 - 1 = -1. \tag{46}$$

Therefore the operator A has at least one fixed point in $(B_R \setminus \overline{B}_{k_5}) \cap (P \times P)$ and so (1) has at least a positive solution. This completes the proof. □

Theorem 10. *Suppose that (H1), (H4), and (H5) hold. Then (1) has at least one positive solution.*

Proof. We first show that there exists $R > k_5$ such that

$$(x,y) \neq \lambda A(x,y), \quad \forall (x,y) \in \partial B_R \cap (P \times P), \; \lambda \in [0,1]. \tag{47}$$

Suppose there exist $(x,y) \in \partial B_R \cap (P \times P)$, $\lambda \in [0,1]$ with $(x,y) = \lambda A(x,y)$, then $x(t) \le A_1(x,y)(t)$, $y(t) \le A_2(x,y)(t)$ for $t \in [0,1]$. From (i), (ii) of (H4), we have

$$\begin{aligned} x(t) &\le \int_0^1 G_1(t,s) F_1\Bigl(s, \int_0^1 G_1(s,\tau)(x(\tau) - w(\tau))\,d\tau, \\ &\int_0^1 G_2(s,\tau)(x(\tau) - w(\tau))\,d\tau, \\ &\int_0^1 G_1(s,\tau) \times (y(\tau) - w(\tau))\,d\tau, \\ &\int_0^1 G_2(s,\tau)(y(\tau) - w(\tau))\,d\tau\Bigr) ds \le \int_0^1 G_1(t,s) \\ &\cdot \beta\left(\int_0^1 (G_1(s,\tau) + G_2(s,\tau))(y(\tau) - w(\tau))\,d\tau\right) ds \\ &= \int_0^1 G_1(t,s) \\ &\cdot \beta\left(\int_0^1 \frac{G_1(s,\tau) + G_2(s,\tau)}{K} K(y(\tau) - w(\tau))\,d\tau\right) ds \\ &\le \int_0^1 G_1(t,s) \\ &\cdot \int_0^1 \frac{G_1(s,\tau) + G_2(s,\tau)}{K} \beta(Ky(\tau))\,d\tau\,ds. \end{aligned} \tag{48}$$

From (ii) of (H4), we get

$$\begin{aligned} Ky(t) &\le K \int_0^1 G_1(t,s) F_2\Bigl(s, \int_0^1 G_1(s,\tau) \\ &\cdot (x(\tau) - w(\tau))\,d\tau, \int_0^1 G_2(s,\tau) \\ &\cdot (x(\tau) - w(\tau))\,d\tau, \int_0^1 G_1(s,\tau) \\ &\times (y(\tau) - w(\tau))\,d\tau, \int_0^1 G_2(s,\tau) \\ &\cdot (y(\tau) - w(\tau))\,d\tau\Bigr) ds \le K \int_0^1 G_1(t,s) \\ &\cdot \gamma\Bigl(\int_0^1 (G_1(s,\tau) + G_2(s,\tau)) \\ &\cdot (x(\tau) - w(\tau))\,d\tau\Bigr) ds. \end{aligned} \tag{49}$$

From (49) and (i), (iii) of (H4) we have

$$\begin{aligned} \beta(Ky(t)) &\le \beta\left(K \int_0^1 G_1(t,s) \gamma\left(\int_0^1 (G_1(s,\tau) + G_2(s,\tau))(x(\tau) - w(\tau))\,d\tau\right) ds\right) \\ &\le \int_0^1 \beta\left(\frac{G_1(t,s)}{K} K^2 \gamma\left(\int_0^1 (G_1(s,\tau) + G_2(s,\tau))(x(\tau) - w(\tau))\,d\tau\right)\right) ds \\ &\le \int_0^1 \frac{G_1(t,s)}{K} \beta\left(K^2 \gamma\left(\int_0^1 (G_1(s,\tau) + G_2(s,\tau))(x(\tau) - w(\tau))\,d\tau\right)\right) ds \\ &\le K \int_0^1 G_1(t,s) \left(\int_0^1 (G_1(s,\tau) + G_2(s,\tau))(x(\tau) - w(\tau))\,d\tau + d_2\right) ds \\ &\le K \int_0^1 G_1(t,s) \int_0^1 (G_1(s,\tau) + G_2(s,\tau))(x(\tau) - w(\tau))\,d\tau\,ds + d_3. \end{aligned} \tag{50}$$

Then substitute (50) into (48) and we obtain

$$\begin{aligned} x(t) &\le \int_0^1 G_1(t,s)\int_0^1 \frac{G_1(s,\tau)+G_2(s,\tau)}{K}\beta(Ky(\tau))\,d\tau\,ds \\ &\le \int_0^1 G_1(t,s)\int_0^1 \frac{G_1(s,\tau)+G_2(s,\tau)}{K}\left(K\int_0^1 G_1(\tau,r)\int_0^1 (G_1(r,l)+G_2(r,l))\times(x(l)-w(l))\,dl\,dr+d_3\right)d\tau\,ds \\ &\le \int_0^1 G_1(t,s)\int_0^1 (G_1(s,\tau)+G_2(s,\tau))\int_0^1 G_1(\tau,r)\int_0^1 (G_1(r,l)+G_2(r,l))\times(x(l)-w(l))\,dl\,dr\,d\tau\,ds+d_4. \end{aligned} \tag{51}$$

Multiplying by $\varphi(t)$ for (51) and integrating over $[0,1]$, from (35), we obtain

$$\int_0^1 \varphi(t)x(t)\,dt \le k_2^2(k_2+k_4)^2\int_0^1 \varphi(t)x(t)\,dt + \Gamma(\alpha)k_2d_4. \tag{52}$$

Consequently, we have

$$\int_0^1 \varphi(t)x(t)\,dt \le \frac{\Gamma(\alpha)k_2d_4}{1-k_2^2(k_2+k_4)^2}. \tag{53}$$

Note that $x\in P_0$ (note $x=\lambda A_1(x,y)$ and $A_1(x,y)\in P_0$) and we have

$$\begin{aligned} \int_0^1 \varphi(t)t^{\alpha-1}\|x(t)\|\,dt &\le \int_0^1 \varphi(t)x(t)\,dt \\ &\le \frac{\Gamma(\alpha)k_2d_4}{1-k_2^2(k_2+k_4)^2}, \\ \|x\| &\le \frac{k_2d_4}{k_1-k_1k_2^2(k_2+k_4)^2}. \end{aligned} \tag{54}$$

From (50) and Lemma 3 we have

$$\begin{aligned} \beta(Ky(t)) &\le K\int_0^1 G_1(t,s)\int_0^1 (G_1(s,\tau)+G_2(s,\tau)) \\ &\quad\cdot(x(\tau)-w(\tau))\,d\tau\,ds+d_3 \le K\|x\| \\ &\quad\cdot\int_0^1 G_1(t,s)\int_0^1 (G_1(s,\tau)+G_2(s,\tau))\,d\tau\,ds+d_3 \\ &\le K\|x\|\int_0^1 s(1-s)^{\alpha-2}\int_0^1 \frac{1}{\Gamma(\alpha)}\left(\tau(1-\tau)^{\alpha-2}\right. \\ &\quad\left.+(\alpha-1)s^{\alpha-3}\tau(1-\tau)^{\alpha-2}\right)d\tau\,ds+d_3 \\ &\le \frac{K\|x\|}{\alpha^2(\alpha-1)^2\Gamma(\alpha)^2} \\ &\quad+\frac{\sqrt{\pi}2^{3-2\alpha}\Gamma(\alpha-1)K\|x\|}{(\alpha-1)\Gamma(\alpha-1/2)\alpha^2\Gamma(\alpha)^2}+d_3, \end{aligned} \tag{55}$$

so $\beta(Ky(t))$ is bounded. Note $\lim_{z\to+\infty}\beta(z)=+\infty$, and thus there exists $N_3\ge 0$ such that $\|Ky\|\le N_3$. Therefore if $(x,y)\in \partial B_R\cap(P\times P)$, $\lambda\in[0,1]$ with $(x,y)=\lambda A(x,y)$ then $\|x\|\le k_2d_4/(k_1-k_1k_2^2(k_2+k_4)^2)$ and $\|y\|\le N_3/K$. Thus if we take $R>\max\{k_5, k_2d_4/(k_1-k_1k_2^2(k_2+k_4)^2), N_3/K\}$ then (47) is true. Lemma 8 implies

$$i(A, B_R\cap(P\times P), P\times P)=1. \tag{56}$$

Let $B_{k_5}:=\{x\in P:\|x\|<k_5\}$ and consider $x\in\partial B_{k_5}\cap P$. It follows from (H5) that

$$\begin{aligned} A_1(x,y)(t_0) &= \int_0^1 G_1(t_0,s)F_1\left(s,\int_0^1 G_1(s,\tau)\right. \\ &\quad\cdot(x(\tau)-w(\tau))\,d\tau,\int_0^1 G_2(s,\tau) \\ &\quad\cdot(x(\tau)-w(\tau))\,d\tau,\int_0^1 G_1(s,\tau) \\ &\quad\cdot(y(\tau)-w(\tau))\,d\tau,\int_0^1 G_2(s,\tau) \\ &\quad\left.\cdot(y(\tau)-w(\tau))\,d\tau\right)ds \ge \int_\theta^1 G_1(t_0,s) \\ &\quad\cdot F_1\left(s,\int_0^1 G_1(s,\tau)(x(\tau)-w(\tau))\,d\tau,\int_0^1 G_2(s,\tau)\right. \\ &\quad\cdot(x(\tau)-w(\tau))\,d\tau,\int_0^1 G_1(s,\tau) \\ &\quad\cdot(y(\tau)-w(\tau))\,d\tau,\int_0^1 G_2(s,\tau) \\ &\quad\left.\cdot(y(\tau)-w(\tau))\,d\tau\right)ds = \int_\theta^1 G_1(t_0,s) \\ &\quad\cdot\left[f_1\left(s,\int_0^1 G_1(s,\tau)(x(\tau)-w(\tau))\,d\tau,\right.\right. \\ &\quad\int_0^1 G_2(s,\tau)(x(\tau)-w(\tau))\,d\tau, \end{aligned}$$

$$\int_0^1 G_1(s,\tau)\left(y(\tau)-w(\tau)\right)d\tau,$$
$$\left.\left.\int_0^1 G_2(s,\tau)\left(y(\tau)-w(\tau)\right)d\tau\right)+M\right]ds \tag{57}$$
$$\geq \int_\theta^1 G_1(t_0,s)\,Q(s)\,ds > \frac{M}{(\alpha-1)\Gamma(\alpha)},$$

and, hence, $\|A_1(x,y)\| \geq A_1(x,y)(t_0) > \|x\|$. Similarly $\|A_2(x,y)\| \geq A_2(x,y)(t_0) > \|y\|$ for $y \in \partial B_{k_5} \cap P$. Therefore $\|A(x,y)\| > \|(x,y)\|$ for all $x, y \in \partial B_{k_5} \cap (P \times P)$. Thus

$$(x,y) \neq A(x,y) + \lambda(\phi,\phi), \quad \forall (x,y) \in \partial B_{k_5} \cap (P\times P),\ \phi \in P,\ \lambda \in [0,1]. \tag{58}$$

It follows from Lemma 7 that

$$i\left(A, B_{k_5} \cap (P\times P), P\times P\right) = 0. \tag{59}$$

From (56) and (59), we have

$$i\left(A, \left(B_R \setminus \overline{B}_{k_5}\right) \cap (P\times P), (P\times P)\right) = 1 - 0 = 1. \tag{60}$$

Therefore the operator A has at least one fixed point on $(B_R \setminus \overline{B}_{k_5}) \cap (P\times P)$ and so (1) has at least one positive solution, which completes the proof. □

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The research is supported by the National Natural Science Foundation of China (Grant no. 11601048), Natural Science Foundation of Chongqing (Grant no. cstc2016jcyjA0181), the Science and Technology Research Program of Chongqing Municipal Education Commission (Grant no. KJ1703050), and Natural Science Foundation of Chongqing Normal University (Grant nos. 16XYY24, 15XLB011).

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