Research Article

Fixed-Point Theorems for Systems of Operator Equations and Their Applications to the Fractional Differential Equations

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We study the existence and uniqueness of positive solution for a class of nonlinear binary operator equations systems by means of the cone theory and monotone iterative technique, under more general conditions. Also, we give the iterative sequence of the solution and the error estimation of the system. Moreover, we use this new result to study the existence and uniqueness of the solutions for fractional differential equations systems involving integral boundary value conditions in ordered Banach spaces as an application. The results obtained in this paper are more general than many previous results and complement them.

1. Introduction

In this paper, using the cone theory and monotone iterative technique, we consider the following nonlinear nonmonotone binary equations systems in real Banach spaces,

$$\begin{aligned}x &= A(x, x), \\x &= B(x, x),\end{aligned}\quad (1)$$

where $A, B: D \times D \rightarrow E$, D is a subset of real Banach spaces. There have appeared a series of research results concerning the nonlinear operator equation $x = A(x, x)$, $x = Ax$ [1–7], the sum of several classes of mixed-monotone operator equations, and the nonlinear equations systems (1) [8], in recent years. The techniques they used are cone and semiorde [9, 10], the Granas fixed-point index theory [11], the equivalent classes (which are called components) of a real Banach space [12], the Ishikawa iteration process [13, 14], etc.

In [15], Zhang investigated the existence and uniqueness of solutions for a class of nonlinear operator equations $x = Ax$ in ordered Banach space, by using the cone theory and Banach contraction mapping principle. The assumption they used is

$$-B^{n_0}(x - y) \leq Ax - Ay \leq B^{n_0}(x - y), \quad \forall x, y \in E, \quad x \geq y, \quad (2)$$

where P is a generating normal cone, $A: E \rightarrow E$ is a nonlinear operator, $B: E \rightarrow E$ is a positive linear bounded operator with $r(B) < 1$ (where $r(B)$ is the spectral radius of B), and n_0 is a positive integer.

In [16], Zhang investigated the existence and uniqueness theorems of fixed points to a class of mixed-monotone operators with convexity and concavity, in which suppose that there exist $v > \theta$, $c > 1/2$ such that $\theta < A(v, \theta) \leq v$ and

$$A(\theta, v) \geq cA(v, \theta). \quad (3)$$

But in [15, 16], they did not consider the nonmonotone binary operator equations systems $x = A(x, x)$, $x = B(x, x)$, where $A, B: D \times D \rightarrow E$ are two nonlinear operators and D is the order interval in E . In this article, the existence and uniqueness of positive solution for nonlinear nonmonotone binary operator equations systems are established under more general conditions, even not supposing the generating of cone P . Compared with [16], we do not need the assumption of convexity-concavity. Moreover, the operators we consider are T^{m_0} -increasing in x and the different upper and lower bounds in formula (6) are more general; the results here cannot be obtained by using Banach contraction mapping principle. Also, different from the results in [15], the

iterative sequence of the solution and the error estimation of the system are obtained.

As an application, we study the existence and uniqueness of positive solutions and iterative approximation of the unique solution for the following fractional differential equations involving integral boundary value problems:

$$\begin{aligned}
 -D_{0+}^{\alpha} u(t) &= f_1(t, u(t), v(t), D_{0+}^{\alpha_k} u(s), D_{0+}^{\delta_k} v(s)), \\
 t &\in I, \\
 -D_{0+}^{\alpha} v(t) &= f_2(t, v(t), u(t), D_{0+}^{\alpha_k} v(s), D_{0+}^{\delta_k} u(s)), \\
 t &\in I, \quad n-1 < \alpha \leq n, \\
 u(0) &= D_{0+}^{\gamma_1} u(0) = D_{0+}^{\gamma_2} u(0) = \dots = D_{0+}^{\gamma_{n-2}} u(0) \\
 &= 0, \\
 v(0) &= D_{0+}^{\gamma_1} v(0) = D_{0+}^{\gamma_2} v(0) = \dots = D_{0+}^{\gamma_{n-2}} v(0) \\
 &= 0, \\
 D_{0+}^{\beta_1} u(1) &= \int_0^{\eta} h(s) D_{0+}^{\beta_2} u(s) dA(s) \\
 &\quad + \int_0^1 a(s) D_{0+}^{\beta_3} u(s) dA(s), \\
 D_{0+}^{\beta_1} v(1) &= \int_0^{\eta} h(s) D_{0+}^{\beta_2} v(s) dA(s) \\
 &\quad + \int_0^1 a(s) D_{0+}^{\beta_3} v(s) dA(s),
 \end{aligned} \tag{4}$$

where $n \geq 2$, $D_{0+}^{\alpha} u$, $D_{0+}^{\alpha_k} u$, $D_{0+}^{\delta_k} u$ ($k = 1, 2, \dots, n-2$), $D_{0+}^{\gamma_k} u$, ($k = 1, 2, \dots, n-2$), $D_{0+}^{\beta_i} u$ ($i = 1, 2, 3$) is the standard Riemann-Liouville derivatives and $n-1 < \alpha \leq n$, $k-1 < \alpha_k, \delta_k, \gamma_k \leq k$ ($k = 1, 2, \dots, n-2$), $n-k-1 < \alpha - \alpha_k \leq n-k$, $n-k-1 < \alpha - \delta_k \leq n-k$, $n-k-1 < \alpha - \gamma_k \leq n-k$, ($k = 1, 2, \dots, n-2$), $\beta_1 \geq \beta_2$, $\beta_1 \geq \beta_3$, $\alpha - \beta_1 > 1$, $\beta - \beta_1 > 1$, $\beta_i - \alpha_{n-2} \geq 0$, $\beta_i - \delta_{n-2} \geq 0$ ($i = 1, 2, 3$), $f: I \times E^4 \rightarrow E$ is continuous and $a, h \in C(I, \mathbb{R}^+)$, A is a function of bounded variation, $\int_0^{\eta} D_{0+}^{\beta_2} v(s) dA(s)$, and $\int_0^1 D_{0+}^{\beta_3} v(s) dA(s)$ denote the Riemann-Stieltjes integral with respect to A .

Recently, fractional differential equations, arising in the mathematical modeling of systems and processes, have drawn more and more attention of the research community due to their numerous applications in various fields of science such as engineering, chemistry, physics, and mechanics. Boundary value problems of fractional differential equations have been investigated for many authors [17–22]. Now, there are many papers dealing with the problem for different kinds of boundary value conditions such as multipoint boundary condition [23, 24], integral boundary condition [25–31], and many other boundary conditions [32]. From the application, we can see that the fixed-point theorems in this paper have extensive applied background. The results presented here are more general and complement many previous known results.

2. Preliminaries

Now we present briefly some definitions, lemmas, and basic results that are to be used in the article; for convenience of the reader, we refer the reader to [3, 4, 10, 33, 34] for more details.

Suppose that $(E, \|\cdot\|)$ is a real Banach space, and θ is the zero element of E . Recall that a nonempty closed convex set $P \subset E$ is a cone if it satisfies (1) $x \in P, \lambda \geq 0 \implies \lambda x \in P$; (2) $x \in P, -x \in P \implies x = \theta$. The real Banach space E can be partially ordered by a cone $P \subset E$, i.e., $x \leq y$ if and only if $y - x \in P$. If $x \leq y$ and $x \neq y$, then we denote $x < y$ or $y > x$. Let $C[I, E] = \{x(t): I \rightarrow E \mid x(t) \text{ is continuous}\}$. Then $C[I, E]$ is a Banach space with the norm $\|x\|_c = \max_{t \in I} \|x(t)\|$, for $x \in C[I, E]$.

The cone P is called normal if there exists a constant $N > 0$ such that for all $x, y \in E$, $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$, and the smallest N is called the normality constant of P . If $x_1, x_2 \in E$, $x_1 \leq x_2$, the set $[x_1, x_2] = \{x \in E \mid x_1 \leq x \leq x_2\}$ is called the order interval between x_1 and x_2 .

Definition 1 ([3, 4]). Let D be a subset of a real ordered Banach space E ; $A: D \times D \rightarrow E$ is said to be a mixed-monotone operator if $A(x, y)$ is increasing in x and decreasing in y ; i.e., $x_i, y_i \in D$ ($i = 1, 2$), $x_1 \leq x_2$, $y_1 \geq y_2$ imply $A(x_1, y_1) \leq A(x_2, y_2)$. The element $x \in D$ is called a fixed point of A if $A(x, x) = x$.

3. Main Results

Theorem 2. Let E be a real Banach space, P be a normal cone in E , and $D = [u_0, v_0] = \{x \in E \mid u_0 \leq x \leq v_0\}$ be the order interval in E . Assume that $A, B: D \times D \rightarrow E$ are two nonlinear operators, $T, L: E \rightarrow E$ are two positive linear bounded operators and satisfy the following conditions:

$$(J_0) \quad u_0 \leq A(u_0, v_0), B(v_0, u_0) \leq v_0.$$

(J_1) For all $x \in D$, $A(x, y)$ and $B(x, y)$ are decreasing in y , i.e., for any $x \in D$, $y_1, y_2 \in D$, $y_1 \leq y_2$ implies $A(x, y_1) \geq A(x, y_2)$, $B(x, y_1) \geq B(x, y_2)$; and there exist two positive numbers $M_i > 0$ ($i = 1, 2$) such that for all $y \in D$, $x_1, x_2 \in D$, $x_1 \leq x_2$,

$$A(x_2, y) - A(x_1, y) \geq -T^{m_0}(x_2 - x_1), \tag{5}$$

$$B(x_2, y) - B(x_1, y) \geq -T^{m_0}(x_2 - x_1);$$

(J_2) $I + T^{m_0}$ is reversible (I is the identity operator) and $(I + T^{m_0})x \geq \theta \implies x \in P$.

(J_3) $TL = LT$ and there exist two positive integers m_0, n_0 such that $r[(I + T^{m_0})^{-1}][r(L^{n_0}) + r(T^{m_0})] < 1$ (where $r(\cdot)$ is the spectral radius of linear bounded operator) and

$$\begin{aligned}
 -T^{m_0}(y - x) &\leq B(y, x) - A(x, y) \leq L^{n_0}(y - x), \\
 \forall x, y \in D, \quad x &\leq y.
 \end{aligned} \tag{6}$$

Then the nonlinear operator equations system (1) has a unique solution (x^*, x^*) in $D \times D$. And for any initial values $x_0, y_0 \in D$, $x_0 \leq y_0$, by constructing successively the sequences as follows:

$$\begin{aligned} x_n &= (I + T^{m_0})^{-1} [A(x_{n-1}, y_{n-1}) + T^{m_0} x_{n-1}], \\ y_n &= (I + T^{m_0})^{-1} [B(y_{n-1}, x_{n-1}) + T^{m_0} y_{n-1}], \end{aligned} \quad (7)$$

$$n = 1, 2, \dots,$$

we have $x_n \rightarrow x^*$, $y_n \rightarrow x^*$ in E , as $n \rightarrow \infty$. Moreover, for any $r[(I + T^{m_0})^{-1}][r(L^{n_0}) + r(T^{m_0})] < \delta < 1$, there exists n_1 such that

$$\begin{aligned} \|x_n - x^*\| &\leq 2N\delta^n \|v_0 - u_0\|, \\ \|y_n - x^*\| &\leq 2N\delta^n \|v_0 - u_0\|, \end{aligned} \quad (8)$$

$$n \geq n_1.$$

Proof. Since $I + T^{m_0}$ is reversible, let

$$\begin{aligned} F(x, y) &= (I + T^{m_0})^{-1} [A(x, y) + T^{m_0} x], \\ G(y, x) &= (I + T^{m_0})^{-1} [B(y, x) + T^{m_0} y], \end{aligned} \quad (9)$$

$$x, y \in D.$$

Then (7) can be written as

$$\begin{aligned} x_n &= F(x_{n-1}, y_{n-1}), \\ y_n &= G(y_{n-1}, x_{n-1}), \end{aligned} \quad (10)$$

$$n = 1, 2, \dots$$

By (J_1) , (J_2) , it is easy to prove that F and G satisfy the following conditions:

(A_1) By $(I + T^{m_0}) x \geq \theta \implies x \in P$, we know $F, G: D \times D \rightarrow E$ are two mixed-monotone operators.

(A_2) For all $x, y \in D$,

$$\begin{aligned} G(y, x) - F(x, y) &= (I + T^{m_0})^{-1} [B(y, x) + T^{m_0} y] \\ &\quad - (I + T^{m_0})^{-1} [A(x, y) + T^{m_0} x] \\ &= (I + T^{m_0})^{-1} [(B(y, x) - A(x, y)) + T^{m_0} (y - x)]. \end{aligned} \quad (11)$$

Combining with (J_3) , it is easy to prove that

$$\theta \leq G(y, x) - F(x, y) \leq H(y - x), \quad (12)$$

$$\forall x, y \in D, \quad x \leq y,$$

where $H \triangleq (I + T^{m_0})^{-1}(L^{n_0} + T^{m_0})$ and I is the identity operator.

(A_3) Also, by (J_0) , we have

$$\begin{aligned} F(u_0, v_0) &= (I + T^{m_0})^{-1} [A(u_0, v_0) + T^{m_0} u_0] \\ &\geq (I + T^{m_0})^{-1} [u_0 + T^{m_0} u_0] = u_0, \end{aligned} \quad (13)$$

$$\begin{aligned} G(v_0, u_0) &= (I + T^{m_0})^{-1} [B(v_0, u_0) + T^{m_0} v_0] \\ &\leq (I + T^{m_0})^{-1} [v_0 + T^{m_0} v_0] = v_0. \end{aligned} \quad (14)$$

Thus, combining with (12), we have

$$u_0 \leq F(u_0, v_0) \leq G(v_0, u_0) \leq v_0. \quad (15)$$

Let $u_n = F(u_{n-1}, v_{n-1})$, $v_n = G(v_{n-1}, u_{n-1})$ ($n = 1, 2, \dots$). Thus, by (15), we know

$$u_0 \leq u_1 \leq v_1 \leq v_0. \quad (16)$$

Therefore, by (A_1) and (A_2) , using mathematical induction, we can easily prove that

$$u_0 \leq u_1 \leq \dots \leq u_n \leq \dots \leq v_n \leq \dots \leq v_1 \leq v_0. \quad (17)$$

Firstly, we prove that

$$\theta \leq v_n - u_n \leq H^n(v_0 - u_0), \quad n = 1, 2, \dots \quad (18)$$

By (A_2) , we can easily prove that

$$\theta \leq v_1 - u_1 \leq G(v_0, u_0) - F(u_0, v_0) \leq H(v_0 - u_0), \quad (19)$$

i.e., (18) holds for $n = 1$. Suppose that (18) holds for $n = k$, i.e.,

$$\theta \leq v_k - u_k \leq H^k(v_0 - u_0). \quad (20)$$

Then, for $n = k + 1$, by (A_1) and (A_2) we know

$$\begin{aligned} u_{k+1} &= F(u_k, v_k) \leq G(v_k, u_k) = v_{k+1}, \\ \theta &\leq v_{k+1} - u_{k+1} = G(v_k, u_k) - F(u_k, v_k) \\ &\leq H(v_k - u_k) \leq H^{k+1}(v_0 - u_0). \end{aligned} \quad (21)$$

By (19)-(21), using mathematical induction, we know (18) holds.

Next we prove that $\{x_n\}$ is a Cauchy sequence. From $TL = LT$, we have $(I + T^{m_0})^{-1}(L^{n_0} + T^{m_0}) = (L^{n_0} + T^{m_0})(I + T^{m_0})^{-1}$. Then

$$\begin{aligned} r[(I + T^{m_0})^{-1}(L^{n_0} + T^{m_0})] &\leq r[(I + T^{m_0})^{-1}]r(L^{n_0} + T^{m_0}) \\ &\leq r[(I + T^{m_0})^{-1}][r(L^{n_0}) + r(T^{m_0})] < 1. \end{aligned} \quad (22)$$

Consequently, there exists $\delta > 0$ such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|H^n\|^{1/n} &= r(H) = r[(I + T^{m_0})^{-1}(L^{n_0} + T^{m_0})] \\ &\leq r[(I + T^{m_0})^{-1}][r(L^{n_0}) + r(T^{m_0})] \\ &< \delta < 1. \end{aligned} \quad (23)$$

Thus, there exists n_1 such that

$$\|H^n\| < \delta^n, \quad n \geq n_1. \quad (24)$$

Then by (17), we have

$$\theta \leq u_n \leq u_{n+p} \leq v_{n+p} \leq v_n, \quad n, p = 1, 2, \dots \quad (25)$$

Consequently, by (18) and (25), we have

$$\begin{aligned}\theta &\leq u_{n+p} - u_n \leq v_n - u_n \leq H^n(v_0 - u_0), \\ \theta &\leq v_n - v_{n+p} \leq v_n - u_n \leq H^n(v_0 - u_0),\end{aligned}\quad (26)$$

$$n, p = 1, 2, \dots$$

Therefore, by (24), (26) and the normality of cone P , we have

$$\begin{aligned}\|u_{n+p} - u_n\| &\leq N \|H^n(v_0 - u_0)\| \leq N\delta^n \|v_0 - u_0\|, \\ \|v_n - v_{n+p}\| &\leq N \|H^n(v_0 - u_0)\| \leq N\delta^n \|v_0 - u_0\|,\end{aligned}\quad (27)$$

$$n \geq n_1, \quad p = 1, 2, \dots,$$

where N is the normality constant of P . Consequently, $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences. Since E is complete, thus there exist $u^*, v^* \in D$, such that

$$\begin{aligned}\lim_{n \rightarrow \infty} u_n &= u^*, \\ \lim_{n \rightarrow \infty} v_n &= v^*.\end{aligned}\quad (28)$$

And by (17), we know

$$u_n \leq u^* \leq v^* \leq v_n, \quad n = 0, 1, 2, \dots \quad (29)$$

Thus, $u^*, v^* \in D$. By (18) and (29), we have

$$\begin{aligned}\theta &\leq v^* - u^* \leq v_n - u_n \leq H^n(v_0 - u_0), \\ n &= 0, 1, 2, \dots\end{aligned}\quad (30)$$

Thus, by (26), (30) and the normality of cone P , we have

$$\|v^* - u^*\| \leq N \|H^n(v_0 - u_0)\| \leq N\delta^n \|v_0 - u_0\| \rightarrow 0, \quad (31)$$

$$n \rightarrow \infty.$$

Consequently, $u^* = v^*$. Let $x^* := u^* = v^*$; by (A_1) and (A_2) , we have

$$\begin{aligned}u_{n+1} &= F(u_n, v_n) \leq F(x^*, x^*) \leq G(x^*, x^*) \\ &\leq G(v_n, u_n) = v_{n+1}, \quad n = 1, 2, \dots\end{aligned}\quad (32)$$

Let $n \rightarrow \infty$, by (32) and the normality of P we have $F(x^*, x^*) = G(x^*, x^*) = x^*$. Therefore, by the definitions of F and G , we have $x^* = A(x^*, x^*)$, $x^* = B(x^*, x^*)$; i.e., (x^*, x^*) is the solution of operator equation (1).

Finally, we prove that (x^*, x^*) is the unique solution of operator equations systems (1) in $D \times D$. In fact, suppose $(\bar{x}, \bar{x})$ is another solution of equations systems (1) in $D \times D$, then by (A_1) , using mathematical induction, we can easily see that $u_n \leq \bar{x} \leq v_n$ ($n = 1, 2, \dots$). Thus, by (28) and the normality of P , we have $\bar{x} = x^*$. Therefore, the operator equations systems (1) have a unique solution (x^*, x^*) in $D \times D$.

Now for any initial points $x_0, y_0 \in D$, $x_0 \leq y_0$, we construct successively the sequences

$$\begin{aligned}x_n &= F(x_{n-1}, y_{n-1}), \\ y_n &= G(y_{n-1}, x_{n-1}),\end{aligned}\quad (33)$$

$$n = 1, 2, \dots$$

Since $x_0, y_0 \in D$, i.e.,

$$u_0 \leq x_0 \leq y_0 \leq v_0. \quad (34)$$

Suppose

$$u_{n-1} \leq x_{n-1} \leq y_{n-1} \leq v_{n-1}. \quad (35)$$

Then

$$\begin{aligned}u_n &= F(u_{n-1}, v_{n-1}) \leq x_n = F(x_{n-1}, y_{n-1}) \leq y_n \\ &= G(y_{n-1}, x_{n-1}) \leq v_n = G(v_{n-1}, u_{n-1}).\end{aligned}\quad (36)$$

Thus, by mathematical induction, we have

$$u_n \leq x_n \leq y_n \leq v_n, \quad n = 0, 1, 2, \dots \quad (37)$$

By (18) and (37), we have

$$\theta \leq x_n - u_n \leq v_n - u_n \leq H^n(v_0 - u_0), \quad (38)$$

$$\theta \leq x^* - u_n \leq v_n - u_n \leq H^n(v_0 - u_0). \quad (39)$$

Thus,

$$\begin{aligned}\|x_n - x^*\| &\leq \|x_n - u_n\| + \|x^* - u_n\| \\ &\leq 2N \|H^n(v_0 - u_0)\| \leq 2N\delta^n \|v_0 - u_0\|,\end{aligned}\quad (40)$$

$$n \geq n_1.$$

In the same way, we can prove that

$$\|y_n - x^*\| \leq 2N\delta^n \|v_0 - u_0\|, \quad n \geq n_1. \quad (41)$$

Consequently, by (40) and (41) we know that (8) holds. $\square$

Taking $m_0 = n_0 = 1$ in Theorem 2, then we get the following corollary.

Corollary 3. Let E be a real Banach space, P be a normal cone in E , and $D = [u_0, v_0] = \{x \in E \mid u_0 \leq x \leq v_0\}$ be the order interval in E . Assume that $A, B: D \times D \rightarrow E$ are two nonlinear operators, $T, L: E \rightarrow E$ are two positive linear bounded operators and satisfy the following conditions:

(Q₀) $u_0 \leq A(u_0, v_0)$, $B(v_0, u_0) \leq v_0$.

(Q₁) For all $x \in D$, $A(x, y)$ and $B(x, y)$ are decreasing in y , i.e., for any $x \in D$, $y_1, y_2 \in D$, $y_1 \leq y_2$ implies $A(x, y_1) \geq A(x, y_2)$, $B(x, y_1) \geq B(x, y_2)$; and there exist two positive numbers $M_i > 0$ ($i = 1, 2$) such that for all $y \in D$, $x_1, x_2 \in D$, $x_1 \leq x_2$,

$$\begin{aligned}A(x_2, y) - A(x_1, y) &\geq -T(x_2 - x_1), \\ B(x_2, y) - B(x_1, y) &\geq -T(x_2 - x_1);\end{aligned}\quad (42)$$

(Q₂) $TL = LT$, $r[(I + T)^{-1}][r(L) + r(T)] < 1$ (where $r(\cdot)$ is the spectral radius of linear bounded operator) and

$$\begin{aligned}-T(y - x) &\leq B(y, x) - A(x, y) \leq L(y - x), \\ \forall x, y \in D, x &\leq y.\end{aligned}\quad (43)$$

Then the nonlinear operator equations system (1) has a unique solution (x^*, x^*) in $D \times D$. And for any initial values $x_0, y_0 \in D$, $x_0 \leq y_0$, by constructing successively the sequences as follows:

$$\begin{aligned} x_n &= (I + T)^{-1} [A(x_{n-1}, y_{n-1}) + Tx_{n-1}], \\ y_n &= (I + T)^{-1} [B(y_{n-1}, x_{n-1}) + Ty_{n-1}], \end{aligned} \quad (44)$$

$n = 1, 2, \dots,$

we have $x_n \rightarrow x^*$, $y_n \rightarrow x^*$ in E , as $n \rightarrow \infty$. Moreover, for any $r[(I + T)^{-1}][r(L) + r(T)] < \delta < 1$, there exists n_1 such that

$$\begin{aligned} \|x_n - x^*\| &\leq 2N\delta^n \|v_0 - u_0\|, \quad n \geq n_1, \\ \|y_n - x^*\| &\leq 2N\delta^n \|v_0 - u_0\|, \quad n \geq n_1. \end{aligned} \quad (45)$$

4. An Application for Fractional Differential Equations Involving Integral Boundary Value Problems

Now we present briefly some definitions, lemmas, and basic results that are to be used in the article for convenience of the reader. We refer the reader to [35–39] for more details.

Definition 4 ([35, 36, 38, 39]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u: (0, +\infty) \rightarrow \mathbb{R}$ is given by

$$I_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds \quad (46)$$

provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Definition 5 ([35, 36, 38, 39]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $u: (0, +\infty) \rightarrow \mathbb{R}$ is given by

$$D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_0^t \frac{u(s)}{(t-s)^{\alpha-n+1}} ds, \quad (47)$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number α , provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Lemma 6 ([35, 36, 38, 39]). (1) If $u \in L^1(0, 1)$ and $\alpha > \beta > 0$, then

$$\begin{aligned} I_{0+}^{\alpha} I_{0+}^{\beta} u(t) &= I_{0+}^{\alpha+\beta} u(t), \\ D_{0+}^{\beta} I_{0+}^{\alpha} u(t) &= I_{0+}^{\alpha-\beta} u(t), \\ D_{0+}^{\beta} I_{0+}^{\beta} u(t) &= u(t). \end{aligned} \quad (48)$$

(2) If $u \in L^1(0, 1)$ and $\alpha > 0$, then $D_{0+}^{\alpha} u(t) = 0$ has unique solution

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \dots + c_n t^{\alpha-n}, \quad (49)$$

where $c_i \in \mathbb{R}$ ($i = 0, 1, 2, \dots, n$), $n = [\alpha] + 1$.

Lemma 7 ([35, 36, 38, 39]). Let $\alpha > 0$ and let $f(x)$ be integrable; then

$$I_{0+}^{\alpha} D_{0+}^{\alpha} f(u) = f(u) + c_1 u^{\alpha-1} + c_2 u^{\alpha-2} + \dots + c_n u^{\alpha-n}, \quad (50)$$

where $c_i \in \mathbb{R}$ ($i = 1, 2, \dots, n$) and n is the smallest integer greater than or equal to α .

Lemma 8 ([37]). Let $f \in L^1(0, 1) \cap C(0, 1)$, $n-1 < \alpha \leq n$. Assume that the following condition is satisfied:

$$(H) \quad \frac{\Gamma(\alpha - \beta_1)}{\Gamma(\alpha - \beta_2)} \int_0^{\eta} h(t) t^{\alpha-\beta_2} dA(t) + \frac{\Gamma(\alpha - \beta_1)}{\Gamma(\alpha - \beta_3)} \int_0^1 a(t) t^{\alpha-\beta_3} dA(t) < 1; \quad (51)$$

then

$$\begin{aligned} D_{0+}^{\alpha-\alpha_{n-2}} u(t) + f(t) &= 0, \quad n-1 < \alpha \leq n, \quad 0 < t < 1, \\ D_{0+}^{\gamma_{n-2}-\alpha_{n-2}} u(0) &= 0, \\ D_{0+}^{\beta_1-\alpha_{n-2}} u(1) &= \int_0^{\eta} h(s) D_{0+}^{\beta_2-\alpha_{n-2}} u(s) dA(s) \\ &\quad + \int_0^1 h(s) D_{0+}^{\beta_3-\alpha_{n-2}} u(s) dA(s) \end{aligned} \quad (52)$$

has a unique positive solution

$$u(t) = \int_0^1 G(t, s) f(s) ds, \quad t \in [0, 1], \quad (53)$$

where

$$\begin{aligned} G(t, s) &= G_1(t, s) \\ &\quad + \frac{t^{\alpha-\alpha_{n-2}-1}}{\Gamma(\alpha - \beta_2)} \Delta^{-1} \int_0^{\eta} h(t) G_2(t, s) dA(t) \\ &\quad + \frac{t^{\alpha-\alpha_{n-2}-1}}{\Gamma(\alpha - \beta_3)} \Delta^{-1} \int_0^1 a(t) G_3(t, s) dA(t), \end{aligned} \quad (54)$$

in which

$$\begin{aligned} \delta_1 &:= \int_0^{\eta} h(t) t^{\alpha-\beta_2-1} dA(t), \\ \delta_2 &:= \int_0^{\eta} a(t) t^{\alpha-\beta_3-1} dA(t), \end{aligned}$$

$$\begin{aligned}
\Delta &= \frac{1}{\Gamma(\alpha - \beta_1)} - \frac{1}{\Gamma(\alpha - \beta_2)} \delta_1 - \frac{1}{\Gamma(\alpha - \beta_3)} \delta_2, \\
G_1(t, s) &= \frac{1}{\Gamma(\alpha - \alpha_{n-2})} \\
&\cdot \begin{cases} t^{\alpha - \alpha_{n-2} - 1} (1 - s)^{\alpha - \beta_1 - 1} - (t - s)^{\alpha - \alpha_{n-2} - 1}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha - \alpha_{n-2} - 1} (1 - s)^{\alpha - \beta_1 - 1}, & 0 \leq t \leq s \leq 1, \end{cases} \\
G_2(t, s) &= \frac{1}{\Gamma(\alpha - \alpha_{n-2})} \\
&\cdot \begin{cases} t^{\alpha - \beta_2 - 1} (1 - s)^{\alpha - \beta_1 - 1} - (t - s)^{\alpha - \beta_2 - 1}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha - \beta_2 - 1} (1 - s)^{\alpha - \beta_1 - 1}, & 0 \leq t \leq s \leq 1, \end{cases} \\
G_3(t, s) &= \frac{1}{\Gamma(\alpha - \alpha_{n-2})} \\
&\cdot \begin{cases} t^{\alpha - \beta_3 - 1} (1 - s)^{\alpha - \beta_1 - 1} - (t - s)^{\alpha - \beta_3 - 1}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha - \beta_3 - 1} (1 - s)^{\alpha - \beta_1 - 1}, & 0 \leq t \leq s \leq 1. \end{cases}
\end{aligned} \tag{55}$$

Lemma 9 ([37]). *If the condition (H) in Lemma 8 is satisfied, the Green function $G(t, s)$ has the following properties:*

- (1) $G(t, s) > 0$, for all $t, s \in (0, 1)$.
- (2) For any $t, s \in [0, 1]$, we have

$$\theta \leq t^{\alpha - \alpha_{n-2} - 1} l_1(s) \leq G(t, s) \leq L_1 t^{\alpha - \alpha_{n-2} - 1} \leq L_1, \tag{56}$$

where

$$\begin{aligned}
l_1(s) &= \frac{\Delta^{-1}}{\Gamma(\alpha - \beta_2)} \int_0^\eta h(t) G_2(t, s) dA(t) \\
&+ \frac{\Delta^{-1}}{\Gamma(\alpha - \beta_3)} \int_0^1 a(t) G_3(t, s) dA(t), \\
L_1 &= \frac{1}{\Gamma(\alpha - \alpha_{n-2})} \\
&+ \frac{\Delta^{-1}}{\Gamma(\alpha - \beta_2)} \frac{1}{\Gamma(\alpha - \alpha_{n-2})} \int_0^\eta h(t) dA(t) \\
&+ \frac{\Delta^{-1}}{\Gamma(\alpha - \beta_3)} \frac{1}{\Gamma(\alpha - \alpha_{n-2})} \int_0^1 a(t) dA(t).
\end{aligned} \tag{57}$$

In the following, we need the following assumptions:

(H_0) $\max_{t \in I} \int_0^1 |\bar{G}(t, \tau)| d\tau < 1$, where

$$\begin{aligned}
\bar{G}(t, \tau) &= (r_1 + r_2) G(t, \tau) \\
&+ \frac{r_3}{\Gamma(\alpha_{n-2} - \alpha_k)} \left(\int_\tau^1 G(t, s) (s - \tau)^{\alpha_{n-2} - \alpha_k - 1} ds \right) \\
&+ \frac{r_4}{\Gamma(\delta_{n-2} - \delta_k)} \left(\int_\tau^1 G(t, s) (s - \tau)^{\delta_{n-2} - \delta_k - 1} ds \right);
\end{aligned} \tag{58}$$

(H_1) $f_i \in C[I \times E^4, E]$ and there exists $v_0 \in C[I, E]$ such that

$$\begin{aligned}
&\int_0^1 G(t, s) f_2(s, v_0(s), \theta, I_{0+}^{\alpha_{n-2} - \alpha_k} v_0(s), \theta) ds \\
&\leq v_0(t), \quad \forall t \in I;
\end{aligned} \tag{59}$$

(H_2) For all $x_i, y_i \in E$ ($i = 1, 2, 3, 4$), $y_1 \geq x_1$, $y_2 \leq x_2$, $y_3 \geq x_3$, $y_4 \leq x_4$,

$$\begin{aligned}
&f_i(t, y_1, y_2, y_3, y_4) - f_i(t, x_1, x_2, x_3, x_4) \geq 0, \\
&\forall t \in I, \quad i = 1, 2;
\end{aligned} \tag{60}$$

(H_3) There exist four constants $r_i > 0$ ($i = 1, 2, 3, 4$) such that for any $t \in I$, $x_i, y_i \in E$ ($i = 1, 2, 3, 4$) with $x_1 \leq y_1$, $x_2 \geq y_2$, $x_3 \leq y_3$, $x_4 \geq y_4$,

$$\begin{aligned}
&0 \leq f_2(t, y_1, y_2, y_3, y_4) - f_1(t, x_1, x_2, x_3, x_4) \\
&\leq r_1(y_1 - x_1) + r_2(x_2 - y_2) + r_3(y_3 - x_3) \\
&+ r_4(x_4 - y_4).
\end{aligned} \tag{61}$$

Theorem 10. *Let E be a real Banach space and P be a normal cone in E . Assume that the conditions (H) and (H_0) – (H_3) are satisfied. Then the system of nonlinear differential equations (50) has unique positive symmetry solution $(w^*, w^*) \in D \times D$, where $D = [\theta, v_0] \subset C[I, E]$. Moreover, for any initial functions $x_0, y_0 \in D$, there exist monotone iteration sequences $\{x_n\}, \{y_n\}$, such that $x_n \rightarrow w^*$, $y_n \rightarrow w^*$ in $C[I, E]$, as $n \rightarrow \infty$, where*

$$\begin{aligned}
x_n(t) &= \int_0^1 G(t, s) f_1(s, x_{n-1}(s), y_{n-1}(s), \\
&I_{0+}^{\alpha_{n-2} - \alpha_k} x_{n-1}(s), I_{0+}^{\delta_{n-2} - \delta_k} y_{n-1}(s)) ds, \\
y_n(t) &= \int_0^1 G(t, s) f_2(s, y_{n-1}(s), x_{n-1}(s), \\
&I_{0+}^{\alpha_{n-2} - \alpha_k} y_{n-1}(s), I_{0+}^{\delta_{n-2} - \delta_k} x_{n-1}(s)) ds, \\
&t \in I, \quad n = 1, 2, 3, \dots
\end{aligned} \tag{62}$$

Proof. It is well known that $(u, v) \in C[I, E] \times C[I, E]$ is a solution of the system (50) if and only if $(u, v) \in C[I, E] \times C[I, E]$ is a solution of the system of nonlinear integral equations

$$\begin{aligned}
u(t) &= \int_0^1 G(t, s) f_1(s, u(s), v(s), I_{0+}^{\alpha_{n-2} - \alpha_k} u(s), \\
&I_{0+}^{\delta_{n-2} - \delta_k} v(s)) ds, \\
v(t) &= \int_0^1 G(t, s) f_2(s, v(s), u(s), I_{0+}^{\alpha_{n-2} - \alpha_k} v(s), \\
&I_{0+}^{\delta_{n-2} - \delta_k} u(s)) ds, \\
&t \in I.
\end{aligned} \tag{63}$$

Consider the operators $A, B: D \times D \longrightarrow C[I, E]$ as follows:

$$\begin{aligned} A(u, v)(t) &= \int_0^1 G(t, s) f_1(s, u(s), v(s), I_{0+}^{\alpha_{n-2}-\alpha_k} u(s), \\ &\quad I_{0+}^{\delta_{n-2}-\delta_k} v(s)) ds, \\ B(v, u)(t) &= \int_0^1 G(t, s) f_2(s, v(s), u(s), I_{0+}^{\alpha_{n-2}-\alpha_k} v(s), \\ &\quad I_{0+}^{\delta_{n-2}-\delta_k} u(s)) ds, \end{aligned} \quad (64)$$

$$u, v \in D, \quad t \in I.$$

By (H_2) , for all $(u_1, v_1), (u_2, v_2) \in D \times D$, $u_1 \leq u_2$, $v_1 \geq v_2$, $t \in I$, we obtain

$$\begin{aligned} A(u_2, v_2)(t) - A(u_1, v_1)(t) &= \int_0^1 G(t, s) f_1(s, u_2(s), \\ &\quad v_2(s), I_{0+}^{\alpha_{n-2}-\alpha_k} u_2(s), I_{0+}^{\delta_{n-2}-\delta_k} v_2(s)) ds - \int_0^1 G(t, \\ &\quad s) f_1(s, u_1(s), v_1(s), I_{0+}^{\alpha_{n-2}-\alpha_k} u_1(s), \\ &\quad I_{0+}^{\delta_{n-2}-\delta_k} v_1(s)) ds \geq 0, \end{aligned} \quad (65)$$

$$\begin{aligned} B(u_2, v_2)(t) - B(u_1, v_1)(t) &= \int_0^1 G(t, s) f_2(s, u_2(s), \\ &\quad v_2(s), I_{0+}^{\alpha_{n-2}-\alpha_k} u_2(s), I_{0+}^{\delta_{n-2}-\delta_k} v_2(s)) ds - \int_0^1 G(t, \\ &\quad s) f_2(s, u_1(s), v_1(s), I_{0+}^{\alpha_{n-2}-\alpha_k} u_1(s), \\ &\quad I_{0+}^{\delta_{n-2}-\delta_k} v_1(s)) ds \geq 0. \end{aligned} \quad (66)$$

Consequently, $A, B: D \times D \longrightarrow C[I, E]$ are mixed-monotone. By (H_1) , for all $t \in I$, we obtain

$$\begin{aligned} A(\theta, v_0)(t) &= \int_0^1 G(t, s) f_1(s, 0, v_0(s), 0, I_{0+}^{\delta_{n-2}-\delta_k} v_0(s)) ds \\ &\geq 0, \end{aligned} \quad (67)$$

$$\begin{aligned} B(v_0, \theta)(t) &= \int_0^1 G(t, s) f_2(s, v_0(s), 0, I_{0+}^{\alpha_{n-2}-\alpha_k} v_0(s), 0) ds \\ &\leq v_0(t). \end{aligned} \quad (68)$$

By (H_3) , for any $t \in I$, $u, v \in D$ with $u \leq v$,

$$\begin{aligned} B(v, u)(t) - A(u, v)(t) &= \int_0^1 G(t, s) \\ &\quad \cdot f_2(s, v(s), u(s), I_{0+}^{\alpha_{n-2}-\alpha_k} v(s), I_{0+}^{\delta_{n-2}-\delta_k} u(s)) ds \\ &\quad - \int_0^1 G(t, s) \end{aligned}$$

$$\begin{aligned} &\cdot f_1(s, u(s), v(s), I_{0+}^{\alpha_{n-2}-\alpha_k} u(s), I_{0+}^{\delta_{n-2}-\delta_k} v(s)) ds \\ &\geq 0. \end{aligned} \quad (69)$$

$$\begin{aligned} B(v, u)(t) - A(u, v)(t) &= \int_0^1 G(t, s) \\ &\quad \cdot f_2(s, v(s), u(s), I_{0+}^{\alpha_{n-2}-\alpha_k} v(s), I_{0+}^{\delta_{n-2}-\delta_k} u(s)) ds \\ &\quad - \int_0^1 G(t, s) \\ &\quad \cdot f_1(s, u(s), v(s), I_{0+}^{\alpha_{n-2}-\alpha_k} u(s), I_{0+}^{\delta_{n-2}-\delta_k} v(s)) ds \\ &\leq \int_0^1 G(t, s) [r_1(v(s) - u(s)) + r_2(v(s) - u(s)) \\ &\quad + r_3(I_{0+}^{\alpha_{n-2}-\alpha_k} v(s) - I_{0+}^{\alpha_{n-2}-\alpha_k} u(s)) \\ &\quad + r_4(I_{0+}^{\delta_{n-2}-\delta_k} v(s) - I_{0+}^{\delta_{n-2}-\delta_k} u(s))] ds \\ &= \int_0^1 (r_1 + r_2) G(t, s) (v(s) - u(s)) ds + \int_0^1 r_3 G(t, s) \\ &\quad \cdot I_{0+}^{\alpha_{n-2}-\alpha_k} (v(s) - u(s)) ds + \int_0^1 r_4 G(t, s) \\ &\quad \cdot I_{0+}^{\delta_{n-2}-\delta_k} (v(s) - u(s)) ds \\ &= \int_0^1 (r_1 + r_2) G(t, s) (v(s) - u(s)) ds + \int_0^1 r_3 G(t, s) \\ &\quad \cdot \left(\frac{1}{\Gamma(\alpha_{n-2} - \alpha_k)} \int_0^s (s - \tau)^{\alpha_{n-2}-\alpha_k-1} \right. \\ &\quad \cdot (v(\tau) - u(\tau)) d\tau \Big) ds + \int_0^1 r_4 G(t, s) \\ &\quad \cdot \left(\frac{1}{\Gamma(\delta_{n-2} - \delta_k)} \int_0^s (s - \tau)^{\delta_{n-2}-\delta_k-1} \right. \\ &\quad \cdot (v(\tau) - u(\tau)) d\tau \Big) ds \\ &= \int_0^1 (r_1 + r_2) G(t, s) (v(s) - u(s)) ds \\ &\quad + \int_0^1 \frac{r_3}{\Gamma(\alpha_{n-2} - \alpha_k)} \left(\int_\tau^1 G(t, s) \right. \\ &\quad \cdot (s - \tau)^{\alpha_{n-2}-\alpha_k-1} ds \Big) (v(\tau) - u(\tau)) d\tau \\ &\quad + \int_0^1 \frac{r_4}{\Gamma(\delta_{n-2} - \delta_k)} \left(\int_\tau^1 G(t, s) (s - \tau)^{\delta_{n-2}-\delta_k-1} ds \right) \\ &\quad \cdot (v(\tau) - u(\tau)) d\tau \\ &= L(v - u)(t), \end{aligned} \quad (70)$$

where

$$\begin{aligned} Lu(t) = & \int_0^1 \left[(r_1 + r_2) G(t, \tau) \right. \\ & + \frac{r_3}{\Gamma(\alpha_{n-2} - \alpha_k)} \left(\int_\tau^1 G(t, s) (s - \tau)^{\alpha_{n-2} - \alpha_k - 1} ds \right) \\ & + \frac{r_4}{\Gamma(\delta_{n-2} - \delta_k)} \left(\int_\tau^1 G(t, s) (s - \tau)^{\delta_{n-2} - \delta_k - 1} ds \right) \Big] \\ & \cdot u(\tau) d\tau, \quad \forall t \in I. \end{aligned} \quad (71)$$

Set

$$\begin{aligned} \overline{G}(t, \tau) \\ = (r_1 + r_2) G(t, \tau) \\ + \frac{r_3}{\Gamma(\alpha_{n-2} - \alpha_k)} \left(\int_\tau^1 G(t, s) (s - \tau)^{\alpha_{n-2} - \alpha_k - 1} ds \right) \\ + \frac{r_4}{\Gamma(\delta_{n-2} - \delta_k)} \left(\int_\tau^1 G(t, s) (s - \tau)^{\delta_{n-2} - \delta_k - 1} ds \right), \end{aligned} \quad (72)$$

$\forall t \in I;$

then $Lu(t) = \int_0^1 \overline{G}(t, \tau) u(\tau) d\tau$. Consequently, for any $t \in I$, for any $u, v \in D$ with $u \leq v$,

$$0 \leq B(v, u)(t) - A(u, v)(t) \leq L(v - u)(t). \quad (73)$$

In the following, we prove $r(L) < 1$. In fact, by (H_0) , since $\max_{t \in I} \int_0^1 |\overline{G}(t, s)| ds < 1$, there exists a constant $m_1 : 0 < m_1 < 1$ such that $\int_0^1 |G(t, s)| ds \leq m_1 < 1$, for any $t \in I$. Thus, for all $t \in I, u \in D$,

$$\begin{aligned} \|(Lu)(t)\| &= \left\| \int_0^1 \overline{G}(t, s) u(s) ds \right\| \\ &\leq \int_0^1 \|\overline{G}(t, s) u(s)\| ds \\ &\leq \int_0^1 |\overline{G}(t, s)| ds \|u\|_c \\ &\leq m_1 \|u\|_c, \end{aligned} \quad (74)$$

$$\begin{aligned} \|(L^2 u)(t)\| &= \left\| \int_0^1 \overline{G}(t, s) (Lu)(s) ds \right\| \\ &\leq \int_0^1 \|\overline{G}(t, s) (Lu)(s)\| ds \\ &\leq \int_0^1 |\overline{G}(t, s)| \|(Lu)(s)\| ds \\ &\leq \left(\int_0^1 |\overline{G}(t, s)| ds \right) m_1 \|u\|_c \\ &\leq m_1^2 \|u\|_c. \end{aligned} \quad (75)$$

By mathematical induction, we can easily prove that for all natural number n ,

$$\|(L^n u)(t)\| \leq m_1^n \|u\|_c, \quad t \in I; \quad (76)$$

therefore,

$$\|L^n u\|_c = \max_{t \in I} \|(L^n u)(t)\| \leq m_1^n \|u\|_c, \quad (77)$$

consequently,

$$\|L^n\| \leq m_1^n, \quad (78)$$

thus, $r(L) = \lim_{n \rightarrow \infty} \|L^n\|^{1/n} \leq m_1 < 1$.

Thus, all conditions of Corollary 3 are satisfied; therefore, the conclusions of Theorem 10 hold. Consequently, the proof of Theorem 10 is completed. $\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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