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Research Article

Antiperiodic Boundary Value Problems for Impulsive Fractional Functional Differential Equations via Conformable Derivative

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In this paper, by using the lower and upper solution method and the monotone iterative technique, we investigate the existence of solutions to antiperiodic boundary value problems for impulsive fractional functional equations via a recent novel concept of conformable fractional derivative. An example is given to illustrate our theoretical results.

1. Introduction

In the past decades, fractional differential equations play important roles in describing many phenomena and processes occurring in engineering and scientific disciplines; for instance, see [1–7]. Besides the research of fractional differential equations, the impulsive differential equation is found to be an effective tool to study some problems of medicine, engineering, biology, and physics [8–10]. In various fields, such as physics, engineering, and chemistry, many models come down to antiperiodic boundary value problems, so there have been many papers focused on the subject of fractional differential equations with impulsive antiperiodic boundary value conditions; one can refer to [11–14].

Recently, Khalil et al. [15] introduced the conformable fractional derivative, which was a new well-behaved definition, depending just on the basic limit definition of the derivative. For details and applications of this concept, see [16–18]. Fractional differential equations via conformable fractional derivatives have recently received considerable attention; see [19–22] and the references therein. In [23], the authors studied the following periodic boundary value problem for impulsive conformable fractional integrodifferential equations:

$$\begin{aligned} {}_{t_k}D^{\alpha}x(t) &= f(t, x(t), (Fx)(t), (Sx)(t)), \\ &\qquad t \in J = [0,T], \quad t \neq t_k, \\ \Delta x|_{t=t_k} &= I_k(x(t_k)), \quad k = 1, 2, \ldots, m, \\ x(0) &= x(T), \end{aligned} \tag{1}$$

where ${}_aD^{\alpha}$ denotes the conformable fractional derivative of order $0 < \alpha \leq 1$ starting from $a \in \{t_0, \ldots, t_m\}$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $f \in C(J \times \mathbb{R}^3, \mathbb{R})$:

$$\begin{aligned} (Fx)(t) &= \int_0^t l(t,s)\, x(s)\, ds, \\ (Sx)(t) &= \int_0^T h(t,s)\, x(s)\, ds, \\ & l \in C(D, \mathbb{R}^+), \ D = \left\{(t,s) \in J^2 : t \geq s\right\}, \ h \in C\left(J^2, \mathbb{R}^+\right), \ I_k \in C(\mathbb{R}, \mathbb{R}), \ \Delta x(t_k) = x(t_k^+) - x(t_k^-). \end{aligned} \tag{2}$$

They obtained the existence of solutions for (1) by using the monotone iterative method.

Motivated by the above-mentioned work and a recent paper [24], in this article, we discuss the existence of solutions to antiperiodic boundary value problems for impulsive conformable fractional functional differential equations:

$$\begin{aligned} {}_{t_k}D^{\alpha}x(t) &= f(t, x(t), x(w(t))), \\ &\qquad t \in J = [0,T], \quad t \neq t_k, \\ \Delta x|_{t=t_k} &= I_k(x(t_k)), \quad k = 1,2,\ldots,m, \\ x(0) &= -x(T), \\ x(t) &= x(0), \quad t \in [-r,0], \end{aligned} \tag{3}$$

where $f \in C(J \times \mathbb{R}^2, \mathbb{R})$, ${}_aD^{\alpha}$ denotes the conformable fractional derivative of order $0 < \alpha \leq 1$ starting from $a \in \{t_0, \ldots, t_m\}$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $w \in C(J, J^+)$, $t - r \leq w(t)$, $t \in J$ and $t_k < w(t) \leq t$, $t \in (t_k, t_{k+1}]$, $J^+ = [-r, T]$, $r > 0$, $I_k \in C(\mathbb{R}, \mathbb{R})$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$.

To our knowledge, the work on the antiperiodic boundary value problems for impulsive fractional conformable functional differential equations is not to be initiated yet. By applying the method of lower and upper solutions coupled with the monotone iterative technique, we obtain the existence of extreme solutions for problem (3).

The rest of this paper is arranged as follows. Section 2 contains some preliminary notations, definitions, and basic results about conformable fractional calculus. In Section 3, we establish comparison principle and define the upper and lower solutions, and then we obtain the existence of extreme solutions for problem (3) by means of the monotone iterative technique. Finally, in Section 4, an example is given to show the effectiveness of the results obtained.

Remark 1. If $\alpha = 1$, then BVP (3) is reduced to BVP (1) in [24].

2. Preliminaries

Let $J^- = J^+ \setminus \{t_1, t_2, \ldots, t_m\}$, $J_0 = [t_0, t_1]$, $J_k = (t_k, t_{k+1}]$ for $k = 1, 2, \ldots, m$ be subinterval of J:

$$\begin{aligned} &PC(J^+, \mathbb{R}) = \{x : J^+ \longrightarrow \mathbb{R} : x(t) \\ &\quad \text{is continuous everywhere except for some } t_k \text{ at which } x(t_k^-) \\ &\quad \text{and } x(t_k^+) \text{ exist and } x(t_k^-) = x(t_k), \ k = 1,2,\ldots,m\}. \end{aligned} \tag{4}$$

And let $E = \{x \in PC(J^+, \mathbb{R}) : x(t) = x(0), t \in [-r, 0]\}$ with norm

$$\|x\|_E = \sup_{t \in J^+} |x(t)|, \tag{5}$$

and then E is a Banach space. A function $x \in E$ is called a solution of problem (3) if it satisfies (3).

For $a, b \in J$ with $a \leq b$, new notations are introduced in [23] as follows:

$$e^{M/\alpha}(a,b) = \begin{cases} e^{(M/\alpha)(b-a)^{\alpha}}, & a, b \in (t_i, t_{i+1}], \quad i = 0,1,\ldots,m, \\ e^{(M/\alpha)(b-t_i)^{\alpha}} \cdot e^{(M/\alpha)(t_i-a)^{\alpha}}, & a \in (t_{i-1}, t_i], \quad b \in (t_i, t_{i+1}], \ i = 1,\ldots,m, \\ e^{(M/\alpha)(b-t_q)^{\alpha}} \cdot \prod\limits_{a<t_{i-1}<t_i<b} e^{(M/\alpha)(t_i-t_{i-1})^{\alpha}} \cdot e^{(M/\alpha)(t_p-a)^{\alpha}}, & a < t_{i-1} < t_i < b, \quad i = 2,\ldots,m, \end{cases} \tag{6}$$

where $t_q = \max\{t_k; a < t_k < b\}$, $t_p = \min\{t_k; a < t_k < b\}$ for some $k \in \{1, 2, \ldots, m\}$ and $\prod_0(\cdot) = 1$.

Let $h : J \longrightarrow \mathbb{R}$ be a function given by

$$h(t) = \begin{cases} h_0(t), & t \in J_0, \\ h_1(t), & t \in J_1, \\ \vdots \\ h_m(t), & t \in J_m. \end{cases} \tag{7}$$

The impulsive integral notation is defined as

$$\begin{aligned} \int_a^b h(s)\,\widehat{d}s &= \int_a^{t_p} h_{p-1}(s)\,ds + \int_{t_p}^{t_{p+1}} h_p(s)\,ds + \cdots \\ &\quad + \int_{t_q}^{b} h_q(s)\,ds, \quad a, b \in J, \end{aligned} \tag{8}$$

where $a \leq t_p < \cdots < t_q \leq b$.

Property 2 (see [17]). Let $a \leq c \leq b \leq d$ be nonnegative real numbers. The following relations hold:

(i) $e^{M/\alpha}(a,c)e^{M/\alpha}(c,b) = e^{M/\alpha}(a,b)$,

(ii) $e^{M/\alpha}(a,b)e^{M/\alpha}(c,d) = e^{M/\alpha}(a,d)e^{M/\alpha}(c,b)$.

Definition 3 (see [17]). The conformable fractional derivative starting from a point a of a function $f : [a, \infty) \longrightarrow \mathbb{R}$ of order $0 < \alpha \leq 1$ is defined by

$${}_aD^{\alpha}f(t) = \lim_{\varepsilon \longrightarrow 0} \frac{f\left(t + \varepsilon(t-a)^{1-\alpha}\right) - f(t)}{\varepsilon}, \tag{9}$$

provided that the limit exists.

Definition 4 (see [17]). Let $\alpha \in (0,1]$. The conformable fractional integral starting from a point a of a function $f : [a,\infty) \longrightarrow \mathbb{R}$ of order α is defined as

$$ {}_aI^{\alpha} f(t) = \int_a^t (s-a)^{\alpha-1} f(s)\, ds. \tag{10} $$

Remark 5. If f is differentiable, then ${}_aD^{\alpha} f(t) = (t-a)^{1-\alpha} f'(t)$. In addition, if ${}_aD^{\alpha} f$ exists on $[a,\infty)$, then we say that f is α-differentiable on $[a,\infty)$.

Lemma 6 (see [17]). *Let $\alpha \in (0,1]$, $k_1, k_2, p, \lambda \in \mathbb{R}$, and the functions f, g be α-differentiable on $[a,\infty)$. Then*

(i) ${}_aD^{\alpha}(k_1 f + k_2 g) = k_1\, {}_aD^{\alpha}(f) + k_2\, {}_aD^{\alpha}(g)$

(ii) ${}_aD^{\alpha}(t-a)^p = p(t-a)^{p-\alpha}$

(iii) ${}_aD^{\alpha}\lambda = 0$ *for all constant functions* $f(t) = \lambda$

(iv) ${}_aD^{\alpha}(fg) = f\, {}_aD^{\alpha} g + g\, {}_aD^{\alpha} f$

(v) ${}_aD^{\alpha}(f/g) = (g\, {}_aD^{\alpha} f - f\, {}_aD^{\alpha} g)/g^2$ *for all functions* $g(t) \neq 0$

Theorem 7 (see [23]). *Let an interval $[c,d] \subset [a,\infty)$ and let $f : [a,\infty) \longrightarrow \mathbb{R}$ be given function satisfying the following:*

(i) *f is continuous on $[c,d]$*

(ii) *f is α-differentiable for some $\alpha \in (0,1)$*

Then there exists a constant $e \in (c,d)$, such that ${}_aD^{\alpha} f(e) = (f(d) - f(c))/((1/\alpha)(d-a)^{\alpha} - (1/\alpha)(c-a)^{\alpha})$.

3. Main Results

We now consider the following antiperiodic boundary value problem:

$$ \begin{aligned} {}_{t_k}D^{\alpha} x(t) + Mx(t) &= \sigma(t) - Nx(w(t)), \\ &\qquad t \neq t_k,\ t \in J, \\ \Delta x|_{t=t_k} &= -L_k x(t_k) + I_k(\eta(t_k)) \\ &\quad + L_k \eta(t_k), \\ &\qquad k = 1, 2, \ldots, m, \\ x(0) &= -x(T), \\ x(t) &= x(0), \quad t \in [-r, 0], \end{aligned} \tag{11} $$

where $M > 0$, $N \geq 0$, $0 \leq L_k < 1$ are constants and $\sigma \in PC(J,\mathbb{R})$.

Lemma 8. *$x \in PC(J,\mathbb{R})$ is a solution of BVP (11) if and only if $x \in E$ is a solution of the impulsive integral equation:*

$$ x(t) = \begin{cases} \displaystyle\int_0^T G_1(t,s)\,[\sigma(s) - Nx(w(s))]\,\hat{d}s + \sum_{k=1}^{m} \left[G_2(t,t_k)\left[-L_k x(t_k) + I_k(\eta(t_k)) + L_k \eta(t_k)\right]\right], & t \in J, \\ \displaystyle\int_0^T G_1(0,s)\,[\sigma(s) - Nx(w(s))]\,\hat{d}s + \sum_{k=1}^{m} \left[G_2(0,t_k)\left[-L_k x(t_k) + I_k(\eta(t_k)) + L_k \eta(t_k)\right]\right], & t \in [-r, 0], \end{cases} \tag{12} $$

where

$$ G_1(t,s) = \begin{cases} \dfrac{(s-t_h)^{\alpha-1} e^{-M/\alpha}(t_h,t)\, e^{M/\alpha}(t_h,s)}{1 + e^{-M/\alpha}(0,T)}, & 0 \leq s < t \leq T, \\ \dfrac{-(s-t_h)^{\alpha-1} e^{-M/\alpha}(0,t)\, e^{-M/\alpha}(t_h,T)\, e^{M/\alpha}(t_h,s)}{1 + e^{-M/\alpha}(0,T)}, & 0 \leq t \leq s \leq T, \end{cases} \tag{13} $$

and

$$ G_2(t,s) = \begin{cases} \dfrac{e^{-M/\alpha}(s,t)}{1 + e^{-M/\alpha}(0,T)}, & 0 \leq s < t \leq T, \\ \dfrac{-e^{-M/\alpha}(0,t)\, e^{-M/\alpha}(s,T)}{1 + e^{-M/\alpha}(0,T)}, & 0 \leq t \leq s \leq T, \end{cases} \tag{14} $$

with $t_h = \max\{t_k; k = 0, 1, \ldots, m$ and $t_k \leq s\}$.

Proof. For convenience, let

$$ \begin{aligned} B(t) &= \sigma(t) - Nx(w(t)), \\ \tau(t_k) &= -L_k x(t_k) + I_k(\eta(t_k)) + L_k \eta(t_k). \end{aligned} \tag{15} $$

If $x \in PC(J,\mathbb{R})$ is a solution of BVP (11), then for $t \in J_0 = [t_0, t_1]$, we have by Lemma 6 that

$$ \begin{aligned} &{}_{t_0}D^{\alpha}\left(e^{(M/\alpha)(t-t_0)^{\alpha}} x(t)\right) \\ &\quad = e^{(M/\alpha)(t-t_0)^{\alpha}}\, {}_{t_0}D^{\alpha} x(t) + e^{(M/\alpha)(t-t_0)^{\alpha}} Mx(t). \end{aligned} \tag{16} $$

Thus, multiplying by $e^{M/\alpha}(t_0,t)$ both sides of the first equation of (11), we obtain

$$ {}_{t_0}D^{\alpha}\left(e^{M/\alpha}(t_0,t)\,x(t)\right) = e^{M/\alpha}(t_0,t)\,B(t). \tag{17}$$

The conformable fractional integral of order α from t_0 to t ($t \in J_0$) of (17) yields

$$\begin{aligned} x(t) &= x(t_0)\,e^{-M/\alpha}(t_0,t) \\ &\quad + \int_{t_0}^{t} (s-t_0)^{\alpha-1} e^{-M/\alpha}(t_0,t)\,e^{M/\alpha}(t_0,s)\,B(s)\,ds, \\ &\qquad t \in J_0. \end{aligned} \tag{18}$$

For $t \in J_1 = (t_1,t_2]$, multiplying by $e^{M/\alpha}(t_1,t)$ both sides of the first equation of (11), we get

$$ {}_{t_1}D^{\alpha}\left(e^{M/\alpha}(t_1,t)\,x(t)\right) = e^{M/\alpha}(t_1,t)\,B(t). \tag{19}$$

Applying the conformable fractional integral of order α to both sides of (19) for $t \in J_1$, we obtain

$$\begin{aligned} x(t) &= x(t_1^+)\,e^{-M/\alpha}(t_1,t) \\ &\quad + \int_{t_1}^{t} (s-t_1)^{\alpha-1} e^{-M/\alpha}(t_1,t)\,e^{M/\alpha}(t_1,s)\,B(s)\,ds. \end{aligned} \tag{20}$$

By using Property 2, $x(t_1^+) = x(t_1) + \tau(t_1)$, and

$$\begin{aligned} x(t_1) &= x(t_0)\,e^{-M/\alpha}(t_0,t_1) \\ &\quad + \int_{t_0}^{t_1} (s-t_0)^{\alpha-1} e^{-M/\alpha}(t_0,t_1)\,e^{M/\alpha}(t_0,s)\,B(s)\,ds, \end{aligned} \tag{21}$$

(20) implies that

$$\begin{aligned} x(t) &= x(t_0)\,e^{-M/\alpha}(t_0,t) \\ &\quad + \int_{t_0}^{t_1} (s-t_0)^{\alpha-1} e^{-M/\alpha}(t_0,t)\,e^{M/\alpha}(t_0,s)\,B(s)\,ds \\ &\quad + \tau(t_1)\,e^{-M/\alpha}(t_1,t) \\ &\quad + \int_{t_1}^{t} (s-t_1)^{\alpha-1} e^{-M/\alpha}(t_1,t)\,e^{M/\alpha}(t_1,s)\,B(s)\,ds. \end{aligned} \tag{22}$$

In the same way, for $t \in J_k$, we derive

$$\begin{aligned} x(t) &= x(t_0)\,e^{-M/\alpha}(t_0,t) + \sum_{t_0<t_k<t} \int_{t_{k-1}}^{t_k} (s-t_{k-1})^{\alpha-1} \\ &\quad \cdot e^{-M/\alpha}(t_{k-1},t)\,e^{M/\alpha}(t_{k-1},s)\,B(s)\,ds \\ &\quad + \sum_{t_0<t_k<t} \tau(t_k)\,e^{-M/\alpha}(t_k,t) + \int_{t_l}^{t} (s-t_l)^{\alpha-1} \\ &\quad \cdot e^{-M/\alpha}(t_l,t)\,e^{M/\alpha}(t_l,s)\,B(s)\,ds, \end{aligned} \tag{23}$$

where $t_l = \max\{t_k; k = 0,1,\ldots,m \text{ and } t_k < t\}$.

Note that $t_0 = 0$, and we impose $t = T$ in (23). From antiperiodic boundary value condition $x(0) = -x(T)$, we deduce that

$$\begin{aligned} x(0) &= -\frac{1}{1+e^{-M/\alpha}(0,T)} \left[\sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} (s-t_{k-1})^{\alpha-1}\right. \\ &\quad \cdot e^{-M/\alpha}(t_{k-1},T)\,e^{M/\alpha}(t_{k-1},s)\,B(s)\,ds + \int_{t_m}^{T} (s \\ &\quad - t_m)^{\alpha-1} e^{-M/\alpha}(t_m,T)\,e^{M/\alpha}(t_m,s)\,B(s)\,ds \\ &\quad \left. + \sum_{k=1}^{m} \tau(t_k)\,e^{-M/\alpha}(t_k,T)\right]. \end{aligned} \tag{24}$$

Substituting (24) into (23), we have

$$\begin{aligned} x(t) &= -\frac{e^{-M/\alpha}(0,t)}{1+e^{-M/\alpha}(0,T)} \left[\sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} (s-t_{k-1})^{\alpha-1}\right. \\ &\quad \cdot e^{-M/\alpha}(t_{k-1},T)\,e^{M/\alpha}(t_{k-1},s)\,B(s)\,ds + \int_{t_m}^{T} (s \\ &\quad - t_m)^{\alpha-1} e^{-M/\alpha}(t_m,T)\,e^{M/\alpha}(t_m,s)\,B(s)\,ds \\ &\quad \left. + \sum_{k=1}^{m} \tau(t_k)\,e^{-M/\alpha}(t_k,T)\right] + \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (s \\ &\quad - t_{k-1})^{\alpha-1} e^{-M/\alpha}(t_{k-1},t)\,e^{M/\alpha}(t_{k-1},s)\,B(s)\,ds \\ &\quad + \sum_{0<t_k<t} \tau(t_k)\,e^{-M/\alpha}(t_k,t) + \int_{t_l}^{t} (s-t_l)^{\alpha-1} \\ &\quad \cdot e^{-M/\alpha}(t_l,t)\,e^{M/\alpha}(t_l,s)\,B(s)\,ds \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{1+e^{-M/\alpha}(0,T)}\Bigg[-\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(s-t_{k-1})^{\alpha-1} \\
&\cdot e^{-M/\alpha}(0,t)\,e^{-M/\alpha}(t_{k-1},T)\,e^{M/\alpha}(t_{k-1},s)\,B(s)\,ds \\
&-\int_{t_m}^{T}(s-t_m)^{\alpha-1}e^{-M/\alpha}(0,t)\,e^{-M/\alpha}(t_m,T) \\
&\cdot e^{M/\alpha}(t_m,s)\,B(s)\,ds-\sum_{k=1}^{m}\tau(t_k)\,e^{-M/\alpha}(0,t) \\
&\cdot e^{-M/\alpha}(t_k,T)+\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(s-t_{k-1})^{\alpha-1} \\
&\cdot e^{-M/\alpha}(0,T)\,e^{-M/\alpha}(t_{k-1},t)\,e^{M/\alpha}(t_{k-1},s)\,B(s)\,ds \\
&+\sum_{0<t_k<t}\tau(t_k)\,e^{-M/\alpha}(0,T)\,e^{-M/\alpha}(t_k,t)+\int_{t_l}^{t}(s \\
&-t_l)^{\alpha-1}e^{-M/\alpha}(t_l,t)\,e^{M/\alpha}(t_l,s)\,B(s)\,ds \\
&+\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(s-t_{k-1})^{\alpha-1}e^{-M/\alpha}(t_{k-1},t) \\
&\cdot e^{M/\alpha}(t_{k-1},s)\,B(s)\,ds+\sum_{0<t_k<t}\tau(t_k)\,e^{-M/\alpha}(t_k,t) \\
&+\int_{t_l}^{t}(s-t_l)^{\alpha-1}e^{-M/\alpha}(t_l,t)\,e^{M/\alpha}(t_l,s)\,B(s)\,ds\Bigg]
\end{aligned}
\tag{25}
$$

By using Property 2 (ii), we get

$$
\begin{aligned}
x(t) = \frac{1}{1+e^{-M/\alpha}(0,T)}\Bigg[&-\int_t^T(s-t_h)^{\alpha-1}e^{-M/\alpha}(0,t) \\
&\cdot e^{-M/\alpha}(t_h,T)\,e^{M/\alpha}(t_h,s)\,B(s)\,\widehat{d}s-\sum_{t<t_k<T}\tau(t_k) \\
&\cdot e^{-M/\alpha}(0,t)\,e^{-M/\alpha}(t_k,T)+\sum_{0<t_k<t}\tau(t_k) \\
&\cdot e^{-M/\alpha}(t_k,t)+\int_0^t(s-t_h)^{\alpha-1}e^{-M/\alpha}(t_h,t) \\
&\cdot e^{M/\alpha}(t_h,s)\,B(s)\,\widehat{d}s\Bigg],\quad t\in[0,T],
\end{aligned}
\tag{26}
$$

which implies that (12) holds.

Conversely, assume $x(t)$ is a solution of (12); then by direct calculus, we can easily obtain that $x(t)$ satisfies fractional impulsive antiperiodic boundary value problem (11). The proof is completed. □

Denote $a = \max\{t_1-t_0, t_2, -t_1, \ldots, t_{m+1}-t_m\}$. Now we establish the comparsion result.

Lemma 9. *Assume that $\alpha\in(0,1]$, $M>0$, $N, N_1\geq 0$, $0\leq L_k<1$, $k=1,2,\ldots,m$. If*

$$
\begin{aligned}
&\frac{e^{M/\alpha}(0,T)}{\alpha\left(1+e^{-M/\alpha}(0,T)\right)}(m+1)\,a^{\alpha}N \\
&\qquad+\frac{1}{1+e^{-M/\alpha}(0,T)}\sum_{k=1}^{m}L_k<1,
\end{aligned}
\tag{27}
$$

then (11) has a unique solution.

Proof. For each $x\in E$, we define an operator $\mathscr{A}$ by

$$
(\mathscr{A}x)(t)=\begin{cases}\displaystyle\int_0^T G_1(t,s)\left[\sigma(s)-Nx(w(s))\right]\widehat{d}s+\sum_{k=1}^{m}\left[G_2(t,t_k)\left[-L_kx(t_k)+I_k(\eta(t_k))+L_k\eta(t_k)\right]\right., & t\in J,\\ \displaystyle\int_0^T G_1(0,s)\left[\sigma(s)-Nx(w(s))\right]\widehat{d}s+\sum_{k=1}^{m}\left[G_2(0,t_k)\left[-L_kx(t_k)+I_k(\eta(t_k))+L_k\eta(t_k)\right]\right., & t\in[-r,0],\end{cases}
\tag{28}
$$

where G_1, G_2 are given by Lemma 8; then $\mathscr{A}x\in E$.

It is easy to check that

$$
e^{M/\alpha}(c,d)\leq e^{M/\alpha}(0,T),\quad \text{for } 0\leq c\leq d\leq T,
\tag{29}
$$

and

$$
e^{-M/\alpha}(c,d)\leq 1,\quad \text{for } 0\leq c\leq d\leq T.
\tag{30}
$$

For $t\in(t_j,t_{j+1}]$, $j\in\{0,1,\ldots,m\}$, and by (29)-(30), we obtain

$$
\begin{aligned}
\int_0^T|G_1(t,s)|\,\widehat{d}s &= \int_0^t|G_1(t,s)|\,\widehat{d}s+\int_t^T|G_1(t,s)|\,\widehat{d}s \\
&\leq \frac{e^{M/\alpha}(0,T)}{1+e^{-M/\alpha}(0,T)}\Bigg[\int_0^{t_1}s^{\alpha-1}ds+\cdots \\
&\quad+\int_{t_{j-1}}^{t_j}\left(s-t_{j-1}\right)^{\alpha-1}ds+\int_{t_j}^{t}\left(s-t_j\right)^{\alpha-1}ds \\
&\quad+\int_t^T\left(s-t_j\right)^{\alpha-1}ds\Bigg]=\frac{e^{M/\alpha}(0,T)}{\alpha\left(1+e^{-M/\alpha}(0,T)\right)}\left[t_1^{\alpha}\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\cdots+\left(t_j-t_{j-1}\right)^{\alpha}+\left(T-t_j\right)^{\alpha}\Big] \\
&=\frac{e^{M/\alpha}(0,T)}{\alpha\left(1+e^{-M/\alpha}(0,T)\right)}\Big(t_1^{\alpha}+\cdots+\left(t_j-t_{j-1}\right)^{\alpha} \\
&+\left[\left(t_{j+1}-t_j\right)+\left(t_{j+2}-t_{j+1}\right)+\cdots+\left(T-t_m\right)\right]^{\alpha}\Big) \\
&\le\frac{e^{M/\alpha}(0,T)}{\alpha\left(1+e^{-M/\alpha}(0,T)\right)}\Big(t_1^{\alpha}+\cdots+\left(t_j-t_{j-1}\right)^{\alpha} \\
&+\left(t_{j+1}-t_j\right)^{\alpha}+\left(t_{j+2}-t_{j+1}\right)^{\alpha}+\cdots+\left(T-t_m\right)^{\alpha}\Big) \\
&\le\frac{e^{M/\alpha}(0,T)}{\alpha\left(1+e^{-M/\alpha}(0,T)\right)}(m+1)a^{\alpha}.
\end{aligned}
\tag{31}
$$

Moreover, we get

$$
\sup_{(t,s)\in J^2}\left|G_2(t,s)\right|\le\frac{1}{1+e^{-M/\alpha}(0,T)}. \tag{32}
$$

Thus, $\forall x, y\in E$, we obtain

$$
\begin{aligned}
\|\mathscr{A}x-\mathscr{A}y\|_E &= \sup_{t\in J}\Bigg|\int_0^T G_1(t,s)\left[\sigma(s)-Nx(w(s))\right]\widehat{d}s \\
&+\sum_{k=1}^{m}\left[G_2(t,t_k)\left[-L_kx(t_k)+I_k(\eta(t_k))+L_k\eta(t_k)\right]\right. \\
&-\int_0^T G_2(t,s)\left[\sigma(s)-Ny(w(s))\right]\widehat{d}s \\
&-\sum_{k=1}^{m}\left[G_2(t,t_k)\left[-L_ky(t_k)+I_k(\eta(t_k))+L_k\eta(t_k)\right]\right]\Bigg| \\
&=\sup_{t\in J}\Bigg|\int_0^T G_1(t,s)N\left(y(w(s))-x(w(s))\right)\widehat{d}s \\
&+\sum_{k=1}^{m}L_kG_2(t,t_k)\left(y(x_k)-x(t_k)\right)\Bigg| \\
&\le\sup_{t\in J}\left\{N\int_0^T\left|G_1(t,s)\right|\widehat{d}s+\sum_{k=1}^{m}\left|G_2(t,t_k)\right|L_k\right\}\|x \\
&-y\|_E\le\left(\frac{e^{M/\alpha}(0,T)}{\alpha\left(1+e^{-M/\alpha}(0,T)\right)}(m+1)a^{\alpha}N\right. \\
&\left.+\frac{1}{1+e^{-M/\alpha}(0,T)}\sum_{k=1}^{m}L_k\right)\|x-y\|_E.
\end{aligned}
\tag{33}
$$

By (27) and Banach fixed point theorem, $\mathscr{A}$ has a unique fixed point which is the unique solution of problem (11). The proof is completed. □

Lemma 10. *Let $0<\alpha\le 1$. Suppose that $x\in E$ satisfies*

$$
\begin{aligned}
&{}_{t_k}D^{\alpha}x(t)+Mx(t)+Nx(w(t))\le 0,\quad t\in J^-, \\
&\Delta x(t_k)\le -L_kx(t_k), \\
&k=1,2,\ldots,m, \\
&x(0)\le 0, \\
&x(t)=x(0), \\
&t\in[-r,0],
\end{aligned}
\tag{34}
$$

where constants $M>0$, $N,N_1\ge 0$, $0<L_k\le 1$. Assume in addition that

$$
\frac{a^{\alpha}}{\alpha}(m+1)(M+N)+\sum_{k=1}^{m}L_k\le 1. \tag{35}
$$

Then $x(t)\le 0$ for all $t\in J^+$.

Proof. Suppose, to the contrary, that there must be $t^*\in(0,T]$ such that $x(t^*)>0$. Let $t_*\in[0,t^*)$ with $t_*\in J_i$, $i=0,1,\ldots,m$, such that $x(t_*)=\inf\{x(t):t\in[0,t^*]\}=-b<0$. It is easy to show that

$$
{}_{t_k}D^{\alpha}x(t)\le-\left[Mx(t)+Nx(w(t))\right]\le b(M+N). \tag{36}
$$

Suppose $t^*\in(t_j,t_{j+1}]$ for $j\in\{0,1,\ldots,m\}$. It is easy to know that $t_*<t^*$; then $i\le j$. By Theorem 7, we have

$$
\begin{aligned}
&x(t^*)-x(t_j)\le x(t^*)-x(t_j^+)-L_jx(t_j) \\
&\quad=\frac{1}{\alpha}\left(t^*-t_j\right)^{\alpha}\,{}_{t_j}D^{\alpha}x(r_j)-L_jx(t_j) \\
&\quad\le b\left[\frac{a^{\alpha}}{\alpha}(M+N)+L_j\right],\quad r_j\in\left(t_j,t^*\right), \\
&x(t_j)-x(t_{j-1})\le x(t_j)-x(t_{j-1}^+)-L_jx(t_{j-1}) \\
&\quad=\frac{1}{\alpha}\left(t_j-t_{j-1}\right)^{\alpha}\,{}_{t_{j-1}}D^{\alpha}x(r_{j-1})-L_{j-1}x(t_{j-1}) \\
&\quad\le b\left[\frac{a^{\alpha}}{\alpha}(M+N)+L_{j-1}\right],\quad r_{j-1}\in\left(t_{j-1},t_j\right), \\
&\cdots \\
&x(t_{i+1})-x(t_*) \\
&\quad=\left(\frac{1}{\alpha}(t_{i+1}-t_i)^{\alpha}-\frac{1}{\alpha}(t^*-t_i)^{\alpha}\right){}_{t_i}D^{\alpha}x(r_*) \\
&\quad\le b\left[\frac{a^{\alpha}}{\alpha}(M+N)\right],\quad r_*\in(t_*,t_{i+1}).
\end{aligned}
\tag{37}
$$

Summing up the above inequalities, we obtain

$$
x(t^*)-x(t_*)\le b\left[\frac{a^{\alpha}}{\alpha}(m+1)(M+N)+\sum_{k=1}^{m}L_k\right], \tag{38}
$$

which implies that

$$\begin{aligned} 0 &< x(t^*) \\ &\le x(t_*) + b\left[\frac{a^\alpha}{\alpha}(m+1)(M+N) + \sum_{k=1}^{m} L_k\right] \\ &= -b + b\left[\frac{a^\alpha}{\alpha}(m+1)(M+N) + \sum_{k=1}^{m} L_k\right]. \end{aligned} \tag{39}$$

Thus

$$\frac{a^\alpha}{\alpha}(m+1)(M+N) + \sum_{k=1}^{m} L_k > 1, \tag{40}$$

which contradicts (35). The proof is completed. □

Definition 11. The functions $\mu_0, \nu_0 \in E$ are said to be related lower and upper solutions for BVP (3) if

$$\begin{aligned} &\mu_0(t) \le \nu_0(t), \quad t \in J^+, \\ &{}_{t_k}D^\alpha \mu_0(t) \le f(t, \mu_0, \mu_0(w(t))), \quad t \in J^-, \\ &\Delta\mu_0(t_k) \le I_k(\mu_0(t_k)), \quad k = 1, 2, \ldots, m, \\ &\mu_0(0) \le -\nu_0(T), \\ &\mu_0(t) \le \mu_0(0), \quad t \in [-r, 0]. \end{aligned} \tag{41}$$

And

$$\begin{aligned} &{}_{t_k}D^\alpha \nu_0(t) \ge f(t, \nu_0, \nu_0(w(t))), \quad t \in J^-, \\ &\Delta\nu_0(t_k) \ge I_k(\nu_0(t_k)), \quad k = 1, 2, \ldots, m, \\ &\nu_0(0) \ge -\mu_0(T), \\ &\nu_0(t) \ge \nu_0(0), \quad t \in [-r, 0]. \end{aligned} \tag{42}$$

For $\mu_0, \nu_0 \in E$, we write $\mu_0 \le \nu_0$ if $\mu_0(t) \le \nu_0(t)$ for all $t \in J^+$, and

$$[\mu_0, \nu_0] = \{x \in E : \mu_0(t) \le x(t) \le \nu_0(t),\ t \in J^+\}. \tag{43}$$

Theorem 12. *Let $\mu_0, \nu_0 \in E$ be coupled lower and upper solutions of BVP (3). Suppose that the following conditions hold:*

(H1) The function $f \in PC(J \times \mathbb{R}^2, \mathbb{R})$ satisfies

$$f(t, x, y) - f(t, \overline{x}, \overline{y}) \ge -M(x - \overline{x}) - N(y - \overline{y}), \tag{44}$$

for $\mu_0(t) \le \overline{x}(t) \le x(t) \le \nu_0(t)$, $\mu_0(w(t)) \le \overline{y}(t) \le y(t) \le \nu_0(w(t))$, $t \in J^-$

(H2) All functions $I_k \in C(\mathbb{R}, \mathbb{R})$ satisfy

$$I_k(x(t_k)) - I_k(y(t_k)) \ge -L_k(x(t_k) - y(t_k)), \tag{45}$$

where $\mu_0(t_k) \le y(t_k) \le x(t_k) \le \nu_0(t_k)$, $k = 1, 2, \ldots, m$

(H3) Constants $M > 0$, $N \ge 0$, $0 \le L_k < 1$ in (H1) and (H2) satisfy (27) and (35).

Then there exists monotone sequence $\{\mu_n\}, \{\nu_n\} \subset E$ which converges uniformly to the minimal and maximal solutions x_*, x^* of BVP (3), respectively, such that

$$\begin{aligned} &\mu_0 \le \mu_1 \le \cdots \le \mu_n \le x_* \le x \le x^* \le \nu_n \le \cdots \le \nu_1 \\ &\quad \le \nu_0, \end{aligned} \tag{46}$$

on J, where x is a solution of BVP (3) such that $\mu_0(t) \le x(t) \le \nu_0(t)$ on J^+.

Proof. We construct two sequences $\{\mu_n\}$, $\{\nu_n\}$ which are the solutions of

$$\begin{aligned} &{}_{t_k}D^\alpha \mu_n(t) + M\mu_n(t) + N\mu_n(w(t)) \\ &\quad = f(t, \mu_{n-1}(t), \mu_{n-1}(w(t))) + M\mu_{n-1}(t) \\ &\qquad + N\mu_{n-1}(w(t)), \quad t \in J^-, \\ &\Delta\mu_n(t_k) = -L_k\mu_n(t_k) + I_k(\mu_{n-1}(t_k)) + L_k\mu_{n-1}(t_k), \\ &\qquad k = 1, 2, \ldots, m, \\ &\mu_n(0) = -\nu_{n-1}(T), \\ &\mu_n(t) = \mu_n(0), \quad t \in [-r, 0], \end{aligned} \tag{47}$$

and

$$\begin{aligned} &{}_{t_k}D^\alpha \nu_n(t) + M\nu_n(t) + N\nu_n(w(t)) \\ &\quad = f(t, \nu_{n-1}(t), \nu_{n-1}(w(t))) + M\nu_{n-1}(t) \\ &\qquad + N\nu_{n-1}(w(t)), \quad t \in J^-, \\ &\Delta\nu_n(t_k) = -L_k\nu_n(t_k) + I_k(\nu_{n-1}(t_k)) + L_k\nu_{n-1}(t_k), \\ &\qquad k = 1, 2, \ldots, m, \\ &\nu_n(0) = -\mu_{n-1}(T), \\ &\nu_n(t) = \nu_n(0), \quad t \in [-r, 0]. \end{aligned} \tag{48}$$

It follows from (H3) and Lemma 9 that problem (47) has a unique solution. Similarly, we also conclude that problem (48) has a unique solution too.

We prove that these sequences satisfy the following properties:

(i) $\mu_{n-1} \le \mu_n$, $\nu_n \le \nu_{n-1}$, $n \ge 1$

(ii) if $\mu_{n-1} \le \nu_{n-1}$, then $\mu_n \le \nu_n$, $n \ge 1$

To prove (i), put $p(t) = \mu_0(t) - \mu_1(t)$; then we get by Definition 11 and (47) that

$$\begin{aligned} {}_{t_k}D^\alpha p(t) &= {}_{t_k}D^\alpha \mu_0(t) - {}_{t_k}D^\alpha \mu_1(t) \\ &\le f(t, \mu_0, \mu_0(w(t))) - [-M\mu_1(t) - N\mu_1(w(t)) \\ &\quad + f(t, \mu_0, \mu_0((t))) + M\mu_0(t) + N\mu_0(w(t))] \\ &= -Mp(t) - Np(w(t)), \quad t \in J^-, \end{aligned} \tag{49}$$

and

$$\begin{aligned}\Delta p(t_k) &= \Delta\mu_0(t_k) - \Delta\mu_1(t_k)\\ &\le I_k(\mu_0(t_k))\\ &\quad - [-L_k\mu_1(t_k) + I_k(\mu_0(t_k)) + L_k\mu_0(t_k)]\\ &= -L_k p(t_k), \quad k = 1,2,\ldots,m,\\ p(0) &= \mu_0(0) - \mu_1(0) = \mu_0(0) + \nu_0(T) \le 0,\end{aligned} \tag{50}$$

and

$$p(t) = p(0), \quad t \in [-r, 0]. \tag{51}$$

By using (H3) and Lemma 10, we deduce that $p(t) \le 0$, which implies $\mu_0(t) \le \mu_1(t)$ for all $t \in J^+$, i.e., $\mu_0 \le \mu_1$. Analogously, $\nu_1 \le \nu_0$. By mathematics induction, we can obtain that $\{\mu_n\}$ is a nondecreasing sequence and $\{\nu_n\}$ is a nonincreasing sequence.

To prove (ii), we first show that $\mu_1 \le \nu_1$, if $\mu_0 \le \nu_0$. Let $q = \mu_1 - \nu_1$, using (47), (48), and (H1), we have

$$\begin{aligned}&q(0) = \mu_1(0) - \nu_1(0) = -\nu_0(T) + \mu_0(T) \le 0,\\ &q(t) = q(0), \quad t \in [-r, 0],\\ &{}_{t_k}D^\alpha q(t) + Mq(t) + Nq(w(t)) = {}_{t_k}D^\alpha \mu_1(t)\\ &\quad + M\mu_1(t) + N\mu_1(w(t)) - \left[{}_{t_k}D^\alpha \nu_1(t) + M\nu_1(t)\right.\\ &\quad \left.+ N\nu_1(w(t))\right] = [f(t, \mu_0(t), \mu_0(w(t)))\\ &\quad + M\mu_0(t) + N\mu_0(w(t))] - [f(t, \nu_0(t), \nu_0(w(t)))\\ &\quad + M\nu_0(t) + N\nu_0(w(t))] = f(t, \mu_0(t), \mu_0(w(t)))\\ &\quad - f(t, \nu_0(t), \nu_0(w(t))) - M(\nu_0(t) - \mu_0(t))\\ &\quad - N(\nu_0(w(t)) - \mu_0(w(t))) \le 0, \quad t \in J^-,\end{aligned} \tag{52}$$

and by (H2)

$$\begin{aligned}\Delta q(t_k) &= [-L_k\mu_1(t_k) + I_k(\mu_0(t_k)) + L_k\mu_0(t_k)]\\ &\quad - [-L_k\nu_1(t_k) + I_k(\nu_0(t_k)) + L_k\nu_0(t_k)]\\ &\le -L_k q(t_k), \quad k = 1,2,\ldots,m.\end{aligned} \tag{53}$$

Thus, from (H3) and Lemma 10, we obtain $q(t) \le 0$, which yields $\mu_1 \le \nu_1$. Still by mathematical induction, we have $\mu_n \le \nu_n$, $n \ge 1$.

Following (i) and (ii) above, we have

$$\mu_0 \le \mu_1 \le \cdots \le \mu_n \le \cdots \le \nu_n \le \cdots \nu_1 \le \nu_0, \quad \text{for all } n \in \mathbb{N}. \tag{54}$$

Obviously, $\{\mu_n\}$ and $\{\nu_n\}$ satisfy (47) and (48), respectively. Thus, there exist x_* and x^* on J, such that $\lim_{n\longrightarrow\infty}\mu_n = x_*$ and $\lim_{n\longrightarrow\infty}\nu_n = x^*$ uniformly on J. It is easy to see that x_* and x^* are solutions of BVP (3) in $[\mu_0, \nu_0]$.

Finally, we prove x_* and x^* are extreme solutions of BVP (3) in $[\mu_0, \nu_0]$. Assume that $x(t)$ is any solution of problem (3), which satisfies $\mu_0(t) \le x(t) \le \nu_0(t)$, $t \in J^+$. In the following, we will prove that if $\mu_n(t) \le x(t) \le \nu_n(t)$ for some positive integral n, then there holds $\mu_{n+1}(t) \le x(t) \le \nu_{n+1}(t)$ on J^+. Let $p = \mu_{n+1} - x$; then for $t \in J^+$ we have

$$\begin{aligned}&p(0) = \mu_{n+1}(0) - x(0) = -\nu_n(T) + x(T) \le 0,\\ &p(t) = \mu_{n+1}(t) - x(t) = \mu_{n+1}(0) - x(0) = p(0),\\ &\qquad t \in [-r, 0],\\ &{}_{t_k}D^\alpha p(t) = {}_{t_k}D^\alpha \mu_{n+1}(t) - {}_{t_k}D^\alpha x(t) = -M\mu_{n+1}(t)\\ &\quad - N\mu_{n+1}(w(t)) + f(t, \mu_n(t), \mu_n(w(t)))\\ &\quad + M\mu_n(t) + N\mu_n(w(t)) - f(t, x(t), x(w(t)))\\ &\quad = -M(\mu_{n+1}(t) - x(t)) - N(\mu_{n+1}(w(t))\\ &\quad - x(w(t))) - [f(t, x(t), x(w(t)))\\ &\quad - f(t, \mu_n(t), \mu_n(w(t))) + M(x(t) - \mu_n(t))\\ &\quad + N(x(w(t)) - \mu_n(w(t)))] \le -Mp(t)\\ &\quad - Np(w(t)), \quad t \in J^-,\end{aligned} \tag{55}$$

and

$$\begin{aligned}\Delta p(t_k) &= \Delta\mu_{n+1}(t_k) - \Delta x(t_k) = -L_k\mu_{n+1}(t_k)\\ &\quad + I_k(\mu_n(t_k)) + L_k\mu_n(t_k) - I_k(x(t_k))\\ &= -L_k(\mu_{n+1}(t_k) - x(t_k)) - [I_k(x(t_k))\\ &\quad - I_k(\mu_n(t_k)) + L_k(x(t_k) - \mu_n(t_k))] \le -L_k p(t_k),\\ &\quad k = 1,2,\ldots,m.\end{aligned} \tag{56}$$

By Lemma 10, we have $p(t) \le 0$ for all $t \in J^+$, which leads to $\mu_{n+1} \le x$ on $t \in J^+$. Similarly, we can show that $x \le \nu_{n+1}$ on J^+. By mathematical induction, we obtain that $\mu_n \le x \le \nu_n$ for each n. Taking $n \longrightarrow \infty$, we have $x_*(t) \le x(t) \le x^*(t)$ for $t \in J^+$. The proof is finished. □

Remark 13. Let $\alpha = 1$; then Theorem 12 is generalization of the result of Theorem 3.1 in [24].

4. An Example

Example 1. Consider the problem

$$\begin{aligned} {}_{t_k}D^{2/3}x(t) &= -\frac{1}{6}\sin\left(x(t)-\frac{1}{2}\right)-\frac{3}{5}x(w(t)), \\ &\qquad t\in\left[0,\frac{1}{2}\right]\setminus\left\{\frac{1}{4}\right\}, \\ \Delta x\left(\frac{1}{4}\right) &= \frac{1}{20}x\left(\frac{1}{4}\right), \\ x(0) &= -x\left(\frac{1}{2}\right), \\ x(t) &= x(0), \quad t\in[-1,0], \end{aligned} \tag{57}$$

where

$$\begin{aligned} w(t) &= \left\{\frac{5}{6}t, t\in\left[0,\frac{1}{4}\right], \frac{3}{2}t-\frac{1}{6}, t\in\left(\frac{1}{4},\frac{1}{3}\right], t, t \right. \\ &\quad \left. \in\left[\frac{1}{3},\frac{1}{2}\right]\right. . \end{aligned} \tag{58}$$

Thus

$$\begin{aligned} &f(t,x(t),x(w(t)))-f(t,\overline{x}(t),\overline{x}(w(t))) \\ &\quad = -\frac{1}{6}\sin\left(x(t)-\frac{1}{2}\right)-\frac{1}{6}\sin\left(\overline{x}(t)-\frac{1}{2}\right) \\ &\qquad -\frac{3}{5}x(w(t))+\frac{3}{5}\overline{x}(w(t)) \\ &\quad \geq -\frac{1}{6}(x(t)-\overline{x}(t))-\frac{3}{5}(x(w(t))-\overline{x}(w(t))) \end{aligned} \tag{59}$$

for all $t\in J$, $x(t), x(w(t)), \overline{x}(t), \overline{x}(w(t)))\in\mathbb{R}$, $x(t)\geq\overline{x}(t)$, $x(w(t))\geq\overline{x}(w(t)))$.

Let $M=1/6$, $N=3/5$; then condition (H1) holds. Since $I_1(x)=(1/20)x$, condition (H2) also holds with $L_1=1/20$. We easily know that $\alpha=2/3$, $T=1/2$, $m=1$, $t_1=1/4$, $a=1/4$, $r=1$, $w\in C(J,J^+)$, $t-r\leq w(t)$, $t\in[0,1/2]$, and $t_1<w(t)\leq t$ for $t\in(t_1,1/2]$. Moreover, we have

$$\begin{aligned} &e^{M/\alpha}\left(0,\frac{1}{2}\right) = e^{(M/\alpha)(1/2-1/4)^\alpha}+e^{(M/\alpha)(1/4-0)^\alpha} \\ &\quad = 2e^{1/4\cdot(1/4)^{2/3}} = 2.2086, \\ &\frac{e^{M/\alpha}(0,1/2)}{\alpha\left(1+e^{-M/\alpha}(0,1/2)\right)}(m+1)a^\alpha N \\ &\quad + \frac{1}{1+e^{-M/\alpha}(0,1/2)}\sum_{k=1}^{m}L_k = 0.5612+0.0178 \\ &\quad = 0.5790<1, \end{aligned} \tag{60}$$

and

$$\begin{aligned} &\frac{a^\alpha}{\alpha}(m+1)(M+N)+\sum_{k=1}^{m}L_k = 0.9128+0.05 \\ &\quad = 0.9628<1, \end{aligned} \tag{61}$$

which implies that hypothesis (H_3) is satisfied.

Setting

$$\begin{aligned} &\mu_0(t) = \left\{-\frac{5}{8}, t\in\left[-1,\frac{1}{4}\right], -\frac{2}{3}, t\in\left(\frac{1}{4},\frac{1}{2}\right], \right. \\ &\nu_0(t) \\ &\quad = \left\{-\frac{1}{4}t+\frac{2}{3}, t\in\left[-1,\frac{1}{4}\right], -\frac{1}{4}t+\frac{3}{4}, t\in\left(\frac{1}{4},\frac{1}{2}\right]\right. . \end{aligned} \tag{62}$$

Obviously,

$$\begin{aligned} &\mu_0(t)\leq\nu_0(t) \quad \text{for } t\in J^+=\left[-1,\frac{1}{2}\right], \\ &\mu_0(t)=\mu_0(0), \\ &\nu_0(t)\geq\nu_0(0), \\ &\qquad t\in[-1,0], \\ &\mu_0(0) = -\frac{5}{8} = -\nu_0\left(\frac{1}{2}\right), \\ &\nu_0(0) = \frac{2}{3} = -\mu_0\left(\frac{1}{2}\right), \\ &\Delta\mu_0\left(\frac{1}{4}\right) = -\frac{2}{3}-\left(-\frac{5}{8}\right) = -\frac{1}{24}\leq-\frac{1}{32} \\ &\quad = \frac{1}{20}\mu_0\left(\frac{1}{4}\right), \end{aligned} \tag{63}$$

and

$$\Delta\nu_0\left(\frac{1}{4}\right) = \frac{3}{4}-\frac{2}{3} = \frac{1}{12} > \frac{29}{960} = \frac{1}{20}\nu_0\left(\frac{1}{4}\right). \tag{64}$$

We see that

$$\begin{aligned} &f(t,\mu_0(t),\mu_0(w(t))) \\ &\quad = -\frac{1}{6}\sin\left(\mu_0(t)-\frac{1}{2}\right)-\frac{3}{5}\mu_0(w(t)) \\ &\quad = \begin{cases} -\frac{1}{6}\sin\left(-\frac{9}{8}\right)+\frac{3}{8}, & t\in\left[0,\frac{1}{4}\right], \\ -\frac{1}{6}\sin\left(-\frac{7}{6}\right)+\frac{2}{5}, & t\in\left(\frac{1}{4},\frac{1}{2}\right] \end{cases} > 0 \\ &\quad = {}_{t_k}D^{2/3}\mu_0(t), \quad t\in\left[0,\frac{1}{2}\right]\setminus\left\{\frac{1}{4}\right\}. \end{aligned} \tag{65}$$

Moreover, we have

$$\begin{aligned}
f(t,\nu_0(t),\nu_0(w(t))) &= -\frac{1}{6}\sin\left(\nu_0(t)-\frac{1}{2}\right)-\frac{3}{5}\nu_0(w(t)) \\
&= \begin{cases} -\frac{1}{6}\sin\left(-\frac{1}{4}t+\frac{1}{6}\right)-\frac{3}{5}\left(-\frac{5}{24}t+\frac{5}{9}\right), & t\in\left[0,\frac{1}{4}\right], \\ -\frac{1}{6}\sin\left(-\frac{1}{4}t+\frac{1}{4}\right)-\frac{3}{5}\left[-\frac{1}{4}\left(\frac{3}{2}t-\frac{1}{6}\right)+\frac{3}{4}\right], & t\in\left(\frac{1}{4},\frac{1}{3}\right], \\ -\frac{1}{6}\sin\left(-\frac{1}{4}t+\frac{1}{4}\right)-\frac{3}{5}\left(-\frac{1}{4}t+\frac{3}{4}\right), & t\in\left(\frac{1}{3},\frac{1}{2}\right] \end{cases} \\
&\le \begin{cases} -\frac{1}{6}\sin\left(\frac{5}{48}\right)+\frac{1}{32}-\frac{1}{3}, & t\in\left[0,\frac{1}{4}\right], \\ -\frac{1}{6}\sin\left(\frac{1}{6}\right)+\frac{3}{40}-\frac{19}{40}, & t\in\left(\frac{1}{4},\frac{1}{3}\right], \\ -\frac{1}{6}\sin\left(\frac{1}{8}\right)+\frac{3}{40}-\frac{9}{20}, & t\in\left(\frac{1}{3},\frac{1}{2}\right] \end{cases} \\
&= \begin{cases} -0.3194, & t\in\left[0,\frac{1}{4}\right], \\ -0.4276, & t\in\left(\frac{1}{4},\frac{1}{3}\right], \\ -0.3958, & t\in\left(\frac{1}{3},\frac{1}{2}\right], \end{cases}
\end{aligned} \tag{66}$$

$$\begin{aligned}
{}_0D^{2/3}\nu_0(t) &= {}_0D^{2/3}\left(-\frac{1}{4}t+\frac{2}{3}\right) = -\frac{1}{4}t^{1/3} \ge -\frac{1}{4}\left(\frac{1}{4}\right)^{1/3} \\
&= -0.1575, \quad t\in\left[0,\frac{1}{4}\right],
\end{aligned}$$

and

$$\begin{aligned}
{}_{1/4}D^{2/3}\nu_0(t) &= {}_{1/4}D^{2/3}\left(-\frac{1}{4}t+\frac{3}{4}\right) = -\frac{1}{4}\left(t-\frac{1}{4}\right)^{1/3} \\
&\ge -\frac{1}{4}\left(\frac{1}{4}\right)^{1/3} = -0.1575, \\
&\qquad t\in\left(\frac{1}{4},\frac{1}{2}\right],
\end{aligned} \tag{67}$$

which implies that

$$\begin{aligned}
f(t,\nu_0(t),\nu_0(w(t))) &\le {}_{t_k}D^{2/3}\nu_0(t), \\
&\qquad t\in\left[0,\frac{1}{2}\right]\setminus\left\{\frac{1}{4}\right\}.
\end{aligned} \tag{68}$$

Hence, μ_0 and ν_0 are coupled lower and upper solutions of (57). Thus, applying Theorem 12, we have the existence of monotone sequences that approximate the extremal solutions of (57) in the interval $[-1,1/2]$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Authors' Contributions

The authors contributed equally to this paper. All authors read and approved the final manuscript.

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