Hindawi
Journal of Function Spaces
Volume 2018, Article ID 8056276, 14 pages
https://doi.org/10.1155/2018/8056276

Research Article

The Conjugate Gradient Viscosity Approximation Algorithm for Split Generalized Equilibrium and Variational Inequality Problems

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Received 16 July 2018; Revised 28 September 2018; Accepted 15 October 2018; Published 1 November 2018

Academic Editor: Liguang Wang

In this paper, we study a kind of conjugate gradient viscosity approximation algorithm for finding a common solution of split generalized equilibrium problem and variational inequality problem. Under mild conditions, we prove that the sequence generated by the proposed iterative algorithm converges strongly to the common solution. The conclusion presented in this paper is the generalization, extension, and supplement of the previously known results in the corresponding references. Some numerical results are illustrated to show the feasibility and efficiency of the proposed algorithm.

1. Introduction

Let H_1 and H_2 be real Hilbert spaces with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $\{x_n\}$ be a sequence in H_1; then $x_n \longrightarrow x$ (respectively, $x_n \rightharpoonup x$) will denote strong (respectively, weak) convergence of the sequence $\{x_n\}$. Assume $w_\omega(x_k) = \{x : \exists x_{k_j} \rightharpoonup x\}$ to stand for the weak ω-limit set of x_k.

The split feasibility problem (SFP) originally introduced by Censor and Elfving [1] is to find

$$x \in C \text{ such that } Ax \in Q, \tag{1}$$

where $A : H_1 \longrightarrow H_2$ is a bounded linear operator. It serves as a model for many inverse problems where constraints are imposed on the solutions in the domain of a linear operator as well as in these operator's ranges. The applications of the split feasibility problem are very comprehensive such as CT in medicine, intelligence antennas, and the electronic warning systems in military, the development of fast image processing technology and HDTV, etc. Many authors generalize SFP to a lot of important problems, such as multiple-sets split feasibility problem, split equality fixed point problem, split variational inequality problem, split variational inclusion problem, and split equilibrium problem, and the theories and algorithms are studied and details can be seen in [2–15] and references therein.

The fixed point problem (FPP) for the mapping T is to find $x \in C$ such that

$$Tx = x. \tag{2}$$

We denote $\mathrm{Fix}(T) := \{x \in C : Tx = x\}$ the set of solution of FPP.

Let $B : C \longrightarrow H_1$ be a nonlinear mapping. The variational inequality problem (VIP) is to find $x \in C$ such that

$$\langle Bx, y - x\rangle \geq 0, \quad \forall y \in C. \tag{3}$$

The solution set of VIP is denoted by $\mathrm{VI}(C, B)$. It is well known that if B is strongly monotone and Lipschitz continuous mapping on C, then VIP has a unique solution.

For finding a common problem of $\mathrm{Fix}(T) \cap \mathrm{VI}(C, B)$, Takahashi and Toyoda [16] introduced the following iterative scheme:

$$\begin{gathered} x_0 \quad \text{chosen arbitrary}, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) TP_C(x_n - \lambda_n Bx_n), \quad \forall n \geq 0, \end{gathered} \tag{4}$$

where B is ρ-inverse-strongly monotone, $\{\alpha_n\}$ is a sequence in $(0,1)$, and $\{\lambda_n\}$ is a sequence in $(0,2\rho)$. They showed that if $\mathrm{Fix}(T)\cap \mathrm{VI}(C,B)\neq\emptyset$, then the sequence $\{x_n\}$ generated by (4) converges weakly to $z_0\in \mathrm{Fix}(T)\cap \mathrm{VI}(C,B)$.

On the other hand, there are several numerical methods for solving variational inequalities and related optimization problems; see [5, 17–24] and the references therein.

In 2013, Kazmi and Rizvi [25] introduced the split generalized equilibrium problem (SGEP). Let $F_1,h_1 : C\times C\longrightarrow R$ and $F_2,h_2 : Q\times Q\longrightarrow R$ be nonlinear bifunctions and $A : H_1\longrightarrow H_2$ be a bounded linear operator; then the split generalized equilibrium problem (SGEP) is to find $x^*\in C$ such that

$$F_1(x^*,x)+h_1(x^*,x)\geq 0,\quad \forall x\in C \tag{5}$$

and such that

$$y^*=Ax^*\in Q \text{ solves } F_2(y^*,y)+h_2(y^*,y)\geq 0,\quad \forall y\in Q. \tag{6}$$

Denote the solution sets of generalized equilibrium problem (GEP) (5) and GEP (6) by $\mathrm{GEP}(F_1,h_1)$ and $\mathrm{GEP}(F_2,h_2)$, respectively. The solution set of SGEP is denoted by $\Gamma=\{x^*\in \mathrm{GEP}(F_1,h_1) : Ax^*\in \mathrm{GEP}(F_2,h_2)\}$. They proposed the following iterative method for finding a common solution of split generalized equilibrium and fixed point problem.

Let $\{x_n\}$ and $\{u_n\}$ be the sequences generated by $x_0\in C$ and

$$\begin{aligned} u_n&=T_{r_n}^{(F_1,h_1)}\left(x_n+\delta A^*\left(T_{r_n}^{(F_2,h_2)}-I\right)Ax_n\right),\\ x_{n+1}&=\alpha_n\gamma f(x_n)+\beta_n x_n\\ &\quad+\left(\left(1-\beta_n\right)I-\alpha_n D\right)\frac{1}{s_n}\int_0^{s_n}T(s)u_n ds, \end{aligned} \tag{7}$$

where $S=\{T(s) : 0\leq s<\infty\}$ is a nonexpansive semigroup on C and $Fix(S)\cap\Gamma\neq\emptyset$, s_n is a positive real sequence which diverges to $+\infty$, $\{\alpha_n\}$, $\{\beta_n\}\subset(0,1)$, $\{r_n\}\subset(0,\infty)$, and $\delta\in(0,1/L)$, L is the spectral radius of the operator A^*A, A^* is the adjoint of A, and

$$\begin{aligned} T_r^{(F_1,h_1)}(x)=\Big\{&z\in C : F_1(z,y)+h_1(z,y)\\ &+\frac{1}{r}\langle y-z,z-x\rangle\geq 0,\ \forall y\in C\Big\}, \end{aligned} \tag{8}$$

$$\begin{aligned} T_s^{(F_2,h_2)}(w)=\Big\{&d\in Q : F_2(d,e)+h_2(d,e)\\ &+\frac{1}{s}\langle e-d,d-w\rangle\geq 0,\ \forall e\in Q\Big\}. \end{aligned} \tag{9}$$

Under suitable conditions, they proved a strong convergence theorem for the sequence generated by the proposed iterative scheme. But the calculation of integral is generally not easy. Therefore, it is necessary to reconsider the algorithm for solving this kind of problem.

Motivated by Kazmi and Rivi [25] as well as Che and Li [2], we introduce and study a kind of conjugate gradient viscosity approximation algorithm for finding a common solution of split generalized equilibrium problem and variational inequality problem. Under mild conditions, we prove that the sequence generated by the proposed iterative algorithm converges strongly to the common solution of $\mathrm{VI}(C,B)$ and SGEP. The results presented in this paper are the generalization, extension, and supplement of the previously known results in the corresponding references. Numerical results show the feasibility and efficiency of the proposed algorithm.

2. Preliminaries

In this section, we introduce some concepts and results which are needed in sequel.

A mapping $T : H_1\longrightarrow H_1$ is called

(1) contraction, if there exists a constant $\alpha\in(0,1)$ such that

$$\|Tx-Ty\|\leq\alpha\|x-y\|,\quad \forall x,y\in H_1. \tag{10}$$

If $\alpha=1$, then T is called nonexpansive.

(2) monotone, if

$$\langle Tx-Ty,x-y\rangle\geq 0,\quad \forall x,y\in H_1. \tag{11}$$

(3) η-strongly monotone, if there exists a positive constant η such that

$$\langle Tx-Ty,x-y\rangle\geq\eta\|x-y\|^2,\quad \forall x,y\in H_1. \tag{12}$$

(4) α-inverse strongly monotone (α-ism), if there exists a constant $\alpha>0$ such that

$$\langle Tx-Ty,x-y\rangle\geq\alpha\|Tx-Ty\|^2,\quad \forall x,y\in H_1. \tag{13}$$

(5) firmly nonexpansive, if

$$\langle Tx-Ty,x-y\rangle\geq\|Tx-Ty\|^2,\quad \forall x,y\in H_1. \tag{14}$$

A mapping P_C is said to be metric projection of H_1 onto C if, for every point $x\in H_1$, there exists a unique nearest point in C denoted by P_Cx such that

$$\|x-P_Cx\|\leq\|x-y\|,\quad \forall y\in C. \tag{15}$$

It is well known that P_C is a nonexpansive mapping and is characterized by the following properties:

$$\|P_Cx-P_Cy\|^2\leq\langle x-y,P_Cx-P_Cy\rangle,\quad \forall x,y\in H_1, \tag{16}$$

$$\langle x-P_Cx,y-P_Cx\rangle\leq 0,\quad \forall x\in H_1,\ y\in C, \tag{17}$$

$$\|x-y\|^2\geq\|x-P_Cx\|^2+\|y-P_Cx\|^2,\quad \forall x\in H_1,\ y\in C \tag{18}$$

and

$$\begin{aligned} &\|(x-y)-(P_Cx-P_Cy)\|^2\\ &\quad\geq\|x-y\|^2-\|P_Cx-P_Cy\|^2,\quad \forall x,y\in H_1. \end{aligned} \tag{19}$$

A linear bounded operator B is strongly positive if there exists a constant $\overline{\gamma} > 0$ with the property

$$\langle Bx, x\rangle \geq \overline{\gamma}\,\|x\|^2, \quad \forall x \in H_1. \tag{20}$$

A mapping $T : H_1 \longrightarrow H_1$ is said to be averaged if and only if it can be written as the average of the identity mapping and a nonexpansive mapping; i.e.,

$$T = (1-\alpha)\,I + \alpha S, \tag{21}$$

where $\alpha \in (0,1)$, $S : H_1 \longrightarrow H_1$ is nonexpansive, and I is the identity operator on H_1. More precisely, we say that T is α-averaged. We note that averaged mapping is nonexpansive. Furthermore, firmly nonexpansive mapping (in particular, projection on nonempty closed and convex subset) is averaged.

Let B be a monotone mapping of C into H_1. In the context of the variational inequality problem, the characterization of projection (17) implies the following relation:

$$u \in \mathrm{VI}(C,B) \Longleftrightarrow u = P_C(u - \lambda Bu), \quad \lambda > 0. \tag{22}$$

In the proof of our results, we need the following assumptions and lemmas.

Lemma 1 (see [26]). *If $x, y, z \in H_1$, then*

(i) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$.

(ii) For any $\lambda \in [0,1]$,

$$\begin{aligned}\|\lambda x + (1-\lambda)\,y\|^2 &= \lambda\,\|x\|^2 + (1-\lambda)\,\|y\|^2 \\ &\quad - \lambda(1-\lambda)\,\|x-y\|^2.\end{aligned} \tag{23}$$

(iii) For $a, b, c \in [0,1]$ with $a+b+c=1$,

$$\begin{aligned}\|ax+by+cz\|^2 &= a\,\|x\|^2 + b\,\|y\|^2 + c\,\|z\|^2 \\ &\quad - ab\,\|x-y\|^2 - ac\,\|x-z\|^2 \\ &\quad - bc\,\|y-z\|^2.\end{aligned} \tag{24}$$

Assumption 2. Let $F : C \times C \longrightarrow R$ be a bifunction satisfying the following assumption:

(i) $F(x,x) \geq 0$, $\forall x \in C$.

(ii) F is monotone; i.e., $F(x,y) + F(y,x) \leq 0$, $\forall x \in C$.

(iii) F is upper hemicontinuous; i.e., for each $x, y, z \in C$,

$$\limsup_{t \longrightarrow 0} F(tz + (1-t)\,x, y) \leq F(x,y). \tag{25}$$

(iv) For each $x \in C$ fixed, the function $y \longmapsto F(x,y)$ is convex and lower semicontinuous.

Let $h : C \times C \longrightarrow R$ such that

(i) $h(x,x) \geq 0$, $\forall x \in C$

(ii) for each $y \in C$ fixed, the function $x \longrightarrow h(x,y)$ is upper semicontinuous

(iii) for each $x \in C$ fixed, the function $y \longrightarrow h(x,y)$ is convex and lower semicontinuous

And assume that, for fixed $r > 0$ and $z \in C$, there exists a nonempty compact convex subset K of H_1 and $x \in C \cap K$ such that

$$F(y,x) + h(y,x) + \frac{1}{r}\langle y-x, x-z\rangle < 0, \quad \forall y \in C \setminus K. \tag{26}$$

Lemma 3 (see [25]). *Assume that $F_1, h_1 : C \times C \longrightarrow R$ satisfy Assumption 2. Let $r > 0$ and $x \in H_1$. Then, there exists $z \in C$ such that*

$$F_1(z,y) + h_1(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \quad \forall y \in C. \tag{27}$$

Lemma 4 (see [1]). *Assume that the bifunctions $F_1, h_1 : C \times C \longrightarrow R$ satisfy Assumption 2 and h_1 is monotone. For $r > 0$ and for all $x \in H_1$, define a mapping $T_r^{(F_1,h_1)} : H_1 \longrightarrow C$ as (8). Then, the following conclusions hold.*

(1) $T_r^{(F_1,h_1)}$ is single-valued.

(2) $T_r^{(F_1,h_1)}$ is firmly nonexpansive.

(3) $Fix(T_r^{(F_1,h_1)}) = GEP(F_1,h_1)$.

(4) $GEP(F_1,h_1)$ is compact and convex.

Furthermore, assume that $F_2, h_2 : Q \times Q \longrightarrow R$ satisfy Assumption 2. For $s > 0$ and, for all $w \in H_2$, $T_s^{(F_2,h_2)} : H_2 \longrightarrow Q$ is defined as (9). By Lemma 4, we easily observe that $T_s^{(F_2,h_2)}$ is single-valued and firmly nonexpansive, $\mathrm{GEP}(F_2,h_2,Q)$ is compact and convex, and $\mathrm{Fix}(T_s^{(F_2,h_2)}) = \mathrm{GEP}(F_2,h_2,Q)$, where $\mathrm{GEP}(F_2,h_2,Q)$ is the solution set of the following generalized equilibrium problem, which is to find $y^* \in Q$ such that $F_2(y^*,y) + h_2(y^*,y) \geq 0$, $\forall y \in Q$.

We observe that $\mathrm{GEP}(F_2,h_2) \subset \mathrm{GEP}(F_2,h_2,Q)$. Further, it is easy to prove that Γ is a closed and convex set.

Lemma 5 (see [27, 28]). *Assume that $T : H_1 \longrightarrow H_1$ is nonexpansive operator. For all $(x,y) \in H_1 \times H_1$, the following inequality is true*

$$\begin{aligned}&\langle (x - T(x)) - (y - T(y)), T(y) - T(x)\rangle \\ &\quad \leq \frac{1}{2}\,\|(T(x)-x) - (T(y)-y)\|^2.\end{aligned} \tag{28}$$

And for all $(x,y) \in H_1 \times Fix(T)$, one has

$$\langle x - T(x), y - T(x)\rangle \leq \frac{1}{2}\,\|T(x)-x\|^2. \tag{29}$$

Lemma 6 (see [29]). *Assume A is a strongly positive linear bounded operator on Hilbert space H_1 with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.*

Lemma 7 (see [30, 31]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)\,a_n + \gamma_n\delta_n + \beta_n, \quad n \geq 0, \tag{30}$$

where $\{\gamma_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ and $\{\delta_n\}$ is a sequence in R, such that

(i) $\sum_{n=0}^{\infty}\gamma_n = \infty$.
(ii) $\limsup_{n\longrightarrow\infty}\delta_n \le 0$ *or* $\sum_{n=0}^{\infty}|\gamma_n\delta_n| < \infty$.
(iii) $\sum_{n=0}^{\infty}\beta_n < \infty$.
Then $\lim_{n\longrightarrow\infty}a_n = 0$.

According to [28], it is easy to prove the following Lemma.

Lemma 8. *Let* $F_1, h_1 : C\times C \longrightarrow R$ *be bifunctions satisfying Assumption 2 and, for* $r > 0$, *the mapping* $T_r^{(F_1,h_1)}$ *is defined as (8). Let* $x, y \in H_1$, *and* $r_1, r_2 > 0$. *Then*

$$\left\|T_{r_2}^{(F_1,h_1)}y - T_{r_1}^{(F_1,h_1)}x\right\| \le \|y-x\| + \left|\frac{r_2-r_1}{r_2}\right|\left\|T_{r_2}^{(F_1,h_1)}y - y\right\|. \tag{31}$$

Lemma 9 (see [32, 33]). *Let* $T : H \longrightarrow H$ *be given. We have the following:*
(i) T *is nonexpansive, iff the complement* $I - T$ *is* $1/2$*-ism*
(ii) if T *is* ν*-ism, then, for* $\gamma > 0$, γT *is* ν/γ*-ism*
(iii) T *is averaged, iff the complement* $I - T$ *is* ν*-ism for some* $\nu > 1/2$; *indeed, for* $\alpha \in (0,1)$, T *is* α*-averaged, iff* $I - T$ *is* $1/2\alpha$*-ism*

Lemma 10 (see [4, 32]). *Let the operators* $S, T, V : H \longrightarrow H$ *be given.*
(i) If $T = (1-\alpha)S + \alpha V$, *where* S *is averaged,* V *is nonexpansive, and* $\alpha \in (0,1)$, *then* T *is averaged.*
(ii) T *is firmly nonexpansive, iff the complement* $I - T$ *is firmly nonexpansive.*
(iii) If $T = (1-\alpha)S + \alpha V$, *where* S *is firmly nonexpansive,* V *is nonexpansive, and* $\alpha \in (0,1)$, *then* T *is averaged.*
(iv) The composite of finitely many averaged mappings is averaged. That is, if each of the mappings $\{T_i\}_{i=1}^N$ *is averaged, then so is the composite* $T_1, \ldots, T_N$. *In particular, if* T_1 *is* α_1*-averaged and* T_2 *is* α_2*-averaged, where* $\alpha_1, \alpha_2 \in (0,1)$, *then the composite* T_1T_2 *is* α*-averaged, where* $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.
(v) If the mappings $\{T_i\}_{i=1}^N$ *are averaged and have a common fixed point, then*

$$\bigcap_{i=1}^{N} Fix(T_i) = Fix(T_1, \ldots, T_N). \tag{32}$$

In the following, we give the relation between the projection operator and average mapping.

Lemma 11. *Assume that the variational inequality problem (3) is solvable. If* B *is* β*-ism from* C *into* H_1, *then* $P_C(I - \lambda B)$ *is* $(2\beta+\lambda)/4\beta$*-averaged.*

Proof. Note that B is β-ism, which implies that λB is β/λ-ism; i.e., $I-(I-\lambda B)$ is β/λ-ism. By Lemma 9(iii), we can see that $I-\lambda B$ is $\lambda/2\beta$-averaged. Since the projection P_C is $1/2$-averaged, it is easy to see from Lemma 10 that the composite $P_C(I-\lambda B)$ is $(2\beta+\lambda)/4\beta$-averaged for $0 < \lambda < 2\beta$ according to

$$\frac{1}{2} + \frac{\lambda}{2\beta} - \frac{1}{2}\cdot\frac{\lambda}{2\beta} = \frac{2\beta+\lambda}{4\beta}, \tag{33}$$

which completes the proof. □

As a result we have that, for each n, $P_C(I-\lambda_n B)$ is $(2\beta+\lambda_n)/4\beta$-averaged. Therefore, we can write

$$\begin{aligned} P_C(I-\lambda_n B) &= \frac{2\beta-\lambda_n}{4\beta}I + \frac{2\beta+\lambda_n}{4\beta}T_n \\ &= (1-b_n)I + b_nT_n, \end{aligned} \tag{34}$$

where T_n is nonexpansive and $b_n = (2\beta+\lambda_n)/4\beta \in [1/2, 1]$.

Lemma 12 (see [28]). *Let* $\{x_n\}$ *and* $\{z_n\}$ *be bounded sequence in a Banach space* X *and let* $\{\beta_n\}$ *be a sequence in* $[0,1]$ *with* $0 < \liminf_{n\longrightarrow\infty}\beta_n \le \limsup_{n\longrightarrow\infty}\beta_n < 1$. *Suppose* $x_{n+1} = (1-\beta_n)z_n + \beta_n x_n$ *for all integers* $n \ge 0$ *and* $\limsup_{n\longrightarrow\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0$. *Then* $\lim_{n\longrightarrow\infty}\|z_n - x_n\| = 0$.

3. Main Results

In this section, we give the main results of this paper. First, we describe the algorithm for finding a common solution of split generalized equilibrium and variational inequality problems.

Throughout the rest of this paper, let f be a contraction of H_1 into itself with coefficient $\eta \in (0,1)$, A be a bounded linear operator, B be a β-inverse strongly monotone mapping from C into H_1, and D be a strongly positive linear bounded self-adjoint operator on H_1 with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}/\eta$.

Now, we give the description of the algorithm.

Algorithm 13. Let $x_0 \in H_1$ be arbitrary and $\alpha > 0$. Assume that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$, $\{\lambda_n\} \subset (0, 2\beta)$, $\{r_n\} \subset (0,\infty)$, and $\xi \in (0, 1/L)$, where L is the spectral radius of the operator AA^* and A^* is the adjoint of A. Calculate sequences $\{u_n\}$, $\{y_n\}$, and $\{x_n\}$ by the following iteration formula.

$$\begin{aligned} u_n &= T_{r_n}^{(F_1,h_1)}\left(x_n + \xi A^*\left(T_{r_n}^{(F_2,h_2)} - I\right)Ax_n\right), \\ y_n &= u_n + \alpha d_{n+1}, \\ x_{n+1} &= \alpha_n\gamma f(x_n) + \beta_n x_n + \left((1-\beta_n)I - \alpha_n D\right)y_n, \end{aligned} \tag{35}$$

where $d_{n+1} = (1/\alpha)(T_nu_n - u_n) + \gamma_n d_n$, $d_0 = (1/\alpha)(T_0u_0 - u_0)$ and T_n is defined by (34).

As follows, we propose the convergence analysis of Algorithm 13.

Theorem 14. *Let* H_1 *and* H_2 *be two real Hilbert spaces and* $C \subset H_1$, $Q \subset H_2$ *be nonempty closed convex subsets. Let* $F_1, h_1 : C\times C \longrightarrow R$ *and* $F_2, h_2 : Q\times Q \longrightarrow R$ *satisfy Assumption 2;* h_1, h_2 *are monotone and* F_2 *is upper semicontinuous in the first argument. Assume that* $\Omega := VI(C,B)\cap\Gamma \ne \emptyset$ *and* $\{x_n\}$, $\{u_n\}$, *and* $\{y_n\}$ *are generated by (35). Suppose that* $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$, $\{\gamma_n\} \subset (0, 1/2)$, $\{\lambda_n\} \subset (0, 2\beta)$, $\{r_n\} \subset (0,\infty)$ *satisfy the following conditions:*
(C1) $\lim_{n\longrightarrow\infty}\alpha_n = 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$.
(C2) $0 < \liminf_{n\longrightarrow\infty}\beta_n \le \limsup_{n\longrightarrow\infty}\beta_n < 1$.
(C3) $\gamma_n = o(\alpha_n)$.
(C4) $\liminf_{n\longrightarrow\infty}\lambda_n > 0$, $\lim_{n\longrightarrow\infty}|\lambda_{n+1} - \lambda_n| = 0$.
(C5) $\lim_{n\longrightarrow\infty}(|r_{n+1} - r_n|/r_{n+1}) = 0$.
(C6) $\{T_nu_n - u_n\}$ *is bounded.*

Then the sequence $\{x_n\}$ converges strongly to $q \in \Omega$, where $q = P_\Omega(I - D + \gamma f)q$, which is the unique solution of the variational inequality problem

$$\langle (D - \gamma f) q, x - q \rangle \geq 0, \quad \forall x \in \Omega, \tag{36}$$

or equivalently, q is the unique solution to the minimization problem

$$\min_{x \in \Omega} \frac{1}{2} \langle Dx, x \rangle - h(x), \tag{37}$$

where h is a potential function for γf such that $h'(x) = \gamma f(x)$ for $x \in H_1$.

Proof. Some equalities and inequalities in the following can be obtained according to the proof of Theorem 1 in [25]. However, we give the detailed proof process in order to read handily.

From (C1) and (C2), without loss of generality we assume that $\alpha_n \leq (1 - \beta_n)\|D\|^{-1}$ for all $n \in N$. By Lemma 6, we have $\|I - \rho D\| \leq 1 - \rho\overline{\gamma}$ if $0 < \rho \leq \|D\|^{-1}$. Now suppose that $\|I - D\| \leq 1 - \overline{\gamma}$. Since D is a strongly positive linear bounded self-adjoint operator on H_1, we obtain

$$\|D\| = \sup\{|\langle Dx, x\rangle| : x \in H_1, \|x\| = 1\}. \tag{38}$$

Notice that

$$\begin{aligned} \langle ((1 - \beta_n) I - \alpha_n D) x, x \rangle &= 1 - \beta_n - \alpha_n \langle Dx, x \rangle \\ &\geq 1 - \beta_n - \alpha_n \|D\| \geq 0, \end{aligned} \tag{39}$$

which shows that $(1 - \beta_n)I - \alpha_n D$ is positive definite. Furthermore, we have

$$\begin{aligned} &\|(1 - \beta_n) I - \alpha_n D\| \\ &\quad = \sup\{|\langle ((1 - \beta_n) I - \alpha_n D) x, x\rangle| : x \in H_1, \|x\| \\ &\quad = 1\} = \sup\{1 - \beta_n - \alpha_n \langle Dx, x\rangle : x \in H_1, \|x\| = 1\} \\ &\quad \leq 1 - \beta_n - \alpha_n \overline{\gamma}. \end{aligned} \tag{40}$$

Since f is a contraction mapping with constant $\eta \in (0, 1)$, for all $x, y \in H_1$, we have

$$\begin{aligned} &\|P_\Omega (I - D + \gamma f)(x) - P_\Omega (I - D + \gamma f)(y)\| \\ &\quad \leq \|(I - D + \gamma f)(x) - (I - D + \gamma f)(y)\| \\ &\quad \leq \|I - D\| \|x - y\| + \gamma \|f(x) - f(y)\| \\ &\quad \leq (1 - \overline{\gamma}) \|x - y\| + \gamma\eta \|x - y\| \\ &\quad \leq (1 - (\overline{\gamma} - \gamma\eta)) \|x - y\|, \end{aligned} \tag{41}$$

which implies that $P_\Omega(I - D + \gamma f)$ is a contraction mapping from H_1 into itself. It follows from the Banach contraction principle that there exists an element $q \in \Omega$ such that $q = P_\Omega(I - D + \gamma f)q$.

Step 1 (we show that $\{x_n\}$ is bounded). Let $x^* \in \Omega$; i.e., $x^* \in \Gamma$; we have $x^* = T_{r_n}^{(F_1,h_1)} x^*$ and $Ax^* = T_{r_n}^{(F_2,h_2)} Ax^*$.

In the following, we compute

$$\begin{aligned} \|u_n - x^*\|^2 &= \|T_{r_n}^{(F_1,h_1)} (x_n + \xi A^* (T_{r_n}^{(F_2,h_2)} - I) Ax_n) \\ &\quad - x^*\|^2 = \|T_{r_n}^{(F_1,h_1)} (x_n + \xi A^* (T_{r_n}^{(F_2,h_2)} - I) Ax_n) \\ &\quad - T_{r_n}^{(F_1,h_1)} x^*\|^2 \leq \|x_n + \xi A^* (T_{r_n}^{(F_2,h_2)} - I) Ax_n \\ &\quad - x^*\|^2 = \|x_n - x^*\|^2 + \xi^2 \|A^* (T_{r_n}^{(F_2,h_2)} - I) Ax_n\|^2 \\ &\quad + 2\xi \langle x_n - x^*, A^* (T_{r_n}^{(F_2,h_2)} - I) Ax_n \rangle. \end{aligned} \tag{42}$$

Thus, we have

$$\begin{aligned} &\|u_n - x^*\|^2 \\ &\quad \leq \|x_n - x^*\|^2 \\ &\qquad + \xi^2 \langle (T_{r_n}^{(F_2,h_2)} - I) Ax_n, AA^* (T_{r_n}^{(F_2,h_2)} - I) Ax_n \rangle \\ &\qquad + 2\xi \langle x_n - x^*, A^* (T_{r_n}^{(F_2,h_2)} - I) Ax_n \rangle. \end{aligned} \tag{43}$$

On the other hand, we have

$$\begin{aligned} &\xi^2 \langle (T_{r_n}^{(F_2,h_2)} - I) Ax_n, AA^* (T_{r_n}^{(F_2,h_2)} - I) Ax_n \rangle \\ &\quad \leq L\xi^2 \|(T_{r_n}^{(F_2,h_2)} - I) Ax_n\|^2. \end{aligned} \tag{44}$$

and

$$\begin{aligned} &2\xi \langle x_n - x^*, A^* (T_{r_n}^{(F_2,h_2)} - I) Ax_n \rangle \\ &\quad = 2\xi \langle A(x_n - x^*), (T_{r_n}^{(F_2,h_2)} - I) Ax_n \rangle \\ &\quad = 2\xi \langle A(x_n - x^*) + (T_{r_n}^{(F_2,h_2)} - I) Ax_n \\ &\qquad - (T_{r_n}^{(F_2,h_2)} - I) Ax_n, (T_{r_n}^{(F_2,h_2)} - I) Ax_n \rangle \\ &\quad = 2\xi \{\langle T_{r_n}^{(F_2,h_2)} Ax_n - Ax^*, (T_{r_n}^{(F_2,h_2)} - I) Ax_n \rangle \\ &\qquad - \|(T_{r_n}^{(F_2,h_2)} - I) Ax_n\|^2\} \\ &\quad \leq 2\xi \left\{\frac{1}{2} \|(T_{r_n}^{(F_2,h_2)} - I) Ax_n\|^2 \right. \\ &\qquad \left. - \|(T_{r_n}^{(F_2,h_2)} - I) Ax_n\|^2\right\} \\ &\quad \leq -\xi \|(T_{r_n}^{(F_2,h_2)} - I) Ax_n\|^2, \end{aligned} \tag{45}$$

where the first inequality is derived from (29).

From (43)-(45), we have

$$\begin{aligned} \|u_n - x^*\|^2 &\leq \|x_n - x^*\|^2 \\ &\quad + \xi (L\xi - 1) \|(T_{r_n}^{(F_2,h_2)} - I) Ax_n\|^2. \end{aligned} \tag{46}$$

Noticing that $\xi \in (0, 1/L)$, we obtain

$$\|u_n - x^*\|^2 \leq \|x_n - x^*\|^2. \tag{47}$$

As follows, we prove that $\{d_n\}$ is bounded. The proof is by induction. It is true trivially for $n = 0$. Let $M_1 = \max\{\|d_0\|, (2/\alpha)\sup_{n\in N}\|T_n u_n - u_n\|\}$. From (C6), it is shown that $M_1 < \infty$. Assume that $\|d_n\| \le M_1$ for some n; we prove that it holds for $n+1$. According to the triangle inequality, we obtain

$$\begin{aligned}\|d_{n+1}\| &= \left\|\frac{1}{\alpha}(T_n u_n - u_n) + \gamma_n d_n\right\| \\ &\le \frac{1}{\alpha}\|T_n u_n - u_n\| + \gamma_n\|d_n\| \le \frac{1}{\alpha}\cdot\frac{\alpha}{2}M_1 + \frac{M_1}{2} \\ &= M_1,\end{aligned} \tag{48}$$

which implies that $\|d_n\| \le M_1$ for all $n \in N$; i.e., $\{d_n\}$ is bounded.

It is easy to see that $x^* \in VI(C,B)$ according to $x^* \in \Omega$. By (22), we have $P_C(I - \lambda B)x^* = x^*$, which, together with (34), implies that

$$b_n T_n x^* = P_C(I - \lambda B)x^* - (1 - b_n)x^* = b_n x^*; \tag{49}$$

that is,

$$T_n x^* = x^*. \tag{50}$$

By the definition of $\{y_n\}$, (47), and $\{T_n\}$ being nonexpansive, we have

$$\begin{aligned}\|y_n - x^*\| &= \|u_n + \alpha d_{n+1} - x^*\| \\ &= \|T_n(u_n) + \alpha\gamma_n d_n - T_n x^*\| \\ &\le \|u_n - x^*\| + \alpha\gamma_n M_1 \\ &\le \|x_n - x^*\| + \alpha\gamma_n M_1.\end{aligned} \tag{51}$$

As a result, it follows from (51), Lemma 6,and the fact that $\alpha_n \longrightarrow 0$ and $\gamma_n = o(\alpha_n)$ that when n is large enough,

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|\alpha_n(\gamma f(x_n) - Dx^*) + \beta_n(x_n - x^*) \\ &\quad + ((1-\beta_n)I - \alpha_n D)(y_n - x^*)\| \le \alpha_n\|\gamma f(x_n) \\ &\quad - Dx^*\| + \beta_n\|x_n - x^*\| + (1 - \beta_n - \alpha_n\overline{\gamma}) \\ &\quad \cdot(\|x_n - x^*\| + \alpha\gamma_n M_1) \le \alpha_n\gamma\|f(x_n) - f(x^*)\| \\ &\quad + \alpha_n\|\gamma f(x^*) - Dx^*\| + \beta_n\|x_n - x^*\| + (1 - \beta_n \\ &\quad - \alpha_n\overline{\gamma})(\|x_n - x^*\| + \alpha\gamma_n M_1) \le \alpha_n\gamma\eta\|x_n - x^*\| \\ &\quad + \alpha_n\|\gamma f(x^*) - Dx^*\| + (1 - \alpha_n\overline{\gamma})\|x_n - x^*\| + (1 \\ &\quad - \beta_n - \alpha_n\overline{\gamma})\alpha\gamma_n M_1 \le (1 - \alpha_n(\overline{\gamma} - \gamma\eta))\|x_n - x^*\| \\ &\quad + \alpha_n(\overline{\gamma} - \gamma\eta)\frac{\alpha M_1 + \|\gamma f(x^*) - Dx^*\|}{\overline{\gamma} - \gamma\eta} \\ &\le \max\left\{\|x_n - x^*\|, \frac{\alpha M_1 + \|\gamma f(x^*) - Dx^*\|}{\overline{\gamma} - \gamma\eta}\right\},\end{aligned} \tag{52}$$

where the third inequality is true because $\beta_n \in (0,1)$, $\alpha_n \longrightarrow 0$, and $\gamma_n = o(\alpha_n)$. As a result,

$$(1 - \beta_n - \alpha_n\overline{\gamma})\alpha\gamma_n M_1 \le \alpha\alpha_n M_1, \tag{53}$$

when n is large enough.

Hence, $\{x_n\}$ is bounded and so are $\{u_n\}$ and $\{y_n\}$.

Step 2 (we show that $\lim_{n\longrightarrow\infty}\|x_{n+1} - x_n\| = 0$). Since $T_{r_{n+1}}^{(F_1,h_1)}$ and $T_{r_{n+1}}^{(F_2,h_2)}$ both are firmly nonexpansive, for $\xi \in (0, 1/L)$, the mapping $T_{r_{n+1}}^{(F_1,h_1)}(I + \xi A^*(T_{r_{n+1}}^{(F_2,h_2)} - I)A)$ is nonexpansive; see [34, 35]. Noticing that $u_n = T_{r_n}^{(F_1,h_1)}(x_n + \xi A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n)$ and $u_{n+1} = T_{r_{n+1}}^{(F_1,h_1)}(x_{n+1} + \xi A^*(T_{r_{n+1}}^{(F_2,h_2)} - I)Ax_{n+1})$, we have from Lemma 8 that

$$\begin{aligned}&\|u_{n+1} - u_n\| \\ &\le \left\|T_{r_{n+1}}^{(F_1,h_1)}\left(x_{n+1} + \xi A^*\left(T_{r_{n+1}}^{(F_2,h_2)} - I\right)Ax_{n+1}\right)\right. \\ &\quad \left. - T_{r_{n+1}}^{(F_1,h_1)}\left(x_n + \xi A^*\left(T_{r_{n+1}}^{(F_2,h_2)} - I\right)Ax_n\right)\right\| \\ &\quad + \left\|T_{r_{n+1}}^{(F_1,h_1)}\left(x_n + \xi A^*\left(T_{r_{n+1}}^{(F_2,h_2)} - I\right)Ax_n\right)\right. \\ &\quad \left. - T_{r_n}^{(F_1,h_1)}\left(x_n + \xi A^*\left(T_{r_n}^{(F_2,h_2)} - I\right)Ax_n\right)\right\| \le \|x_{n+1} \\ &\quad - x_n\| + \left\|x_n + \xi A^*\left(T_{r_{n+1}}^{(F_2,h_2)} - I\right)Ax_n\right. \\ &\quad \left. - \left(x_n + \xi A^*\left(T_{r_n}^{(F_2,h_2)} - I\right)Ax_n\right)\right\| + \left|1 - \frac{r_n}{r_{n+1}}\right| \\ &\quad \cdot\left\|T_{r_{n+1}}^{(F_1,h_1)}\left(x_n + \xi A^*\left(T_{r_{n+1}}^{(F_2,h_2)} - I\right)Ax_n\right)\right. \\ &\quad \left. - \left(x_n + \xi A^*\left(T_{r_{n+1}}^{(F_2,h_2)} - I\right)Ax_n\right)\right\| \le \|x_{n+1} - x_n\| \\ &\quad + \xi\|A\|\left\|T_{r_{n+1}}^{(F_2,h_2)}Ax_n - T_{r_n}^{(F_2,h_2)}Ax_n\right\| + \zeta_n \le \|x_{n+1} \\ &\quad - x_n\| + \xi\|A\|\left|1 - \frac{r_n}{r_{n+1}}\right|\left\|T_{r_{n+1}}^{(F_2,h_2)}Ax_n - Ax_n\right\| + \zeta_n \\ &= \|x_{n+1} - x_n\| + \xi\|A\|\sigma_n + \zeta_n,\end{aligned} \tag{54}$$

where $\sigma_n := |1 - r_n/r_{n+1}|\|T_{r_{n+1}}^{(F_2,h_2)}Ax_n - Ax_n\|$ and $\zeta_n := |1 - r_n/r_{n+1}|\|T_{r_{n+1}}^{(F_1,h_1)}(x_n + \xi A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n) - (x_n + \xi A^*(T_{r_{n+1}}^{(F_2,h_2)} - I)Ax_n)\|$.

Furthermore, one has

$$\begin{aligned}\|y_{n+1} - y_n\| &= \|u_{n+1} + \alpha d_{n+2} - (u_n + \alpha d_{n+1})\| \\ &\le \|T_{n+1}u_{n+1} - T_n u_n\| \\ &\quad + \alpha\|\gamma_{n+1}d_{n+1} - \gamma_n d_n\| \\ &\le \|T_{n+1}u_{n+1} - T_{n+1}u_n\| \\ &\quad + \|T_{n+1}u_n - T_n u_n\| \\ &\quad + \alpha\|\gamma_{n+1}d_{n+1} - \gamma_n d_n\| \\ &\le \|u_{n+1} - u_n\| + \|T_{n+1}u_n - T_n u_n\| \\ &\quad + \alpha M_1(\gamma_{n+1} + \gamma_n)\end{aligned} \tag{55}$$

It follows from (34) that

$$\begin{aligned}
\|T_{n+1}u_n - T_n u_n\| &= \left\| \frac{P_C(I-\lambda_{n+1}B) - (1-b_{n+1})I}{b_{n+1}} u_n \right. \\
&\quad \left. - \frac{P_C(I-\lambda_n B) - (1-b_n)I}{b_n} u_n \right\| \\
&= \left\| \frac{4\beta P_C(I-\lambda_{n+1}B) - (2\beta-\lambda_{n+1})I}{2\beta+\lambda_{n+1}} u_n \right. \\
&\quad \left. - \frac{4\beta P_C(I-\lambda_n B) - (2\beta-\lambda_n)I}{2\beta+\lambda_n} u_n \right\| = \left\| \frac{4\beta P_C(I-\lambda_{n+1}B)}{2\beta+\lambda_{n+1}} u_n \right. \\
&\quad \left. - \frac{4\beta P_C(I-\lambda_n B)}{2\beta+\lambda_n} u_n + \frac{(2\beta-\lambda_n)}{2\beta+\lambda_n} u_n - \frac{(2\beta-\lambda_{n+1})}{2\beta+\lambda_{n+1}} u_n \right\| \\
&\le \left\| \frac{4\beta\left[(2\beta+\lambda_n)P_C(I-\lambda_{n+1}B) - (2\beta+\lambda_{n+1})P_C(I-\lambda_n B)\right]}{(2\beta+\lambda_{n+1})(2\beta+\lambda_n)} \right. \\
&\quad \left. \cdot u_n \right\| + \left\| \frac{4\beta(\lambda_{n+1}-\lambda_n)}{(2\beta+\lambda_{n+1})(2\beta+\lambda_n)} u_n \right\| \\
&\le \left\| \frac{4\beta(\lambda_n-\lambda_{n+1})P_C(I-\lambda_{n+1}B)}{(2\beta+\lambda_{n+1})(2\beta+\lambda_n)} u_n \right\| \\
&\quad + \left\| \frac{4\beta(2\beta+\lambda_{n+1})\left[P_C(I-\lambda_{n+1}B) - P_C(I-\lambda_n B)\right]}{(2\beta+\lambda_{n+1})(2\beta+\lambda_n)} u_n \right\| \\
&\quad + \left\| \frac{4\beta(\lambda_{n+1}-\lambda_n)}{(2\beta+\lambda_{n+1})(2\beta+\lambda_n)} u_n \right\| \le \frac{1}{\beta}|\lambda_n - \lambda_{n+1}| \\
&\quad \cdot \left(\|P_C(I-\lambda_{n+1}B)u_n\| + 4\beta\|Bu_n\| \right. \\
&\quad \left. + \|u_n\|\right) \le M_2|\lambda_{n+1}-\lambda_n|,
\end{aligned} \tag{56}$$

where $M_2 = \sup_{n\in N}\{(1/\beta)(\|P_C(I-\lambda_{n+1}B)u_n\| + 4\beta\|Bu_n\| + \|u_n\|)\}$.

Hence from (54)-(56), we get

$$\begin{aligned}
\|y_{n+1} - y_n\| &\le \|x_{n+1} - x_n\| + \xi\|A\|\sigma_n + \zeta_n \\
&\quad + M_2|\lambda_{n+1}-\lambda_n| + \alpha M_1(\gamma_{n+1}+\gamma_n).
\end{aligned} \tag{57}$$

Set $x_{n+1} = (1-\beta_n)z_n + \beta_n x_n$; it follows that

$$\begin{aligned}
z_n &= \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} \\
&= \frac{\alpha_n \gamma f(x_n) + ((1-\beta_n)I - \alpha_n D)y_n}{1-\beta_n}.
\end{aligned} \tag{58}$$

As a result,

$$\begin{aligned}
&\|z_{n+1} - z_n\| \\
&\quad = \left\| \frac{\alpha_{n+1}\gamma f(x_{n+1}) + ((1-\beta_{n+1})I - \alpha_{n+1}D)y_{n+1}}{1-\beta_{n+1}} \right. \\
&\quad \left. - \frac{\alpha_n \gamma f(x_n) + ((1-\beta_n)I - \alpha_n D)y_n}{1-\beta_n} \right\| \\
&\quad = \left\| \frac{\alpha_{n+1}(\gamma f(x_{n+1}) - Dy_{n+1})}{1-\beta_{n+1}} \right. \\
&\quad \left. - \frac{\alpha_n(\gamma f(x_n) - Dy_n)}{1-\beta_n} + y_{n+1} - y_n \right\| \\
&\quad \le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1}) - Dy_{n+1}\| \\
&\quad + \frac{\alpha_n}{1-\beta_n}\|\gamma f(x_n) - Dy_n\| + \|y_{n+1} - y_n\| \\
&\quad \le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1}) - Dy_{n+1}\| \\
&\quad + \frac{\alpha_n}{1-\beta_n}\|\gamma f(x_n) - Dy_n\| + \|x_{n+1} - x_n\| + \xi\|A\| \\
&\quad \cdot \sigma_n + \zeta_n + M_2|\lambda_{n+1} - \lambda_n| + \alpha M_1(\gamma_{n+1} + \gamma_n).
\end{aligned} \tag{59}$$

Letting $n \longrightarrow \infty$, from (C1)-(C5), we have

$$\limsup_{n\longrightarrow\infty}\left(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|\right) \le 0. \tag{60}$$

By Lemma 12, we obtain

$$\lim_{n\longrightarrow\infty}\|z_n - x_n\| = 0. \tag{61}$$

Further,

$$\lim_{n\longrightarrow\infty}\|x_{n+1} - x_n\| = \lim_{n\longrightarrow\infty}(1-\beta_n)\|z_n - x_n\| = 0. \tag{62}$$

Step 3 (we show that $\lim_{n\longrightarrow\infty}\|u_n - x_n\| = 0$). Since $x^* \in \Omega$, $x^* = T_{r_n}^{(F_1,h_1)}x^*$, and $T_{r_n}^{(F_1,h_1)}$ is firmly nonexpansive, we obtain

$$\begin{aligned}
\|u_n - x^*\|^2 &= \left\|T_{r_n}^{(F_1,h_1)}\left(x_n + \xi A^*\left(T_{r_n}^{(F_2,h_2)} - I\right)Ax_n\right) \right. \\
&\quad \left. - x^*\right\|^2 = \left\|T_{r_n}^{(F_1,h_1)}\left(x_n + \xi A^*\left(T_{r_n}^{(F_2,h_2)} - I\right)Ax_n\right) \right. \\
&\quad \left. - T_{r_n}^{(F_1,h_1)}x^*\right\|^2 \le \left\langle u_n - x^*, x_n + \xi A^*\left(T_{r_n}^{(F_2,h_2)} - I\right) \right. \\
&\quad \left. \cdot Ax_n - x^*\right\rangle = \frac{1}{2}\left\{\|u_n - x^*\|^2 + \|x_n \right. \\
&\quad + \xi A^*\left(T_{r_n}^{(F_2,h_2)} - I\right)Ax_n - x^*\|^2 - \left\|(u_n - x^*) \right. \\
&\quad \left. - \left[x_n + \xi A^*\left(T_{r_n}^{(F_2,h_2)} - I\right)Ax_n - x^*\right]\right\|^2\right\} = \frac{1}{2}\left\{\|u_n \right. \\
&\quad - x^*\|^2 + \|x_n - x^*\|^2 - \|u_n - x_n\|^2 + 2\xi\left\langle u_n \right. \\
&\quad \left. - x^*, A^*\left(T_{r_n}^{(F_2,h_2)} - I\right)Ax_n\right\rangle\right\}.
\end{aligned} \tag{63}$$

Hence, we obtain

$$\begin{aligned}
&\|u_n - x^*\|^2 \\
&\quad \le \|x_n - x^*\|^2 - \|u_n - x_n\|^2 \\
&\quad + 2\xi\|A(u_n - x^*)\|\left\|\left(T_{r_n}^{(F_2,h_2)} - I\right)Ax_n\right\|.
\end{aligned} \tag{64}$$

Furthermore,

$$\begin{aligned}\|y_n - x^*\|^2 &= \|u_n + \alpha d_{n+1} - x^*\|^2 \\ &\le \|u_n - x^*\|^2 + 2\alpha\gamma_n \langle y_n - x^*, d_n\rangle \\ &\le \|u_n - x^*\|^2 + M_3\gamma_n,\end{aligned} \tag{65}$$

where $M_3 = \sup_{n\in N} 2\alpha\langle y_n - x^*, d_n\rangle$.

By Lemma 1 (*iii*), (46), and (65), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|\alpha_n\gamma f(x_n) + \beta_n x_n \\ &\quad + ((1-\beta_n)I - \alpha_n D)y_n - x^*\|^2 \\ &= \|\alpha_n(\gamma f(x_n) - Dx^*) + \beta_n(x_n - x^*) \\ &\quad + ((1-\beta_n)I - \alpha_n D)(y_n - x^*)\|^2 \\ &\le (\alpha_n\|\gamma f(x_n) - Dx^*\| + \beta_n\|x_n - x^*\| \\ &\quad + (1-\beta_n-\alpha_n\overline{\gamma})\|y_n - x^*\|)^2 \\ &= \left(\alpha_n\overline{\gamma}\left\|\frac{1}{\overline{\gamma}}(\gamma f(x_n) - Dx^*)\right\| + \beta_n\|x_n - x^*\| \right. \\ &\quad \left. + (1-\beta_n-\alpha_n\overline{\gamma})\|y_n - x^*\|\right)^2 \le \alpha_n\overline{\gamma}\frac{1}{\overline{\gamma}^2}\|\gamma f(x_n) \\ &\quad - Dx^*\|^2 + \beta_n\|x_n - x^*\|^2 + (1-\beta_n-\alpha_n\overline{\gamma})\|y_n \\ &\quad - x^*\|^2 \le \frac{\alpha_n}{\overline{\gamma}}\|\gamma f(x_n) - Dx^*\|^2 + \beta_n\|x_n - x^*\|^2 \\ &\quad + (1-\beta_n-\alpha_n\overline{\gamma})\|y_n - x^*\|^2 \le \frac{\alpha_n}{\overline{\gamma}}\|\gamma f(x_n) \\ &\quad - Dx^*\|^2 + \beta_n\|x_n - x^*\|^2 + (1-\beta_n-\alpha_n\overline{\gamma}) \\ &\quad \cdot \left(\|x_n - x^*\|^2 + \xi(L\xi - 1)\left\|\left(T_{r_n}^{(F_2,h_2)} - I\right)Ax_n\right\|^2 \right. \\ &\quad \left. + M_3\gamma_n\right) = \frac{\alpha_n}{\overline{\gamma}}\|\gamma f(x_n) - Dx^*\|^2 + \beta_n\|x_n - x^*\|^2 \\ &\quad + (1-\beta_n-\alpha_n\overline{\gamma})\|x_n - x^*\|^2 - (1-\beta_n-\alpha_n\overline{\gamma})\xi(1 \\ &\quad - L\xi)\left\|\left(T_{r_n}^{(F_2,h_2)} - I\right)Ax_n\right\|^2 + (1-\beta_n-\alpha_n\overline{\gamma})M_3\gamma_n.\end{aligned} \tag{66}$$

As a result,

$$\begin{aligned}&(1-\beta_n-\alpha_n\overline{\gamma})\xi(1-L\xi)\left\|\left(T_{r_n}^{(F_2,h_2)} - I\right)Ax_n\right\|^2 \\ &\quad \le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\ &\qquad + \frac{\alpha_n}{\overline{\gamma}}\|\gamma f(x_n) - Dx^*\|^2 + (1-\beta_n-\alpha_n\overline{\gamma})M_3\gamma_n \\ &\quad \le (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_n - x_{n+1}\| \\ &\qquad + \frac{\alpha_n}{\overline{\gamma}}\|\gamma f(x_n) - Dx^*\|^2 + (1-\beta_n-\alpha_n\overline{\gamma})M_3\gamma_n.\end{aligned} \tag{67}$$

According to $\alpha_n \longrightarrow 0$, $\gamma_n = o(\alpha_n)$, $(1-\beta_n-\alpha_n\overline{\gamma})\xi(1-L\xi) > 0$, and $\lim_{n\longrightarrow\infty}\|x_{n+1} - x_n\| = 0$, we obtain

$$\lim_{n\longrightarrow\infty}\left\|\left(T_{r_n}^{(F_2,h_2)} - I\right)Ax_n\right\| = 0. \tag{68}$$

From (65), (64), and Lemma 1 (*iii*), we obtain

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|\alpha_n\gamma f(x_n) + \beta_n x_n \\ &\quad + ((1-\beta_n)I - \alpha_n D)y_n - x^*\|^2 \\ &= \|\alpha_n(\gamma f(x_n) - Dx^*) + \beta_n(x_n - x^*) \\ &\quad + ((1-\beta_n)I - \alpha_n D)(y_n - x^*)\|^2 \\ &\le (\alpha_n\|\gamma f(x_n) - Dx^*\| + \beta_n\|x_n - x^*\| \\ &\quad + (1-\beta_n-\alpha_n\overline{\gamma})\|y_n - x^*\|)^2 \\ &= \left(\alpha_n\overline{\gamma}\left\|\frac{1}{\overline{\gamma}}(\gamma f(x_n) - Dx^*)\right\| + \beta_n\|x_n - x^*\| \right. \\ &\quad \left. + (1-\beta_n-\alpha_n\overline{\gamma})\|y_n - x^*\|\right)^2 \le \alpha_n\overline{\gamma}\frac{1}{\overline{\gamma}^2}\|\gamma f(x_n) \\ &\quad - Dx^*\|^2 + \beta_n\|x_n - x^*\|^2 + (1-\beta_n-\alpha_n\overline{\gamma})\|y_n \\ &\quad - x^*\|^2 \le \frac{\alpha_n}{\overline{\gamma}}\|\gamma f(x_n) - Dx^*\|^2 + \beta_n\|x_n - x^*\|^2 \\ &\quad + (1-\beta_n-\alpha_n\overline{\gamma})\|y_n - x^*\|^2 \le \frac{\alpha_n}{\overline{\gamma}}\|\gamma f(x_n) \\ &\quad - Dx^*\|^2 + \beta_n\|x_n - x^*\|^2 + (1-\beta_n-\alpha_n\overline{\gamma}) \\ &\quad \cdot \left(\|u_n - x^*\|^2 + M_3\gamma_n\right) \le \frac{\alpha_n}{\overline{\gamma}}\|\gamma f(x_n) - Dx^*\|^2 \\ &\quad + \beta_n\|x_n - x^*\|^2 + (1-\beta_n-\alpha_n\overline{\gamma})\left(\|x_n - x^*\|^2 \right. \\ &\quad \left. - \|u_n - x_n\|^2 \right. \\ &\quad \left. + 2\xi\|A(u_n - x^*)\|\left\|\left(T_{r_n}^{(F_2,h_2)} - I\right)Ax_n\right\| + M_3\gamma_n\right) \\ &\quad = \frac{\alpha_n}{\overline{\gamma}}\|\gamma f(x_n) - Dx^*\|^2 + \beta_n\|x_n - x^*\|^2 + (1-\beta_n \\ &\quad - \alpha_n\overline{\gamma})\|x_n - x^*\|^2 - (1-\beta_n-\alpha_n\overline{\gamma})\|u_n - x_n\|^2 \\ &\quad + 2\xi(1-\beta_n-\alpha_n\overline{\gamma})\|A(u_n - x^*)\| \\ &\quad \cdot \left\|\left(T_{r_n}^{(F_2,h_2)} - I\right)Ax_n\right\| + (1-\beta_n-\alpha_n\overline{\gamma})M_3\gamma_n.\end{aligned} \tag{69}$$

Therefore, one has

$$\begin{aligned}&(1-\beta_n-\alpha_n\overline{\gamma})\|u_n - x_n\|^2 \le \frac{\alpha_n}{\overline{\gamma}}\|\gamma f(x_n) - Dx^*\|^2 \\ &\quad + (1-\alpha_n\overline{\gamma})\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\ &\quad + 2\xi(1-\beta_n-\alpha_n\overline{\gamma})\|A(u_n - x^*)\|\end{aligned}$$

$$\begin{aligned}
&\cdot \left\|\left(T_{r_n}^{(F_2,h_2)} - I\right) Ax_n\right\| + \left(1 - \beta_n - \alpha_n\overline{\gamma}\right) M_3\gamma_n \\
&\leq \frac{\alpha_n}{\overline{\gamma}} \left\|\gamma f\left(x_n\right) - Dx^*\right\|^2 + \left\|x_n - x^*\right\|^2 \\
&- \left\|x_{n+1} - x^*\right\|^2 + 2\xi\left(1 - \beta_n - \alpha_n\overline{\gamma}\right) \left\|A\left(u_n - x^*\right)\right\| \\
&\cdot \left\|\left(T_{r_n}^{(F_2,h_2)} - I\right) Ax_n\right\| + \left(1 - \beta_n - \alpha_n\overline{\gamma}\right) M_3\gamma_n \\
&\leq \frac{\alpha_n}{\overline{\gamma}} \left\|\gamma f\left(x_n\right) - Dx^*\right\|^2 + \left\|x_n - x_{n+1}\right\| \\
&\cdot \left(\left\|x_n - x^*\right\| + \left\|x_{n+1} - x^*\right\|\right) + 2\xi\left(1 - \beta_n - \alpha_n\overline{\gamma}\right) \\
&\cdot \left\|A\left(u_n - x^*\right)\right\| \left\|\left(T_{r_n}^{(F_2,h_2)} - I\right) Ax_n\right\| \\
&+ \left(1 - \beta_n - \alpha_n\overline{\gamma}\right) M_3\gamma_n.
\end{aligned} \tag{70}$$

According to $\alpha_n \longrightarrow 0$, $\gamma_n = o(\alpha_n)$, (62), and (68), we obtain

$$\lim_{n\longrightarrow\infty} \left\|u_n - x_n\right\| = 0. \tag{71}$$

Step 4 (we show that $\lim_{n\longrightarrow\infty}\|T_nu_n - u_n\| = 0$). From (58), we have

$$z_n - y_n = \frac{\alpha_n}{1 - \beta_n}\gamma f\left(x_n\right) - \frac{\alpha_n}{1 - \beta_n} Dy_n. \tag{72}$$

Hence,

$$\left\|z_n - y_n\right\| \leq \frac{\alpha_n\gamma}{1 - \beta_n} \left|f\left(x_n\right)\right| + \frac{\alpha_n}{1 - \beta_n} \left\|Dy_n\right\|. \tag{73}$$

From (C1) and (C2), one has

$$\lim_{n\longrightarrow\infty} \left\|z_n - y_n\right\| = 0. \tag{74}$$

By (74) and (61), we have

$$\lim_{n\longrightarrow\infty} \left\|x_n - y_n\right\| = 0. \tag{75}$$

Combining (71) and (75), one has

$$\lim_{n\longrightarrow\infty} \left\|u_n - y_n\right\| = 0. \tag{76}$$

Noting that

$$y_n - u_n = T_nu_n - u_n + \alpha\gamma_nd_n, \tag{77}$$

one has

$$T_nu_n - u_n = y_n - u_n - \alpha\gamma_nd_n. \tag{78}$$

It follows from (76) and $\alpha_n \longrightarrow 0$, $\gamma_n = o(\alpha_n)$ that

$$\lim_{n\longrightarrow\infty} \left\|T_nu_n - u_n\right\| = 0. \tag{79}$$

We claim that

$$\limsup_{n\longrightarrow\infty} \left\langle\left(D - \gamma f\right) q, q - x_n\right\rangle \leq 0, \tag{80}$$

where q is the unique solution of the variational inequality $\langle(D - \gamma f)q, x - q\rangle \geq 0,\ \forall x \in \Omega$.

To show this inequality, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\begin{aligned}
&\limsup_{n\longrightarrow\infty} \left\langle\left(D - \gamma f\right) q, q - x_n\right\rangle \\
&\quad = \lim_{i\longrightarrow\infty} \left\langle\left(D - \gamma f\right) q, q - x_{n_i}\right\rangle.
\end{aligned} \tag{81}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence of $\{x_{n_i}\}$ which converges weakly to $z \in C$. Without loss of generality, we can assume that $x_{n_i} \rightharpoonup z$.

Step 5 (we show that $z \in \Omega$). First, we show that $z \in \mathrm{VI}(C, B)$.

Let $M : H_1 \longrightarrow 2^{H_1}$ be a set-valued mapping defined by

$$Mv = \begin{cases} Bv + N_Cv, & v \in C, \\ \emptyset, & v \notin C, \end{cases} \tag{82}$$

where $N_Cv := \{z \in H_1 : \langle v - u, z\rangle \geq 0,\ \forall u \in C\}$ is the normal cone to C at $v \in C$. Then M is maximal monotone and $0 \in Mv$ if and only if $v \in \mathrm{VI}(C, B)$ (see [36]). Let $(v, u) \in G(M)$. Therefore, we have

$$u \in Mv = Bv + N_Cv, \tag{83}$$

and so

$$u - Bv \in N_Cv. \tag{84}$$

According to $u_n \in C$, we obtain

$$\left\langle v - u_n, u - Bv\right\rangle \geq 0. \tag{85}$$

On the other hand, according to

$$P_C\left(\left(I - \lambda_nB\right) u_n\right) = u_n + b_n\left(T_nu_n - u_n\right), \tag{86}$$

where $b_n = (2\beta + \lambda_n)/4\beta$, for $\forall n \in N$, and $v \in H_1$, we have

$$\begin{aligned}
&\left\langle v - u_n - b_n\left(T_nu_n - u_n\right), u_n + b_n\left(T_nu_n - u_n\right)\right. \\
&\quad \left. - \left(u_n - \lambda_nBu_n\right)\right\rangle \geq 0.
\end{aligned} \tag{87}$$

Therefore,

$$\begin{aligned}
&\left\langle v - u_n, b_n\left(T_nu_n - u_n\right) + \lambda_nBu_n\right\rangle - \left\|b_n\left(T_nu_n - u_n\right)\right\|^2 \\
&\quad - \left\langle b_n\left(T_nu_n - u_n\right), \lambda_nBu_n\right\rangle \geq 0.
\end{aligned} \tag{88}$$

As a result,

$$\begin{aligned}
&\left\langle v - u_n, \frac{b_n}{\lambda_n}\left(T_nu_n - u_n\right) + Bu_n\right\rangle \\
&\quad - \left\langle b_n\left(T_nu_n - u_n\right), Bu_n\right\rangle \geq 0.
\end{aligned} \tag{89}$$

Furthermore, according to (85) and (89), for $\forall n \in N$, one has

$$\begin{aligned}\langle v-u_n,u\rangle &\geq \langle v-u_n,Bv\rangle \\ &\quad -\left\langle v-u_n,\frac{b_n}{\lambda_n}(T_nu_n-u_n)+Bu_n\right\rangle \\ &\quad +\langle b_n(T_nu_n-u_n),Bu_n\rangle \\ &= \langle v-u_n,Bv-Bu_n\rangle \\ &\quad -\left\langle v-u_n,\frac{b_n}{\lambda_n}(T_nu_n-u_n)\right\rangle \\ &\quad +\langle b_n(T_nu_n-u_n),Bu_n\rangle .\end{aligned} \tag{90}$$

Replacing n by n_i, one has

$$\begin{aligned}\left\langle v-u_{n_i},u\right\rangle &\geq \left\langle v-u_{n_i},Bv-Bu_{n_i}\right\rangle \\ &\quad -\left\langle v-u_{n_i},\frac{b_{n_i}}{\lambda_{n_i}}\left(T_{n_i}u_{n_i}-u_{n_i}\right)\right\rangle \\ &\quad +\left\langle b_{n_i}\left(T_{n_i}u_{n_i}-u_{n_i}\right),Bu_{n_i}\right\rangle .\end{aligned} \tag{91}$$

Since $\|u_n-x_n\|\longrightarrow 0$ and $x_{n_i}\rightharpoonup z$, we have $u_{n_i}\rightharpoonup z$. Noting that B is β-ism, from (91) and (79), we have

$$\langle v-z,u\rangle \geq 0. \tag{92}$$

Since M is maximal monotone, one has $z\in M^{-1}0$. Hence $z\in VI(C,B)$.

Next, we prove $z\in\Gamma$.

According to Algorithm 13, we have

$$u_{n_i}=T_{r_{n_i}}^{(F_1,h_1)}\left(x_{n_i}+\xi A^*\left(T_{r_{n_i}}^{(F_2,h_2)}-I\right)Ax_{n_i}\right). \tag{93}$$

By (8), for any $w\in C$, one has

$$\begin{aligned}0 &\leq F_1\left(u_{n_i},w\right)+h_1\left(u_{n_i},w\right)+\frac{1}{r_{n_i}}\left\langle w-u_{n_i},u_{n_i}\right. \\ &\quad \left.-\left(x_{n_i}+\xi A^*\left(T_{r_{n_i}}^{(F_2,h_2)}-I\right)Ax_{n_i}\right)\right\rangle = F_1\left(u_{n_i},w\right) \\ &\quad +h_1\left(u_{n_i},w\right)+\frac{1}{r_{n_i}}\left\langle w-u_{n_i},u_{n_i}-x_{n_i}\right\rangle-\frac{\xi}{r_{n_i}}\left\langle Aw\right. \\ &\quad \left.-Au_{n_i},\left(T_{r_{n_i}}^{(F_2,h_2)}-I\right)Ax_{n_i}\right\rangle \leq F_1\left(u_{n_i},w\right) \\ &\quad +h_1\left(u_{n_i},w\right)+\frac{1}{r_{n_i}}\left\|w-u_{n_i}\right\|\left\|u_{n_i}-x_{n_i}\right\|+\frac{\xi}{r_{n_i}}\left\|Aw\right. \\ &\quad \left.-Au_{n_i}\right\|\left\|\left(T_{r_{n_i}}^{(F_2,h_2)}-I\right)Ax_{n_i}\right\|\end{aligned} \tag{94}$$

According to the monotonicity of F_1, we have

$$\begin{aligned}&F_1\left(w,u_{n_i}\right) \\ &\quad \leq h_1\left(u_{n_i},w\right)+\frac{1}{r_{n_i}}\left\|w-u_{n_i}\right\|\left\|u_{n_i}-x_{n_i}\right\| \\ &\quad +\frac{\xi}{r_{n_i}}\left\|Aw-Au_{n_i}\right\|\left\|\left(T_{r_{n_i}}^{(F_2,h_2)}-I\right)Ax_{n_i}\right\|\end{aligned} \tag{95}$$

From Assumption 2 (iv) on F and (ii) on h, (68), and (71), one has

$$F_1(w,z)\leq h_1(z,w). \tag{96}$$

It follows from the monotonicity of h_1 that

$$F_1(w,z)+h_1(w,z)\leq 0,\quad \forall w\in C. \tag{97}$$

For any $t\in(0,1]$ and $w\in C$, let $w_t=tw+(1-t)z$. Since $z\in C$ and C is convex, we obtain that $w_t\in C$. Hence

$$F_1(w_t,z)+h_1(w_t,z)\leq 0. \tag{98}$$

From Assumption 2 (i), (iv) on F and (i), (iii) on h, we have

$$\begin{aligned}0 &\leq F_1(w_t,w_t)+h_1(w_t,w_t) \\ &\leq t\left[F_1(w_t,w)+h_1(w_t,w)\right] \\ &\quad +(1-t)\left[F_1(w_t,z)+h_1(w_t,z)\right] \\ &\leq t\left[F_1(w_t,w)+h_1(w_t,w)\right],\end{aligned} \tag{99}$$

which implies that

$$F_1(w_t,w)+h_1(w_t,w)\geq 0,\quad \forall w\in C. \tag{100}$$

Letting $t\longrightarrow 0$ and by Assumption 2 (iii) on F and (ii) on h, we obtain

$$F_1(z,w)+h_1(z,w)\geq 0,\quad \forall w\in C; \tag{101}$$

that is, $z\in \mathrm{GEP}(F_1,h_1)$.

As follows, we prove $Az\in \mathrm{GEP}(F_2,h_2)$.

Since A is a bounded linear operator, one has $Ax_{n_i}\rightharpoonup Az$.

Now, set $\zeta_{n_i}=Ax_{n_i}-T_{r_{n_i}}^{(F_2,h_2)}Ax_{n_i}$. It follows from (68) that $\lim_{i\longrightarrow\infty}\zeta_{n_i}=0$. Since $Ax_{n_i}-\zeta_{n_i}=T_{r_{n_i}}^{(F_2,h_2)}Ax_{n_i}$, by (9) we have

$$\begin{aligned}&F_2\left(Ax_{n_i}-\zeta_{n_i},\tilde{z}\right)+h_2\left(Ax_{n_i}-\zeta_{n_i},\tilde{z}\right) \\ &\quad +\frac{1}{r_{n_i}}\left\langle\tilde{z}-\left(Ax_{n_i}-\zeta_{n_i}\right),\left(Ax_{n_i}-\zeta_{n_i}\right)-Ax_{n_i}\right\rangle \\ &\quad \geq 0,\quad \forall\tilde{z}\in Q.\end{aligned} \tag{102}$$

Furthermore, one has

$$\begin{aligned}&F_2\left(Ax_{n_i}-\zeta_{n_i},\tilde{z}\right)+h_2\left(Ax_{n_i}-\zeta_{n_i},\tilde{z}\right) \\ &\quad +\frac{1}{r_{n_i}}\left\langle\tilde{z}-Ax_{n_i}+\zeta_{n_i},-\zeta_{n_i}\right\rangle\geq 0,\quad \forall\tilde{z}\in Q.\end{aligned} \tag{103}$$

From the upper semicontinuity of $F_2(x, y)$ and $h_2(x, y)$ on x, we have

$$F_2(Az, \tilde{z}) + h_2(Az, \tilde{z}) \geq 0, \quad \forall \tilde{z} \in Q, \tag{104}$$

which means that $Az \in \mathrm{GEP}(F_2, h_2)$. As a result, $z \in \Gamma$.

Therefore, $z \in \Omega$.

Since $q = P_\Omega(I - D + \gamma f)q$ is the unique solution of the variational inequality problem $\langle (D-\gamma f)q, x-q\rangle \geq 0$, $\forall x \in \Omega$, by (81) and $z \in \Omega$, we have

$$\limsup_{n\longrightarrow\infty} \langle (D - \gamma f) q, q - x_n\rangle \leq 0. \tag{105}$$

Step 6 (finally, we show that $\{x_n\}$ converges strongly to q). It is obvious that

$$\begin{aligned}\|y_n - q\| &= \|T_n u_n - q + \alpha\gamma_n d_n\| \\ &\leq \|T_n u_n - T_n q\| + \alpha\gamma_n \|d_n\| \\ &\leq \|u_n - q\| + \alpha M_1 \gamma_n = \|u_n - q\| + M_4\gamma_n \\ &\leq \|x_n - q\| + M_4\gamma_n,\end{aligned} \tag{106}$$

where $M_4 = \alpha M_1$. And the first inequality is true because $q \in \Omega$ and $T_n q = q$ according to the same reasoning to equality (50). The last inequality is obtained by $q \in \Omega$ and the same reasoning to inequality (47). Thus, from Lemma 1 (i), we have

$$\begin{aligned}\|x_{n+1} - q\|^2 &= \|\alpha_n\gamma f(x_n) + \beta_n x_n \\ &\quad + ((1-\beta_n) I - \alpha_n D) y_n - q\|^2 \\ &= \|\alpha_n(\gamma f(x_n) - Dq) + \beta_n(x_n - q) \\ &\quad + ((1-\beta_n) I - \alpha_n D)(y_n - q)\|^2 \leq \|\beta_n(x_n - q) \\ &\quad + ((1-\beta_n) I - \alpha_n D)(y_n - q)\|^2 + 2\alpha_n \langle \gamma f(x_n) \\ &\quad - Dq, x_{n+1} - q\rangle \leq (\beta_n \|x_n - q\| \\ &\quad + (1 - \beta_n - \alpha_n\overline{\gamma}) \|y_n - q)\|)^2 + 2\alpha_n\gamma \langle f(x_n) \\ &\quad - f(q), x_{n+1} - q\rangle + 2\alpha_n \langle \gamma f(q) - Dq, x_{n+1} - q\rangle \\ &\leq ((1 - \alpha_n\overline{\gamma}) \|x_n - q\| + (1 - \beta_n - \alpha_n\overline{\gamma}) M_4\gamma_n)^2 \\ &\quad + 2\alpha_n\gamma\eta \|x_n - q\| \|x_{n+1} - q\| + 2\alpha_n \langle \gamma f(q) \\ &\quad - Dq, x_{n+1} - q\rangle \leq (1 - \alpha_n\overline{\gamma})^2 \|x_n - q\|^2 + M_5\gamma_n \\ &\quad + \alpha_n\gamma\eta \|x_n - q\|^2 + \alpha_n\gamma\eta \|x_{n+1} - q\|^2 \\ &\quad + 2\alpha_n \langle \gamma f(q) - Dq, x_{n+1} - q\rangle,\end{aligned} \tag{107}$$

where $M_5 = \sup_{n\in N}\{2(1 - \alpha_n\overline{\gamma})(1 - \beta_n - \alpha_n\overline{\gamma})M_4\|x_n - q\| + (1 - \beta_n - \alpha_n\overline{\gamma})^2 M_4^2\gamma_n\}$.

As a result,

$$\begin{aligned}&(1 - \alpha_n\gamma\eta) \|x_{n+1} - q\|^2 \\ &\quad \leq \left(1 - 2\alpha_n\overline{\gamma} + \alpha_n^2\overline{\gamma}^2 + \alpha_n\gamma\eta\right) \|x_n - q\|^2 + M_5\gamma_n \\ &\qquad + 2\alpha_n \langle \gamma f(q) - Dq, x_{n+1} - q\rangle,\end{aligned} \tag{108}$$

which implies that

$$\begin{aligned}\|x_{n+1} - q\|^2 &\leq \left(1 - \frac{2(\overline{\gamma} - \gamma\eta)\alpha_n}{1 - \alpha_n\gamma\eta}\right) \|x_n - q\|^2 \\ &\quad + \frac{2(\overline{\gamma} - \gamma\eta)\alpha_n}{1 - \alpha_n\gamma\eta} \left(\frac{\alpha_n\overline{\gamma}^2 \|x_n - q\|^2 + M_5(\gamma_n/\alpha_n)}{2(\overline{\gamma} - \gamma\eta)}\right. \\ &\quad \left. + \frac{1}{\overline{\gamma} - \gamma\eta} \langle \gamma f(q) - Dq, x_{n+1} - q\rangle\right) = (1 - t_n) \\ &\quad \cdot \|x_n - q\|^2 + t_n\delta_n,\end{aligned} \tag{109}$$

where $t_n = 2(\overline{\gamma} - \gamma\eta)\alpha_n/(1 - \alpha_n\gamma\eta)$, $\delta_n = (\alpha_n\overline{\gamma}^2\|x_n - q\|^2 + M_5(\gamma_n/\alpha_n))/2(\overline{\gamma} - \gamma\eta) + (1/(\overline{\gamma} - \gamma\eta))\langle \gamma f(q) - Dq, x_{n+1} - q\rangle$.

According to (80), (C_1), (C_3), and $\eta\gamma < \overline{\gamma}$, we have $\sum_{n=0}^{\infty} t_n = \infty$ and $\limsup_{n\longrightarrow\infty}\delta_n \leq 0$.

By Lemma 7, $x_n \longrightarrow q$, which completes the proof. □

4. Consequently Results

In the above section, we discuss the iterative algorithm and prove the strong convergence theorem for finding a common solution of split generalized equilibrium and variational inequality problems. In this section, we give some corollaries, which can find a common solution of the special issues obtained from split generalized equilibrium and variational inequality problems.

If $h_1 = h_2 = 0$, then SGEP (5)-(6) reduces to the following split equilibrium problem (SEP).

Let $F_1 : C \times C \longrightarrow R$ and $F_2 : Q \times Q \longrightarrow R$ be nonlinear bifunctions and $A : H_1 \longrightarrow H_2$ be a bounded linear operator; then SEP is to find $x^* \in C$ such that

$$F_1(x^*, x) \geq 0, \quad \forall x \in C, \tag{110}$$

and such that

$$y^* = Ax^* \in Q \text{ solves } F_2(y^*, y) \geq 0, \quad \forall y \in Q. \tag{111}$$

The solution set of SEP (110)-(111) is denoted by Γ_1. And

$$\begin{aligned}T_r^{F_1}(x) = \Big\{&z \in C : F_1(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \\ &\geq 0, \ \forall y \in C\Big\},\end{aligned} \tag{112}$$

$$\begin{aligned}T_s^{F_2}(w) = \Big\{&d \in Q : F_2(d, e) + \frac{1}{s}\langle e - d, d - w\rangle \\ &\geq 0, \ \forall e \in Q\Big\}.\end{aligned} \tag{113}$$

According to Theorem 14, we can obtain the following corollary.

Corollary 15. *Let H_1 and H_2 be two real Hilbert spaces and $C \subset H_1$, $Q \subset H_2$ be nonempty closed convex subsets. Let $F_1 : C \times C \longrightarrow R$ and $F_2 : Q \times Q \longrightarrow R$ satisfy Assumption 2 and F_2 is upper semicontinuous in the first argument. Assume that*

$\Omega := \mathrm{VI}(C,B) \cap \Gamma_1 \neq \emptyset$, $x_0 \in H_1$ and $\{u_n\}$, $\{y_n\}$, and $\{x_n\}$ are generated by the following iterative scheme:

$$\begin{aligned} u_n &= T_{r_n}^{F_1}\left(x_n + \xi A^*\left(T_{r_n}^{F_2} - I\right)Ax_n\right), \\ y_n &= u_n + \alpha d_{n+1}, \\ x_{n+1} &= \alpha_n \gamma f(x_n) + \beta_n x_n + \left(\left(1-\beta_n\right)I - \alpha_n D\right)y_n, \end{aligned} \tag{114}$$

where $d_{n+1} = (1/\alpha)(T_n u_n - u_n) + \gamma_n d_n$, $d_0 = (1/\alpha)(T_0 u_0 - u_0)$, $\alpha > 0$, and T_n is defined by (34). Suppose that $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$, $\{\gamma_n\} \subset (0,1/2)$, $\{\lambda_n\} \subset (0,2\beta)$, $\{r_n\} \subset (0,\infty)$ satisfy conditions (C1)-(C6) in Theorem 14. Then sequence $\{x_n\}$ converges strongly to $q \in \Omega$, where $q = P_\Omega(I - D + \gamma f)q$, which is the unique solution of the variational inequality problem

$$\left\langle \left(D - \gamma f\right)q, x - q\right\rangle \geq 0, \quad \forall x \in \Omega, \tag{115}$$

or equivalently, q is the unique solution to the minimization problem

$$\min_{x\in Q} \frac{1}{2}\langle Dx, x\rangle - h(x), \tag{116}$$

where h is a potential function for γf such that $h'(x) = \gamma f(x)$ for $x \in H_1$.

Furthermore, if $F_1 = F_2 = F$, $H_1 = H_2 = H$, and $A = 0$, then SEP (110)-(111) reduce to the following equilibrium problem (EP).

Let $F : C \times C \longrightarrow R$ be nonlinear bifunction; then EP is to find $x^* \in C$ such that

$$F(x^*, x) \geq 0, \quad \forall x \in C. \tag{117}$$

The solution set of EP (117) is denoted by Γ_2. And

$$\begin{aligned} T_r^F(x) = \Big\{ z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \\ \geq 0, \ \forall y \in C \Big\}. \end{aligned} \tag{118}$$

According to Corollary 15, let $D = I$, and we can obtain the following corollary.

Corollary 16. *Let H be real Hilbert space and $C \subset H$ be nonempty closed convex subset. Let $F : C \times C \longrightarrow R$ satisfy Assumption 2. Assume that $\Omega := \mathrm{VI}(C,B) \cap \Gamma_2 \neq \emptyset$, $x_0 \in H_1$, and $\{u_n\}$, $\{y_n\}$, and $\{x_n\}$ are generated by the following iterative scheme:*

$$\begin{aligned} u_n &= T_{r_n}^F(x_n), \\ y_n &= u_n + \alpha d_{n+1}, \\ x_{n+1} &= \alpha_n \gamma f(x_n) + \beta_n x_n + \left(1 - \beta_n - \alpha_n\right)y_n, \end{aligned} \tag{119}$$

where $d_{n+1} = (1/\alpha)(T_n u_n - u_n) + \gamma_n d_n$, $d_0 = (1/\alpha)(T_0 u_0 - u_0)$, $\alpha > 0$, and T_n is defined by (34). Suppose that $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$, $\{\gamma_n\} \subset (0,1/2)$, $\{\lambda_n\} \subset (0,2\beta)$, $\{r_n\} \subset (0,\infty)$ satisfy conditions (C1)-(C6) in Theorem 14. Then sequence $\{x_n\}$ converges strongly to $q \in \Omega$, where $q = P_\Omega \gamma f(q)$.

Table 1: The final value and the cpu time for different initial points.

Init.	Fina.	Sec.
$x = (0.7060, 0.0318)^T$	$\overline{x} = (4.9924, 3.9939)^T$	1.14
$x = (5.4722, 1.3862)^T$	$\overline{x} = (4.9924, 3.9939)^T$	1.35
$x = (89.0903, 95.9291)^T$	$\overline{x} = (4.9924, 3.9939)^T$	1.36

5. Numerical Examples

In this section, we show some insight into the behavior of Algorithm 13. The whole codes are written in Matlab 7.0. All the numerical results are carried out on a personal Lenovo Thinkpad computer with Intel(R) Core(TM) i7-6500U CPU 2.50GHz and RAM 8.00GB.

Example 1. In the variational inequality problem (3) as well as the split generalized equilibrium problem (5) and (6), let $H_1 = H_2 = R^2$, $C = \{(x_1, x_2)^T \in R^2 \mid 1 \leq x_1 \leq 5, 0.5 \leq x_2 \leq 4\}$, $Q = \{(y_1, y_2)^T \in R^2 \mid 2 \leq y_1 \leq 10, 3 \leq y_2 \leq 24\}$, $F_1(x,y) = h_1(x,y) = e^T(x-y), \forall x, y \in C$, $F_2(x,y) = h_2(x,y) = 5e^T(x-y), \forall x, y \in Q$, and $A = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$, where $e = (1,1)^T$.

It is easy to see that the optimal solution of Example 1 is $x^* = (5,4)^T$. Now, we use Algorithm 13 to compute the solution of the problem. Let $\alpha = 1/2$, $\xi = 1/7$, $D = \begin{pmatrix} 9/10 & 0 \\ 0 & 9/10 \end{pmatrix}$, $\gamma = 1.2$, $r_n = 1/10 + 1/2n$, $\lambda_n = 1/n$, $\alpha_n = 1/\sqrt{n}$, $\beta_n = 1/10 + 1/n^2$, $\gamma_n = 1/n^2$, and $f(x) = x/2$, $\forall x \in H_1$. The stopping criterion is $\|x_{n+1} - x_n\| \leq \epsilon$. For $\epsilon = 10^{-7}$ and different initial points which are presented randomly, such as

$$\begin{aligned} x &= \mathrm{rand}(2,1), \\ x &= 10 * \mathrm{rand}(2,1), \\ x &= 1000 * \mathrm{rand}(2,1), \end{aligned} \tag{120}$$

separately. Table 1 shows the initial value, the final value, and the cpu time for the above three cases. We denote Init., Fina., and Sec. the initial value, the final value, and the cpu time in seconds, respectively.

From Table 1, we can see that the final value $\overline{x}$ is not influenced by the initial value.

To show the changing tendency of the final value $\overline{x}$ for different ϵ, Table 2 gives the different ϵ, the initial value, the final value, and the cpu time in seconds.

Now, we further express the status that the final value $\overline{x}$ tends to optimal solution x^* through Figures 1 and 2. We carry out 100 experiments for different ϵ from 10^{-7} to 10^{-9}. From Figures 1 and 2, we know that the final value $\overline{x}$ tends to the optimal solution x^* when ϵ tends to 0, which illustrates the efficiency of Algorithm 13.

6. Conclusion

In this paper, we study the split generalized equilibrium problem and variational inequality problem. For finding their common solution, we propose a kind of conjugate gradient viscosity approximation algorithm. Under mild conditions, we prove that the sequence generated by the

Table 2: The final value and the cpu time for different ϵ.

ϵ	Init.	Fina.	Sec.
10^{-5}	$x = (8.0028, 1.4189)^T$	$\overline{x} = (4.9640, 3.9712)^T$	0.08
10^{-6}	$x = (7.4313, 3.9223)^T$	$\overline{x} = (4.9835, 3.9868)^T$	0.30
10^{-7}	$x = (5.4722, 1.3862)^T$	$\overline{x} = (4.9924, 3.9939)^T$	1.35

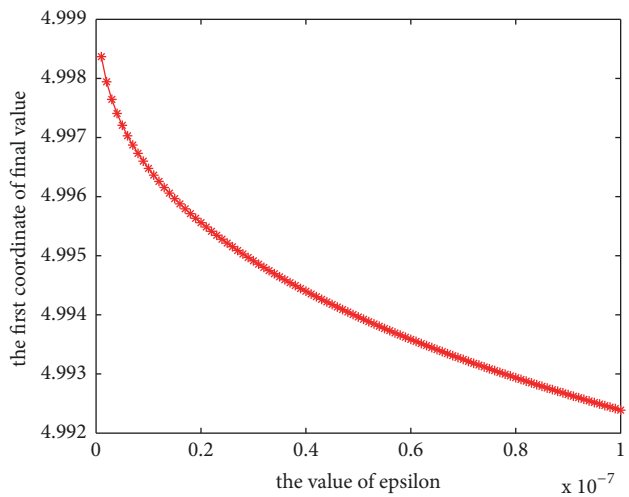

Figure 1: The behaviors of first coordinate of $\overline{x}$ for different ϵ.

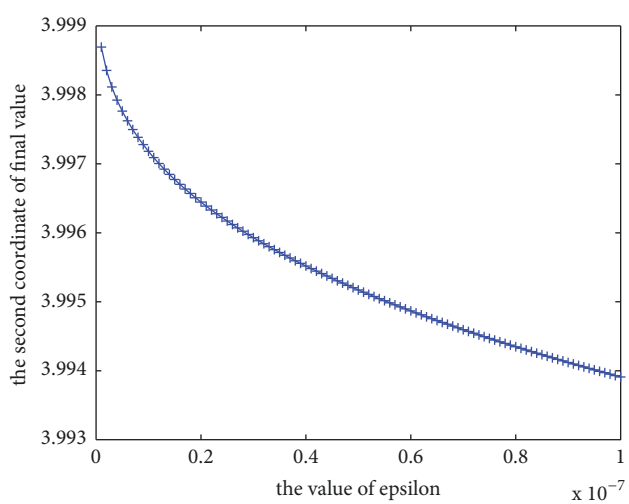

Figure 2: The behaviors of the second coordinate of $\overline{x}$ for different ϵ.

iterative algorithm converges strongly to the common solution. In comparison to [25], the authors introduce and study an effective iterative algorithm to approximate a common solution of a split generalized equilibrium problem and a fixed point problem. Under suitable conditions, they proved a strong convergence theorem for the sequence generated by the iterative scheme. Furthermore, we can study the iterative algorithm for finding a common solution of the split generalized equilibrium problem, variational inequality problem, and fixed point problem.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This project is supported by the Natural Science Foundation of China (Grants nos. 11401438 and 11571120) and Shandong Provincial Natural Science Foundation (Grants nos. ZR2013FL032 and ZR2017LA002) and the Project of Shandong Province Higher Educational Science and Technology Program (Grant no. J14LI52).

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