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Research Article

Orlicz Mean Dual Affine Quermassintegrals

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Our main aim is to generalize the mean dual affine quermassintegrals to the Orlicz space. Under the framework of dual Orlicz-Brunn-Minkowski theory, we introduce a new affine geometric quantity by calculating the first Orlicz variation of the mean dual affine quermassintegrals and call it the Orlicz mean dual affine quermassintegral. The fundamental notions and conclusions of the mean dual affine quermassintegrals and the Minkowski and Brunn-Minkowski inequalities for them are extended to an Orlicz setting. The related concepts and inequalities of dual Orlicz mixed volumes are also included in our conclusions. The new Orlicz isoperimetric inequalities in special case yield the L_p-dual Minkowski inequality and Brunn-Minkowski inequality for the mean dual affine quermassintegrals, which also imply the dual Orlicz-Minkowski inequality and dual Orlicz-Brunn-Minkowski inequality.

1. Introduction

The radial addition $K \tilde{+} L$ of star sets (compact sets that are star-shaped at o and contain o) K and L can be defined by

$$K \tilde{+} L = \{x \tilde{+} y : x \in K, y \in L\}, \tag{1}$$

where $x \tilde{+} y = x + y$ if x, y, and o are collinear and $x \tilde{+} y = o$, otherwise, or by

$$\rho(K \tilde{+} L, \cdot) = \rho(K, \cdot) + \rho(L, \cdot), \tag{2}$$

where $\rho(K, \cdot)$ denotes the radial function of star set K, which is defined by

$$\rho(K, u) = \max\{c \geq 0 : cu \in K\}, \tag{3}$$

for $u \in S^{n-1}$, where S^{n-1} is the surface of the unit sphere. Hints as to the origins of the radial addition can be found in [1, p. 235]. If $\rho(K, \cdot)$ is positive and continuous, K will be called a star body. Let $\mathcal{S}^n$ denote the set of star bodies about the origin in $\mathbb{R}^n$. When combined with volume, radial addition gives rise to another substantial appendage to the classical theory, called the dual Brunn-Minkowski theory. Radial addition is the basis for the dual Brunn-Minkowski theory (see, e.g., [2–10] for recent important contributions). The original theory is originated from Lutwak [11]. He introduced the concept of dual mixed volume which laid the foundation of the dual Brunn-Minkowski theory. The dual theory can count among its successes the solutions of the Busemann-Petty problem in [3, 4, 9, 12, 13]. For $p \neq 0$, $x \in \mathbb{R}^n$, and $K, L \in \mathcal{S}^n$, the p-radial addition $K \tilde{+}_p L$ is defined by (see [14])

$$\rho\left(K \tilde{+}_p L, x\right)^p = \rho(K, x)^p + \rho(L, x)^p. \tag{4}$$

The L_p-harmonic radial combination for star bodies was introduced: If $K, L \in \mathcal{S}^n$, $u \in \mathcal{S}^{n-1}$, and $p \geq 1$, then the L_p-harmonic radial addition defined by Lutwak [8] is

$$\rho\left(K \tilde{+}_p L, u\right)^{-p} = \rho(K, u)^{-p} + \rho(L, u)^{-p}. \tag{5}$$

For convex bodies, the L_p-harmonic addition was first investigated by Firey [15].

If K is a nonempty closed (not necessarily bounded) convex set in $\mathbb{R}^n$, then

$$h(K,x)=\max\{x\cdot y: y\in K\}, \tag{6}$$

for $x\in\mathbb{R}^n$, which defined the support function $h(K,x)$ of K. A nonempty closed convex set is uniquely determined by its support function. L_p-addition and inequalities are the fundamental and core content in the L_p-Brunn-Minkowski theory. In recent years, a new extension of L_p-Brunn-Minkowski theory is to Orlicz-Brunn-Minkowski theory, initiated by Lutwak et al. [16, 17]. Gardner et al. [18] introduced the Orlicz addition for the first time, constructed a general framework for the Orlicz-Brunn-Minkowski theory, and made the relation to Orlicz spaces and norms clear. The Orlicz addition of convex bodies was also introduced from different angles and the L_p-Brunn-Minkowski inequality was extended to the Orlicz-Brunn-Minkowski inequality (see [19]). The Orlicz centroid inequality for star bodies was introduced in [20]. The other articles advancing the theory can be found in literatures [7, 21–25].

Just as the L_p-Brunn-Minkowski theory is extended to the Orlicz Brunn-Minkowski theory, it has recently turned to a study extending from L_p-dual Brunn-Minkowski theory to dual Orlicz Brunn-Minkowski theory. The dual Orlicz-Brunn-Minkowski theory has also attracted mathematicians' attention [14, 26–28]. In 2014, Zhu et al. [29] introduced the Orlicz harmonic radial sum $K\widetilde{+}_\phi L$ of two star bodies K and L, defined by

$$\rho\left(K\widetilde{+}_\phi L,u\right)=\sup\left\{\lambda>0:\phi\left(\frac{\rho(K,u)}{\lambda}\right)+\phi\left(\frac{\rho(L,u)}{\lambda}\right)\le\phi(1)\right\}, \tag{7}$$

where $u\in S^{n-1}$, $\phi:(0,\infty)\to(0,\infty)$ is a convex and decreasing function such that $\phi(0)=\infty$, $\lim_{t\to\infty}\phi(t)=0$, and $\lim_{t\to 0}\phi(t)=\infty$. Let $\mathscr{C}$ denote the class of the convex and decreasing functions ϕ. When $p\ge 1$ and $\phi(t)=t^{-p}$, the Orlicz harmonic addition $\widetilde{+}_\phi$ becomes the L_p-harmonic radial addition $\widetilde{+}_p$. The dual Orlicz mixed volume with respect to Orlicz harmonic radial addition, denoted by $\widetilde{V}_\phi(K,L)$, is defined by

$$\begin{aligned}\widetilde{V}_\phi(K,L)&:=\frac{\phi_+'(1)}{n}\lim_{\varepsilon\to0^+}\frac{V\left(K\widetilde{+}_\phi\varepsilon\cdot L\right)-V(K)}{\varepsilon}\\&=\frac{1}{n}\int_{S^{n-1}}\phi\left(\frac{\rho(L,u)}{\rho(K,u)}\right)\rho(K,u)^n\,dS(u),\end{aligned} \tag{8}$$

where $K\widetilde{+}_\phi\varepsilon\cdot L$ is the Orlicz linear combination of K and L, $dS(u)$ denotes the surface area measure of the unit sphere S^{n-1}, and $\phi_+'(1)$ denotes the value of the right derivative of convex function ϕ at point 1.

The dual affine quermassintegrals were defined, for a convex body $K\in\mathscr{S}^n$, by letting $\widetilde{\Phi}_0(K):=V(K)$, $\widetilde{\Phi}_n(K):=\omega_n$, and for $0<j<n$ (see, e.g., [30], p. 515)

$$\widetilde{\Phi}_{n-j}(K):=\omega_n\left[\int_{G_{n,j}}\left(\frac{\operatorname{vol}_j(K\cap\xi)}{\omega_j}\right)^n d\mu_j(\xi)\right]^{1/n}, \tag{9}$$

where $G_{n,j}$ denotes the Grassmann manifold of j-dimensional subspaces in $\mathbb{R}^n$, μ_j denotes the gauge Haar measure on $G_{n,j}$, $\operatorname{vol}_j(K\cap\xi)$ denotes the j-dimensional volume of intersection of K on j-dimensional subspace $\xi\subset\mathbb{R}^n$, and ω_j denotes the volume of j-dimensional unit ball. Gardner [31] showed the Brunn-Minkowski inequality for the dual affine quermassintegrals. If $K,L\in\mathscr{S}^n$ and $0\le j\le n-1$, then

$$\widetilde{\Phi}_j(K\widetilde{+}L)^{1/(n-j)}\le\widetilde{\Phi}_j(K)^{1/(n-j)}+\widetilde{\Phi}_j(L)^{1/(n-j)}, \tag{10}$$

with equality if and only if K is a dilate of L, modulo a set of measure zero. In analogy to (9), one may also define mean dual affine quermassintegrals by (see, e.g., [30], p. 516)

$$\overline{\Phi}_{n-j}(K):=\omega_n\left[\int_{A_{n,j}}\left(\frac{\operatorname{vol}_j(K\cap\xi)}{\omega_j}\right)^{n+1}d\nu_j(\xi)\right]^{1/(n+1)}, \tag{11}$$

for a convex body and $0<j<n$ and by letting $\overline{\Phi}_0(K):=V(K)$ and $\overline{\Phi}_n(K):=\omega_n$. Here, $A_{n,j}$ denotes the space of the j-dimensional affine subspace in $\mathbb{R}^n$ and ν_j denotes the gauge Haar measure on $A_{n,j}$. They are related to the dual affine quermassintegrals by (see [32], p. 373).

$$\overline{\Phi}_{n-j}(K)=\frac{\omega_n}{\omega_j}\left(\int_K\widetilde{\Phi}_{n-j}(K-x)^n\,dx\right)^{1/(n+1)}. \tag{12}$$

Obviously, $\overline{\Phi}_{n-j}(K)$ is invariant under unimodular affine transformations of K.

In the paper, our main aim is to generalize the mean dual affine quermassintegrals to the Orlicz space. Under the framework of dual Orlicz-Brunn-Minkowski theory, we introduce a new affine geometric quantity such as Orlicz mean dual affine quermassintegrals. The fundamental notions and conclusions of the mean dual affine quermassintegrals and the Minkowski and Brunn-Minkowski inequalities for the mean dual affine quermassintegrals are extended to an Orlicz setting. The new Orlicz-Minkowski and Brunn-Minkowski inequalities for the Orlicz mean dual affine quermassintegrals in special case yield the L_p-dual Minkowski inequality and Brunn-Minkowski inequalities for the mean dual affine quermassintegrals, which also imply the dual Orlicz-Minkowski inequality and Brunn-Minkowski inequalities for general volumes.

Following the basic spirit of Alexandroff [33], Fenchel and Jessen [34] introduction of mixed quermassintegrals, and introduction of Lutwak's L_p-mixed quermassintegrals (see [8, 35]), the study is based on the first-order Orlicz variation of the dual affine quermassintegrals. In Section 3, we prove that the Orlicz first-order variation of the mean dual affine quermassintegrals can be expressed as follows: for $\phi \in \mathscr{C}$, $\varepsilon > 0$, $0 < j \leq n$, and $K, L \in \mathscr{S}^n$,

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0^+} \overline{\Phi}_{n-j}\left(K \widetilde{+}_\phi \varepsilon \cdot L\right) = \frac{j}{\phi'_+(1)} \overline{\Phi}_{n-j}(K)^{-n} \overline{\Phi}_{\phi,n-j}(K,L)^{n+1}. \tag{13}$$

Putting $j = n$ in (13), then we have the well-known result.

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0^+} V\left(K \widetilde{+}_\phi \varepsilon \cdot L\right) = \frac{n}{\phi'_+(1)} \widetilde{V}_\phi(K,L). \tag{14}$$

In (13), we find a new geometric quantity. Based on this, we extract the required geometric quantity, denoted by $\overline{\Phi}_{\phi,n-j}(K,L)$, and call it as Orlicz mean dual affine quermassintegrals, defined by

$$\overline{\Phi}_{\phi,n-j}(K,L) := \left(\frac{\phi'_+(1)}{j \cdot \overline{\Phi}_{n-j}(K)^{-n}} \cdot \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0^+} \overline{\Phi}_{n-j}\left(K \widetilde{+}_\phi \varepsilon \cdot L\right)\right)^{1/(n+1)}, \tag{15}$$

where $\phi \in \mathscr{C}$, $0 < j \leq n$, and $K, L \in \mathscr{S}^n$. We also prove that the new affine geometric quantity $\overline{\Phi}_{\phi,n-j}(K,L)$ has an integral representation.

$$\overline{\Phi}_{\phi,n-j}(K,L) = \omega_n \left[\int_{A_{n,j}} \frac{\widetilde{V}_\phi^{(j)}(K \cap \xi, L \cap \xi)}{\mathrm{vol}_j(K \cap \xi)} \left(\frac{\mathrm{vol}_j(K \cap \xi)}{\omega_j}\right)^{n+1} d\nu_j(\xi)\right]^{1/(n+1)}, \tag{16}$$

where $\widetilde{V}_\phi^{(j)}(K \cap \xi, L \cap \xi)$ denotes the Orlicz dual mixed volume of j-dimensional star bodies $K \cap \xi$ and $L \cap \xi$ in j-dimensional subspace ξ.

Obviously, the Orlicz mean dual affine quermassintegrals are an extension of the mean dual affine quermassintegrals; a very natural question is raised: is there a Minkowski type isoperimetric inequality for the Orlicz mean dual affine quermassintegrals? In Section 4, we give a positive answer to this question and establish the dual Orlicz-Minkowski inequality for the new affine geometric quantity. For $\phi \in \mathscr{C}$, $0 < j \leq n$, and $K, L \in \mathscr{S}^n$, we prove the Orlicz-Minkowski inequality for the Orlicz mean dual affine quermassintegrals.

$$\left(\frac{\overline{\Phi}_{\phi,n-j}(K,L)}{\overline{\Phi}_{n-j}(K)}\right)^{n+1} \geq \phi\left(\left(\frac{\overline{\Phi}_{n-j}(L)}{\overline{\Phi}_{n-j}(K)}\right)^{1/j}\right). \tag{17}$$

If ϕ is strictly convex, equality holds if and only if K and L are dilates. For $j = n$, (17) becomes the following dual Orlicz-Minkowski inequality established by Zhu et al. [29]:

$$\widetilde{V}_\phi(K,L) \geq V(K)\,\phi\left(\left(\frac{V(L)}{V(K)}\right)^{1/n}\right). \tag{18}$$

If ϕ is strictly convex, equality holds if and only if K and L are dilates.

In Section 5, on the basis of the dual Minkowski inequality for the Orlicz mean dual affine quermassintegrals, we establish a dual Orlicz-Brunn-Minkowski inequality for the dual mixed mean affine quermassintegrals. If $K, L \in \mathscr{S}^n$, $0 < j \leq n$, and $\phi \in \mathscr{C}$, then for any $\varepsilon > 0$

$$\phi(1) \geq \phi\left(\left(\frac{\overline{\Phi}_{n-j}(K)}{\overline{\Phi}_{n-j}\left(K \widetilde{+}_\phi \varepsilon \cdot L\right)}\right)^{1/j}\right) + \varepsilon \cdot \phi\left(\left(\frac{\overline{\Phi}_{n-j}(L)}{\overline{\Phi}_{n-j}\left(K \widetilde{+}_\phi \varepsilon \cdot L\right)}\right)^{1/j}\right). \tag{19}$$

If ϕ is strictly convex, equality holds if and only if K and L are dilates. For $j = n$ and $\varepsilon = 1$, (19) becomes the following dual Orlicz-Brunn-Minkowski inequality established by Zhu et al. [29]. If $K, L \in \mathscr{S}^n$ and $\phi \in \mathscr{C}$, then

$$\phi(1) \geq \phi\left(\left(\frac{V(K)}{V\left(K \widetilde{+}_\phi L\right)}\right)^{1/n}\right) + \phi\left(\left(\frac{V(L)}{V\left(K \widetilde{+}_\phi L\right)}\right)^{1/n}\right). \tag{20}$$

If ϕ is strictly convex, equality holds if and only if K and L are dilates. Moreover, for $\varepsilon = 1$, $\phi(t) = t^{-p}$, and $p \geq 1$, (19) becomes the L_p-dual Brunn-Minkowski inequality for the mean dual affine quermassintegrals. If $K, L \in \mathscr{S}^n$, $\varepsilon > 0$, $0 < j \leq n$, $p \geq 1$, and $\phi \in \mathscr{C}$, then

$$\overline{\Phi}_{n-j}\left(K \widetilde{+}_p L\right)^{-p/j} \geq \overline{\Phi}_{n-j}(K)^{-p/j} + \overline{\Phi}_{n-j}(L)^{-p/j}, \tag{21}$$

with equality if and only if K and L are dilates. When $j = n$, (21) becomes Lutwak's dual Brunn-Minkowski inequality (36).

2. Preliminaries

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n$. A body in $\mathbb{R}^n$ is a compact set equal to the closure of its interior. For a compact set $K \subset \mathbb{R}^n$, we write $V(K)$ for the (n-dimensional) Lebesgue measure of K and call this the volume of K. Associated with a compact subset K of $\mathbb{R}^n$, which is star-shaped with respect to the origin and contains the origin, its radial function is $\rho(K,\cdot) : S^{n-1} \to [0,\infty)$, defined by $\rho(K,u) = \max\{\lambda \geq 0 : \lambda u \in K\}$. Note that the class (star sets) is closed under unions, intersection, and intersection with subspace. The radial function is homogeneous of degree -1; that is, $\rho(K,rx) = r^{-1}\rho(K,x)$, for all $x \in \mathbb{R}^n$ and $r > 0$. Let $\widetilde{\delta}$ denote the radial Hausdorff metric, as follows; if $K, L \in \mathcal{S}^n$, then (see, e.g., [30])

$$\widetilde{\delta}(K,L) = \left|\rho(K,u) - \rho(L,u)\right|_{\infty}. \tag{22}$$

From the definition of the radial function, it follows immediately that for $g \in GL(n)$ the radial function of the image $gK = \{gy : y \in K\}$ of K is given by

$$\rho(gK,x) = \rho\left(K, g^{-1}x\right), \tag{23}$$

for all $x \in \mathbb{R}^n$.

2.1. Dual Mixed Volumes and L_p-Dual Mixed Volumes. If $K_1, \ldots, K_n \in \mathcal{S}^n$, the dual mixed volume $\widetilde{V}(K_1, \ldots, K_n)$ defined by (see [11]) is as follows:

$$\widetilde{V}(K_1,\ldots,K_n) = \frac{1}{n}\int_{S^{n-1}} \rho(K_1,u)\cdots\rho(K_n,u)\,dS(u). \tag{24}$$

If $K_1 = \cdots = K_{n-i} = K$, $K_{n-i+1} = \cdots = K_n = L$, the dual mixed volume $\widetilde{V}(K_1, \ldots, K_n)$ is written as $\widetilde{V}_i(K,L)$. If $K_1 = \cdots = K_n = K$, the dual mixed volume $\widetilde{V}(K_1,\ldots,K_n)$ is written as $V(K)$. Obviously, For $K \in \mathcal{S}^n$, we have

$$V(K) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^n\,dS(u), \tag{25}$$

and (see [11])

$$\widetilde{V}_1(K,L) = \lim_{\varepsilon\to 0^+} \frac{V\left(K \widetilde{+} \varepsilon\cdot L\right) - V(K)}{\varepsilon} = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n-1}\rho(L,u)\,dS(u). \tag{26}$$

The fundamental inequality for dual mixed volumes stated that if $K, L \in \mathcal{S}^n$, then

$$\widetilde{V}_1(K,L)^n \leq V(K)^{n-1}V(L), \tag{27}$$

with equality if and only if K and L are dilates. The Brunn-Minkowski inequality for the radial addition is the following: If $K, L \in \mathcal{S}^n$, then

$$V\left(K \widetilde{+} L\right)^{1/n} \leq V(K)^{1/n} + V(L)^{1/n}, \tag{28}$$

with equality if and only if K and L are dilates.

The following result follows immediately from the definition of L_p-radial addition, with $p \neq 0$.

$$\frac{p}{n}\lim_{\varepsilon\to 0^+} \frac{V\left(K \widetilde{+}_p \varepsilon\cdot L\right) - V(L)}{\varepsilon} = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n-i-p}\rho(L,u)^p\,dS(u). \tag{29}$$

Let $K, L \in \mathcal{S}^n$ and $p < 0$; we define L_p-dual mixed volume of star bodies K and L, $\widetilde{V}_p(K,L)$, by

$$\widetilde{V}_p(K,L) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n-p}\rho(L,u)^p\,dS(u). \tag{30}$$

This integral representation (30), together with the Hölder inequality, yields the p-dual Minkowski inequality (see [36]): If $K, L \in \mathcal{S}^n$ and $p < 0$, then

$$\widetilde{V}_p(K,L)^n \geq V(K)^{n-p}V(L)^p, \tag{31}$$

with equality if and only if K and L are dilates. The definition of L_p-radial addition, together with (31), yields Gardner's Brunn-Minkowski inequality for p-radial addition (see [37]). If $K, L \in \mathcal{S}^n$ and $p < 0$, then

$$V\left(K \widetilde{+}_p L\right)^{p/n} \geq V(K)^{p/n} + V(L)^{p/n}, \tag{32}$$

with equality if and only if K and L are dilates.

2.2. L_p-Harmonic Mixed Volumes. The following result follows immediately form (5) with $p \geq 1$.

$$-\frac{p}{n}\lim_{\varepsilon\to 0^+} \frac{V\left(K \widetilde{+}_p \varepsilon\cdot L\right) - V(L)}{\varepsilon} = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n+p}\rho(L,u)^{-p}\,dS(u). \tag{33}$$

Let $K, L \in \mathcal{S}^n$ and $p \geq 1$; the L_p-harmonic mixed volumes of star bodies K and L denotes $\widetilde{V}_{-p}(K,L)$, defined by (see [35])

$$\widetilde{V}_{-p}(K,L) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n+p}\rho(L,u)^{-p}\,dS(u). \tag{34}$$

This integral representation (34), together with the Hölder inequality, yields Lutwak's L_p-dual Minkowski inequality as follows: If $K, L \in \mathcal{S}^n$ and $p \geq 1$, then

$$\widetilde{V}_{-p}(K,L)^n \geq V(K)^{n+p}V(L)^{-p}, \tag{35}$$

with equality if and only if K and L are dilates. This integral representation (34), together with the definition of p-harmonic addition, yields Lutwak's L_p-Brunn-Minkowski inequality for harmonic p-addition (see [35]). If $K, L \in \mathcal{S}^n$ and $p \geq 1$, then

$$V\left(K \widetilde{+}_p L\right)^{-p/n} \geq V(K)^{-p/n} + V(L)^{-p/n}, \tag{36}$$

with equality if and only if K and L are dilates.

2.3. Orlicz Harmonic Addition and Orlicz Harmonic Linear Combination

Definition 1. Let $m \geq 2$, $\phi \in \mathcal{C}$, $K_j \in \mathcal{S}^n$, $j = 1, \ldots, m$, define the Orlicz harmonic addition of $K_1, \ldots, K_m$, denoted by $K_1 \widetilde{+}_\phi \cdots \widetilde{+}_\phi K_m$, defined by

$$\begin{aligned} &\rho\left(K_1 \widetilde{+}_\phi \cdots \widetilde{+}_\phi K_m, x\right) \\ &\quad = \sup\left\{\lambda > 0 : \sum_{j=1}^{m} \phi\left(\frac{\rho\left(K_j, x\right)}{\lambda}\right) \leq \phi(1)\right\}, \end{aligned} \tag{37}$$

for all $x \in \mathbb{R}^n$.

Equivalently, the Orlicz harmonic addition $K_1 \widetilde{+}_\phi \cdots \widetilde{+}_\phi K_m$ can be defined implicitly by

$$\begin{aligned} &\phi\left(\frac{\rho(K_1, x)}{\rho\left(K_1 \widetilde{+}_\phi \cdots \widetilde{+}_\phi K_m, x\right)}\right) + \cdots \\ &\quad + \phi\left(\frac{\rho(K_m, x)}{\rho\left(K_1 \widetilde{+}_\phi \cdots \widetilde{+}_\phi K_m, x\right)}\right) = \phi(1), \end{aligned} \tag{38}$$

for all $x \in \mathbb{R}^n$.

The Orlicz harmonic linear combination on the case $m = 2$ is defined.

Definition 2. Orlicz harmonic linear combination $\widetilde{+}_\phi(K, L, \alpha, \beta)$ for $K, L \in \mathcal{S}^n$, $\phi \in \mathcal{C}$, and $\alpha, \beta \geq 0$ (both not zero) is defined by

$$\begin{aligned} &\alpha \cdot \phi\left(\frac{\rho(K, x)}{\rho\left(\widetilde{+}_\phi(K, L, \alpha, \beta), x\right)}\right) + \beta \\ &\quad \cdot \phi\left(\frac{\rho(L, x)}{\rho\left(\widetilde{+}_\phi(K, L, \alpha, \beta), x\right)}\right) = \phi(1), \end{aligned} \tag{39}$$

for all $x \in \mathbb{R}^n$.

When $\phi(t) = t^{-p}$ and $p \geq 1$, then Orlicz harmonic linear combination $\widetilde{+}_\phi(K, L, \alpha, \beta)$ changes to the L_p-harmonic linear combination $\alpha \lozenge K \widetilde{+}_p \beta \lozenge L$ (see [9]). Moreover, we shall write $K \widetilde{+}_\phi \varepsilon \cdot L$ instead of $\widetilde{+}_\phi(K, L, 1, \varepsilon)$, for $\varepsilon \geq 0$, and assume throughout that this is defined by (39), where $\alpha = 1$, $\beta = \varepsilon$, and $\phi \in \mathcal{C}$, and write $\widetilde{+}_\phi(K, L, 1, 1)$ as $K \widetilde{+}_\phi L$.

3. Orlicz Mean Dual Affine Quermassintegrals

In order to define Orlicz mean dual affine quermassintegrals, we need the following lemmas.

Lemma 3. *If $K, L \in \mathcal{S}^n$ and $\phi \in \mathcal{C}$, then for $\varepsilon > 0$*

$$\begin{aligned} \phi(1) V\left(K \widetilde{+}_\phi \varepsilon \cdot L\right) &= \widetilde{V}_\phi\left(K \widetilde{+}_\phi \varepsilon \cdot L, K\right) + \varepsilon \\ &\quad \cdot \widetilde{V}_\phi\left(K \widetilde{+}_\phi \varepsilon \cdot L, L\right). \end{aligned} \tag{40}$$

Proof. From (8) and (39), we have for any $Q \in \mathcal{S}^n$

$$\begin{aligned} &\widetilde{V}_\phi(Q, K) + \varepsilon \cdot \widetilde{V}_\phi(Q, L) = \frac{1}{n} \\ &\quad \cdot \int_{S^{n-1}} \left(\varphi\left(\frac{\rho(K, u)}{\rho(Q, u)}\right) + \varepsilon \cdot \phi\left(\frac{\rho(L, u)}{\rho(Q, u)}\right)\right) \\ &\quad \cdot \rho(Q, u)^n \, dS(u) = \frac{\phi(1)}{n} \\ &\quad \cdot \int_{S^{n-1}} \rho(Q, u)^n \, dS(u) = \phi(1) V(Q). \end{aligned} \tag{41}$$

Putting $Q = K \widetilde{+}_\phi \varepsilon \cdot L$ in (41), (41) easily becomes (40). □

Lemma 4 (see [29]). *If $K, L \in \mathcal{S}^n$ and $\phi \in \mathcal{C}$, then for $\varepsilon > 0$*

$$K \widetilde{+}_\phi \varepsilon \cdot L \longrightarrow K, \tag{42}$$

in the radial Hausdorff metric as $\varepsilon \to 0^+$.

Lemma 5. *If $K, L \in \mathcal{S}^n$, $\varepsilon > 0$, and $\phi \in \mathcal{C}$, then*

$$\left(K \widetilde{+}_\phi \varepsilon \cdot L\right) \cap \xi = (K \cap \xi) \widetilde{+}_\phi \varepsilon \cdot (L \cap \xi). \tag{43}$$

Proof. Suppose $\xi \in A_{n,j}$ and $S^{j-1} = S^{n-1} \cap \xi$. For $u \in S^{j-1}$ and $Q \in \mathcal{S}^n$, we have

$$\rho(Q, u) = \rho(Q \cap \xi, u). \tag{44}$$

Hence

$$\begin{aligned} &\phi\left(\frac{\rho(K \cap \xi, u)}{\rho\left(\left(K \widetilde{+}_\phi \varepsilon \cdot L\right) \cap \xi, u\right)}\right) \\ &\quad + \varepsilon\phi\left(\frac{\rho(L \cap \xi, u)}{\rho\left(\left(K \widetilde{+}_\phi \varepsilon \cdot L\right) \cap \xi, u\right)}\right) = \phi(1). \end{aligned} \tag{45}$$

On the other hand

$$\begin{aligned} &\phi\left(\frac{\rho(K \cap \xi, u)}{\rho\left((K \cap \xi) \widetilde{+}_\phi \varepsilon \cdot (L \cap \xi), u\right)}\right) \\ &\quad + \varepsilon\phi\left(\frac{\rho(L \cap \xi, u)}{\rho\left((K \cap \xi) \widetilde{+}_\phi \varepsilon \cdot (L \cap \xi), u\right)}\right) = \phi(1). \end{aligned} \tag{46}$$

Therefore $(K \widetilde{+}_\phi \varepsilon\cdot L)\cap\xi$ and $(K\cap\xi)\widetilde{+}_\phi \varepsilon\cdot(L\cap\xi)$ are the same star body in ξ. □

Definition 6. If $\phi\in\mathscr{C}$, $0<j\le n$, and $K,L\in\mathscr{S}^n$, then Orlicz mean dual affine quermassintegral of K and L, denoted by $\overline{\Phi}_{\phi,n-j}(K,L)$, is defined by

$$\overline{\Phi}_{\phi,n-j}(K,L) := \omega_n\left[\int_{A_{n,j}}\frac{\widetilde{V}_\phi^{(j)}(K\cap\xi,L\cap\xi)}{\mathrm{vol}_j(K\cap\xi)}\left(\frac{\mathrm{vol}_j(K\cap\xi)}{\omega_j}\right)^{n+1}d\nu_j(\xi)\right]^{1/(n+1)}. \tag{47}$$

Specifically, we agreed on the following:

$$\overline{\Phi}_{\phi,0}(K,L)=\left(\frac{\widetilde{V}_\phi(K,L)}{V(K)}\right)^{1/(n+1)}V(K). \tag{48}$$

In order to define the Orlicz mean dual affine quermassintegrals, we need also to calculate the first Orlicz variation of the mean dual affine quermassintegrals.

Lemma 7. *If $\phi\in\mathscr{C}$, $0<j\le n$, and $K,L\in\mathscr{S}^n$, then for any $\varepsilon>0$*

$$\begin{aligned}&\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0^+}\overline{\Phi}_{n-j}\left(K\widetilde{+}_\phi\varepsilon\cdot L\right)\\&\quad=\frac{j}{\phi'_+(1)}\overline{\Phi}_{n-j}(K)^{1-n}\,\overline{\Phi}_{\phi,n-j}(K,L)^n.\end{aligned} \tag{49}$$

Proof. On the one hand, from (8), we have

$$\begin{aligned}&\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0^+}\int_{A_{n,j}}\mathrm{vol}_j\left(\left(K\widetilde{+}_\phi\varepsilon\cdot L\right)\cap\xi\right)^{n+1}d\nu_j(\xi)\\&=\lim_{\varepsilon\to0^+}\int_{A_{n,j}}\frac{\mathrm{vol}_j\left(\left(K\widetilde{+}_\phi\varepsilon\cdot L\right)\cap\xi\right)^{n+1}-\mathrm{vol}_j(K\cap\xi)^{n+1}}{\varepsilon}d\nu_j(\xi)\\&=(n+1)\lim_{\varepsilon\to0^+}\int_{A_{n,j}}\Big(\mathrm{vol}_j(K\cap\xi)^n\\&\quad\cdot\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0^+}\mathrm{vol}_j\left(\left(K\widetilde{+}_\phi\varepsilon\cdot L\right)\cap\xi\right)\Big)d\nu_j(\xi)\\&=\frac{(n+1)j}{\phi'_+(1)}\int_{A_{n,j}}\mathrm{vol}_j(K\cap\xi)^n\,\widetilde{V}_\phi^{(j)}(K\cap\xi,L\cap\xi)\,d\nu_j(\xi).\end{aligned} \tag{50}$$

On the other hand, from (11), (47), and (50), we obtain

$$\begin{aligned}&\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0^+}\overline{\Phi}_{n-j}\left(K\widetilde{+}_\phi\varepsilon\cdot L\right)=\frac{\omega_n}{\omega_j}\cdot\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0^+}\\&\quad\cdot\left[\int_{A_{n,j}}\mathrm{vol}_j\left(\left(K\widetilde{+}_\phi\varepsilon\cdot L\right)\cap\xi\right)^{n+1}d\nu_j(\xi)\right]^{1/(n+1)}\\&\quad=\frac{\omega_n}{(n+1)\,\omega_j}\left(\int_{A_{n,j}}\mathrm{vol}_j(K\cap\xi)^{n+1}d\nu_j(\xi)\right)^{-n/(n+1)}\\&\quad\cdot\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0^+}\int_{A_{n,j}}\mathrm{vol}_j\left(\left(K\widetilde{+}_\phi\varepsilon\cdot L\right)\cap\xi\right)^{n+1}d\nu_j(\xi)\\&\quad=\frac{j}{\phi'_+(1)}\frac{\omega_n}{\omega_j}\left(\int_{A_{n,j}}\mathrm{vol}_j(K\cap\xi)^n\,d\nu_j(\xi)\right)^{-n/(n+1)}\\&\quad\cdot\int_{A_{n,j}}\frac{\widetilde{V}_\phi^{(j)}(K\cap\xi,L\cap\xi)}{\mathrm{vol}_j(K\cap\xi)}\mathrm{vol}_j(K\cap\xi)^{n+1}d\nu_j(\xi)\\&\quad=\frac{j}{\phi'_+(1)}\overline{\Phi}_{n-j}(K)^{-n}\,\overline{\Phi}_{\phi,n-j}(K,L)^{n+1}.\end{aligned} \tag{51}$$

□

Lemma 8. *If $K,L\in\mathscr{S}^n$, $0<j\le n$, and $\phi\in\mathscr{C}$, then*

$$\overline{\Phi}_{\phi,n-j}(K,K)=\phi(1)^{1/(n+1)}\,\overline{\Phi}_{n-j}(K). \tag{52}$$

Proof. The definition of the Orlicz mean dual affine quermassintegrals, together with (8) and (11), gives (52).

If $\phi(t)=t^{-p}$, $p\ge1$, then $\overline{\Phi}_{\phi,n-j}(K,L)=\overline{\Phi}_{-p,n-j}(K,L)$ and call the L_p-dual mixed mean affine quermassintegral of K and L, and

$$\overline{\Phi}_{-p,n-j}(K,L)=\omega_n\left[\int_{A_{n,j}}\frac{\widetilde{V}_{-p}^{(j)}(K\cap\xi,L\cap\xi)}{\mathrm{vol}_j(K\cap\xi)}\left(\frac{\mathrm{vol}_j(K\cap\xi)}{\omega_j}\right)^{n+1}d\nu_j(\xi)\right]^{1/(n+1)}, \tag{53}$$

where $\widetilde{V}_{-p}^{(j)}(K\cap\xi,L\cap\xi)$ denotes the L_p-dual mixed volume of j-dimensional star bodies $K\cap\xi$ and $L\cap\xi$ in j-dimensional subspace ξ. □

Lemma 9 (see [29]). *If $K,L\in\mathscr{S}^n$, $\phi\in\mathscr{C}$, and any $g\in GL(n)$, then for $\varepsilon>0$*

$$g\left(K\widetilde{+}_\phi\varepsilon\cdot L\right)=(gK)\,\widetilde{+}_\phi\,\varepsilon\cdot(gL). \tag{54}$$

In the following, we will prove that Orlicz mean dual affine quermassintegral $\overline{\Phi}_{\phi,n-j}(K,L)$ is invariant under simultaneous unimodular centro-affine transformation.

Lemma 10. *If $K, L \in \mathcal{S}^n$, $0 < j \le n$, $\phi \in \mathcal{C}$, and any $g \in SL(n)$, then*

$$\overline{\Phi}_{\phi,n-j}(gK, gL) = \overline{\Phi}_{\phi,n-j}(K, L). \tag{55}$$

Proof. Suppose that $\xi \in A_{n,j}$ and $S^{j-1} = S^{n-1} \cap \xi$. For any $g \in SL(n)$, $u \in S^{j-1}$, and $Q \in \mathcal{S}^n$, we have

$$\rho(gQ, u) = \rho(gQ \cap \xi, u). \tag{56}$$

When $x \in \mathbb{R}^n \setminus \{0\}$, let

$$\langle x \rangle = \frac{x}{\|x\|}. \tag{57}$$

From (23), we obtain

$$\begin{aligned}
\widetilde{V}_\phi^{(j)}(gK \cap \xi, gL \cap \xi) &= \frac{1}{j}\int_{S^{n-1}\cap\xi} \phi\left(\frac{\rho(gL \cap \xi, u)}{\rho(gK \cap \xi, u)}\right) \cdot \rho(gK \cap \xi, u)^j \, dS(u) = \frac{1}{j} \\
&\cdot \int_{S^{n-1}} \phi\left(\frac{\rho(L, \langle g^{-1}u\rangle)}{\rho(K, \langle g^{-1}u\rangle)}\right) \cdot \rho(K, \langle g^{-1}u\rangle)^j \, dS(\langle g^{-1}u\rangle) = \frac{1}{j} \\
&\cdot \int_{S^{n-1}\cap\xi} \phi\left(\frac{\rho(L \cap \xi, \langle g^{-1}u\rangle)}{\rho(K \cap \xi, \langle g^{-1}u\rangle)}\right) \cdot \rho(K \cap \xi, \langle g^{-1}u\rangle)^j \, dS(\langle g^{-1}u\rangle) \\
&= \widetilde{V}_\phi^{(j)}(K \cap \xi, L \cap \xi).
\end{aligned} \tag{58}$$

On the other hand, from Definition 6 and (58), we have

$$\begin{aligned}
\overline{\Phi}_{\phi,n-j}(gK, gL) &= \omega_n \left[\int_{A_{n,j}} \frac{\widetilde{V}_\phi^{(j)}(gK \cap \xi, gL \cap \xi)}{\mathrm{vol}_j(gK \cap \xi)} \left(\frac{\mathrm{vol}_j(gK \cap \xi)}{\omega_j}\right)^{n+1} d\nu_j(\xi)\right]^{1/(n+1)} \\
&= \omega_n \left[\int_{A_{n,j}} \frac{\widetilde{V}_\phi^{(j)}(K \cap \xi, L \cap \xi)}{\mathrm{vol}_j(K \cap \xi)} \left(\frac{\mathrm{vol}_j(K \cap \xi)}{\omega_j}\right)^{n+1} d\nu_j(\xi)\right]^{1/(n+1)} = \overline{\Phi}_{\phi,n-j}(K, L).
\end{aligned} \tag{59}$$

□

Next, we also can give another proof directly.

Proof. From Lemmas 7 and 9, we have, for $g \in SL(n)$,

$$\begin{aligned}
\overline{\Phi}_{\phi,n-j}(gK, gL) &= \left(\frac{\phi_+'(1)}{j\overline{\Phi}_{n-j}(gK)^{-n}} \cdot \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} \overline{\Phi}_{n-j}(gK \widetilde{+}_\phi \varepsilon \cdot gL)\right)^{1/(n+1)} \\
&= \left(\frac{\phi_+'(1)}{j\overline{\Phi}_{n-j}(gK)^{-n}} \cdot \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} \overline{\Phi}_{n-j}(g(K \widetilde{+}_\phi \varepsilon \cdot L))\right)^{1/(n+1)} \\
&= \left(\frac{\phi_+'(1)}{j\overline{\Phi}_{n-j}(K)^{-n}} \cdot \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} \overline{\Phi}_{n-j}(K \widetilde{+}_\phi \varepsilon \cdot L)\right)^{1/(n+1)} = \overline{\Phi}_{\phi,n-j}(K, L).
\end{aligned} \tag{60}$$

□

Here, we point out the connections between the Orlicz mean dual affine quermassintegrals and the dual affine quermassintegrals. From (13) and in view of the connections between the mean dual affine quermassintegrals and the dual affine quermassintegrals, we have the following: for $\phi \in \mathcal{C}$, $\varepsilon > 0$, $0 < j \le n$, and $K, L \in \mathcal{S}^n$,

$$\overline{\Phi}_{\phi,n-j}(K, L)^{n+1} = \frac{\phi_+'(1)\,\omega_n^n}{j\omega_j^n} \cdot \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} \overline{\Phi}_{n-j}(K \widetilde{+}_\phi \varepsilon \cdot L) \cdot \left(\int_K \widetilde{\Phi}_{n-j}(K - x)^n \, dx\right)^{n/(n+1)}. \tag{61}$$

We also need the following lemma to prove our main results.

Lemma 11 (Jensen's inequality). *Let μ be a probability measure on a space X and $g : X \to I \subset \mathbb{R}$ is a μ-integrable function, where I is a possibly infinite interval. If $\psi : I \to \mathbb{R}$ is a convex function, then*

$$\int_X \psi(g(x))\, d\mu(x) \ge \psi\left(\int_X g(x)\, d\mu(x)\right). \tag{62}$$

If ψ is strictly convex, equality holds if and only if $g(x)$ is constant for μ-almost all $x \in X$ (see [38, p.165]).

4. Orlicz-Minkowski Inequality for Orlicz Mean Dual Quermassintegrals

Theorem 12. *If $K, L \in \mathcal{S}^n$, $\phi \in \mathcal{C}$, and $0 < j \leq n$, then*

$$\left(\frac{\overline{\Phi}_{\phi,n-j}(K,L)}{\overline{\Phi}_{n-j}(K)}\right)^{n+1} \geq \phi\left(\left(\frac{\overline{\Phi}_{n-j}(L)}{\overline{\Phi}_{n-j}(K)}\right)^{1/j}\right). \quad (63)$$

If ϕ is strictly convex, equality holds if and only if K and L are dilates.

Proof. When $j = n$, (63) becomes the dual Orlicz-Minkowski inequality; hence we assume $0 < j < n$. Since

$$\int_{A_{n,j}} d\nu(\xi) = \int_{A_{n,j}} \frac{\operatorname{vol}_j(K\cap\xi)^{n+1}}{\int_{A_{n,j}} \operatorname{vol}_j(K\cap\xi)^{n+1}\, d\nu_j(\xi)} d\nu_j(\xi) = 1, \quad (64)$$

the above equation defines a Borel probability measure ν on $A_{n,j}$; namely,

$$d\nu(\xi) = \frac{\operatorname{vol}_j(K\cap\xi)^{n+1}}{\int_{A_{n,j}} \operatorname{vol}_j(K\cap\xi)^{n+1}\, d\nu_j(\xi)} d\nu_j(\xi). \quad (65)$$

From (11), (47), and (65) and using dual Orlicz-Minkowski inequality, Jensen inequality, and Hölder inequality, we obtain

$$\begin{aligned}
\left(\frac{\overline{\Phi}_{\phi,n-j}(K,L)}{\overline{\Phi}_{n-j}(K)}\right)^{n+1} &= \frac{\int_{A_{n,j}} \left(\widetilde{V}_\phi^{(j)}(K\cap\xi, L\cap\xi)/\operatorname{vol}_j(K\cap\xi)\right)\left(\operatorname{vol}_j(K\cap\xi)/\omega_j\right)^{n+1} d\nu_j(\xi)}{\int_{A_{n,j}} \left(\operatorname{vol}_j(K\cap\xi)/\omega_j\right)^{n+1} d\nu_j(\xi)} \\
&= \int_{A_{n,j}} \frac{\widetilde{V}_\phi^{(j)}(K\cap\xi, L\cap\xi)}{\operatorname{vol}_j(K\cap\xi)} d\nu \geq \int_{A_{n,j}} \phi\left(\left(\frac{\operatorname{vol}_j(L\cap\xi)}{\operatorname{vol}_j(K\cap\xi)}\right)^{1/j}\right) d\nu \\
&\geq \phi\left(\int_{A_{n,j}} \left(\frac{\operatorname{vol}_j(L\cap\xi)}{\operatorname{vol}_j(K\cap\xi)}\right)^{1/j} d\nu\right) = \phi\left(\frac{\int_{A_{n,j}} \operatorname{vol}_j(K\cap\xi)^{(j(n+1)-1)/j} \operatorname{vol}_j(L\cap\xi)^{1/j} d\nu_j(\xi)}{\int_{A_{n,j}} \operatorname{vol}_j(K\cap\xi)^{n+1} d\nu_j(\xi)}\right) \\
&\geq \phi\left(\frac{\left(\int_{A_{n,j}} \operatorname{vol}_j(K\cap\xi)^{n+1} d\nu_j(\xi)\right)^{(j(n+1)-1)/j(n+1)} \left(\int_{A_{n,j}} \operatorname{vol}_j(L\cap\xi)^{n+1} d\nu_j(\xi)\right)^{1/j(n+1)}}{\int_{A_{n,j}} \operatorname{vol}_j(K\cap\xi)^{n+1} d\nu_j(\xi)}\right) \\
&= \phi\left(\left(\frac{\overline{\Phi}_{n-j}(L)}{\overline{\Phi}_{n-j}(K)}\right)^{1/j}\right).
\end{aligned} \quad (66)$$

Next, we discuss the equal condition of (63). If ϕ is strictly convex, suppose that K and L are dilates; that is, there exists $\lambda > 0$ such that $L = \lambda K$. Hence

$$\begin{aligned}
\left(\frac{\overline{\Phi}_{\phi,n-j}(K,L)}{\overline{\Phi}_{n-j}(K)}\right)^{n+1} &= \left(\frac{\overline{\Phi}_{\phi,n-j}(K,\lambda K)}{\overline{\Phi}_{n-j}(K)}\right)^{n+1} \\
&= \left(\frac{\phi(\lambda)^{1/(n+1)}\, \overline{\Phi}_{n-j}(K)}{\overline{\Phi}_{n-j}(K)}\right)^{n+1} \\
&= \phi(\lambda) \\
&= \phi\left(\left(\frac{\overline{\Phi}_{n-j}(\lambda K)}{\overline{\Phi}_{n-j}(K)}\right)^{1/j}\right) \\
&= \phi\left(\left(\frac{\overline{\Phi}_{n-j}(L)}{\overline{\Phi}_{n-j}(K)}\right)^{1/j}\right).
\end{aligned} \quad (67)$$

This implies that theequality in (63) holds.

On the other hand, suppose the equality holds in (63); then these three inequalities in the above proof must satisfy the equal sign. Since the first inequality in the above proof is the dual Orlicz-Minkowski inequality,

$$\frac{\widetilde{V}_\phi^{(j)}(K\cap\xi, L\cap\xi)}{\operatorname{vol}_j(K\cap\xi)} \geq \phi\left(\left(\frac{\operatorname{vol}_j(L\cap\xi)}{\operatorname{vol}_j(K\cap\xi)}\right)^{1/j}\right). \quad (68)$$

Form the equality condition of dual Orlicz-Minkowski inequality, if the equality holds, then $K\cap\xi$ and $L\cap\xi$ must be dilates. The second inequality in the above proof is Jensen inequality.

$$\begin{aligned}
&\int_{A_{n,j}} \phi\left(\left(\frac{\operatorname{vol}_j(L\cap\xi)}{\operatorname{vol}_j(K\cap\xi)}\right)^{1/j}\right) d\nu \\
&\quad \geq \phi\left(\int_{A_{n,j}} \left(\frac{\operatorname{vol}_j(L\cap\xi)}{\operatorname{vol}_j(K\cap\xi)}\right)^{1/j} d\nu\right).
\end{aligned} \quad (69)$$

From the equality condition of Jensen inequality, if ϕ is strictly convex and the equality holds, then $\text{vol}_j(L \cap \xi)/\text{vol}_j(K \cap \xi)$ must be a constant; this yields that $K \cap \xi$ and $L \cap \xi$ must be dilates. In this proof, the third inequality is obtained by applying the Hölder inequality. From the equality condition of Hölder inequality, this yields that equality holds and $\text{vol}_j(K\cap\xi)$ and $\text{vol}_j(L\cap\xi)$ must be proportional; namely, $K \cap \xi$ and $L \cap \xi$ are dilates. From the combinations of these equal conditions, it follows that equality in (63) holds, if ϕ is strictly convex, and equality holds if and only if K and L are dilates. □

Corollary 13. *If $K, L \in \mathcal{S}^n$, $p \geq 1$, and $0 < j \leq n$, then*

$$\left(\frac{\overline{\Phi}_{-p,n-j}(K,L)}{\overline{\Phi}_{n-j}(K)}\right)^{n+1} \geq \left(\frac{\overline{\Phi}_{n-j}(L)}{\overline{\Phi}_{n-j}(K)}\right)^{-p/j}, \tag{70}$$

with equality if and only if K and L are dilates.

Proof. This follows immediately from (63) with $\phi(t) = t^{-p}$ and $1 < p < \infty$. □

Corollary 14. *If $K, L \in \mathcal{S}^n$ and $\phi \in \mathcal{C}$, then*

$$\widetilde{V}_{\phi}(K,L) \geq V(K)\,\phi\left(\left(\frac{V(L)}{V(K)}\right)^{1/n}\right). \tag{71}$$

If ϕ is strictly convex, equality holds if and only if K and L are dilates.

Proof. This follows immediately from (63) with $j = n$. □

The following uniqueness is a direct consequence of the Orlicz-Minkowski inequality for the Orlicz mean dual affine quermassintegrals.

Theorem 15. *If $\phi \in \mathcal{C}$ and is strictly convex, $0 < j \leq n$ and $\mathcal{M} \subset \mathcal{S}^n$ such that $K, L \in \mathcal{M}$. If*

$$\overline{\Phi}_{\phi,n-j}(M,K) = \overline{\Phi}_{\phi,n-j}(M,L), \quad \forall M \in \mathcal{M} \tag{72}$$

or

$$\frac{\overline{\Phi}_{\phi,n-j}(K,M)}{\overline{\Phi}_{n-j}(K)} = \frac{\overline{\Phi}_{\phi,n-j}(L,M)}{\overline{\Phi}_{n-j}(L)}, \quad \forall M \in \mathcal{M} \tag{73}$$

then $K = L$.

Proof. Suppose that (72) holds. Taking K for M, then, from Lemma 8 and (63), we obtain

$$\begin{aligned}\phi(1)\,\overline{\Phi}_{n-j}(K)^{n+1} &= \overline{\Phi}_{\phi,n-j}(K,L)^{n+1} \\ &\geq \overline{\Phi}_{n-j}(K)^{n+1}\,\phi\left(\left(\frac{\overline{\Phi}_{n-j}(L)}{\overline{\Phi}_{n-j}(K)}\right)^{1/j}\right),\end{aligned} \tag{74}$$

with equality if and only if K and L are dilates. Hence

$$\phi(1) \geq \phi\left(\left(\frac{\overline{\Phi}_{n-j}(L)}{\overline{\Phi}_{n-j}(K)}\right)^{1/j}\right), \tag{75}$$

with equality if and only if K and L are dilates. Since ϕ is decreasing function on $(0, \infty)$, it follows that

$$\overline{\Phi}_{n-j}(K) \leq \overline{\Phi}_{n-j}(L), \tag{76}$$

with equality if and only if K and L are dilates. On the other hand, if taking L for M, we similarly get $\overline{\Phi}_{n-j}(K) \geq \overline{\Phi}_{n-j}(L)$, with equality if and only if K and L are dilates. Hence $\overline{\Phi}_{n-j}(K) = \overline{\Phi}_{n-j}(L)$, and K and L are dilates; it follows that K and L must be equal.

Suppose that (73) holds. Taking L for M, then, from Lemma 8 and (63), we obtain

$$\phi(1) = \frac{\overline{\Phi}_{\phi,n-j}(K,L)^{n+1}}{\overline{\Phi}_{n-j}(K)^{n+1}} \geq \phi\left(\left(\frac{\overline{\Phi}_{n-j}(L)}{\overline{\Phi}_{n-j}(K)}\right)^{1/j}\right), \tag{77}$$

with equality if and only if K and L are dilates. Hence

$$\phi(1) \geq \phi\left(\left(\frac{\overline{\Phi}_{n-j}(L)}{\overline{\Phi}_{n-j}(K)}\right)^{1/j}\right), \tag{78}$$

with equality if and only if K and L are dilates. Since ϕ is decreasing function on $(0, \infty)$, it follows that

$$\overline{\Phi}_{n-j}(K) \leq \overline{\Phi}_{n-j}(L), \tag{79}$$

with equality if and only if K and L are dilates. On the other hand, if we take K for M, we similarly get $\overline{\Phi}_{n-j}(K) \geq \overline{\Phi}_{n-j}(L)$, with equality if and only if K and L are dilates. Hence $\overline{\Phi}_{n-j}(K) = \overline{\Phi}_{n-j}(L)$, and K and L are dilates; it follows that K and L must be equal. □

Corollary 16 (see [29]). *If $\phi \in \mathcal{C}$ and is strictly convex, $0 < j \leq n$ and $\mathcal{M} \subset \mathcal{S}^n$ such that $K, L \in \mathcal{M}$. If*

$$\widetilde{V}_{\phi}(M,K) = \widetilde{V}_{\phi}(M,L), \quad \forall M \in \mathcal{M} \tag{80}$$

or

$$\frac{\widetilde{V}_{\phi}(K,M)}{V(K)} = \frac{\widetilde{V}_{\phi}(L,M)}{V(L)}, \quad \forall M \in \mathcal{M} \tag{81}$$

then $K = L$.

Proof. This follows immediately from Theorem 15 with $j = n$. □

5. Orlicz-Brunn-Minkowski Inequality for Mean Dual Affine Quermassintegrals

Lemma 17. *If $K, L \in \mathcal{S}^n$, $0 < j \le n$, and $\phi \in \mathcal{C}$, then for any $\varepsilon > 0$*

$$\phi(1) = \left(\frac{\overline{\Phi}_{\phi,n-j}\left(K \widetilde{+}_\phi \varepsilon \cdot L, K\right)}{\overline{\Phi}_{n-j}\left(K \widetilde{+}_\phi \varepsilon \cdot L\right)} \right)^{n+1} + \varepsilon \cdot \left(\frac{\overline{\Phi}_{\phi,n-j}\left(K \widetilde{+}_\phi \varepsilon \cdot L, L\right)}{\overline{\Phi}_{n-j}\left(K \widetilde{+}_\phi \varepsilon \cdot L\right)} \right)^{n+1}. \tag{82}$$

Proof. From (8) and Lemmas 3 and 5, we have

$$\begin{aligned} &\widetilde{V}_\phi\left(\left(K \widetilde{+}_\phi \varepsilon \cdot L\right) \cap \xi, K \cap \xi\right) + \varepsilon \widetilde{V}_\phi\left(\left(K \widetilde{+}_\phi \varepsilon \cdot L\right) \cap \xi, L \cap \xi\right) = \widetilde{V}_\phi\left((K \cap \xi) \widetilde{+}_\phi \varepsilon \cdot (L \cap \xi), K \cap \xi\right) \\ &+ \varepsilon \widetilde{V}_\phi\left((K \cap \xi) \widetilde{+}_\phi \varepsilon \cdot (L \cap \xi), L \cap \xi\right) \\ &= \widetilde{V}_\phi\left((K \cap \xi) \widetilde{+}_\phi \varepsilon \cdot (L \cap \xi), (K \cap \xi) \widetilde{+}_\phi \varepsilon \cdot (L \cap \xi)\right) = \phi(1) \operatorname{vol}_j\left((K \cap \xi) \widetilde{+}_\phi \varepsilon \cdot (L \cap \xi)\right) \\ &= \phi(1) \operatorname{vol}_j\left(\left(K \widetilde{+}_\phi \varepsilon \cdot L\right) \cap \xi\right). \end{aligned} \tag{83}$$

Let $Q = K \widetilde{+}_\phi \varepsilon \cdot L$; from (83) and (47), we have

$$\begin{aligned} \overline{\Phi}_{\phi,n-j}(Q,K)^{n+1} + \varepsilon \cdot \overline{\Phi}_{\phi,n-j}(Q,L)^{n+1} &= \omega_n^{n+1} \int_{A_{n,j}} \frac{\widetilde{V}_\phi^{(j)}(Q \cap \xi, K \cap \xi) + \varepsilon \cdot \widetilde{V}_\phi^{(j)}(Q \cap \xi, L \cap \xi)}{\operatorname{vol}_j(Q \cap \xi)} \left(\frac{\operatorname{vol}_j(Q \cap \xi)}{\omega_j} \right)^{n+1} d\nu_j(\xi) \\ &= \phi(1)\, \omega_n^{n+1} \int_{A_{n,j}} \left(\frac{\operatorname{vol}_j(Q \cap \xi)}{\omega_j} \right)^{n+1} d\nu_j(\xi) = \phi(1)\, \overline{\Phi}_{n-j}(Q)^{n+1}. \end{aligned} \tag{84}$$

The proof is complete. □

Lemma 18. *Let $K, L \in \mathcal{S}^n$, $\varepsilon > 0$, and $\phi \in \mathcal{C}$.*

(1) *If K and L are dilates, then K and $K \widetilde{+}_\phi \varepsilon \cdot L$ are dilates.*

(2) *If K and $K \widetilde{+}_\phi \varepsilon \cdot L$ are dilates, then K and L are dilates.*

Proof. Suppose that there exists a constant $\lambda > 0$ such that $L = \lambda K$; we have

$$\phi\left(\frac{\rho(K,u)}{\rho\left(K \widetilde{+}_\phi \varepsilon \cdot L, u\right)} \right) + \varepsilon \phi\left(\frac{\lambda \rho(K,u)}{\rho\left(K \widetilde{+}_\phi \varepsilon \cdot L, u\right)} \right) = \phi(1). \tag{85}$$

On the other hand, there exists a unique constant $\delta > 0$ such that

$$\phi\left(\frac{\rho(K,u)}{\rho(\delta K,u)} \right) + \varepsilon \phi\left(\frac{\lambda \rho(K,u)}{\rho(\delta K,u)} \right) = \phi(1), \tag{86}$$

where δ satisfies that

$$\phi\left(\frac{1}{\delta} \right) + \varepsilon \phi\left(\frac{\varepsilon}{\delta} \right) = \phi(1). \tag{87}$$

This shows that $K \widetilde{+}_\phi \varepsilon \cdot L = \delta K$.

Suppose that there exists a constant $\lambda > 0$ such that $K \widetilde{+}_\phi \varepsilon \cdot L = \lambda K$. Then

$$\phi\left(\frac{1}{\lambda} \right) + \varepsilon \phi\left(\frac{\rho(L,u)}{\rho\left(K \widetilde{+}_\phi \varepsilon \cdot L, u\right)} \right) = \phi(1). \tag{88}$$

This shows that

$$\frac{\rho(L,u)}{\rho\left(K \widetilde{+}_\phi \varepsilon \cdot L, u\right)} \tag{89}$$

is a constant. This yields that $K \widetilde{+}_\phi \varepsilon \cdot L$ and L are dilates. Namely, K and L are dilates. □

Theorem 19. *If $K, L \in \mathcal{S}^n$, $\varepsilon > 0$, $0 < j \le n$, and $\phi \in \mathcal{C}$, then*

$$\phi(1) \ge \phi\left(\left(\frac{\overline{\Phi}_{n-j}(K)}{\overline{\Phi}_{n-j}\left(K \widetilde{+}_\phi \varepsilon \cdot L\right)} \right)^{1/j} \right) + \varepsilon \cdot \phi\left(\left(\frac{\overline{\Phi}_{n-j}(L)}{\overline{\Phi}_{n-j}\left(K \widetilde{+}_\phi \varepsilon \cdot L\right)} \right)^{1/j} \right). \tag{90}$$

If ϕ is strictly convex, equality holds if and only if K and L are dilates.

Proof. From Lemma 17 and (63), we obtain

$$\begin{aligned} \phi(1) &= \left(\frac{\overline{\Phi}_{\phi,n-j}\left(K \widetilde{+}_\phi \varepsilon \cdot L, K\right)}{\overline{\Phi}_{n-j}\left(K \widetilde{+}_\phi \varepsilon \cdot L\right)} \right)^{n+1} + \varepsilon \cdot \left(\frac{\overline{\Phi}_{\phi,n-j}\left(K \widetilde{+}_\phi \varepsilon \cdot L, L\right)}{\overline{\Phi}_{n-j}\left(K \widetilde{+}_\phi \varepsilon \cdot L\right)} \right)^{n+1} \\ &\ge \phi\left(\left(\frac{\overline{\Phi}_{n-j}(K)}{\overline{\Phi}_{n-j}\left(K \widetilde{+}_\phi \varepsilon \cdot L\right)} \right)^{1/j} \right) + \varepsilon \cdot \phi\left(\left(\frac{\overline{\Phi}_{n-j}(L)}{\overline{\Phi}_{n-j}\left(K \widetilde{+}_\phi \varepsilon \cdot L\right)} \right)^{1/j} \right). \end{aligned} \tag{91}$$

If ϕ is strictly convex, from equality condition of the Orlicz-Minkowski inequality, the equality holds if and only if K and $K \tilde{+}_\phi \varepsilon \cdot L$ are dilates, and L and $K \tilde{+}_\phi \varepsilon \cdot L$ are dilates and, combining with Lemma 18, this yields that if ϕ is strictly convex, equality holds in (90) if and only if K and L are dilates. □

Corollary 20. *If $K, L \in \mathcal{S}^n$, $p \geq 1$, $\varepsilon > 0$, and $0 < j \leq n$, then*

$$\overline{\Phi}_{n-j}\left(K \tilde{+}_p \varepsilon \cdot L\right)^{-p/j} \geq \overline{\Phi}_{n-j}(K)^{-p/j} + \varepsilon \cdot \overline{\Phi}_{n-j}(L)^{-p/j}, \tag{92}$$

with equality if and only if K and L are dilates.

Proof. This follows immediately from (90) with $\phi(t) = t^{-p}$ and $p \geq 1$. □

For $j = n$ and $\varepsilon = 1$, (92) becomes Lutwak's L_p dual Brunn-Minkowski inequality (36).

Corollary 21. *If $K, L \in \mathcal{S}^n$ and $\phi \in \mathcal{C}$, then*

$$1 \geq \phi\left(\left(\frac{V(K)}{V\left(K \tilde{+}_\phi L\right)}\right)^{1/n}\right) + \left(\left(\frac{V(L)}{V\left(K \tilde{+}_\phi L\right)}\right)^{1/n}\right). \tag{93}$$

If ϕ is strictly convex, equality holds if and only if K and L are dilates.

Proof. This follows immediately from (90) with $\varepsilon = 1$ and $j = n$. □

Theorem 22. *Orlicz-Minkowski inequality for the Orlicz mean dual affine quermassintegrals is equivalent to the Orlicz Brunn-Minkowski inequality for the mean dual affine quermassintegrals. Namely, if $\phi \in \mathcal{C}$, $0 < j \leq n$, and $K, L \in \mathcal{S}^n$, then*

$$\left(\frac{\overline{\Phi}_{\phi,n-j}(K,L)}{\overline{\Phi}_{n-j}(K)}\right)^{n+1} \geq \phi\left(\left(\frac{\overline{\Phi}_{n-j}(L)}{\overline{\Phi}_{n-j}(K)}\right)^{1/j}\right) \Longleftrightarrow \phi(1) \geq \phi\left(\left(\frac{\overline{\Phi}_{n-j}(K)}{\overline{\Phi}_{n-j}\left(K \tilde{+}_\phi L\right)}\right)^{1/j}\right) + \phi\left(\left(\frac{\overline{\Phi}_{n-j}(L)}{\overline{\Phi}_{n-j}\left(K \tilde{+}_\phi L\right)}\right)^{1/j}\right). \tag{94}$$

If ϕ is strictly convex, equality holds if and only if K and L are dilates.

Proof.

⇐: Let

$$K_\varepsilon = K \tilde{+}_\phi \varepsilon \cdot L. \tag{95}$$

From Lemmas 4 and 7 and using the Orlicz-Brunn-Minkowski inequality (90), we obtain

$$\begin{aligned}
&\frac{j}{\phi'_-(1)} \overline{\Phi}_{n-j}(K)^{-n} \overline{\Phi}_{\phi,n-j}(K,L)^{n+1} \\
&= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0^+} \overline{\Phi}_{n-j}(K_\varepsilon) = \lim_{\varepsilon\to0^+} \frac{\overline{\Phi}_{n-j}(K_\varepsilon) - \overline{\Phi}_{n-j}(K)}{\varepsilon} \\
&= \lim_{\varepsilon\to0^+} \frac{1 - \overline{\Phi}_{n-j}(K)/\overline{\Phi}_{n-j}(K_\varepsilon)}{\phi(1) - \phi\left(\left(\overline{\Phi}_{n-j}(K)/\overline{\Phi}_{n-j}(K_\varepsilon)\right)^{1/j}\right)} \\
&\quad \cdot \frac{\phi(1) - \phi\left(\left(\overline{\Phi}_{n-j}(K)/\overline{\Phi}_{n-j}(K_\varepsilon)\right)^{1/j}\right)}{\varepsilon} \\
&\quad \cdot \overline{\Phi}_{n-j}(K_\varepsilon) \\
&= \lim_{t\to1^+} \frac{1-t}{\phi(1) - \phi(t^{1/j})} \\
&\quad \cdot \lim_{\varepsilon\to0^+} \frac{\phi(1) - \phi\left(\left(\overline{\Phi}_{n-j}(K)/\overline{\Phi}_{n-j}(K_\varepsilon)\right)^{1/j}\right)}{\varepsilon} \\
&\quad \cdot \lim_{\varepsilon\to0^+} \overline{\Phi}_{n-j}(K_\varepsilon) \\
&\geq \frac{j}{\phi'_+(1)} \cdot \lim_{\varepsilon\to0^+} \phi\left(\left(\frac{\overline{\Phi}_{n-j}(L)}{\overline{\Phi}_{n-j}(K_\varepsilon)}\right)^{1/j}\right) \\
&\quad \cdot \lim_{\varepsilon\to0^+} \overline{\Phi}_{n-j}(K_\varepsilon) \\
&= \frac{j}{\phi'_+(1)} \cdot \phi\left(\left(\frac{\overline{\Phi}_{n-j}(L)}{\overline{\Phi}_{n-j}(K)}\right)^{1/j}\right) \cdot \overline{\Phi}_{n-j}(K).
\end{aligned} \tag{96}$$

⇒: From the proof of Theorem 19, we may see that Orlicz-Minkowski inequality for Orlicz mean dual affine quermassintegrals implies also Orlicz-Brunn-Minkowski inequality for the mean dual affine quermassintegrals.

This proof is complete. □

Corollary 23. *If $\phi \in \mathscr{C}$ and $K, L \in \mathscr{S}^n$, then*

$$\frac{\widetilde{V}_{\phi}(K,L)}{V(K)} \geq \phi\left(\left(\frac{V(L)}{V(K)}\right)^{1/n}\right) \Longleftrightarrow$$
$$\phi(1) \geq \phi\left(\left(\frac{V(K)}{V(K \widetilde{+}_{\phi} L)}\right)^{1/n}\right) + \phi\left(\left(\frac{V(L)}{V(K \widetilde{+}_{\phi} L)}\right)^{1/n}\right). \quad (97)$$

If ϕ is strictly convex, equality holds if and only if K and L are dilates.

Proof. This follows immediately from Theorem 22 with $\varepsilon = 1$ and $j = n$. □

Corollary 24. *If $0 < j \leq n$, $p \geq 1$, and $K, L \in \mathscr{S}^n$, then*

$$\left(\frac{\overline{\Phi}_{-p,n-j}(K,L)}{\overline{\Phi}_{n-j}(K)}\right)^{n+1} \geq \left(\frac{\overline{\Phi}_{n-j}(L)}{\overline{\Phi}_{n-j}(K)}\right)^{-p/j} \Longleftrightarrow$$
$$\overline{\Phi}_{n-j}\left(K \widetilde{+}_{\phi} L\right)^{-p/j} \geq \overline{\Phi}_{n-j}(K)^{-p/j} + \overline{\Phi}_{n-j}(L)^{-p/j}. \quad (98)$$

If ϕ is strictly convex, equality holds if and only if K and L are dilates.

Proof. This follows immediately from Theorem 22 with $\phi(t) = t^{-p}$ and $p \geq 1$. □

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Chang-Jian Zhao and Wing-Sum Cheung provided the questions and gave the proof for the main results. They all read and approved the manuscript.

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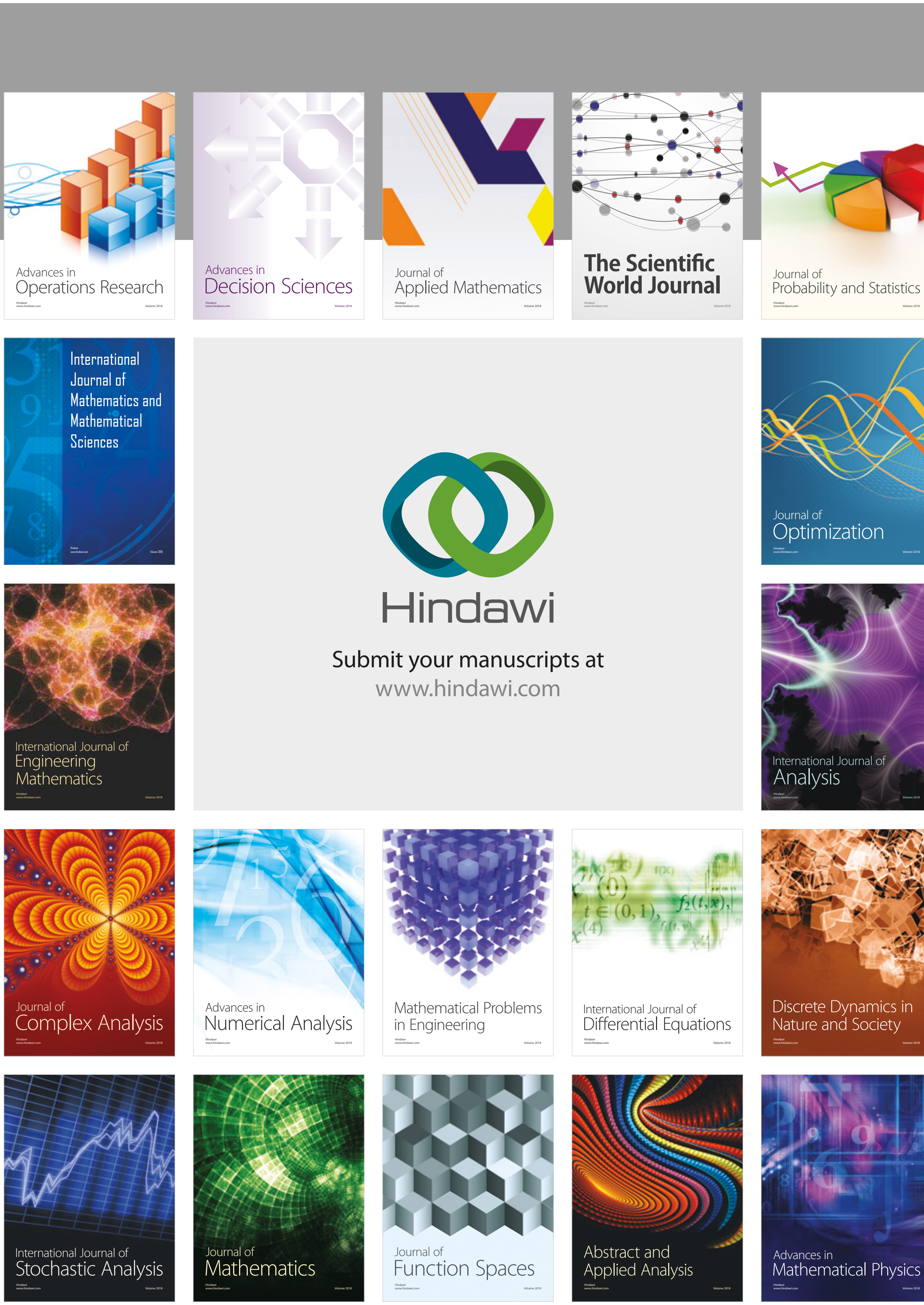
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