

Research Article

A New Approach to the Existence of Quasiperiodic Solutions for Second-Order Asymmetric p -Laplacian Differential Equations

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For $p \geq 2$ and $\phi_p(s) := |s|^{p-2}s$, we propose a new estimate approach to study the existence of Aubry-Mather sets and quasiperiodic solutions for the second-order asymmetric p -Laplacian differential equations $(\phi_p(x'))' + \lambda\phi_p(x^+) - \mu\phi_p(x^-) = \psi(t, x)$, where λ and μ are two positive constants satisfying $\lambda^{-1/p} + \mu^{-1/p} = 2/\omega$ with $\omega \in \mathbb{R}^+$, $\psi(t, x) \in C^{0,1}(\mathbb{S}^p \times \mathbb{R})$ is a continuous function, $2\pi_p$ -periodic in the first argument and continuously differentiable in the second one, $x^\pm = \max\{\pm x, 0\}$, $\pi_p = 2\pi(p-1)^{1/p}/p \sin(\pi/p)$, and $\mathbb{S}^p = \mathbb{R}/2\pi_p\mathbb{Z}$. Using the Aubry-Mather theorem given by Pei, we obtain the existence of Aubry-Mather sets and quasiperiodic solutions under some reasonable conditions. Particularly, the advantage of our approach is that it not only gives a simpler estimation procedure, but also weakens the smoothness assumption on the function $\psi(t, x)$ in the existing literature.

1. Introduction

In recent years, all kinds of nonlinear dynamic behavior, such as the existence of positive solutions [1–16] and sign-changing solutions [17, 18], the existence and uniqueness of solutions [19–25], the existence and multiplicity results [26–30], and the existence of unbounded solutions [31, 32], have been widely investigated for some nonlinear ordinary differential equations and partial differential equations due to the application in many fields such as physics, mechanics, and the engineering technique fields. In the present paper, we deal with the existence of Aubry-Mather sets and quasiperiodic solutions for the second-order differential equations with a p -Laplacian and an asymmetric nonlinear term

$$(\phi_p(x'))' + \lambda\phi_p(x^+) - \mu\phi_p(x^-) = \psi(t, x), \quad (1)$$

where $\phi_p(s) := |s|^{p-2}s$, $p \geq 2$, $x^\pm = \max\{\pm x, 0\}$, and λ and μ are positive constants satisfying

$$\frac{1}{\lambda^{1/p}} + \frac{1}{\mu^{1/p}} = \frac{2}{\omega}, \quad \omega \in \mathbb{R}^+, \quad (2)$$

where $\psi(t, x) \in C^{0,1}(\mathbb{S}^p \times \mathbb{R})$ is a continuous function, $2\pi_p$ -periodic in the first argument and continuously differentiable in the second one, where $\mathbb{S}^p = \mathbb{R}/2\pi_p\mathbb{Z}$. Since the pioneering works of Aubry [33] and Mather [34], the existence of Aubry-Mather sets and quasiperiodic solutions for a variety of differential equations, such as Hamiltonian systems [35–41], and reversible systems [42–44] had been widely investigated due to the application in many fields such as one-dimensional crystal model of solid state physics, differential geometry, and dynamical systems (see [45, 46]).

If $p = 2$, then $\pi_p = \pi$ and (1) reduces to the following piecewise linear equation:

$$x'' + \lambda x^+ - \mu x^- = \psi(t, x) \quad (3)$$

and (2) becomes

$$\frac{1}{\sqrt{\lambda}} + \frac{1}{\sqrt{\mu}} = \frac{2}{\omega}, \quad \omega \in \mathbb{R}^+. \quad (4)$$

The first result is due to Capietto and Liu [38], who proved that the existence of Aubry-Mather sets and quasiperiodic solutions of (3) for some $\omega \in \mathbb{Q}^+$ in (4), provided that

$\psi(t, x) = e(t) - \phi(x)$, $e(t) \in C^2(\mathbb{R}/2\pi\mathbb{Z})$, and the perturbation term $\phi(x) \in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ satisfies some growth conditions. Recently, this result was extended to a much weaker smoothness nonlinearity $\psi(t, x)$. In [41], by using the Aubry-Mather theorem generalized by Pei [37], the present author [41] studied the existence of Aubry-Mather sets and quasiperiodic solutions of (3), under the condition that $\omega \in \mathbb{R}^+$ in (4) and $\psi(t, x) \in C^{0,1}(\mathbb{S}^p \times \mathbb{R})$ can be allowed to be either a bounded function or an unbounded function, which differs from above existing results.

In [39], (3) has been generalized to the following p -Laplacian-like nonlinear differential equation:

$$(p-1)^{-1} (\phi_p(x'))' + \lambda \phi_p(x^+) - \mu \phi_p(x^-) + g(x) = h(t), \quad (5)$$

where $p > 1$, λ and μ are positive constants satisfying (2) with $\omega = n \in \mathbb{N}$, $g(x) \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, and $h(t) \in C^2(\mathbb{S}^p)$ is a $2\pi_p$ -periodic function. They considered the existence of Aubry-Mather sets and quasiperiodic solutions of (5) when $g(x)$ satisfies some further approximate properties at infinity. We notice that in [39], to overcome the barriers of weak smoothness, they made use of exchange of the role of time and angle variables skills and showed the existence of Aubry-Mather sets and quasiperiodic solutions by employing a version of Aubry-Mather theorem obtained by Pei [37]. Moreover, the results in [39] need the smoothness requirement of the perturbation function at least to C^2 smooth in t .

Now a natural question to ask is whether the smoothness of the function $h(t)$ in (5) is further reduced; we can also obtain the same results as [13]. In this paper, we will deal with this interesting problem and answer this question in the form of Theorem 1 with more general case (1) than that of (5). Because of the presence of weak smoothness nonlinearity, the methods of seeking the existence of Aubry-Mather sets and quasiperiodic solutions for problems as [38, 39] do not seem to be applicable to (1). This phenomenon provokes some mathematical difficulties, which make the study of (1) particularly interesting. Our approach here is mainly based on the direct proof of the Poincaré map of the transformed system satisfying monotone twist property and is developed from the present author (see the recent papers [41, 44]) but is more subtle than the ones in [38–40]. More efforts have to be made to estimate the monotone twist property for the Poincaré map of the transformed system, but the procedure is a little simpler than those in [38–40]. One important advantage of our approach is that it does not require any high smoothness assumptions on function $\psi(t, x)$. Our results improve and generalize some results of the previous studies [39, 41] to some extent.

The main result of this paper is the following theorem.

Theorem 1. *Suppose that (2) holds. Moreover, $\psi(t, x) \in C^{0,1}(\mathbb{S}^p \times \mathbb{R})$ satisfies the following conditions:*

(A₁) *The limit is*

$$\lim_{|x| \rightarrow +\infty} \psi_x(t, x) = 0, \quad \text{uniformly in } t \in [0, 2\pi_p]. \quad (6)$$

(A₂) *There exist constants $d \geq 0, \beta > 0$, such that*

$$\text{sgn}(x) [(p-1)\psi(t, x) - x\psi_x(t, x)] > \beta, \quad (7)$$

for $\forall |x| \geq d$ and $\forall t \in [0, 2\pi_p]$.

Then there exists $\varepsilon_0 > 0$, such that, for any $\alpha \in (2\omega\pi_p, 2\omega\pi_p + \varepsilon_0)$, (1) possesses an Aubry-Mather type solution $z_\alpha(t) = (x_\alpha(t), x'_\alpha(t))$ with rotation number α ; that is,

(i) *if $\alpha = k/m$ is rational and $(k, m) = 1$, the solutions $z_\alpha^i(t) = z_\alpha(t + 2\pi_p i)$, $0 \leq i \leq m-1$, are mutually unlinked periodic solutions of period m ;*

(ii) *if α is irrational, the solution $z_\alpha(t)$ is either a usual quasiperiodic solution or a generalized one.*

Remark 2. A solution is called generalized quasiperiodic one if the closed set

$$M_\alpha \equiv \overline{\{z_\alpha(2\pi_p i), i \in \mathbb{Z}\}} \quad (8)$$

is Denjoy's minimal set (see its definition in [47]).

Remark 3. Using the rule of L'Hospital to condition (A₁), it can easily be seen that

$$\lim_{|x| \rightarrow +\infty} \frac{\psi(t, x)}{x} = 0, \quad \text{uniformly in } t \in [0, 2\pi_p]. \quad (\psi_0)$$

Remark 4. We noticed that the perturbations $g(x)$ and $h(t)$ in [39] need to be bounded. But from (A₁) and (A₂) of this paper, it is easy to verify that the perturbation $\psi(t, x)$ can be either a bounded function or an unbounded function. For example, we can set $\psi(t, x)$ to be a bounded function $\arctan x \cdot (1 + \sin^2((\pi/\pi_p)t))$ or an unbounded function $x^{1/3} \cdot (1 + \cos^2((\pi/\pi_p)t))$ for $t \in [0, 2\pi_p]$ when $d = 1$ and $\beta = \pi/4 - 1/2$ in Theorem 1. Moreover, positive constant $\omega = n \in \mathbb{N}$ satisfying (1.2) in [39] has been extended to the case $\omega \in \mathbb{R}^+$ in this paper. Thus, our situation is more general than the results obtained in [39] for $p \geq 2$.

Remark 5. If $p = 2$, let us point out that the results in Theorem 1 have covered the conclusions obtained by Wang [41]. Besides, the estimation process in this paper is much more meticulous than that in [41] since the p -Laplacian $(\phi_p(x'))'$ of a function $x(t)$, with $p > 2$, is no longer linear. Therefore, the results obtained in this paper are natural generalizations and refinements of the results obtained in [41].

The main idea of our proof is acquired from [39, 41]. The proof of Theorem 1 is based on an Aubry-Mather theorem due to Pei [37]. The rest of this manuscript is as follows. In Section 2, we introduce some action-angle variables transformation to transform system (1) into an equivalent integral Hamiltonian system and then present some growth properties on the corresponding action and angle variables functions. In Section 3, we provide some crucial estimates by some lemmas which say that the Poincaré mapping of the new system is monotone twist around the infinity. At last, Section 4 gives the proof of Theorem 1 by using Pei's Aubry-Mather theorem [37].

2. Preliminaries

2.1. *The Action and Angle Variables.* Let $S_p(t) = \sin_p t$ be the solution of

$$(\phi_p(u'))' + \phi_p(u) = 0 \tag{9}$$

satisfying the initial condition $u(0) = 0, u'(0) = 1$. Then it follows from [48] that $S_p(t) = \sin_p t$ is a $2\pi_p$ -periodic C^2 odd function with $\sin_p(\pi_p - t) = \sin_p t$, for $t \in [0, \pi_p/2]$, and $\sin_p(2\pi_p - t) = -\sin_p t$ for $t \in [\pi_p, 2\pi_p]$. Moreover, for $t \in (0, \pi_p/2), S_p(t) > 0, S_p'(t) > 0$ and $S_p : [0, \pi_p/2] \rightarrow [0, (p - 1)^{1/p}]$ can be implicitly given by

$$\int_0^{\sin_p t} \frac{ds}{(1 - s^p/(p - 1))^{1/p}} = t. \tag{10}$$

Introducing a new variable $y = -\phi_p(x')$, then (9) is equivalent to the planar system

$$\begin{aligned} x' &= -\phi_q(y), \\ y' &= \phi_p(x), \end{aligned} \tag{11}$$

where q is the conjugate exponent of $p : p^{-1} + q^{-1} = 1$. Letting $(x, y) = (C_p(t), S_p(t))$ be the unique solution of (11) satisfying $(C_p(0), S_p(0)) = (1, 0)$, then the functions $C_p(t)$ and $S_p(t)$ are much similar to cosine and sine. It follows from [49] that $C_p(t) \in C^2$ and $S_p(t) \in C^1$ are $2\pi_p$ -periodic, and for $\forall n \in \mathbb{Z}, C_p(t) = 0$ iff $t = \pi_p/2 + n\pi_p$, and $S_p(t) = 0$ iff $t = n\pi_p$. Moreover, $C_p'(t) = -\phi_q(S_p(t))$ and $S_p'(t) = \phi_p(C_p(t))$, and $(1/p)|C_p(t)|^{1/p} + (1/q)|S_p(t)|^{1/q} \equiv 1/p$.

Now we consider (1). Set $y = -\phi_p(x')$ in (1); then (1) can be rewritten as a planar system

$$\begin{aligned} x' &= -\phi_q(y), \\ y' &= \lambda\phi_p(x^+) - \mu\phi_p(x^-) - \psi(t, x), \end{aligned} \tag{12}$$

where $q = p/(p - 1)$ is the conjugate exponent of p .

Lemma 6. For $p \geq 2$ and for any $(x_0, y_0) \in \mathbb{R}^2, t_0 \in \mathbb{R}$, the solution

$$z(t) = (x(t, t_0, x_0, y_0), y(t, t_0, x_0, y_0)) \tag{13}$$

of (12) satisfying the initial condition $z(t_0) = (x_0, y_0)$ is unique and exists on the whole t -axis.

Proof. The proof of uniqueness can be established similarly to the proof of Proposition 2 in [50]; the global existence result can be acquired similarly to Lemma 3.1 in [51]. \square

Let $(C(t), S(t))$ be the solution of the following homogeneous system:

$$\begin{aligned} x' &= -\phi_q(y), \\ y' &= \lambda\phi_p(x^+) - \mu\phi_p(x^-), \\ x(0) &= 1, \\ y(0) &= 0. \end{aligned} \tag{14}$$

Then, by using (2) and direct computation, one obtains the following.

Lemma 7. (i) Both $C(t) \in C^2$ and $S(t) \in C^1$ are $2\pi_p/\omega$ -periodic functions, and $C(t)$ can be given by

$$C(t) = \begin{cases} C_p(\lambda^{1/p}t), & 0 \leq |t| \leq \frac{\pi_p}{2\lambda^{1/p}}, \\ C_p\left(\mu^{1/p}\left(t - \frac{\pi_p}{2\lambda^{1/p}} + \frac{\pi_p}{2\mu^{1/p}}\right)\right), & \frac{\pi_p}{2\lambda^{1/p}} \leq t \leq \frac{\pi_p}{\omega}. \end{cases} \tag{15}$$

(ii) $C'(t) = -\phi_q(S(t))$ and $S'(t) = \lambda\phi_p(C(t)^+) - \mu\phi_p(C(t)^-)$.
 (iii) $(1/q)|S(t)|^q + (1/p)(\lambda|C(t)^+|^p + \mu|C(t)^-|^p) = \lambda/p$.

Now we introduce an action-angle variables transformation by the mapping $\Psi : \mathbb{S}^p \times (0, +\infty) \rightarrow \mathbb{R}^2 \setminus \{(0, 0)\}$, where $(x, y) = \Psi(\theta, I)$ defined by the formula

$$\begin{aligned} x &= (\gamma I)^{1/p} C\left(\frac{\theta}{\omega}\right), \\ y &= (\gamma I)^{1/q} S\left(\frac{\theta}{\omega}\right), \end{aligned} \tag{16}$$

where $\gamma = \lambda^{-1}\omega p$ is a constant. This transformation is said to be a generalized symplectic transformation because its Jacobian is equal to 1.

2.2. *Some Properties on Action and Angle Variables Functions.* Under the transformation Ψ and using Lemma 7 (iii), (12) is changed into

$$\begin{aligned} \dot{\theta} &= \Phi_1(t, \theta, I), \\ \dot{I} &= \Phi_2(t, \theta, I), \end{aligned} \tag{17}$$

where $\Phi_1(t, \theta, I) = \omega + x(\theta, I)\psi(t, x(\theta, I))/pI, \Phi_2(t, \theta, I) = \phi_q(y(\theta, I))\psi(t, x(\theta, I))/\omega$.

We notice that the relation between (17) and (12) is that if $\theta(t) = \theta(t; \theta_0, I_0), I(t) = I(t; \theta_0, I_0)$ are the solutions of (17) with the initial value condition $\theta(0) = \theta_0, I(0) = I_0$, then

$$\begin{aligned} x(t) &= x(t; \theta_0, I_0) = x(\theta(t; \theta_0, I_0), I(t; \theta_0, I_0)) \\ &= (\gamma I(t; \theta_0, I_0))^{1/p} C\left(\frac{\theta(t; \theta_0, I_0)}{\omega}\right) \end{aligned} \tag{18}$$

and

$$\begin{aligned} y(t) &= y(t; \theta_0, I_0) = y(\theta(t; \theta_0, I_0), I(t; \theta_0, I_0)) \\ &= (\gamma I(t; \theta_0, I_0))^{1/q} S\left(\frac{\theta(t; \theta_0, I_0)}{\omega}\right) \end{aligned} \tag{19}$$

are the solutions of (12) with initial data $x(0) = x(0; \theta_0, I_0), y(0) = y(0; \theta_0, I_0)$. By Lemma 6, (17) has a unique solution for $I_0 > 0$ and $\theta_0 \in \mathbb{R}$. Moreover, this

solution has continuous derivatives with respect to initial data θ_0 and I_0 .

For notional convenience, hereinafter, we write x, y, θ, I instead of $x(\theta(t; \theta_0, I_0), I(t; \theta_0, I_0)), y(\theta(t; \theta_0, I_0), I(t; \theta_0, I_0)), \theta(t; \theta_0, I_0), I(t; \theta_0, I_0)$, respectively.

Firstly, by some simple calculations, we have the following.

Lemma 8. (i) $\partial x/\partial I = x/pI, \partial y/\partial I = y/qI, \partial x/\partial \theta = -(1/\omega)\phi_q(y), \partial y/\partial \theta = (\lambda\phi_p(x^+) - \mu\phi_p(x^-))/\omega$.

(ii) $|C(t)| \leq \max\{1, (\lambda/\mu)^{1/p}\} := C_\infty, |S(t)| \leq (q\lambda/p)^{1/q} := S_\infty$.

Now we are concerned with the growth estimates with regard to $I(t; \theta_0, I_0)$ and $\theta(t; \theta_0, I_0)$.

Lemma 9. *The limit*

$$\lim_{I_0 \rightarrow +\infty} I(t; \theta_0, I_0) = +\infty \quad (20)$$

holds uniformly on $t \in [0, 2\pi_p]$.

Proof. In view of (ψ_0) and (16), there exist constants $D > 0, K > 0$, such that

$$|I'(t)| = \left| \frac{\phi_q(y(\theta, I))\psi(t, x(\theta, I))}{\omega} \right| \leq DI(t) + K, \quad (21)$$

$\forall I \neq 0.$

Then, by the Gronwall inequality, one has

$$\begin{aligned} e^{-2\pi D} I_0 - \frac{K}{D} (1 - e^{-2\pi D}) &\leq I(t) \\ &\leq e^{2\pi D} I_0 + \frac{K}{D} (e^{2\pi D} - 1) \end{aligned} \quad (22)$$

for all $t \in [0, 2\pi_p]$.

So, by (22), $I(t; \theta_0, I_0) \rightarrow +\infty$ as $I_0 \rightarrow +\infty$ uniformly for $t \in [0, 2\pi_p]$. \square

According to (22), it is easy to see the following.

Corollary 10. $\forall \theta_0 \in \mathbb{R}$ and $\forall t \in [0, 2\pi_p]$, there exist constants $\rho_2 > \rho_1 > 0$ and $\bar{I} > 0$, such that

$$\rho_1 I_0 \leq I(t; \theta_0, I_0) \leq \rho_2 I_0 \quad (23)$$

when $I_0 \geq \bar{I}$.

Lemma 11. $\forall \theta_0 \in \mathbb{R}$ and $\forall t \in [0, 2\pi_p]$, there exists constant $\bar{I} > 0$, such that

$$\frac{\omega}{2} \leq \theta'(t; \theta_0, I_0) \leq 2\omega \quad (24)$$

if $I_0 \geq \bar{I}$.

Proof. Since (ψ_0) holds, then, for every $\varepsilon > 0$, there exists $M = M(\varepsilon) > 0$, such that

$$|\psi(t, x)| \leq \varepsilon |x| \quad (25)$$

if $|x| \geq M$ and $\forall t \in [0, 2\pi_p]$. Hence,

$$\frac{d\theta}{dt} = \omega + \frac{x\psi(t, x)}{pI} \geq \omega - \frac{\varepsilon x^2}{pI}. \quad (26)$$

Thus, by using action-angle variables transformation (16) and Lemma 9, there exists $\bar{I}_1 > 0$ such that $d\theta/dt \geq \omega/2$ if $I_0 \geq \bar{I}_1$.

For if $|x| \leq M$, we assume that $|\psi(t, x)| \leq \psi_\infty$, where $\psi_\infty = \max\{|\psi(t, x)| : t \in [0, 2\pi_p], |x| \leq M\}$, and then

$$\frac{d\theta}{dt} = \omega + \frac{x\psi(t, x)}{pI} \geq \omega - \frac{\psi_\infty |x|}{pI}. \quad (27)$$

So, by (16), Lemma 8 (ii), and Lemma 9, there exists a constant $\bar{I}_2 > 0$, such that $d\theta/dt \geq \omega/2$ if $I_0 \geq \bar{I}_2$.

If we choose $\bar{I} = \max\{\bar{I}_1, \bar{I}_2\}$, then $I_0 \geq \bar{I}$ implies $d\theta/dt \geq \omega/2$.

Exploiting the same arguments, one can show that the inequality on the right side of (i) holds. \square

3. Twist Property and Proof of Theorem 1

Let the Poincaré mapping P of equation (17) be

$$P : (\theta_0, I_0) \mapsto (\theta(2\pi_p, \theta_0, I_0), I(2\pi_p, \theta_0, I_0)). \quad (28)$$

In order to apply the Aubry-Mather theorem developed by Pei [37], we only need to show that the Poincaré mapping P is a monotone twist map around the infinity; that is, it is enough to show $\partial\theta(2\pi; \theta_0, I_0)/\partial I_0 < 0$ if $I_0 \gg 1$. In the following we are going to give its detailed proofs by some lemmas.

Similarly, for notional convenience, hereinafter, we also write x, y, θ, I instead of $x(\theta(t; \theta_0, I_0), I(t; \theta_0, I_0)), y(\theta(t; \theta_0, I_0), I(t; \theta_0, I_0)), \theta(t; \theta_0, I_0), I(t; \theta_0, I_0)$, respectively.

Lemma 12. *The following convergences hold uniformly on $t \in [0, 2\pi_p]$:*

- (i) $x\psi(t, x)/I \rightarrow 0; x^2\psi_x(t, x)/I \rightarrow 0$, as $I_0 \rightarrow +\infty$.
- (ii) $\phi_q(y)\psi(t, x)/I \rightarrow 0; \phi_q(y)x\psi_x(t, x)/I \rightarrow 0$, as $I_0 \rightarrow +\infty$.
- (iii) $x\psi^2(t, x)\phi_q'(y)\phi_p(x)/I^2 \rightarrow 0; x\psi(t, x)\psi_x(t, x)\phi_q^2(y)/I^2 \rightarrow 0; x^2\psi_x(t, x)\psi(t, x)\phi_q'(y)\phi_p(x)/I^2 \rightarrow 0; x^2\psi_x^2(t, x)\phi_q^2(y)/I^2 \rightarrow 0$, as $I_0 \rightarrow +\infty$.

Proof. If (A_1) and (ψ_0) hold, then to each $\varepsilon > 0$ there corresponds a positive number $M = M(\varepsilon) > 0$, such that

$$|\psi_x(t, x)| \leq \frac{\varepsilon}{2\gamma^{2/p}} \quad (29)$$

and

$$|\psi(t, x)| \leq \frac{\varepsilon}{2\gamma^{2/p}} |x| \quad (30)$$

when $|x| \geq M$ and $t \in [0, 2\pi_p]$, where $\gamma = \lambda^{-1}\omega p$ is a constant given in (16).

Denote $V_1(\varepsilon) = \max\{|\psi(t, x)| : t \in [0, 2\pi_p], |x| \leq M\}$, $V_2(\varepsilon) = \max\{|\psi_x(t, x)| : t \in [0, 2\pi_p], |x| \leq M\}$.

(i) By action-angle variables transformation (16) and Lemma 8 (ii) and $p \geq 2$, we have

$$\begin{aligned} \left| \frac{x\psi(t, x)}{I} \right| &\leq \frac{MV_1(\varepsilon)}{I} + \frac{\varepsilon x^2}{2\gamma^{2/p}I} \\ &\leq \frac{MV_1(\varepsilon)}{I} + \frac{\varepsilon}{2I^{1-2/p}}; \\ \left| \frac{x^2\psi_x(t, x)}{I} \right| &\leq \frac{M^2V_2(\varepsilon)}{I} + \frac{\varepsilon x^2}{2\gamma^{2/p}I} \\ &\leq \frac{M^2V_2(\varepsilon)}{I} + \frac{\varepsilon}{2I^{1-2/p}}. \end{aligned} \tag{31}$$

Then, given $\bar{I} > 0$, choosing I_0 so that $I_0 \geq \bar{I}$, by using Corollary 10, provided

$$I(t) \geq \max \left\{ \frac{2MV_1(\varepsilon)}{\varepsilon}; \frac{2M^2V_2(\varepsilon)}{\varepsilon}; 1 \right\}, \tag{32}$$

we have

$$\begin{aligned} \left| \frac{x\psi(t, x)}{I} \right| &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon; \\ \left| \frac{x^2\psi_x(t, x)}{I} \right| &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon. \end{aligned} \tag{33}$$

Since $\varepsilon > 0$ is arbitrary, the proof of (i) is complete.

For (ii), observe that $p \geq 2$, and, combining (16) and Lemma 8 (ii), one has

$$\begin{aligned} \left| \frac{\phi_q(y)\psi(t, x)}{I} \right| &\leq \frac{\varepsilon|y|^{q-1}|x|}{2\gamma^{2/p}I} + \frac{V_1(\varepsilon)|y|^{q-1}}{I} \\ &\leq \frac{\varepsilon}{2}C_\infty S_\infty^{q-1}I^{2/p-1} \\ &\quad + V_1(\varepsilon)\gamma^{1/p}S_\infty^{q-1}I^{-1/q}; \\ \left| \frac{\phi_q(y)x\psi_x(t, x)}{I} \right| &\leq \frac{\varepsilon|y|^{q-1}|x|}{2\gamma^{2/p}I} + \frac{M|y|^{q-1}V_2(\varepsilon)}{I} \\ &\leq \frac{\varepsilon}{2}C_\infty S_\infty^{q-1}I^{2/p-1} \\ &\quad + M\gamma^{1/p}V_2(\varepsilon)S_\infty^{q-1}I^{-1/q}. \end{aligned} \tag{34}$$

Then, given $\bar{I} > 0$, choosing I_0 so that $I_0 \geq \bar{I}$, by using Corollary 10, provided

$$\begin{aligned} I^{1/q}(t) \\ \geq \max \left\{ \frac{2V_1(\varepsilon)S_\infty^{q-1}\gamma^{1/p}}{\varepsilon}; \frac{2MV_2(\varepsilon)S_\infty^{q-1}\gamma^{1/p}}{\varepsilon}; 1 \right\}, \end{aligned} \tag{35}$$

we have

$$\begin{aligned} \left| \frac{\phi_q(y)\psi(t, x)}{I} \right| &< \frac{\sigma\varepsilon}{2} + \frac{\varepsilon}{2}; \\ \left| \frac{\phi_q(y)x\psi_x(t, x)}{I} \right| &< \frac{\sigma\varepsilon}{2} + \frac{\varepsilon}{2}, \end{aligned} \tag{36}$$

where $\sigma = C_\infty S_\infty^{q-1}$. Since $\varepsilon > 0$ is arbitrary, (ii) is proved.

(iii) Set $\eta_1 = qC_\infty^{p+2}S_\infty^{q-2}$, $\eta_2 = qM^{p+2}S_\infty^{q-2}\gamma^{1-2/q}$, $\nu_1 = C_\infty^2S_\infty^{2(q-1)}$, $\nu_2 = MV_2(\varepsilon)\gamma^{2/p}S_\infty^{2(q-1)}$. In view of (16), Lemma 7 (ii), and $p \geq 2$, we can get

$$\begin{aligned} \left| \frac{x\psi^2(t, x)\phi'_q(y)\phi_p(x)}{I^2} \right| &= \left| \frac{(q-1)x\psi^2(t, x)\phi_p(x)y^{q-2}}{I^2} \right| \\ &\leq \frac{q|x|^{p+2}|y|^{q-2}\varepsilon^2}{4\gamma^{4/p}I^2} + \frac{qM^{p+2}|y|^{q-2}}{I^2} \\ &\leq \frac{\varepsilon^2}{4}qI^{4/p-2}C_\infty^{p+2}S_\infty^{q-2} + \frac{qM^{p+2}S_\infty^{q-2}\gamma^{1-2/q}}{I^{1+2/q}} \\ &\leq \frac{\varepsilon^2\eta_1}{2}I^{4/p-2} + \frac{\eta_2}{2I^{1+2/q}}; \\ \left| \frac{x\psi(t, x)\psi_x(t, x)\phi_q^2(y)}{I^2} \right| &\leq \frac{|y|^{2(q-1)}|x|^2\varepsilon^2}{4\gamma^{4/p}I^2} + \frac{MV_1(\varepsilon)V_2(\varepsilon)|y|^{2(q-1)}}{I^2} \\ &\leq \frac{\varepsilon^2}{4}I^{4/p-2}C_\infty^2S_\infty^{2(q-1)} + \frac{MV_1(\varepsilon)V_2(\varepsilon)\gamma^{2/p}S_\infty^{2(q-1)}}{I^{2/q}} \\ &\leq \frac{\varepsilon^2\nu_1I^{4/p-2}}{2} + \frac{V_1(\varepsilon)\nu_2}{2I^{2/q}}; \\ \left| \frac{x^2\psi_x(t, x)\psi(t, x)\phi'_q(y)\phi_p(x)}{I^2} \right| &= \left| \frac{(q-1)x^2\psi(t, x)\psi_x(t, x)\phi_p(x)|y|^{q-2}}{I^2} \right| \\ &\leq \frac{q|x|^{p+2}|y|^{q-2}\varepsilon^2}{4\gamma^{4/p}I^2} + \frac{qM^{p+1}V_1(\varepsilon)V_2(\varepsilon)|y|^{q-2}}{I^2} \\ &\leq \frac{\varepsilon^2}{4}qI^{4/p-2}C_\infty^{p+2}S_\infty^{q-2} \\ &\quad + \frac{qM^{p+2}V_1(\varepsilon)V_2(\varepsilon)S_\infty^{q-2}\gamma^{1-2/q}}{I^{1+2/q}} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{\varepsilon^2 \eta_1}{2} I^{4/p-2} + \frac{V_1(\varepsilon) V_2(\varepsilon) \eta_2}{2I^{1+2/q}}; \\
&\left| \frac{x^2 \psi_x^2(t, x) \phi_q^2(y)}{I^2} \right| \\
&\leq \frac{|x|^2 |y|^{2(q-1)} \varepsilon^2}{4\gamma^{4/p} I^2} + \frac{M^2 V_2(\varepsilon) |y|^{2(q-1)}}{I^2} \\
&\leq \frac{\varepsilon^2}{4} I^{4/p-2} C_\infty^2 S_\infty^{2(q-1)} + \frac{M^2 V_2(\varepsilon) \gamma^{2/p} S_\infty^{2(q-1)}}{I^{2/q}} \\
&\leq \frac{\varepsilon^2 \nu_1 I^{4/p-2}}{2} + \frac{M \nu_2}{2I^{2/q}}.
\end{aligned} \tag{37}$$

Then, given $\bar{I} > 0$, choosing I_0 so that $I_0 \geq \bar{I}$, by using Corollary 10, provided

$$\begin{aligned}
I^{1/q}(t) \geq \max \left\{ \left(\frac{\eta_2}{\varepsilon} \right)^{1/(q+2)} ; \left(\frac{V_1(\varepsilon) \nu_2}{\varepsilon} \right)^{1/2} ; \right. \\
\left. \left(\frac{V_1(\varepsilon) V_2(\varepsilon) \eta_2}{\varepsilon} \right)^{1/(q+2)} ; \left(\frac{M \nu_2}{\varepsilon} \right)^{1/2} ; 1 \right\},
\end{aligned} \tag{38}$$

we have

$$\begin{aligned}
&\left| \frac{x \psi^2(t, x) \phi_q'(y) \phi_p(x)}{I^2} \right| \leq \frac{\eta_1 \varepsilon^2}{2} + \frac{\varepsilon}{2}; \\
&\left| \frac{x \psi(t, x) \psi_x(t, x) \phi_q^2(y)}{I^2} \right| \leq \frac{\nu_1 \varepsilon^2}{2} + \frac{\varepsilon}{2}; \\
&\left| \frac{x^2 \psi_x(t, x) \psi(t, x) \phi_q'(y) \phi_p(x)}{I^2} \right| \leq \frac{\eta_1 \varepsilon^2}{2} + \frac{\varepsilon}{2}; \\
&\left| \frac{x^2 \psi_x^2(t, x) \phi_q^2(y)}{I^2} \right| \leq \frac{\nu_1 \varepsilon^2}{2} + \frac{\varepsilon}{2}.
\end{aligned} \tag{39}$$

Since $\varepsilon > 0$ is arbitrary, the proof of (iii) is finished. \square

$\forall t \in [0, 2\pi_p]$, set

$$\begin{aligned}
b_1(t) &= \frac{\partial \Phi_1}{\partial I} = \frac{-x [(p-1)\psi(t, x) - x\psi_x(t, x)]}{p^2 I^2}, \\
b_2(t) &= \frac{\partial \Phi_1}{\partial \theta} = \frac{-\phi_q(y)}{\omega p I} [\psi(t, x) + x\psi_x(t, x)], \\
b_3(t) &= \frac{\partial \Phi_2}{\partial \theta} \\
&= \frac{[\lambda \phi_p(x^+) - \mu \phi_p(x^-)] \phi_q'(y) \psi(t, x)}{\omega^2} \\
&\quad - \frac{\phi_q^2(y) \psi_x(t, x)}{\omega^2}.
\end{aligned} \tag{40}$$

As a result of Lemma 9, Corollary 10, and Lemma 12, we have the following.

Lemma 13. $\forall t, s \in [0, 2\pi_p]$, the following conclusions hold:

- (i) $b_1(t) = o(1/I_0)$, as $I_0 \rightarrow +\infty$.
- (ii) $b_2(t) = o(1)$, as $I_0 \rightarrow +\infty$.
- (iii) $b_1(t) \cdot b_3(s) = o(1)$, as $I_0 \rightarrow +\infty$.

Let us consider the variational equation of (17) with respect to the initial value I_0 . One can verify that

$$\begin{aligned}
\dot{\theta}_{I_0} &= b_1(t) \frac{\partial I}{\partial I_0} + b_2(t) \frac{\partial \theta}{\partial I_0}, \\
\dot{I}_{I_0} &= -b_2(t) \frac{\partial I}{\partial I_0} + b_3(t) \frac{\partial \theta}{\partial I_0}.
\end{aligned} \tag{41}$$

Lemma 14. For all $t \in (0, 2\pi_p]$, $I_0 \rightarrow +\infty$, one has

- (i) $\theta_{I_0}(t; \theta_0, I_0) \rightarrow 0$;
- (ii) $I_{I_0}(t; \theta_0, I_0) = 1 + o(1)$;
- (iii) $\theta_{\theta_0}(t; \theta_0, I_0) = 1 + o(1)$.

Proof. From variational equations (41) and Lemma 13, one has

$$\begin{aligned}
\theta_{I_0}(t) &= e^{\int_0^t b_2(s) ds} \cdot \int_0^t e^{-\int_0^s b_2(t) dt} b_1(s) \cdot I_{I_0}(s) ds \\
&= (1 + o(1)) \int_0^t b_1(s) \cdot I_{I_0}(s) ds, \\
I_{I_0}(t) &= e^{-\int_0^t b_2(s) ds} \cdot \left(1 + \int_0^t e^{\int_0^s b_2(t) dt} b_3(s) \cdot \theta_{I_0}(s) ds \right) \\
&= 1 + o(1) \\
&\quad + (1 + o(1)) \int_0^t b_3(s) \cdot \left(\int_0^s b_1(t) \cdot I_{I_0}(t) dt \right) ds \\
&= 1 + o(1) + o(1) \int_0^t \int_0^s I_{I_0}(t) dt ds,
\end{aligned} \tag{42}$$

and here we have used $\theta_{I_0}(0) = 0$ and $I_{I_0}(0) = 1$.

Hence, for all $t \in (0, 2\pi_p]$, $I_0 \rightarrow +\infty$, we have $I_{I_0}(t) = 1 + o(1)$ and $\theta_{I_0}(t) = (1 + o(1)) \int_0^t b_1(s) ds \rightarrow 0$. Thus, (i) and (ii) are proved.

To prove (iii), we consider the variational equation of (17) about θ_0 ; one can get

$$\begin{aligned}
\dot{\theta}_{\theta_0} &= b_1(t) \frac{\partial I}{\partial \theta_0} + b_2(t) \frac{\partial \theta}{\partial \theta_0}, \\
\dot{I}_{\theta_0} &= -b_2(t) \frac{\partial I}{\partial \theta_0} + b_3(t) \frac{\partial \theta}{\partial \theta_0}.
\end{aligned} \tag{43}$$

By using a similar argument in (ii), we can also show that $\theta_{\theta_0}(t; \theta_0, I_0) = 1 + o(1)$, $\forall t \in (0, 2\pi_p]$, as $I_0 \rightarrow +\infty$. This completes the proof of Lemma 14. \square

Next, we will develop an estimate of upper bound and lower bound for $b_1(t)$.

Lemma 15. Let $d \geq 0$ satisfy (A_2) .

(i) If $|x(t; \theta_0, I_0)| \leq d$ for all $t \in [0, 2\pi_p]$, then there exists a constant $E_d > 0$, such that $|b_1(t)| \leq E_d/I^2(t)$.

(ii) If $|x(t; \theta_0, I_0)| \geq d$ for all $t \in [0, 2\pi_p]$, then there exists a constant $F_d > 0$, such that $|b_1(t)| \geq F_d/I^2(t)$. Moreover, if $|x(t; \theta_0, I_0)| \geq d$, then $b_1(t) < 0$.

Proof. (i) If $|x(t; \theta_0, I_0)| \leq d$, take $M_d = \max_{|x| \leq d, t \in [0, 2\pi_p]} |(p-1)\psi(x, t) - x\psi_x(x, t)|$. Then

$$\begin{aligned} |b_1(t)| &= \left| \frac{-x[(p-1)\psi(t, x) - x\psi_x(t, x)]}{p^2 I^2} \right| \\ &\leq \frac{|x| M_d}{p^2 I^2(t)} \leq \frac{d M_d}{p^2 I^2(t)}. \end{aligned} \tag{44}$$

Writing $E_d = dM_d/p^2$, we have $|b_1(t)| \leq E_d/I^2(t)$.

(ii) If $|x(t; \theta_0, I_0)| \geq d$, with condition (A_2) , it is easy to know that $(p-1)x\psi(t, x) - x^2\psi_x(t, x) > \beta x$ when $x \geq d$ and $(p-1)x\psi(t, x) - x^2\psi_x(t, x) > -\beta x$ when $x \leq -d$. Hence, $b_1(t) < 0$ and

$$\begin{aligned} |b_1(t)| &= \left| \frac{-x[(p-1)\psi(t, x) - x\psi_x(t, x)]}{p^2 I^2} \right| \\ &\geq \frac{\beta |x|}{p^2 I^2(t)} \geq \frac{d\beta}{p^2 I^2(t)}. \end{aligned} \tag{45}$$

Therefore, setting $F_d = d\beta/p^2$, we obtain $|b_1(t)| \geq F_d/I^2(t)$. The proof is complete. \square

Let $b_1(t) = b_1^+(t) - b_1^-(t)$ with $b_1^+(t) = \max\{\pm b_1(t), 0\}$. To estimate that the integral of $b_1^+(t)$ on $[0, 2\pi]$ is smaller than the integral of $b_1^-(t)$ on $[0, 2\pi]$, we need the following lemma.

Lemma 16. Let $d \geq 0$ be as in Theorem 1. Define $\Delta t = \{t \in [0, 2\pi_p] \mid |x(t; \theta_0, I_0)| \leq d\}$. Then there exist $\bar{I}_0 > 0, T > 0$, such that

$$|\Delta t| \leq \frac{T}{I_0^{1/q}}, \tag{46}$$

for all $I_0 \geq \bar{I}_0$.

Proof. According to Lemma 11, we see that $\Delta t \rightarrow 0$ if and only if $\Delta\theta \rightarrow 0$.

By the action-angle variables transformation (16), it is not difficult to verify that there exists $\tau > 0$ such that $|\tan \Delta\theta| \leq \tau d/I^{1/q}(t)$ when $\Delta\theta \rightarrow 0$. Therefore, by using Corollary 10, we know that there exist $\bar{I}_0 > 0, T > 0$, such that

$$|\Delta t| \leq \frac{T}{I_0^{1/q}}, \tag{47}$$

for all $I_0 \geq \bar{I}_0$. Thus, we prove Lemma 16. \square

The next lemma gives the estimates of $\partial\theta(2\pi_p; \theta_0, I_0)/\partial I_0$ for $I_0 \gg 1$.

Lemma 17. For $I_0 \gg 1$, one gets $\theta_{I_0}(2\pi_p) < 0$.

Proof. The following results immediately from Corollary 10, Lemma 15, and Lemma 16:

$$\begin{aligned} \theta_{I_0}(2\pi_p) &= (1 + o(1)) \int_0^{2\pi_p} b_1(s) ds \\ &= (1 + o(1)) \left(\int_{b_1(t) \leq 0} b_1(s) ds + \int_{b_1(t) \geq 0} b_1(s) ds \right) \\ &\leq -(1 + o(1)) \left(\frac{F_d}{\rho_2^2 I_0^2} (2\pi_p - |\Delta t|) - \frac{E_d}{\rho_1^2 I_0^2} |\Delta t| \right) \\ &\leq (1 + o(1)) \left(\frac{(\rho_1^2 F_d + \rho_2^2 E_d) T}{\rho_1^2 \rho_2^2 I_0^{2+1/q}} - \frac{2\pi_p F_d}{\rho_2^2 I_0^2} \right). \end{aligned} \tag{48}$$

So, if $I_0^{1/q} > (\rho_1^2 F_d + \rho_2^2 E_d) T / 2\pi_p L_d \rho_1^2$, we have $\theta_{I_0}(2\pi_p) < 0$. \square

4. Proof of Theorem 1

Now we start to give the proof of Theorem 1.

Proof of Theorem 1. Based on Lemma 17 and the Aubry-Mather theorem [37], we can see that the Poincaré map P of system (17) is a monotone twist map when $I_0 \gg 1$. At last, using similar arguments as in [37], we may broaden the Poincaré map P to a new map \hat{P} which is a whole monotone twist homeomorphism on the cylinder $\mathbf{S}^1 \times \mathbb{R}$ and agree with P on $\mathbf{S}^1 \times [I_0, +\infty)$ with a fixed constant $I_0 \gg 1$. Hence, the existence of Aubry-Mather sets M_σ of \hat{P} is ensured by the Aubry-Mather theorem due to Pei [37]. Moreover, for some small $\varepsilon_0 > 0$, all those Aubry-Mather sets with rotation number $\alpha \in (2\omega\pi_p, 2\omega\pi_p + \varepsilon_0)$ lie in the domain $\mathbf{S}^1 \times [I_0, +\infty)$. Therefore, they happen to be the Aubry-Mather sets of the Poincaré map of P . From the above discussions, we have showed the existence of Aubry-Mather sets; this implies that (1) has an Aubry-Mather type solution $u_\alpha(t) = (x_\alpha(t), x'_\alpha(t))$ with rotation number α . This completes the proof of Theorem 1. \square

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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