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Research Article

Fixed Point Theorems for Generalized α_s-ψ-Contractions with Applications

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We study the sufficient conditions for the existence of a unique common fixed point of generalized α_s-ψ-Geraghty contractions in an α_s-complete partial b-metric space. We give an example in support of our findings. Our work generalizes many existing results in the literature. As an application of our findings we demonstrate the existence of the solution of the system of elliptic boundary value problems.

1. Introduction and Preliminaries

The Banach contraction mapping principle is very important in modern mathematics. For decades, several authors have studied existence of fixed points by contraction mappings, such as fuzzy mappings and others, and also get some important results, for details we can see [1–5, 5–12]. Now the theory of fixed point has been applied in many applied mathematics [13, 14] besides integral equations and differential equations [15]. For decades, people have done a lot of research on this issue and got a lot of important results [16–20].

As is well known, the existence of the solution of boundary value problems is an important of differential equations. In this paper we study the sufficient conditions for the existence of a unique common fixed point of generalized α_s-ψ-Geraghty contractions in an α_s-complete partial b-metric space. As an application of our findings we demonstrate the existence of the solution of the system of elliptic boundary value problems.

We first give some conceptions of this paper. In 1973, Geraghty studied a generalization of Banach contraction principle. In 2013, Cho introduced the notion of α-Geraghty contractive type mappings and established some unique fixed point theorems for such mappings in complete metric spaces. Popescu defined the concept of triangular α-orbital admissible mappings and proved the unique fixed point theorems for the mentioned mappings which are generalized α-Geraghty contraction type mappings. On the other hand, Karapinar proved the existence of a unique fixed point theorem for a triangular α-admissible mapping which is a generalized α-ψ-Geraghty contraction type mapping. Shukla [21] introduced the concept of partial b-metric space and established some fixed point theorems. We have Figure 1 where arrows stand for inclusions. The inverse inclusions do not hold.

In this paper, we introduce the notion of generalized α_s-ψ-Geraghty contraction type mappings and develop some new common fixed point theorems for such mappings in an α_s-complete partial b-metric space. An example and an application are given to support the theory.

We denote the set of natural numbers, rational numbers, $(-\infty,+\infty)$, $(0,+\infty)$, and $[0,+\infty)$ by $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{R}_0^+$, respectively.

First we recall some definitions and properties of a partial b-metric space.

Shukla generalized the notion of b-metric, as follows.

Definition 1 (see [21]). Let X be a nonempty set and $s \geq 1$ be a real number. A mapping $p_b : X \times X \longrightarrow \mathbb{R}_0^+$ is said

$$\begin{tikzcd} \text{metric space} \arrow[r] \arrow[d] & \text{b-metric space} \arrow[d] \\ \text{partial metric space} \arrow[r] & \text{partial b-metric space} \end{tikzcd}$$

FIGURE 1

to be a partial b-metric if it satisfies following axioms, for all $x, y, z \in X$,

(p_b1) $x = y$ if and only if $p_b(x, y) = p_b(x, x) = p_b(y, y)$

(p_b2) $p_b(x, x) \leq p_b(x, y)$

(p_b3) $p_b(x, y) = p_b(y, x)$;

(p_b4) $p_b(x, y) \leq s[p_b(x, z) + p_b(z, y)] - p_b(z, z)$.

The triplet (X, p_b, s) is called a partial b-metric space.

Remark 1. The self-distance $p_b(x, x)$, referred to the size or weight of x, is a feature used to describe the amount of information contained in x.

Remark 2. Obviously, every partial metric space is a partial b-metric space with coefficient $s = 1$ and every b-metric space is a partial b-metric space with zero self-distance. However, the converse of this fact needs not to hold.

Example 3. Let $X = \mathbb{R}^+$ and $k > 1$; the mapping $p_b : \mathrm{X} \times X \longrightarrow \mathbb{R}^+$ defined by

$$p_b(x, y) = \left\{(x \vee y)^k + |x - y|^k\right\} \quad \text{for all } x, y \in X \tag{1}$$

is a partial b-metric on X with $s = 2^k$. For $x = y$, $p_b(x, x) = x^k \neq 0$, so p_b is not a b-metric on X.

Let $x, y, z \in X$ such that $x > z > y$. Then following inequality always holds:

$$(x - y)^k > (x - z)^k + (z - y)^k. \tag{2}$$

Since $p_b(x, y) = x^k + (x - y)^k$ and $p_b(x, z) + p_b(z, y) - p_b(z, z) = x^k + (x - z)^k + (z - y)^k$,

$$p_b(x, y) > p_b(x, z) + p_b(z, y) - p_b(z, z). \tag{3}$$

This shows that p_b is not a partial metric on X.

Example 4 (see [21]). Let X be a nonempty set and p be a partial metric defined on X. The mapping $p_b : X \times X \longrightarrow \mathbb{R}^+$ defined by

$$p_b(x, y) = [p(x, y)]^q \quad \text{for all } x, y \in X \text{ and } q > 1 \tag{4}$$

defines a partial b-metric.

Definition 5. Let (X, p_b, s) be a partial b-metric space. The mapping $d_{p_b} : X \times X \longrightarrow \mathbb{R}_0^+$ defined by

$$d_{p_b}(x, y) = 2p_b(x, y) - p_b(x, x) - p_b(y, y) \quad \text{for all } x, y \in X \tag{5}$$

defines a metric on X, called induced metric.

In partial b-metric space (X, p_b, s), we immediately have a natural definition for the open balls:

$$B_{p_b}(x; \epsilon) = \{y \in X \mid p_b(x, y) < p_b(x, x) + \epsilon\} \quad \text{for all } x \in X. \tag{6}$$

Remark 6. The open balls in a partial b-metric space (X, p_b, s) may not be open set.

The following example justifies Remark 6.

Example 7. Let $X = \{a, b, c\}$ and define p_b as follows: $p_b(a, a) = p_b(c, c) = 1$, $p_b(b, b) = 1/2$, $p_b(a, b) = p_b(b, a) = 3$, $p_b(a, c) = p(c, a) = 3/2$, $p_b(b, c) = p_b(c, b) = 1$. Then p is a partial b-metric, $c \in B_{p_b}(a; 1)$ but, for any $r > 0$, $B_{p_b}(c; r)$ does not lie in $B_{p_b}(a; 1)$. This implies that $B_{p_b}(a; 1)$ is not an open set in (X, p_b, s).

Definition 8. Let (X, p_b, s) be a partial b-metric space.

(1) A sequence $\{x_n\}_{n \in \mathbb{N}}$ in (X, p_b, s) is called a Cauchy sequence if $\lim_{n,m \longrightarrow \infty} p_b(x_n, x_m)$ exists and is finite.

(2) A partial b-metric space (X, p_b, s) is said to be complete if every Cauchy sequence $\{x_n\}_{n \in \mathbb{N}}$ in X converges with respect to topology induced by its convergence, to a point $\upsilon \in X$ such that

$$p_b(x, x) = \lim_{n,m \longrightarrow \infty} p_b(x_n, x_m). \tag{7}$$

Lemma 9 (see [21]). *Let (X, p_b, s) be a partial b-metric space. Then*

(1) *every Cauchy sequence in (X, d_{p_b}) is also a Cauchy sequence in (X, p_b, s) and vice versa;*

(2) *a partial b-metric (X, p_b, s) is complete if and only if the metric space (X, d_{p_b}) is complete;*

(3) *a sequence $\{x_n\}_{n \in \mathbb{N}}$ in X converges to a point $\upsilon \in X$ if and only if*

$$\lim_{n \longrightarrow \infty} p_b(\upsilon, x_n) = p_b(\upsilon, \upsilon) = \lim_{n \longrightarrow \infty} p_b(x_n, x_m). \tag{8}$$

Remark 10. We know that in a metric space limit of a convergent sequence is always unique but in a partial b-metric space the limit of a convergent sequence may not be unique. Indeed, if $X = \mathbb{R}^+$, let $\sigma > 0$ be any constant. Define $p_b : X \times X \longrightarrow \mathbb{R}^+$ by $p_b(x, y) = x \vee y + \sigma$ for all $x, y \in X$, then (X, p_b, s) is a partial b-metric space with arbitrary coefficient $s \geq 1$. Define the sequence $\{x_n\}$ in X by $x_n = \rho$ for all $n \in \mathbb{N}$. One can note that if $y \geq \rho$ then $p_b(x_n, y) = y + \sigma = p_b(y, y)$; thus $\lim_{n \longrightarrow \infty} p_b(x_n, y) = p_b(y, y)$ for all $y \geq \rho$. Hence, the limit of a convergent sequence is not unique.

Definition 11 (see [22]). Let $T : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow \mathbb{R}_0^+$ be two mappings. We say that T is α-admissible if $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$.

Definition 12 (see [22]). Let $T : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow \mathbb{R}_0^+$ be two mappings. Then T is said to be triangular α-admissible if T satisfies the following conditions:

(*T*1) T is α-admissible.

(*T*2) $\alpha(x,u) \geq 1$ and $\alpha(u,y) \geq 1$ imply $\alpha(x,y) \geq 1$.

Definition 13 (see [23]). Let $T : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow \mathbb{R}_0^+$ be two mappings. Then T is said to be α-orbital admissible if

(*T*3) $\alpha(x,Tx) \geq 1$ implies $\alpha(Tx,T^2x) \geq 1$.

Definition 14 (see [23]). Let $T : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow \mathbb{R}_0^+$ be two mappings. Then T is said to be triangular α-orbital admissible if T is α-orbital admissible and

(*T*4) $\alpha(x,y) \geq 1$ and $\alpha(y,Ty) \geq 1$ imply $\alpha(x,Ty) \geq 1$.

Let Ω denote the class of all mappings $\beta : \mathbb{R}_0^+ \longrightarrow [0,1/s)$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \longrightarrow 1/s$ implies $t_n \longrightarrow 0$.

We let Ψ denote the class of the functions $\psi : [0,\infty) \longrightarrow \mathbb{R}_0^+$ satisfying the following conditions:

(1) ψ is nondecreasing.

(2) ψ is continuous.

(3) $\psi(t) = 0$ if and only if $t = 0$.

2. Main Results

Throughout this paper we let $X = (X, p_b, s)$ be a partial b-metric space, $\alpha_s : X \times X \longrightarrow \mathbb{R}_0^+$ be a mapping, and

$$M(x,y) = \max\left\{p_b(x,y), p_b(x,Sx), p_b(y,Ty), \frac{p_b(x,Ty) + p_b(y,Sx)}{2s}\right\}. \tag{9}$$

Definition 15. The space (X, p_b, s) is said to be α_s-complete if every Cauchy sequence $\{x_n\}$ in X satisfying $\alpha_s(x_n, x_{n+1}) \geq s^2$ for all $n \in \mathbb{N}$ converges in X.

Remark 16. If X is a complete partial b-metric space, then X is also an α_s-complete partial b-metric space but the converse is not true. The following example explains this fact.

Example 17. Let $X = (0,\infty)$ and the partial b-metric $p_b : X \times X \longrightarrow [0,\infty)$ be defined by $p_b(x,y) = (x \vee y)^2$, for all $x, y \in X$. Define $\alpha_2 : X \times X \longrightarrow [0,\infty)$ by

$$\alpha_2(x,y) = \begin{cases} 4e^{|x-y|} & \text{if } x, y \in [1,3]; \\ 0 & \text{otherwise}. \end{cases} \tag{10}$$

It is easy to see that $(X, p_b, 2)$ is not a complete partial b-metric space, but $(X, p_b, 2)$ is an α_2-complete partial b-metric space. Indeed, if $\{x_n\}$ is a Cauchy sequence in X such that $\alpha_2(x_n, x_{n+1}) \geq 4$, for all $n \in \mathbb{N}$, then $x_n \in [1,3]$, for all $n \in \mathbb{N}$. Since $[1,3]$ is a closed subset of $\mathbb{R}$, we see that $([1,3], p_b, 2)$ is a complete partial b-metric space and then there exists $x \in [1,3]$ such that $x_n \longrightarrow x$ as $n \longrightarrow \infty$.

Definition 18. Let $(X, \preceq)$ be a partially ordered set. Two mappings $S, T : X \longrightarrow X$ are said to be weakly increasing mappings, if $S(x) \preceq TS(x)$ and $T(y) \preceq ST(y)$ hold for all $x, y \in X$.

Example 19. Let $X = \mathbb{R}^+$. Define $S, T : X \longrightarrow X$ by

$$S(x) = \begin{cases} x^{1/2} & \text{if } x \in [0,1] \\ x^2 & \text{if } x \in (1,\infty) \end{cases}$$
$$\text{and } T(x) = \begin{cases} x & \text{if } x \in [0,1] \\ 2x & \text{if } x \in (1,\infty) \end{cases} \tag{11}$$

Then, S, T are weakly increasing mappings.

Definition 20. The self-mappings $S, T : X \longrightarrow X$ are said to be α_s-orbital admissible if the following condition holds.

$\alpha_s(x,Sx) \geq s^2$ and $\alpha_s(x,Tx) \geq s^2$ imply $\alpha_s(Sx,TSx) \geq s^2$ and $\alpha_s(Tx,STx) \geq s^2$.

We note that Definitions 13 and 18 are particular cases of Definition 20 (set $S = T$ and define $\alpha_s(x,y) \geq s^2$ whenever $x \preceq y$ or $y \preceq x$, respectively, in Definition 20).

Definition 21. Let $S, T : X \longrightarrow X$ be two mappings. The pair (S, T) is said to be triangular α_s-orbital admissible, if

(i) the self-mappings S, T are α_s-orbital admissible,

(ii) $\alpha_s(x,y) \geq s^2$, $\alpha_s(y,Sy) \geq s^2$ and $\alpha_s(y,Ty) \geq s^2$ imply $\alpha_s(x,Sy) \geq s^2$ and $\alpha_s(x,Ty) \geq s^2$.

Example 22. Let $M = \mathbb{R}_0^+$ and $p_b(r_1, r_2) = (r_1 \vee r_2)^2$ for all $r_1, r_2 \in M$ be a partial b-metric with $s = 2$:

$$S(r) = \begin{cases} r & \text{if } r \in [0,1); \\ 1 & \text{if } r \in [1,\infty), \end{cases}$$
$$T(r) = \begin{cases} r^{1/3} & \text{if } r \in [0,1); \\ 1 & \text{if } r \in [1,\infty). \end{cases} \tag{12}$$

Define $\alpha_s : M \times M \longrightarrow \mathbb{R}_0^+$ by

$$\alpha_s(r_1, r_2) = \begin{cases} 4 + r_2 - r_1 & \text{if } r_1, r_2 \in [0,1); \\ 0 & \text{if } r_1, r_2 \in [1,\infty). \end{cases} \tag{13}$$

Then it is easy to show that the mappings S, T satisfy conditions (i) and (ii) in Definition 21.

Lemma 23. *Let $S, T : X \longrightarrow X$ be two mappings such that the pair (S, T) is triangular α_s-orbital admissible. Assume that there exists $x_0 \in X$ such that $\alpha_s(x_0, Sx_0) \geq s^2$. Define a sequence $\{x_n\}$ in X by $x_{2i+1} = S(x_{2i})$ and $x_{2i+2} = T(x_{2i+1})$, where $i = 0, 1, 2, \ldots$. Then for $n, m \in \mathbb{N} \cup \{0\}$ with $m > n$, we have $\alpha_s(x_n, x_m) \geq s^2$.*

Proof. Since $\alpha_s(x_0, Sx_0) = \alpha_s(x_0, x_1) \geq s^2$ and S, T are α_s-orbital admissible self-mappings,

$$\alpha_s(x_0, Sx_0) \geq s^2 \text{ implies}$$
$$\alpha_s(Sx_0, TSx_0) = \alpha_s(x_1, Tx_1) = \alpha_s(x_1, x_2) \geq s^2$$
$$\alpha_s(x_1, Tx_1) \geq s^2 \text{ implies} \tag{14}$$
$$\alpha_s(Tx_1, STx_1) = \alpha_s(x_2, Sx_2) = \alpha_s(x_2, x_3) \geq s^2$$
$$\alpha_s(x_2, Sx_2) \geq s^2 \text{ implies}$$
$$\alpha_s(Sx_2, TSx_2) = \alpha_s(x_3, Tx_3) = \alpha_s(x_3, x_4) \geq s^2. \tag{15}$$

Applying the above argument repeatedly, we obtain $\alpha_s(x_n, x_m) \geq s^2$ for all $n, m \in \mathbb{N} \cup \{0\}$ with $m = n+1$. Since S, T are triangular α_s-orbital admissible mappings, $\alpha_s(x_n, x_m) \geq s^2$ for all $n, m \in \mathbb{N} \cup \{0\}$ with $m > n$. ☐

Definition 24. We say the self-mapping $S : X \longrightarrow X$ is an α_s-p_b-continuous mapping if whenever $\{x_n\}$ is a sequence in X with $\alpha_s(x_n, x_{n+1}) \geq s^2$ for all $n \in \mathbb{N}$ and $x \in X$ such that $\lim_{n\longrightarrow\infty} p_b(x_n, x) = 0$, then $\lim_{n\longrightarrow\infty} p_b(S(x_n), S(x)) = 0$.

Now, we introduce the concept of generalized α_s-ψ-Geraghty contractions as follows.

Definition 25. The self-mappings S, T defined on (X, p_b, s) are called generalized α_s-ψ-Geraghty contractions with respect to p_b, if there exist $\beta \in \Omega$, $\psi \in \Psi$, and $L \geq 0$ such that

$$\begin{aligned} &\psi(\alpha_s(x, y) p_b(Sx, Ty)) \\ &\quad \leq \beta(\psi(M(x, y))) . \psi(M(x, y)) \\ &\qquad + L\psi(\min\{p_b(x, Sx), p_b(y, Sx)\}), \end{aligned} \tag{16}$$

for $x, y \in X$ satisfying $\alpha_s(x, y) \geq s^2$.

The main result of this section is given by the following:

Theorem 26. *Let (X, p_b, s) be an α_s-complete partial b-metric space. Let $S, T : X \longrightarrow X$ be generalized α_s-ψ-Geraghty contractions satisfying the following conditions:*

(i) *there exists $x_0 \in X$ such that $\alpha_s(x_0, Sx_0) \geq s^2$;*

(ii) *the mappings S, T are triangular α_s-orbital admissible;*

(iii)

(a) *the mappings S, T are α_s-p_b-continuous*

(b) *$\{x_n\}$ is a sequence in X such that $\alpha_s(x_n, x_{n+1}) \geq s^2$ for all $n \in \mathbb{N}$ and $x_n \longrightarrow x^* \in X$ as $n \longrightarrow \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha_s(x_{n(k)}, x^*) \geq s^2$ for all $k \in \mathbb{N}$.*

Then S and T have a common fixed point in X. In addition, if y^* is also a common fixed point of the pair (S, T) such that $\alpha_s(x^*, y^*) \geq s^2$, then $x^* = y^*$.

Proof. Firstly we prove that the self-mappings S, T have at most one common fixed point. Suppose that v and ω are two different common fixed points of S and T. Then $S(v) = v \neq \omega = T(\omega)$. It follows that $p_b(S(v), T(\omega)) = p_b(v, \omega) > 0$, $p_b(v, v) = 0$ and $p_b(\omega, \omega) = 0$. Since $\alpha_s(v, \omega) \geq s^2$, contractive condition (16) implies

$$\begin{aligned} \psi(p_b(v, \omega)) &= \psi(p_b(S(v), T(\omega))) \\ &\leq \psi(\alpha_s(v, \omega) p_b(S(v), T(\omega))) \\ &\leq \beta(\psi(M(v, \omega))) \cdot \psi(M(v, \omega)) \\ &\quad + L\psi(\min\{p_b(v, Sv), p_b(\omega, Sv)\}) \\ &< \psi(M(v, \omega)) = \psi(p_b(v, \omega)). \end{aligned} \tag{17}$$

which is a contradiction. Hence, the pair (S, T) has at most one common fixed point.

(a). By assumption (i) and Lemma 23, we have

$$\alpha_s(x_n, x_{n+1}) \geq s^2, \quad \text{for all } n \in \mathbb{N}. \tag{18}$$

For $i \in \mathbb{N}$, we have

$$\begin{aligned} 0 &< \psi(p_b(x_{2i+1}, x_{2i+2})) \\ &\leq \psi(\alpha_s(x_{2i}, x_{2i+1}) p_b(Sx_{2i}, Tx_{2i+1})) \\ &\leq \beta(\psi(M(x_{2i}, x_{2i+1}))) . \psi(M(x_{2i}, x_{2i+1})) \\ &\quad + L\psi(\min\{p_b(x_{2i}, Sx_{2i}), p_b(x_{2i+1}, Sx_{2i})\}), \end{aligned} \tag{19}$$

where

$$\begin{aligned} M(x_{2i}, x_{2i+1}) &= \max\left\{p_b(x_{2i}, x_{2i+1}), p_b(x_{2i}, Sx_{2i}), p_b(x_{2i+1}, Tx_{2i+1}), \frac{p_b(x_{2i}, Tx_{2i+1}) + p_b(x_{2i+1}, Sx_{2i})}{2s}\right\} \\ &= \max\left\{p_b(x_{2i}, x_{2i+1}), p_b(x_{2i}, x_{2i+1}), p_b(x_{2i+1}, x_{2i+2}), \frac{p_b(x_{2i}, x_{2i+2}) + p_b(x_{2i+1}, x_{2i+1})}{2s}\right\} \\ &\leq \max\left\{p_b(x_{2i}, x_{2i+1}), p_b(x_{2i}, x_{2i+1}), p_b(x_{2i+1}, x_{2i+2}), \frac{p_b(x_{2i}, x_{2i+1}) + p_b(x_{2i+1}, x_{2i+2})}{2s}\right\} \\ &= \max\{p_b(x_{2i}, x_{2i+1}), p_b(x_{2i+1}, x_{2i+2})\}. \end{aligned} \tag{20}$$

If $\max\{p_b(x_{2i}, x_{2i+1}), p_b(x_{2i+1}, x_{2i+2})\} = p_b(x_{2i+1}, x_{2i+2})$, then by (29) we have

$$\begin{aligned} &\psi(p_b(x_{2i+1}, x_{2i+2})) \\ &\quad \le \beta(\psi(p_b(x_{2i+1}, x_{2i+2}))).\psi(p_b(x_{2i+1}, x_{2i+2})) \\ &\quad < \psi(p_b(x_{2i+1}, x_{2i+2})), \end{aligned} \tag{21}$$

which is a contradiction. Thus we conclude that

$$\begin{aligned} &\max\{p_b(x_{2i}, x_{2i+1}), p_b(x_{2i+1}, x_{2i+2})\} \\ &\quad = p_b(x_{2i}, x_{2i+1}). \end{aligned} \tag{22}$$

By (29), we get that $\psi(p_b(x_{2i+1}, x_{2i+2})) < \psi(p_b(x_{2i}, x_{2i+1}))$. Since ψ is nondecreasing, we have

$$p_b(x_{2i+1}, x_{2i+2}) < p_b(x_{2i}, x_{2i+1}). \tag{23}$$

This implies that

$$p_b(x_{n+1}, x_{n+2}) < p_b(x_n, x_{n+1}), \quad \text{for all } n \in \mathbb{N}. \tag{24}$$

Hence, we deduce that the sequence $\{p_b(x_n, x_{n+1})\}$ is non-increasing. Therefore, there exists $r \ge 0$ such that $\lim_{n\to\infty} p_b(x_n, x_{n+1}) = r$. Now, we shall prove that $r = 0$. Suppose that $r > 0$. By (16), we have

$$\begin{aligned} &\psi(p_b(x_{n+1}, x_{n+2})) \\ &\quad \le \psi(\alpha_s(x_n, x_{n+1}) p_b(Sx_n, Tx_{n+1})) \\ &\quad \le \beta(\psi(M(x_n, x_{n+1}))).\psi(M(x_n, x_{n+1})) \\ &\qquad + L\psi(\min\{p_b(x_n, Sx_n), p_b(x_{n+1}, Sx_n)\}), \end{aligned} \tag{25}$$

which implies

$$\begin{aligned} &\psi(p_b(x_{n+1}, x_{n+2})) \\ &\quad \le \beta(\psi(p_b(x_n, x_{n+1}))).\psi(p_b(x_n, x_{n+1})). \end{aligned} \tag{26}$$

Hence

$$\frac{\psi(p_b(x_{n+1}, x_{n+2}))}{\psi(p_b(x_n, x_{n+1}))} \le \beta(\psi(p_b(x_n, x_{n+1}))) < 1. \tag{27}$$

This implies that $\lim_{n\to\infty} \beta(\psi(p_b(x_n, x_{n+1}))) = 1$. Since $\beta \in \Omega$, we have

$$\lim_{n\to\infty} \psi(p_b(x_n, x_{n+1})) = 0, \tag{28}$$

which yields

$$r = \lim_{n\to\infty} p_b(x_n, x_{n+1}) = 0, \tag{29}$$

a contradiction. Now, we claim that $\{x_n\}$ is a Cauchy sequence in (X, p_b, s). Suppose, on thw contrary, that $\{x_n\}$ is not a Cauchy sequence; that is, $\lim_{n,m\to\infty} p_b(x_n, x_m) \ne 0$. Then there exists $\epsilon > 0$ for which we can find two subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ such that n_k is the smallest index for which $n_k > m_k > k$,

$$p_b(x_{m_k}, x_{n_k}) \ge \epsilon. \tag{30}$$

This means that

$$p_b(x_{m_k}, x_{n_{k-1}}) < \epsilon. \tag{31}$$

By the triangle inequality, we have

$$\begin{aligned} \epsilon &\le p_b(x_{m_k}, x_{n_k}) \\ &\le s(p_b(x_{m_k}, x_{n_{k-1}}) + p_b(x_{n_{k-1}}, x_{n_k})) \\ &\quad - p_b(x_{n_{k-1}}, x_{n_{k-1}}) \\ &\le s(p_b(x_{m_k}, x_{n_{k-1}}) + p_b(x_{n_{k-1}}, x_{n_k})). \end{aligned} \tag{32}$$

Thus,

$$\frac{\epsilon}{s} \le p_b(x_{m_k}, x_{n_k}) < \epsilon + p_b(x_{n_{k-1}}, x_{n_k}) \tag{33}$$

for all $k \in \mathbb{N}$. In the view of (33) and (29), we have

$$\epsilon \le \lim_{k\to\infty} p_b(x_{m_k}, x_{n_k}) < s\epsilon. \tag{34}$$

Again by triangle inequality, we have

$$\begin{aligned} p_b(x_{m_k}, x_{n_k}) &\le s(p_b(x_{m_k}, x_{m_{k+1}}) + p_b(x_{m_{k+1}}, x_{n_k})) \\ &\quad - p_b(x_{m_{k+1}}, x_{m_{k+1}}) \\ &\le s(p_b(x_{m_k}, x_{m_{k+1}}) + p_b(x_{m_{k+1}}, x_{n_k})) \\ &\le sp_b(x_{m_k}, x_{m_{k+1}}) + s^2 p_b(x_{m_{k+1}}, x_{n_{k+1}}) \\ &\quad + s^2 p_b(x_{n_{k+1}}, x_{n_k}) - p_b(x_{n_{k+1}}, x_{n_{k+1}}) \\ &\le sp_b(x_{m_k}, x_{m_{k+1}}) + s^2 p_b(x_{m_{k+1}}, x_{n_{k+1}}) \\ &\quad + s^2 p_b(x_{n_{k+1}}, x_{n_k}) \end{aligned} \tag{35}$$

and

$$\begin{aligned} &p_b(x_{m_{k+1}}, x_{n_{k+1}}) \\ &\quad \le s(p_b(x_{m_{k+1}}, x_{m_k}) + p_b(x_{m_k}, x_{n_{k+1}})) \\ &\qquad - p_b(x_{m_k}, x_{m_k}) \\ &\quad \le sp_b(x_{m_{k+1}}, x_{m_k}) + sp_b(x_{m_k}, x_{n_{k+1}}) \\ &\quad \le sp_b(x_{m_{k+1}}, x_{m_k}) + s^2 p_b(x_{m_k}, x_{n_k}) \\ &\qquad + s^2 p_b(x_{n_k}, x_{n_{k+1}}) - p_b(x_{n_k}, x_{n_k}) \\ &\quad \le sp_b(x_{m_{k+1}}, x_{m_k}) + s^2 p_b(x_{m_k}, x_{n_k}) \\ &\qquad + s^2 p_b(x_{n_k}, x_{n_{k+1}}). \end{aligned} \tag{36}$$

By (29) and (34), we deduce that

$$\frac{\epsilon}{s^2} \le \lim_{k\to+\infty} p(x_{m_{k+1}}, x_{n_{k+1}}) < s^3\epsilon. \tag{37}$$

Also by application of triangle inequality, it follows that

$$\frac{\epsilon}{s} \le \lim_{k \longrightarrow \infty} p_b\left(x_{n_{k+1}}, x_{m_k}\right) \le s^2 \epsilon. \tag{38}$$

By Lemma 23, since $\alpha_s(x_{n_{k+1}}, x_{m_k}) \ge s^2$, we have

$$\begin{aligned} \frac{\varepsilon}{s} &= \max\left\{\frac{\varepsilon}{s}, \frac{s\varepsilon}{4}\right\} \le \lim_{k \longrightarrow \infty} \sup M\left(x_{n(k)+1}, x_{m(k)}\right) \\ &\le \max\left\{s^2\varepsilon, \frac{s^2\varepsilon}{4}\right\} = s^2\varepsilon. \end{aligned} \tag{39}$$

Similarly, we can show that

$$\begin{aligned} \frac{\varepsilon}{s} &= \max\left\{\frac{\varepsilon}{s}, \frac{s\varepsilon}{4}\right\} \le \lim_{k \longrightarrow \infty} \inf M\left(x_{n(k)+1}, x_{m(k)}\right) \\ &\le \max\left\{s^2\varepsilon, \frac{s^2\varepsilon}{4}\right\} = s^2\varepsilon. \end{aligned} \tag{40}$$

Thus, concluding above arguments we have

$$\begin{aligned} \psi\left(s^2\varepsilon\right) &\le \psi\Big(\alpha_s\left(x_{n(k)+1}, x_{m(k)}\right) \\ &\quad \cdot \lim_{k \longrightarrow \infty} \sup p_b\left(x_{n(k)+2}, x_{m(k)+1}\right)\Big) \\ &\le \beta\left(\psi\left(\lim_{k \longrightarrow \infty} \sup M\left(x_{n(k)+1}, x_{m(k)}\right)\right)\right) \\ &\quad \cdot \psi\left(\lim_{k \longrightarrow \infty} \sup M\left(x_{n(k)+1}, x_{m(k)}\right)\right) + 0 \\ &\le \beta\left(\psi\left(s^2\varepsilon\right)\right)\psi\left(s^2\varepsilon\right) < \psi\left(s^2\varepsilon\right), \end{aligned} \tag{41}$$

which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence in (X, p_b, s). Since (X, p_b, s) is an α_s-complete partial b-metric space, by Lemma 9(2), (X, d_{p_b}) is an α_s-complete b-metric space. Therefore, the sequence $\{x_n\}$ converges to some $x^* \in (X, d_{p_b})$. By Lemma 9(3), there exists $x^* \in X$ such that $\lim_{n\longrightarrow\infty} d_{p_b}(x_n, x^*) = 0$ if and only if

$$\lim_{n \longrightarrow \infty} p_b\left(x^*, x_n\right) = p_b\left(x^*, x^*\right) = \lim_{n,m \longrightarrow \infty} p_b\left(x_n, x_m\right). \tag{42}$$

Since $d_{p_b}(x, y) = 2p_b(x, y) - p_(x, x) - p_b(y, y)$, thus, considering (29) and axiom (p_b2) with $\lim_{n\longrightarrow\infty} d_{p_b}(x_n, x^*) = 0$, we conclude that

$$\lim_{n \longrightarrow \infty} p_b\left(x_n, x_m\right) = 0. \tag{43}$$

Combining (42) and (43), we have

$$\begin{aligned} \lim_{n \longrightarrow \infty} p_b\left(x^*, x_n\right) &= p_b\left(x^*, x^*\right) = \lim_{n,m \longrightarrow \infty} p_b\left(x_n, x_m\right) \\ &= 0. \end{aligned} \tag{44}$$

Now $\lim_{n\longrightarrow\infty} p_b(x^*, x_n) = 0$ implies that $\lim_{i\longrightarrow\infty} p_b(x_{2i+1}, x^*) = 0$ and $\lim_{i\longrightarrow\infty} p_b(x_{2i+2}, x^*) = 0$. As S and T are α_s-p_b-continuous mappings, we $\lim_{i\longrightarrow\infty} p_b(Sx_{2i+1}, Sx^*) = 0$. Thus

$$\begin{aligned} p_b\left(x^*, Sx^*\right) &= \lim_{i \longrightarrow \infty} p_b\left(x_{2i+2}, Sx^*\right) \\ &= \lim_{i \longrightarrow \infty} p_b\left(Sx_{2i+1}, Sx^*\right) = 0, \end{aligned} \tag{45}$$

and so $x^* = Sx^*$, and, similarly, $x^* = Tx^*$. Therefore S and T have a common fixed point $x^* \in X$.

(b). From (a) we know that the sequence $\{x_n\}$ in X defined by $x_{2i+1} = Sx_{2i}$ and $x_{2i+2} = Tx_{2i+1}$, where $i = 0, 1, 2, \ldots$ with $\alpha_s(x_n, x_{n+1}) \ge s^2$, for all $n \in \mathbb{N}$ converges to $x^* \in X$. There exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha_s(x_{n(k)}, x^*) \ge s^2$ for all k. Therefore,

$$\begin{aligned} \psi\left(p_b\left(x_{2n(k)+1}, Tx^*\right)\right) &= \psi\left(p_b\left(Sx_{2n(k)}, Tx^*\right)\right) \\ &\le \psi\left(\alpha_s\left(x_{2n(k)}, x^*\right) p_b\left(Sx_{2n(k)}, Tx^*\right)\right) \\ &\le \beta\left(\psi\left(M\left(x_{2n(k)}, x^*\right)\right)\right).\psi\left(M\left(x_{2n(k)}, x^*\right)\right) \\ &\quad + L\psi\left(\min\left\{p_b\left(x_{2n(k)}, Sx_{2n(k)}\right), p_b\left(x^*, Sx_{2n(k)}\right)\right\}\right), \end{aligned} \tag{46}$$

which implies

$$\begin{aligned} &\psi\left(p_b\left(x_{2n(k)+1}, Tx^*\right)\right) \\ &\le \beta\left(\psi\left(M\left(x_{2n(k)}, x^*\right)\right)\right).\psi\left(M\left(x_{2n(k)}, x^*\right)\right) \\ &\quad + L\psi\left(\min\left\{p_b\left(x_{2n(k)}, Sx_{2n(k)}\right), p_b\left(x^*, Sx_{2n(k)}\right)\right\}\right), \end{aligned} \tag{47}$$

where

$$\begin{aligned} M\left(x_{2n(k)}, x^*\right) &= \max\Bigg\{p_b\left(x_{2n(k)}, x^*\right), \\ &\quad p_b\left(x_{2n(k)}, Sx_{2n(k)}\right), p_b\left(x^*, Tx^*\right), \\ &\quad \frac{p_b\left(x_{2n(k)}, Sx^*\right) + p_b\left(x^*, Tx_{2n(k)}\right)}{2s}\Bigg\} \\ &\le \max\Bigg\{p_b\left(x_{2n(k)}, x^*\right), p_b\left(x_{2n(k)}, x_{2n(k)+1}\right), \\ &\quad p_b\left(x^*, Tx^*\right), \frac{p_b\left(x_{2n(k)}, Tx^*\right) + p_b\left(x^*, Sx_{2n(k)}\right)}{2s}\Bigg\}. \end{aligned} \tag{48}$$

Since

$$\begin{aligned} &\lim_{k \longrightarrow \infty} \sup \frac{p_b\left(x_{2n(k)}, Tx^*\right) + p_b\left(x^*, Sx_{2n(k)}\right)}{2s} \\ &\quad \le \frac{p_b\left(x^*, Tx^*\right) + p_b\left(x^*, x^*\right)}{2s}, \end{aligned} \tag{49}$$

then by letting $k \longrightarrow \infty$ we have $\lim_{k\longrightarrow\infty} M(x_{2n(k)}, x^*) = p_b(x^*, Tx^*)$. Suppose that $p_b(x^*, Tx^*) > 0$. By (47), we have

$$\frac{\psi\left(p_b\left(x_{2n(k)+1}, Tx^*\right)\right)}{\psi\left(M\left(x_{2n(k)}, x^*\right)\right)} \le \beta\left(\psi\left(M\left(x_{2n(k)}, x^*\right)\right)\right) < 1. \tag{50}$$

Letting $k \longrightarrow \infty$ in above inequality, we obtain that

$$\lim_{k \longrightarrow \infty} \beta\left(\psi\left(M\left(x_{2n(k)}, x^*\right)\right)\right) = 1. \tag{51}$$

So $\lim_{k\longrightarrow\infty} M(x_{2n(k)}, x^*) = 0$. Hence $p_b(x^*, Tx^*) = 0$, and due to (p_b1) and (p_b2) we obtain so $x^* = Tx^*$. Similarly we can show that $x^* = Sx^*$. Thus S and T have a common fixed point $x^* \in X$. □

Remark 27. We note that Theorem 26 is more general than the results established in [24–26].

Example 28. Let $X = [0, 1]$. Define a function $p_b : X \times X \longrightarrow [0, +\infty)$ by $p_b(x, y) = (x \vee y)^2 + (x - y)^2$. Clearly, (X, p_b, s) is a complete partial b-metric space with the constant $s = 4$. Let β be a function on $[0, \infty]$ defined by $\beta(t) = 1/(4 + t)$ for all $t \geq 0$. Then $\beta \in \Omega$. Also, ψ be a function on $[0, +\infty)$ defined by $\psi(t) = t/2$. Then $\psi \in \Psi$. Define the mappings $S, T : X \longrightarrow X$ by

$$T(x) = \begin{cases} \dfrac{2}{245}x, & \text{if } x \in \left[0, \dfrac{1}{2}\right] \\ 1, & \text{if } x \in \left(\dfrac{1}{2}, 1\right] \end{cases} \quad (52)$$

$$\text{and } S(x) = 0 \qquad \forall x \in X.$$

Also, we define the function $\alpha_s : X \times X \longrightarrow [0, \infty)$ by

$$\alpha_s(x, y) = \begin{cases} s^2, & \text{if } 0 \leq x, y \leq \dfrac{1}{2} \\ 0, & \text{otherwise.} \end{cases} \quad (53)$$

If $\{x_n\}$ is a Cauchy sequence such that $\alpha_s(x_n, x_{n+1}) \geq s^2$ for all $n \in \mathbb{N}$, then $\{x_n\} \subseteq [0, 1/2]$. Since $([0, 1/2], p_b)$ is a complete partial b-metric space, then the sequence $\{x_n\}$ converges in $[0, 1/2] \subseteq X$. Thus (X, p_b) is an α_s-complete partial b-metric space. Let $\alpha_s(x, Sx) \geq s^2$ and $\alpha_s(x, Tx) \geq s^2$, and thus $x \in [0, 1/2]$ and Sx, $Tx \in [0, 1/2]$ and so $\alpha_s(Sx, TSx) \geq s^2$ and $\alpha_s(Tx, STx) \geq s^2$. Thus, (S, T) is α_s-orbital admissible. Let $x, y \in X$ be such that $\alpha_s(x, y) \geq^2$, $\alpha_s(y, Sy) \geq 1$ and $\alpha_s(y, Ty) \geq s^2$. Then we have $x, y, Sy, Ty \in [0, 1/2]$, which implies that $\alpha_s(x, Sy) \geq s^2$ and $\alpha_s(x, Ty) \geq s^2$. Therefore (S, T) is triangular α_s-orbital admissible. Let $\{x_n\}$ be a Cauchy sequence such that $x_n \longrightarrow x$ as $n \longrightarrow \infty$ and $\alpha_s(x_n, x_{n+1}) \geq s^2$ for all $n \in \mathbb{N}$. Then $\{x_n\} \subseteq [0, 1/2]$ for all $n \in \mathbb{N}$. So $\lim_{n\longrightarrow\infty} Tx_n = \lim_{n\longrightarrow\infty}(2/245)x_n = (2/245)x = Tx$. Hence T is α_s-continuous. Similarly, we can show that S is α_s-continuous. Let $x_0 = 1/4$. Then

$$\alpha_s\left(\frac{1}{4}, S\left(\frac{1}{4}\right)\right) = \alpha_s\left(\frac{1}{4}, 0\right) \geq s^2. \quad (54)$$

Let $x, y \in X$ such that $\alpha_s(x, y) \geq s^2$. Then $x, y \in [0, 1/2]$ and hence

$$\begin{aligned} \frac{16}{245} &\leq \psi(\alpha_s(x, y)\, p_b(Sx, Ty)) \\ &\leq \beta(\psi(M(x, y))).\psi(M(x, y)) \\ &\quad + L\psi(\min\{p_b(x, Sx), p_b(y, Sx)\}) \leq \frac{1}{9}, \end{aligned} \quad (55)$$

with $L \geq 0$. Thus all conditions of Theorem 26 are satisfied. Hence S and T have a common fixed point ($x = 0$).

3. Consequences

Corollary 29. *Let (X, p_b, s) be an α_s-complete partial b-metric space. Assume that*

(i) *there exist $\beta \in \Omega$ and $L \geq 0$ such that, for all $x, y \in X$ with $\alpha_s(x, y) \geq s^2$ the self-mappings S, T satisfy the following inequality:*

$$\begin{aligned} &\alpha_s(x, y)\, p_b(Sx, Ty) \\ &\quad \leq \beta((M(x, y))).(M(x, y)) \\ &\qquad + L(\min\{p_b(x, Sx), p_b(y, Sx)\}); \end{aligned} \quad (56)$$

(ii) *S, T are triangular α_s-orbital admissible mappings;*

(iii) *there exists $x_0 \in X$ such that $\alpha_s(x_0, Sx_0) \geq s^2$*

(iv)

(a) *S and T are α_s-continuous mappings;*

(b) *$\{x_n\}$ is a sequence in X with $\alpha_s(x_n, x_{n+1}) \geq s^2$ for all $n \in \mathbb{N}$ such that $x_n \longrightarrow x^* \in X$ as $n \longrightarrow \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha_s(x_{n(k)}, x^*) \geq s^2$ for all $k \in \mathbb{N}$.*

Then S and T have a common fixed point $x^* \in X$. In addition, if y^* is also a common fixed point of the pair (S, T) such that $\alpha_s(x^*, y^*) \geq s^2$, then $x^* = y^*$.

Proof. Define $\psi : \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$ by $\psi(t) = t$ for all $t \in \mathbb{R}_0^+$. □

Corollary 30. *Let $Y = (Y, d, s)$ be an α_s-complete b-metric space. Let $S, T : Y \longrightarrow Y$ be a generalized α_s-ψ-Geraghty contractions with respect to d satisfying the following conditions:*

(i) *There exists $y_0 \in Y$ such that $\alpha_s(y_0, Sy_0) \geq s^2$.*

(ii) *The mappings S, T are triangular α_s-orbital admissible.*

(iii)

(a) *The mappings S, T are α_s-d-continuous.*

(b) *$\{y_n\}$ is a sequence in Y such that $\alpha_s(y_n, y_{n+1}) \geq s^2$ for all $n \in \mathbb{N}$ and $y_n \longrightarrow y^* \in Y$ as $n \longrightarrow \infty$, then there exists a subsequence $\{y_{n(k)}\}$ of $\{y_n\}$ such that $\alpha_s(y_{n(k)}, y^*) \geq s^2$ for all $k \in \mathbb{N}$.*

Then S and T have a common fixed point in Y. In addition, if y^* is also a common fixed point of the pair (S, T) such that $\alpha_s(x^*, y^*) \geq s^2$, then $x^* = y^*$.

Proof. Set $p_b(x, x) = 0$ for all $x \in X$ in Theorem 26. □

Corollary 31. *Let $(X, \preceq)$ be a partially ordered set and $(X, \preceq, p_b, s)$ be an ordered complete partial b-metric space. Assume that the weakly increasing mappings $S, T : X \longrightarrow X$ satisfy the following conditions:*

(i) *there exist $\beta \in \Omega\psi \in \Psi$ and $L \geq 0$ such that*

$$\begin{aligned} &\psi(s^2 p_b(Sx, Ty)) \\ &\quad \leq \beta(\psi(M(x, y))).\psi(M(x, y)) \\ &\qquad + L\psi(\min\{p_b(x, Sx), p_b(y, Sx)\}), \end{aligned} \quad (57)$$

for all comparable $x, y \in X$ (i.e. $x \preceq y$ or $y \preceq x$);

(ii) *there exists* $x_0 \in X$ *such that* $x_0 \preceq Sx_0$

(iii)

(a) *either S or T is continuous;*

(b) $\{x_n\}$ *is a nondecreasing sequence such that* $x_n \longrightarrow x^* \in X$ *as* $n \longrightarrow \infty$*, then there exists a subsequence* $\{x_{n(k)}\}$ *of* $\{x_n\}$ *such that* $x_{n(k)} \preceq x^*$ *for all* $k \in \mathbb{N}$.

Then S and T have a common fixed point $x^* \in X$*. In addition, if* y^* *is also a common fixed point of the pair* (S,T) *such that* $x^* \preceq y^*$*, then* $x^* = y^*$.

Proof. Define the relation $\preceq$ on X by

$$\alpha_s(x,y) = \begin{cases} s^2, & \text{if } x \preceq y \text{ or } y \preceq x \\ 0, & \text{otherwise.} \end{cases} \tag{58}$$

Proof follows from the proof of Theorem 26. □

Definition 32. The self-mappings T defined on (X, p_b, s) is called a generalized α_s-ψ-Geraghty contraction if there exist $\beta \in \Omega$, $\psi \in \Psi$, and $L \geq 0$ such that

$$\begin{aligned} &\psi(\alpha_s(x,y)\, p_b(Tx,Ty)) \\ &\quad \leq \beta(\psi(C(x,y))) .\psi(C(x,y)) \\ &\qquad + L\psi(\min\{p_b(x,Tx), p_b(y,Tx)\}), \end{aligned} \tag{59}$$

for $x, y \in X$ satisfying $\alpha_s(x,y) \geq s^2$, where

$$\begin{aligned} C(x,y) = \max\Bigg\{ & p_b(x,y), p_b(x,Tx), p_b(y,Ty), \\ & \frac{p_b(x,Ty) + p_b(y,Tx)}{2s} \Bigg\}. \end{aligned} \tag{60}$$

Corollary 33. *Let* (X, p_b, s) *be an* α_s*-complete partial b-metric space. Let* $T : X \longrightarrow X$ *be a generalized* α_s*-*ψ*-Geraghty contraction satisfying the following conditions:*

(i) *there exists* $x_0 \in X$ *such that* $\alpha_s(x_0, Tx_0) \geq s^2$*;*

(ii) *the mapping T is triangular* α_s*-orbital admissible;*

(iii)

(a) *the mapping T is* α_s*-*p_b*-continuous;*

(b) $\{x_n\}$ *is a sequence in X such that* $\alpha_s(x_n, x_{n+1}) \geq s^2$ *for all* $n \in \mathbb{N}$ *and* $x_n \longrightarrow x^* \in X$ *as* $n \longrightarrow \infty$*, then there exists a subsequence* $\{x_{n(k)}\}$ *of* $\{x_n\}$ *such that* $\alpha_s(x_{n(k)}, x^*) \geq s^2$ *for all* $k \in \mathbb{N}$.

Then T has a fixed point in X. In addition, if y^* *is also a common fixed point of the pair* (S,T) *such that* $\alpha_s(x^*, y^*) \geq s^2$*, then* $x^* = y^*$.

Proof. Set $S = T$ in Theorem 26. □

We extend Definition 25 for all $x, y \in X$ as follows

Definition 34. The self-mappings S, T defined on (X, p_b, s) are called generalized ψ-Geraghty contractions, if there exist $\beta \in \Omega$, $\psi \in \Psi$, and $L \geq 0$ such that

$$\begin{aligned} &\psi(p_b(Sx,Ty)) \\ &\quad \leq \beta(\psi(M(x,y))) .\psi(M(x,y)) \\ &\qquad + L\psi(\min\{p_b(x,Sx), p_b(y,Sx)\}), \end{aligned} \tag{61}$$

for $x, y \in X$.

Theorem 35. *Let* $S, T : X \longrightarrow X$ *be two* p_b*-continuous generalized* ψ*-Geraghty contractions defined on a complete partial b-metric space* (X, p_b, s)*; then S and T have a common fixed point.*

Proof. The arguments follow as the same lines in proof of Theorem 26. □

4. Application

In this section, we present an application on existence of a solution of a pair of elliptic boundary value problems. Let $C(I)$ be the space of all continuous function defined on $I = [0,1]$. Consider the following pair of differential equations:

$$\begin{aligned} -\frac{d^2x}{dt^2} &= f(t, x(t)), \quad t \in [0,1] \\ x(0) &= x(1) = 0. \\ \text{and } -\frac{d^2y}{dt^2} &= K(t, y(t)), \quad t \in [0,1] \\ y(0) &= y(1) = 0, \end{aligned} \tag{62}$$

where $f, K : I \times C(I) \longrightarrow \mathbb{R}$ are continuous functions. The Green function associated with (62) is defined by

$$G(t,s) = \begin{cases} t(1-\tau), & 0 \leq t \leq \tau \leq 1 \\ \tau(1-t), & 0 \leq \tau \leq t \leq 1. \end{cases} \tag{63}$$

Define the function $p_b : C(I) \times C(I) \longrightarrow [0,\infty)$ by

$$\begin{aligned} p_b(x,y) &= \sup_{t\in I} |x(t) - y(t)|^2 + k, \\ &\qquad \text{for all } x, y \in C(I) \text{ and } k > 0. \end{aligned} \tag{64}$$

It is known that $(C(I), p_b)$ is a complete partial b-metric space with constant $s = 4$. Now, define the operators $S, T : C(I) \longrightarrow C(I)$ defined by

$$\begin{aligned} Sx(t) &= \int_0^1 G(t,\tau) f(\tau, x(\tau))\, d\tau \\ \text{and } Tx(t) &= \int_0^1 G(t,\tau) K(\tau, y(\tau))\, d\tau, \end{aligned} \tag{65}$$

for all $t \in I$. Remark that (62) has a solution if and only if operators S and T have a common fixed point.

Theorem 36. *Assume that there exist continuous functions $f, K : I \times C(I) \longrightarrow \mathbb{R}$ such that, for all $x, y \in C(I)$ and $\rho \in \mathbb{R}$, we have*

$$\left|f(t,x)-K(t,y)\right|^2 \leq 64 \ln\left(\frac{M(x(t),y(t))+1}{\rho}\right) \quad \text{for all } t \in I, \tag{66}$$

where $M(x(t), y(t))$ is defined by (9) such that $M(x(t), y(t)) > \rho > 0$

Proof. It is well known that $x^* \in C^2(I)$ is a solution of (62) if and only if $x^* \in C(I)$ is a common solution of the integral equations given by (65). Define the mappings $S, T : C(I) \longrightarrow C(I)$ by (65). Hence the solution of (62) is equivalent to find a common fixed point $x^* \in C(I)$ of T and S. Let $x, y \in C(I)$. By (i), we get

$$\begin{aligned}
&\left|Sx(t)-Ty(t)\right|^2 \\
&= \left[\left|\int_0^1 G(t,\tau)\left[f(\tau,x(\tau))-K(\tau,y(\tau))\right]d\tau\right|\right]^2 \\
&\leq \left[\int_0^1 G(t,\tau)\left|f(\tau,x(\tau))-K(\tau,y(\tau))\right|d\tau\right]^2 \\
&\leq \left[8\int_0^1 G(t,\tau)\sqrt{\ln\left(\frac{M(x(\tau),y(\tau))+1}{\rho}\right)}d\tau\right]^2 \\
&\leq \left[8\int_0^1 G(t,\tau)\sqrt{\ln\left(\frac{M(x(\tau),y(\tau))+1}{\rho}\right)}d\tau\right]^2 \\
&= 8^2 \ln\left(\frac{M(x(\tau),y(\tau))+1}{\rho}\right) \\
&\cdot\left(\sup_{t\in I}\left[\int_0^1 G(t,\tau)\,d\tau\right]^2\right).
\end{aligned} \tag{67}$$

Since $\int_0^1 G(t,\tau)d\tau = -t^2/2 + t/2$ for all $t \in I$, then we have $(\sup_{t\in I}[\int_0^1 G(t,\tau)d\tau]^2) = 1/8^2$, which implies that

$$\left|Sx(t)-Ty(t)\right|^2 \leq \ln\left(\frac{M(x(\tau),y(\tau))+1}{\rho}\right). \tag{68}$$

It can easily be proved that $\mathscr{M}(x,y) = \sup_{\tau\in I}\mathscr{M}(x(\tau),y(\tau))$. Thus,

$$\begin{aligned}
\ln\left(p_b(Sx,Ty)+1\right) &\leq \ln\left(\ln\left(M(x,y)+1\right)+1\right) \\
&= \frac{\ln\left(\ln\left(M(x,y)+1\right)+1\right)}{\ln\left(M(x,y)+1\right)}\ln\left(M(x,y)+1\right).
\end{aligned} \tag{69}$$

Define the functions $\psi : [0,\infty) \longrightarrow [0,\infty)$ and $\beta : [0,\infty) \longrightarrow [0,1/s)$ by

$$\psi(x) = \ln(x+1) \tag{70}$$

$$\text{and } \beta(x) = \begin{cases} \dfrac{\psi(x)}{x}, & \text{if } x \geq 10 \\ 0, & \text{otherwise.} \end{cases} \tag{71}$$

Note that $\psi : [0,\infty) \longrightarrow [0,\infty)$ is continuous, nondecreasing, positive in $(0,\infty)$, $\psi(0) = 0$, and $\psi(x) < x$.

Hence $\beta \in \Omega$ and

$$\psi\left(p_b(Sx,Ty)\right) \leq \beta\left(\psi\left(M(x,y)\right)\right).\psi\left(M(x,y)\right) \tag{72}$$

for all $x, y \in C(I)$. Therefore all assumptions of Theorem 35 are satisfied with $L = 0$. Hence S and T have a common fixed point $x^* \in C(I)$; that is, $Sx^* = Tx^* = x^*$ which is a solution of (62). □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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