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Research Article

On a New Subclass of p-Valent Close-to-Convex Mappings Defined by Two-Sided Inequality

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A new subclass $K_p(\alpha_1, \alpha_2, \beta)$ of p-valent close-to-convex mappings defined by two-sided inequality is introduced. Some sufficient conditions for functions to be in $K_p(\alpha_1, \alpha_2, \beta)$ are given.

1. Introduction

Let $A(p)$ be the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in \mathbb{N}) \tag{1}$$

which are p-valent analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. There and in the following, let $\mathbb{N}$, $\mathbb{C}$, and $\mathbb{R}$ be the sets of positive integers, complex numbers, and real numbers, respectively. A function f analytic in U is said to be close-to-convex if there is a convex function g such that

$$\operatorname{Re}\left\{\frac{f'(z)}{g'(z)}\right\} > 0 \tag{2}$$

for all $z \in U$. The concept of close-to-convex was introduced by Kaplan [1] in 1952. A number of results for close-to-convex functions in U have been obtained by several authors (see, e.g., [2–14]).

A function $f \in A(p)$ is said to be in the class $K_p(\alpha_1, \alpha_2, \beta)$ if it satisfies the following two-sided inequality:

$$-\frac{\alpha_2 \pi}{2} < \arg\left\{\frac{f'(z)}{pz^{p-1}} - \beta\right\} < \frac{\alpha_1 \pi}{2} \quad (z \in U) \tag{3}$$

for $0 < \alpha_1, \alpha_2 \leq 1$ and $0 \leq \beta < 1$. Note that if $f \in K_p(\alpha_1, \alpha_2, \beta)$, then f is p-valent close-to-convex in U. Furthermore, if $f \in K_p(\alpha, \alpha, \beta)$ $(0 < \alpha \leq 1, 0 \leq \beta < 1)$, then (3) becomes

$$\left|\arg\left\{\frac{f'(z)}{pz^{p-1}} - \beta\right\}\right| < \frac{\alpha\pi}{2} \quad (z \in U). \tag{4}$$

A function $f \in K_p(\alpha, \alpha, \beta)$ is called p-valent close-to-convex of order α and type β in U.

Given two functions f and g, which are analytic in U, we say that the function g is subordinate to f and write $g \prec f$ or $g(z) \prec f(z)$ $(z \in U)$, if there exists a Schwarz function w, analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $g(z) = f(w(z))$ $(z \in U)$. In particular, if f is univalent in U, we have the following equivalence:

$$\begin{gathered} g(z) \prec f(z) \\ (z \in U) \Longleftrightarrow g(0) = f(0), \\ g(U) \subset f(U). \end{gathered} \tag{5}$$

Throughout this paper, we let

$$\begin{aligned} &0 < \alpha_1 \le 1, \\ &0 < \alpha_2 \le 1, \\ &0 < \beta < 1, \\ &\gamma = \frac{\alpha_1 - \alpha_2}{\alpha_1 + \alpha_2}, \\ &c = e^{\gamma \pi i}. \end{aligned} \tag{6}$$

In order to prove our main results, we need the following lemmas.

Lemma 1. *The function g defined by*

$$g(z) = \left(\frac{1+cz}{1-z}\right)^{(\alpha_1+\alpha_2)/2} \quad (g(0) = 1) \tag{7}$$

is analytic and univalent convex in U and

$$g(U) \subset \left\{w \in \mathbb{C} : -\frac{\alpha_2 \pi}{2} < \arg w < \frac{\alpha_1 \pi}{2}\right\}. \tag{8}$$

Proof. In view of (6), it is easy to see that $-1 < \gamma < 1$ and the transformation

$$\zeta = e^{-\gamma \pi i/2} w^{2/(\alpha_1+\alpha_2)} \tag{9}$$

maps the convex region

$$G = \left\{w \in \mathbb{C} : -\frac{\alpha_2 \pi}{2} < \arg w < \frac{\alpha_1 \pi}{2}\right\} \tag{10}$$

conformally onto the right-half ζ-plane $-\pi/2 < \arg \zeta < \pi/2$ so that $w = 1$ corresponding to $\zeta = e^{-\gamma \pi i/2}$. Since

$$z = \frac{\zeta - e^{-\gamma \pi i/2}}{\zeta + e^{\gamma \pi i/2}} \tag{11}$$

maps the right-half ζ-plane $\mathrm{Re}(\zeta) > 0$ onto U, from (7), (9), and (11) we find that

$$w = \left(e^{(\gamma \pi i/2)} \zeta\right)^{(\alpha_1+\alpha_2)/2} = \left(\frac{1+cz}{1-z}\right)^{(\alpha_1+\alpha_2)/2} = g(z) \tag{12}$$

maps U conformally onto $G = g(U)$ with $g(0) = 1$. The proof of Lemma 1 is completed. □

Lemma 2 (see [15]). *Let the function g be analytic and univalent in U and let the functions θ and φ be analytic in a domain D containing $g(U)$, with $\varphi(w) \neq 0$ $(w \in g(U))$. Set*

$$\begin{aligned} Q(z) &= zg'(z)\varphi(g(z)), \\ h(z) &= \theta(g(z)) + Q(z) \end{aligned} \tag{13}$$

and suppose that

(i) *Q is univalent starlike in U,*

(ii) $\mathrm{Re}\{zh'(z)/Q(z)\} = \mathrm{Re}\{\theta'(g(z))/\varphi(g(z)) + zQ'(z)/Q(z)\} > 0 (z \in U)$.

If q is analytic in U with $q(0) = g(0)$, $q(U) \subset D$, and

$$\begin{aligned} &\theta(q(z)) + zq'(z)\varphi(q(z)) \\ &\quad \prec \theta(g(z)) + zg'(z)\varphi(g(z)) = h(z) \quad (z \in U), \end{aligned} \tag{14}$$

then $q(z) \prec g(z)$ $(z \in U)$. The function g is the best dominant of (14).

Lemma 3 (see [16]). *Let $p(z)$ be analytic function in U of the form*

$$p(z) = 1 + \sum_{n=m}^{\infty} c_n z^n, \quad c_m \neq 0, \ m \ge 1, \tag{15}$$

with $p(z) \neq 0$ in U. If there exists a point $z_0 \in U$ such that

$$|\arg\{p(z)\}| < \frac{\pi}{2} \tag{16}$$

for $|z| < |z_0|$ and

$$|\arg\{p(z_0)\}| = \frac{\pi}{2}, \tag{17}$$

then

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik, \tag{18}$$

where

$$\begin{aligned} &k \ge \frac{m}{2}\left(a + \frac{1}{a}\right), \quad \text{when } \arg\{p(z_0)\} = \frac{\pi}{2} \\ &k \le -\frac{m}{2}\left(a + \frac{1}{a}\right), \quad \text{when } \arg\{p(z_0)\} = -\frac{\pi}{2}, \end{aligned} \tag{19}$$

where $p(z_0) = \pm ia$ and $a > 0$.

In this paper we shall derive some criteria for a function $f \in A(p)$ to be in the class $K_p(\alpha_1, \alpha_2, \beta)$.

2. Main Results

Our first result is the following theorem.

Theorem 4. *Let $\mu \in \mathbb{C}$ and $s \in \mathbb{R}$. Also let $|s+1| \le 2/(\alpha_1+\alpha_2)$ and $(s+1)\,\mathrm{Re}\,\mu \ge 0$. If the function q is analytic in U with $q(0) = 1$ and $q(z) \neq 0$ $(z \in U)$ and satisfies*

$$\mu(q(z))^{s+1} + zq'(z)(q(z))^s \prec h(z) \quad (z \in U), \tag{20}$$

where

$$\begin{aligned} h(z) = &\left(\frac{1+cz}{1-z}\right)^{(1/2)(s+1)(\alpha_1+\alpha_2)} \\ &\cdot \left(\mu + \frac{\alpha_1+\alpha_2}{2}\left(\frac{z}{1-z} + \frac{cz}{1+cz}\right)\right) \end{aligned} \tag{21}$$

is close-to-convex in U, then

$$-\frac{\alpha_2 \pi}{2} < \arg\{q(z)\} < \frac{\alpha_1 \pi}{2} \quad (z \in U). \tag{22}$$

The bounds α_1 and α_2 in (22) are sharp for the function g defined by (7).

Proof. We choose

$$g(z) = \left(\frac{1+cz}{1-z}\right)^{(\alpha_1+\alpha_2)/2},$$
$$\theta(w) = \mu w^{s+1}, \tag{23}$$
$$\varphi(w) = w^s$$

in Lemma 2. By Lemma 1, the function g is analytic and univalent convex in U and

$$\left|\arg\{g(z)\}\right| < \max\left\{\frac{\alpha_1\pi}{2}, \frac{\alpha_2\pi}{2}\right\} \le \frac{\pi}{2} \quad (z \in U). \tag{24}$$

Clearly, θ and φ are analytic in a domain D containing $g(U)$ and $q(U)$, with $\varphi(w) \neq 0$ when $w \in g(U)$. For

$$-\frac{2}{\alpha_1+\alpha_2} \le s+1 \le \frac{2}{\alpha_1+\alpha_2}, \tag{25}$$

the function Q given by

$$\begin{aligned} Q(z) &= zg'(z)\varphi(g(z)) \\ &= \frac{(\alpha_1+\alpha_2)(1+c)z}{2(1-z)^{1+(1/2)(s+1)(\alpha_1+\alpha_2)}(1+cz)^{1-(1/2)(s+1)(\alpha_1+\alpha_2)}} \end{aligned} \tag{26}$$

is univalent starlike in U because

$$\begin{aligned} &\operatorname{Re}\left\{\frac{zQ'(z)}{Q(z)}\right\} \\ &= 1 + \left(1+\frac{1}{2}(s+1)(\alpha_1+\alpha_2)\right)\operatorname{Re}\left(\frac{z}{1-z}\right) \\ &\quad - \left(1-\frac{1}{2}(s+1)(\alpha_1+\alpha_2)\right)\operatorname{Re}\left(\frac{cz}{1+cz}\right) \\ &> 1 - \frac{1}{2}\left(1+\frac{1}{2}(s+1)(\alpha_1+\alpha_2)\right) \\ &\quad - \frac{1}{2}\left(1-\frac{1}{2}(s+1)(\alpha_1+\alpha_2)\right) = 0 \quad (z \in U). \end{aligned} \tag{27}$$

Further, we have

$$\begin{aligned} \theta(g(z)) + Q(z) &= \mu\left(\frac{1+cz}{1-z}\right)^{(1/2)(s+1)(\alpha_1+\alpha_2)} \\ &\quad + \frac{(\alpha_1+\alpha_2)(1+c)z}{2(1-z)^{1+(1/2)(s+1)(\alpha_1+\alpha_2)}(1+cz)^{1-(1/2)(s+1)(\alpha_1+\alpha_2)}} \\ &= h(z), \end{aligned} \tag{28}$$

where h is given by (21) and so

$$\frac{zh'(z)}{Q(z)} = \mu(s+1) + \frac{zQ'(z)}{Q(z)}. \tag{29}$$

For $(s+1)\operatorname{Re}\mu \ge 0$, it follows from (27) and (29) that

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} > 0 \quad (z \in U). \tag{30}$$

The other conditions of Lemma 2 are also satisfied. Therefore, we conclude that

$$q(z) \prec g(z) = \left(\frac{1+cz}{1-z}\right)^{(\alpha_1+\alpha_2)/2} \quad (z \in U) \tag{31}$$

and the function g is the best dominant of (20). The proof of the theorem is completed. □

Theorem 5. *If $f \in A(p)$ satisfies $f'(z) \neq 0$ $(0 < |z| < 1)$ and*

$$-\beta_2\pi < \arg\left\{\frac{f'(z)}{pz^{p-1}}\left(1+\frac{zf''(z)}{f'(z)}\right) - p\beta\right\} < \beta_1\pi \tag{32}$$

for all $z \in U$, where

$$\beta_j = \frac{\alpha_j}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{\alpha_1+\alpha_2}{2p}\tan\left(\frac{\alpha_j\pi}{2(\alpha_1+\alpha_2)}\right)\right) \quad (j = 1, 2), \tag{33}$$

then $f \in K_p(\alpha_1, \alpha_2, \beta)$. The bounds β_1 and β_2 in (32) are the largest numbers such that (3) holds true.

Proof. For $f \in A(p)$ satisfying $f'(z) \neq 0$ $(0 < |z| < 1)$, we define the function q by

$$\frac{f'(z)}{pz^{p-1}} = \beta + (1-\beta)q(z). \tag{34}$$

Then q is analytic in U with $q(0) = 1$ and $\beta + (1-\beta)q(z) \neq 0$ for all $z \in U$. Taking the logarithmic differentiations in both sides of (34), we have

$$1 + \frac{zf''(z)}{f'(z)} = p + \frac{(1-\beta)zq'(z)}{\beta + (1-\beta)q(z)} \tag{35}$$

and

$$\begin{aligned} &\frac{1}{1-\beta}\left\{\frac{f'(z)}{pz^{p-1}}\left(1+\frac{zf''(z)}{f'(z)}\right) - p\beta\right\} \\ &\quad = pq(z) + zq'(z) \end{aligned} \tag{36}$$

for all $z \in U$. Putting $s = 0$ and $\mu = p$ in Theorem 4 and using (36), we find that if

$$\frac{1}{1-\beta}\left\{\frac{f'(z)}{pz^{p-1}}\left(1+\frac{zf''(z)}{f'(z)}\right) - p\beta\right\} \prec h(z), \tag{37}$$

where

$$\begin{aligned} h(z) &= \left(\frac{1+cz}{1-z}\right)^{(\alpha_1+\alpha_2)/2} \\ &\quad \cdot \left(p + \frac{\alpha_1+\alpha_2}{2}\left(\frac{z}{1-z} + \frac{cz}{1+cz}\right)\right), \end{aligned} \tag{38}$$

is (close-to-convex) univalent in U, then

$$-\frac{\alpha_2\pi}{2} < \arg\{q(z)\} < \frac{\alpha_1\pi}{2} \quad (z \in U), \tag{39}$$

that is, $f \in K_p(\alpha_1, \alpha_2, \beta)$.

For $z = e^{i\theta}$ $(\theta \in \mathbb{R})$, $z \neq 1$, and $z \neq -1/c$, we have

$$\frac{z}{1-z} = -\frac{1}{2} + \frac{i}{2}\cot\frac{\theta}{2}, \qquad \frac{cz}{1+cz} = \frac{1}{2} + \frac{i}{2}\tan\frac{\theta+\gamma\pi}{2} \tag{40}$$

and

$$\frac{1+cz}{1-z} = \frac{1+e^{i(\theta+\gamma\pi)}}{1-e^{i\theta}} = \frac{\cos\left(\left(\theta+\gamma\pi\right)/2\right)}{\sin\left(\theta/2\right)} e^{\alpha_1\pi i/(\alpha_1+\alpha_2)} \neq 0. \tag{41}$$

Now we consider the following two cases.

(i) If

$$K(\theta) = \cos\frac{\theta+\gamma\pi}{2}\sin\frac{\theta}{2} = \frac{1}{2}\left(\sin\left(\theta+\frac{\gamma\pi}{2}\right) - \sin\frac{\gamma\pi}{2}\right) > 0, \tag{42}$$

then we deduce from (38), (40), and (41) that

$$h\left(e^{i\theta}\right) = \left(\frac{\cos\left(\left(\theta+\gamma\pi\right)/2\right)}{\sin\left(\theta/2\right)}\right)^{(\alpha_1+\alpha_2)/2} \cdot e^{(\alpha_1\pi/2)i}\left(p + i\frac{\alpha_1+\alpha_2}{4}\left(\cot\frac{\theta}{2} + \tan\frac{\theta+\gamma\pi}{2}\right)\right), \tag{43}$$

which yields

$$\arg\left\{h\left(e^{i\theta}\right)\right\} = \frac{\alpha_1\pi}{2} + \tan^{-1}\left(\frac{\left(\alpha_1+\alpha_2\right)\cos\left(\gamma\pi/2\right)}{4pK(\theta)}\right). \tag{44}$$

Let $\theta_1 = (\pi/2)(1-\gamma)$. Then

$$0 < K(\theta) \leq K(\theta_1) = \frac{1}{2}\left(1 - \sin\frac{\gamma\pi}{2}\right) \tag{45}$$

and it follows from (44) and (45) that

$$\begin{aligned}\pi &> \arg\left\{h\left(e^{i\theta}\right)\right\} \geq \arg\left\{h\left(e^{i\theta_1}\right)\right\} \\ &= \frac{\alpha_1\pi}{2} + \tan^{-1}\left(\frac{\left(\alpha_1+\alpha_2\right)\cos\left(\gamma\pi/2\right)}{2p\left(1-\sin\left(\gamma\pi/2\right)\right)}\right) \\ &= \frac{\alpha_1\pi}{2} + \tan^{-1}\left(\frac{\alpha_1+\alpha_2}{2p}\tan\left(\frac{\alpha_1\pi}{2\left(\alpha_1+\alpha_2\right)}\right)\right) \\ &= \beta_1\pi > 0.\end{aligned} \tag{46}$$

(ii) If $K(\theta) < 0$, then it follows from (38) to (41) that

$$h\left(e^{i\theta}\right) = \left(-\frac{\cos\left(\left(\theta+\gamma\pi\right)/2\right)}{\sin\left(\theta/2\right)}\right)^{(\alpha_1+\alpha_2)/2} \cdot e^{-(\alpha_2\pi/2)i}\left(p + i\frac{\alpha_1+\alpha_2}{4}\left(\cot\frac{\theta}{2} + \tan\frac{\theta+\gamma\pi}{2}\right)\right), \tag{47}$$

and so

$$\arg\left\{h\left(e^{i\theta}\right)\right\} = -\frac{\alpha_2\pi}{2} + \tan^{-1}\left(\frac{\left(\alpha_1+\alpha_2\right)\cos\left(\gamma\pi/2\right)}{4pK(\theta)}\right). \tag{48}$$

Let $\theta_2 = -(\pi/2)(1+\gamma)$. Then

$$0 > K(\theta) \geq K(\theta_2) = -\frac{1}{2}\left(1 + \sin\frac{\gamma\pi}{2}\right) \tag{49}$$

and from (48) and (49) we have

$$\begin{aligned}-\pi &< \arg\left\{h\left(e^{i\theta}\right)\right\} \leq \arg\left\{h\left(e^{i\theta_2}\right)\right\} \\ &= -\frac{\alpha_2\pi}{2} - \tan^{-1}\left(\frac{\left(\alpha_1+\alpha_2\right)\cos\left(\gamma\pi/2\right)}{2p\left(1+\sin\left(\gamma\pi/2\right)\right)}\right) \\ &= -\frac{\alpha_2\pi}{2} - \tan^{-1}\left(\frac{\alpha_1+\alpha_2}{2p}\tan\left(\frac{\alpha_2\pi}{2\left(\alpha_1+\alpha_2\right)}\right)\right) \\ &= -\beta_2\pi < 0.\end{aligned} \tag{50}$$

Noting that $h(0) = p > 0$, we deduce from (46) and (50) that $h(U)$ properly contains the region $-\beta_2\pi < \arg\{w\} < \beta_1\pi$ in the complex w-plane. Therefore, if a function f satisfies (32), then the subordination relation (37) holds true. This shows that $f \in K_p(\alpha_1, \alpha_2, \beta)$.

Furthermore, for the function g defined by (7), we have

$$\frac{1}{1-\beta}\left\{\frac{g'(z)}{pz^{p-1}}\left(1 + \frac{zg''(z)}{g'(z)}\right) - p\beta\right\} = h(z). \tag{51}$$

Hence, by using (46) and (50), we conclude that the bounds β_1 and β_2 in (32) are the best possible ones. The proof of the theorem is completed. □

Theorem 6. *Let $\delta > 0$. Also let $\alpha \in \mathbb{C}$ and $0 \leq \theta = \arg\{\alpha\} < \pi/2$. If $f \in A(p)$ satisfies*

$$-\frac{\pi}{2} + \theta < \arg\left\{\alpha\left(1 + \frac{zf''(z)}{f'(z)} - p\right) + \left(\frac{f'(z)}{pz^{p-1}}\right)^{\delta}\right\} < \frac{\pi}{2} + \tan^{-1}\frac{|\alpha|\sin\theta}{2\delta + |\alpha|\cos\theta} \tag{52}$$

for $z \in U$, then

$$\left|\arg\left\{\frac{f'(z)}{z^{p-1}}\right\}\right| < \frac{\pi}{2\delta} \quad (z \in U). \tag{53}$$

In particular, if $\delta \geq 1$, then $f \in K_p(1/\delta, 1/\delta, 0)$ or f is p-valent close-to-convex of order $1/\delta$.

Proof. We denote

$$q(z) = \left(\frac{f'(z)}{pz^{p-1}}\right)^{\delta}. \tag{54}$$

Then condition (52) becomes

$$-\frac{\pi}{2}+\theta<\arg\left\{q(z)+\frac{\alpha}{\delta}\frac{zq'(z)}{q(z)}\right\} < \frac{\pi}{2}+\tan^{-1}\frac{|\alpha|\sin\theta}{2\delta+|\alpha|\cos\theta} \quad (z\in U). \tag{55}$$

We want to prove that

$$\left|\arg\{q(z)\}\right|<\frac{\pi}{2} \quad \text{for } z\in U. \tag{56}$$

If there exists a point z_0 ($|z_0|<1$) such that

$$\left|\arg\{q(z)\}\right|<\frac{\pi}{2} \quad (|z|<|z_0|), \qquad \left|\arg\{q(z_0)\}\right|=\frac{\pi}{2}, \tag{57}$$

then, from Lemma 3, we have

$$\frac{z_0q'(z_0)}{q(z_0)}=ik, \tag{58}$$

where $q(z_0)=\pm ai$, $a>0$ and

$$k\geq\frac{(a^2+1)}{(2a)}, \quad \text{when } \arg\{q(z_0)\}=\frac{\pi}{2}, \qquad k\leq\frac{-(a^2+1)}{(2a)}, \quad \text{when } \arg\{q(z_0)\}=-\frac{\pi}{2}. \tag{59}$$

For the case $\arg\{q(z_0)\}=\pi/2$, we have

$$\begin{aligned}\arg\left\{q(z_0)+\frac{\alpha}{\delta}\frac{z_0q'(z_0)}{q(z_0)}\right\}&=\arg\left\{ai+i\frac{k\alpha}{\delta}\right\}\\&=\frac{\pi}{2}+\arg\left\{a+\frac{k\alpha}{\delta}\right\}=\frac{\pi}{2}+\tan^{-1}\frac{\operatorname{Im}\{a+k\alpha/\delta\}}{\operatorname{Re}\{a+k\alpha/\delta\}}\\&=\frac{\pi}{2}+\tan^{-1}\frac{(k|\alpha|/\delta)\sin\theta}{a+(k|\alpha|/\delta)\cos\theta}\\&\geq\frac{\pi}{2}+\tan^{-1}\frac{\left(|\alpha|\left(a^2+1\right)/2a\delta\right)\sin\theta}{a+\left(|\alpha|\left(a^2+1\right)/2a\delta\right)\cos\theta}\\&\geq\frac{\pi}{2}+M(\alpha,\beta),\end{aligned} \tag{60}$$

where

$$M(\alpha,\beta)=\min_{a>0}\tan^{-1}\frac{|\alpha|\left(a^2+1\right)\sin\theta}{(2\delta+|\alpha|\cos\theta)a^2+|\alpha|\cos\theta}. \tag{61}$$

The function

$$g(a)=\frac{|\alpha|\left(a^2+1\right)\sin\theta}{(2\delta+|\alpha|\cos\theta)a^2+|\alpha|\cos\theta}, \quad a>0, \tag{62}$$

has a negative derivative

$$g'(a)=\frac{-4a\delta|\alpha|\sin\theta}{\left((2\delta+|\alpha|\cos\theta)a^2+|\alpha|\cos\theta\right)^2}\leq 0 \quad \text{for } 0\leq\theta<\frac{\pi}{2}. \tag{63}$$

Hence

$$M(\alpha,\beta)=\lim_{a\to\infty}\tan^{-1}g(a)=\tan^{-1}\frac{|\alpha|\sin\theta}{2\delta+|\alpha|\cos\theta}. \tag{64}$$

Therefore, (60) becomes

$$\arg\left\{q(z_0)+\frac{\alpha}{\delta}\frac{z_0q'(z_0)}{q(z_0)}\right\}\geq\frac{\pi}{2}+\tan^{-1}\frac{|\alpha|\sin\theta}{2\delta+|\alpha|\cos\theta}, \tag{65}$$

which contradicts (55). Thus

$$\left|\arg\{q(z)\}\right|<\frac{\pi}{2} \quad \text{for } z\in U. \tag{66}$$

For the case $\arg\{q(z_0)\}=-\pi/2$, we have $k<0$. Applying the same method as the above, we get

$$\begin{aligned}\arg\left\{q(z_0)+\frac{\alpha}{\delta}\frac{z_0q'(z_0)}{q(z_0)}\right\}&=\arg\left\{-ai+i\frac{k\alpha}{\delta}\right\}\\&=-\frac{\pi}{2}+\arg\left\{a-\frac{k\alpha}{\delta}\right\}\leq-\frac{\pi}{2}+\theta.\end{aligned} \tag{67}$$

This contradicts (55). The proof of the theorem is completed. □

Applying the same method as the above we can prove the following theorem.

Theorem 7. *Let $\delta>0$. Also let $\alpha\in\mathbb{C}$ and $-\pi/2<\theta=\arg\{\alpha\}\leq 0$. If $f\in A(p)$ satisfies*

$$\begin{aligned}-\frac{\pi}{2}+\tan^{-1}\frac{|\alpha|\sin\theta}{2\delta+|\alpha|\cos\theta}&<\arg\left\{\alpha\left(1+\frac{zf''(z)}{f'(z)}-p\right)+\left(\frac{f'(z)}{pz^{p-1}}\right)^{\delta}\right\}\\&<\frac{\pi}{2}+\theta\end{aligned} \tag{68}$$

for $z\in U$, then

$$\left|\arg\left\{\frac{f'(z)}{z^{p-1}}\right\}\right|<\frac{\pi}{2\delta} \quad (z\in U). \tag{69}$$

In particular, if $\delta\geq 1$, then $f\in K_p(1/\delta,1/\delta,0)$ or f is p-valent close-to-convex of order $1/\delta$.

Data Availability

For every result in this paper, detailed proof has been given. Readers can understand this paper by reading these detailed proofs carefully. All results in this paper can be released.

Conflicts of Interest

There are no conflicts of interest among the authors.

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