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Research Article

New Fixed Point Theorems and Application of Mixed Monotone Mappings in Partially Ordered Metric Spaces

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We consider the existence of a coupled fixed point for mixed monotone mapping $F : X \times X \longrightarrow X$ satisfying a new contractive inequality which involves an altering distance function in partially ordered metric spaces. We also establish some uniqueness results for coupled fixed points, as well as the existence of fixed points of mixed monotone operators. The presented results generalize and develop some existing results. In addition to an example as well as an application, we establish some uniqueness results for a system of integral equations.

1. Introduction and Preliminaries

In this paper we aim to establish coupled fixed point theorems for a mixed monotone mapping in a metric space endowed with partial order. The concept of the mixed monotone operator was introduced by Guo and Lakshmikantham in [1]. Existence of fixed points in a metric space has been studied for a long time (see [2–10]). The Banach contraction principle, which plays a very important role in nonlinear analysis, is the most famous tool in the study of a fixed point theorem. In the last decade, there has been a trend to weaken the requirement on the contraction by considering metric spaces endowed with partial order. It is of interest to determine if it is still possible to establish the existence of a unique fixed point assuming that the operator considered is monotone in such a setting.

Recently, there have been a lot of generalizations of the Banach contraction-mapping principle in the literature (see [11–28]). Investigation of the existence of fixed points for single-valued mappings in partially ordered metric spaces was initially considered by Ran and Reurings in [29] who proved the existence of a unique fixed point.

Following basically the same approach as the one in [29], Bhaskar and Lakshmikantham in [30] proved a fixed point theorem for a mixed monotone mapping in a metric space endowed with partial order. Bhaskar and Lakshmikantham extended [29, Theorem 2.1] to mixed monotone operators so that they can enlarge, in a unified manner, the class of problems that can be investigated. The authors in [30] also established some uniqueness results for coupled fixed points, as well as the existence of fixed points of F.

Based on the works in [30], Zhao in [31] obtained a more general coupled fixed point theorem for mixed monotone operators. Another generalization of the contraction principle was suggested by Alber and Guerre-Delabriere. In 1997, Alber and Guerre-Delabriere [32] introduced the concept of weak contractions in Hilbert spaces. This concept was extended to metric spaces by Rhoades in [33].

Definition 1. A mapping $T : X \longrightarrow X$, where (X, d) is a metric space, is said to be weakly contractive if

$$d(Tx, Ty) \leq d(x, y) - \psi(d(x, y)), \tag{1}$$

where $x, y \in X$ and ψ is an altering distance function.

The notion of altering distance function was introduced by Khan et al. [34]. An altering distance function is a control function that alters distance between two points in a metric space.

It was shown in [32] that, for Hilbert spaces, weakly contractive maps possess a unique fixed point, without any

additional assumptions, and it was noted that the same is true at least for uniformly smooth and uniformly convex Banach spaces. In [33], Rhoades proved that the theorem remains true in arbitrary complete metric spaces, which was improved and extended by Dutta and Choudhury in [35]. Recently, a new version of the context of ordered metric spaces has been proved by Harjani and Sadarangani in [12]. We refer the readers to [36–41] for more related works.

Motivated by the papers mentioned above, we aim to establish coupled fixed point results for mixed monotone operator F which is *weakly contractive* in partially ordered complete metric spaces. Our main results are Theorems 3, 6, and 8. To the best of our knowledge, there are no similar results in the literature on the existence of the coupled fixed point. Compared with the results obtained in the [32, 33, 35], ψ in our results is not necessary to be an altering distance function, which means that F satisfies a more general contractive condition. On the other hand, our result is still valid for F not necessarily continuous if we require that the underlying metric space X has an additional property. Our main results will generalize and develop the results given in [12].

In Section 2, we give the proof of our main results. In Section 3, as an application of our theorems, we consider the existence of a unique solution to a system of integral equations.

2. Coupled Fixed Point Theorems

Let $(X, \leq)$ be a partially ordered set and let d be a metric on X such that (X, d) is a complete metric space. Further, we endow the product space $X \times X$ with the following partial order:

$$\begin{gathered} \text{for } (x, y), (u, v) \in X \times X, \\ (u, v) \leq (x, y) \Longleftrightarrow \\ x \geq u, y \leq v. \end{gathered} \tag{2}$$

Definition 2 (see [34]). A function $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ is called an altering distance function if the following conditions are satisfied:

(i) φ is continuous and nondecreasing

(ii) $\varphi(t) = 0 \Longleftrightarrow t = 0$

Next we introduce a set of functions

$$\begin{aligned} \Psi := \{\psi \in C([0, +\infty), [0, +\infty)) \mid \psi(0) \\ = 0, \text{ and for any } t > 0, \ \psi(t) > 0\}. \end{aligned} \tag{3}$$

It follows from Definition 2 that if φ is an altering distance function, $\varphi \in \Psi$.

The first main result is the following coupled fixed point result.

Theorem 3. *Assume*

(H_1) $\psi \in \Psi$ *(ψ is not necessary to be an altering distance function)*

(H_2) $F : X \times X \longrightarrow X$ *being a mixed monotone mapping, there exist a constant* $k \in (0, 1)$ *such that*

$$\begin{aligned} &\varphi(d(F(u, v), F(x, y)) + d(F(v, u), F(y, x))) \\ &\quad \leq k\varphi(d(u, x) + d(v, y)) \\ &\qquad - \psi(k[d(u, x) + d(v, y)]) \end{aligned} \tag{4}$$

and, for each $u \geq x, v \leq y$, φ *is an altering distance function which satisfies*

$$\varphi(t + s) \leq \varphi(t) + \varphi(s), \quad \forall t, s \in [0, +\infty) \tag{5}$$

(H_3) *there exist* $(u_0, v_0) \in X \times X$ *such that* $u_0 \leq F(u_0, v_0)$ *and* $v_0 \geq F(v_0, u_0)$

(H_4)

(a) *F is continuous or*

(b) *X has the following properties:*

(i) *If a nondecreasing sequence* $\{u_n\} \longrightarrow u$, *then* $u_n \leq u, \forall n$

(ii) *If a nonincreasing sequence* $\{v_n\} \longrightarrow v$, *then* $v_n \geq v, \forall n$

Then there exist $u, v \in X$, *such that*

$$\begin{gathered} u = F(u, v) \\ \text{and } v = F(v, u). \end{gathered} \tag{6}$$

Proof. Since

$$\begin{gathered} u_0 \leq F(u_0, v_0) \\ \text{and } v_0 \geq F(v_0, u_0), \end{gathered} \tag{7}$$

let $F(u_0, v_0) = u_1$, $F(v_0, u_0) = v_1$, $F(u_1, v_1) = u_2$, and $F(v_1, u_1) = v_2$; we have $u_0 \leq u_1$ $v_0 \geq v_1$. Denote

$$\begin{aligned} F^2(u_0, v_0) &= F(F(u_0, v_0), F(v_0, u_0)) = F(u_1, v_1) \\ &= u_2, \\ F^2(v_0, u_0) &= F(F(v_0, u_0), F(u_0, v_0)) = F(v_1, u_1) \\ &= v_2. \end{aligned} \tag{8}$$

Note that $u_0 \leq u_1$, $v_0 \geq v_1$; it follows from the mixed monotone property of F that

$$\begin{gathered} F(u_1, v_1) \geq F(u_0, v_1) \geq F(u_0, v_0), \\ F(v_1, u_1) \leq F(v_0, u_1) \leq F(v_0, u_0), \end{gathered} \tag{9}$$

which implies

$$\begin{gathered} u_2 = F^2(u_0, v_0) = F(u_1, v_1) \geq F(u_0, v_0) = u_1, \\ v_2 = F^2(v_0, u_0) = F(v_1, u_1) \leq F(v_0, u_0) = v_1. \end{gathered} \tag{10}$$

For $n = 1, 2, \cdots$, let

$$\begin{aligned} u_{n+1} &= F^{n+1}(u_0, v_0) = F(F^n(u_0, v_0), F^n(v_0, u_0)), \\ v_{n+1} &= F^{n+1}(v_0, u_0) = F(F^n(v_0, u_0), F^n(u_0, v_0)). \end{aligned} \tag{11}$$

i.e.,

$$\begin{aligned} u_{n+1} &= F(u_n, v_n), \\ v_{n+1} &= F(v_n, u_n). \end{aligned} \tag{12}$$

It is easy to see that

$$\begin{aligned} u_0 &\le F(u_0, v_0) = u_1 \le F^2(u_0, v_0) = u_2 \le \cdots \\ &\le F^{n+1}(u_0, v_0) \le \cdots, \\ v_0 &\ge F(v_0, u_0) = v_1 \ge F^2(v_0, u_0) = v_2 \ge \cdots \\ &\ge F^{n+1}(v_0, u_0) \ge \cdots, \end{aligned} \tag{13}$$

i.e.,

$$\begin{aligned} &u_0 \le u_1 \le \cdots \le u_n \le \cdots, \\ &v_0 \ge v_1 \ge \cdots \ge v_n \ge \cdots, \\ &\qquad n = 1, 2, \cdots. \end{aligned} \tag{14}$$

By (4) (12), we have

$$\begin{aligned} &\varphi(d(u_{n+1}, u_n) + d(v_{n+1}, v_n)) \\ &\quad = \varphi(d(F(u_n, v_n), F(u_{n-1}, v_{n-1})) \\ &\quad + d(F(v_n, u_n), F(v_{n-1}, u_{n-1}))) \le k\varphi(d(u_n, u_{n-1}) \\ &\quad + d(v_n, v_{n-1})) - \psi(k[d(u_n, u_{n-1}) + d(v_n, v_{n-1})]), \end{aligned} \tag{15}$$

and thus

$$\begin{aligned} &\varphi(d(u_{n+1}, u_n) + d(v_{n+1}, v_n)) \\ &\quad \le k\varphi(d(u_n, u_{n-1}) + d(v_n, v_{n-1})). \end{aligned} \tag{16}$$

From $k \in [0, 1)$, we have

$$\begin{aligned} &\varphi(d(u_{n+1}, u_n) + d(v_{n+1}, v_n)) \\ &\quad \le \varphi(d(u_n, v_{n-1}) + d(v_n, v_{n-1})). \end{aligned} \tag{17}$$

Since φ is continuous and nondecreasing,

$$\begin{aligned} &d(u_{n+1}, u_n) + d(v_{n+1}, v_n) \\ &\quad \le d(u_n, u_{n-1}) + d(v_n, v_{n-1}). \end{aligned} \tag{18}$$

Let $\xi_n = d(u_{n+1}, u_n) + d(v_{n+1}, v_n)$. From $0 \le \xi_{n+1} \le \xi_n$, $\{\xi_n\}$ is a Cauchy sequence, and thus there exist $\xi \ge 0$ such that

$$\lim_{n\longrightarrow\infty} [d(u_{n+1}, u_n) + d(\ (v_{n+1}, v_n)] = \xi. \tag{19}$$

Now we claim $\xi = 0$. In fact, by (14) and (H_1), one has

$$\begin{aligned} \varphi(\xi) &= \lim_{n\longrightarrow\infty} \varphi(\xi_n) \\ &= \lim_{n\longrightarrow\infty} \varphi(d(u_{n+1}, u_n) + d(v_{n+1}, v_n)) \\ &\le k \lim_{n\longrightarrow\infty} \varphi(d(u_n, u_{n-1}) + d(v_n, v_{n-1})) \\ &\quad - \lim_{n\longrightarrow\infty} \psi(k[d(u_n, u_{n-1}) + d(v_n, v_{n-1})]) \\ &= k\varphi(\xi) - \lim_{n\longrightarrow\infty} \psi(k\xi_{n-1}) = k\varphi(\xi) - \lim_{t\longrightarrow k\xi} \psi(t) \\ &\le k\varphi(\xi) \end{aligned} \tag{20}$$

which implies $\varphi(\xi) = 0$, so is ξ.

Next we will show $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences.

Arguing indirectly we suppose $\{u_n\}$ is not a Cauchy sequence. Thus, there exists a positive constant ϵ such that, for any $K > 0$, there exist $n_k > m_k > K$ such that

$$d(u_{n_k}, u_{m_k}) + d(v_{n_k}, v_{m_k}) \ge \epsilon. \tag{21}$$

For m_k, let n_k be the smallest integer satisfying $n_k \ge m_k$ and (21). Thus, one has

$$d(u_{n_k-1}, u_{m_k}) + d(v_{n_k-1}, v_{m_k}) < \epsilon, \tag{22}$$

which implies

$$\begin{aligned} &d(u_{n_k}, u_{m_k}) + d(v_{n_k}, v_{m_k}) \\ &\quad \le d(u_{n_k}, u_{n_k-1}) + d(u_{n_k-1}, u_{m_k}) + d(v_{n_k}, v_{n_k-1}) \\ &\qquad + d(v_{n_k-1}, v_{m_k}) \\ &\quad \le d(u_{n_k}, u_{n_k-1}) + d(v_{n_k}, v_{n_k-1}) + \epsilon = \xi_{n_k-1} + \epsilon, \end{aligned} \tag{23}$$

and, by (21) and $\xi_n \longrightarrow 0$ $(n \longrightarrow \infty)$, we have

$$s_k := d(u_{n_k}, u_{m_k}) + d(v_{n_k}, v_{m_k}) \longrightarrow \epsilon, \quad k \longrightarrow \infty. \tag{24}$$

The triangular inequality gives us

$$\begin{aligned} s_k &\le d(u_{n_k}, u_{n_k+1}) + d(u_{n_k+1}, u_{m_k+1}) \\ &\quad + d(u_{m_k+1}, u_{m_k}) + d(v_{n_k}, v_{n_k+1}) \\ &\quad + d(v_{n_k+1}, v_{m_k+1}) + d(v_{m_k+1}, v_{m_k}) \\ &= \xi_{n_k} + \xi_{m_k} + d(u_{n_k+1}, u_{m_k+1}) + d(v_{n_k+1}, v_{m_k+1}), \end{aligned} \tag{25}$$

and thus

$$\begin{aligned} &\varphi(s_k) \\ &\quad = \varphi(\xi_{n_k} + \xi_{m_k} + d(u_{n_k+1}, u_{m_k+1}) + d(v_{n_k+1}, v_{m_k+1})) \\ &\quad \le \varphi(\xi_{n_k} + \xi_{m_k}) + \varphi(d(u_{n_k+1}, u_{m_k+1})) \\ &\qquad + \varphi(d(v_{n_k+1}, v_{m_k+1})). \end{aligned} \tag{26}$$

By (14),

$$\begin{aligned}
&\varphi\left(d\left(u_{n_k+1}, u_{m_k+1}\right)\right) + \varphi\left(d\left(v_{n_k+1}, v_{m_k+1}\right)\right) \\
&\quad = \varphi\left(d\left(F\left(u_{n_k}, v_{n_k}\right), F\left(u_{m_k}, v_{m_k}\right)\right)\right. \\
&\quad \left. + d\left(F\left(v_{n_k}, u_{n_k}\right), F\left(v_{m_k}, u_{m_k}\right)\right)\right) \\
&\quad \le k\varphi\left(d\left(u_{n_k}, u_{m_k}\right) + d\left(v_{n_k}, v_{m_k}\right)\right) \\
&\quad - \psi\left(k\left[d\left(u_{n_k}, u_{m_k}\right) + d\left(v_{n_k}, v_{m_k}\right)\right]\right) = k\varphi\left(s_k\right) \\
&\quad - \psi\left(ks_k\right).
\end{aligned} \tag{27}$$

From (26) (27), one has

$$\begin{aligned}
\varphi\left(s_k\right) &\le \varphi\left(\xi_{n_k} + \xi_{m_k}\right) + k\varphi\left(s_k\right) - \psi\left(s_k\right) \\
&\le \varphi\left(\xi_{n_k} + \xi_{m_k}\right) + \varphi\left(s_k\right) - \psi\left(s_k\right),
\end{aligned} \tag{28}$$

and, letting $k \longrightarrow \infty$ in the above inequality,

$$\begin{aligned}
\lim_{k\longrightarrow\infty} \varphi\left(s_k\right) &\le \lim_{k\longrightarrow\infty} \left[\varphi\left(\xi_{n_k} + \xi_{m_k}\right) + \varphi\left(s_k\right)\right] \\
&\quad - \lim_{k\longrightarrow\infty} \psi\left(s_k\right).
\end{aligned} \tag{29}$$

From (H_1), note that $\xi_n \longrightarrow 0$, $s_k \longrightarrow \epsilon$; we have

$$\begin{aligned}
\varphi(\epsilon) &\le \varphi(0) + \varphi(\epsilon) - \lim_{k\longrightarrow\infty} \psi\left(s_k\right) \\
&= \varphi(\epsilon) - \lim_{k\longrightarrow\infty} \psi\left(s_k\right) < \varphi(\epsilon),
\end{aligned} \tag{30}$$

which is a contradiction. This shows that $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences. Since X is a complete metric space, there exist $u, v \in X$ such that

$$\begin{aligned}
u_n &\longrightarrow u, \\
v_n &\longrightarrow v, \\
n &\longrightarrow \infty.
\end{aligned} \tag{31}$$

Case (a). Assume F is continuous. Then

$$\begin{aligned}
u &= \lim_{n\longrightarrow\infty} u_n = \lim_{n\longrightarrow\infty} F\left(u_{n-1}, v_{n-1}\right) \\
&= F\left(\lim_{n\longrightarrow\infty} u_{n-1}, \lim_{n\longrightarrow\infty} v_{n-1}\right) = F(u, v), \\
v &= \lim_{n\longrightarrow\infty} v_n = \lim_{n\longrightarrow\infty} F\left(v_{n-1}, u_{n-1}\right) \\
&= F\left(\lim_{n\longrightarrow\infty} v_{n-1}, \lim_{n\longrightarrow\infty} u_{n-1}\right) = F(v, u),
\end{aligned} \tag{32}$$

and, thus, $(u, v) \in X \times X$ is a coupled fixed point of F.

Case (b). Assume X has the following properties:

(i) If a nondecreasing sequence $\{u_n\} \longrightarrow u$, then $u_n \le u, \forall n$

(ii) If a nonincreasing sequence $\{v_n\} \longrightarrow v$, then $v_n \ge v, \forall n$

By (14) (31), we have

$$\begin{aligned}
u_n &\ge u, \\
v_n &\le v, \\
n &= 1, 2, \cdots.
\end{aligned} \tag{33}$$

Again, the triangular inequality gives us

$$\begin{aligned}
d(u, F(u, v)) &\le d\left(u, u_{n+1}\right) + d\left(u_{n+1}, F(u, v)\right) \\
&= d\left(u, u_{n+1}\right) + d\left(F\left(u_n, v_n\right), F(u, v)\right).
\end{aligned} \tag{34}$$

From (H_1) (H_2), observing that φ is nondecreasing and using $u_n \le u$, $v_n \ge v$, we get

$$\begin{aligned}
&\varphi(d(u, F(u, v))) \le \varphi\left(d\left(u, u_{n+1}\right) + d\left(u_{n+1}, F(u, v)\right)\right) \\
&\quad = \varphi\left(d\left(u, u_{n+1}\right)\right) + d\left(F\left(u_n, v_n\right), F(u, v)\right) \ ) \\
&\quad \le \varphi\left(d\left(u, u_{n+1}\right)\right) + \varphi\left(d\left(F\left(u_n, v_n\right), F(u, v)\right)\right) \\
&\quad \le \varphi\left(d\left(u, u_{n+1}\right)\right) + \varphi\left(d\left(F\left(u_n, v_n\right), F(u, v)\right)\right. \\
&\quad \left. + d\left(F\left(v_n, u_n\right), F(v, u)\right)\right) \le \varphi\left(d\left(u, u_{n+1}\right)\right) \\
&\quad + k\varphi\left(d\left(u_n, u\right) + d\left(v_n, v\right)\right) \\
&\quad - \psi\left(k\left(d\left(u_n, u\right) + d\left(v_n, v\right)\right)\right) \longrightarrow 0 \\
&\qquad\qquad (\text{as } n \longrightarrow \infty)
\end{aligned} \tag{35}$$

and we get $\varphi(d(u, F(u, v))) = 0$. Then $u = F(u, v)$. Similarly, we get $v = F(v, u)$. The proof is completed. □

Corollary 4. *Let $(X, \le)$ be a partially ordered set and let d be a metric on X such that (X, d) is a complete metric space. Assume $F : X \times X \longrightarrow X$ is a fixed monotone operator and*

(f_1) *there exist $(u_0, v_0) \in X \times X$ such that $u_0 \le F(u_0, v_0)$ and $v_0 \ge F(v_0, u_0)$*

(f_2) *there exist $\psi \in \Psi, k \in (0, 1)$, for $u, v, x, y \in X$ with $u \ge x$, $v \le y$,*

$$\begin{aligned}
&d\left(F(u, v), F(x, y)\right) + d\left(F(v, u), F(y, x)\right) \\
&\quad \le k\left(d(u, x) + d(v, y)\right) \\
&\quad - \psi\left(k\left(d(u, x) + d(v, y)\right)\right)
\end{aligned} \tag{36}$$

(f_3)

(a) *F is continuous or*

(b) *X has the following properties:*

(i) *If a nondecreasing sequence $\{u_n\} \longrightarrow u$, then $u_n \le u, \forall n$*

(ii) *If a nonincreasing sequence $\{v_n\} \longrightarrow v$, then $v_n \ge v, \forall n$*

Then there exist $u, v \in X$ such that

$$\begin{aligned}
u &= F(u, v) \\
\text{and } v &= F(v, u).
\end{aligned} \tag{37}$$

Proof. Let $\varphi(t) = t, t \in [0, +\infty)$. The proof is finished by Theorem 3. □

Note that, in (36), ψ is not necessary to be an altering distance function.

Corollary 5. *Let $(X, \leq)$ be a partially ordered set and let d be a metric on X such that (X, d) is a complete metric space. Assume $F : X \times X \longrightarrow X$ is a fixed monotone operator and*

(f_1) *there exist $(u_0, v_0) \in X \times X$ such that $u_0 \leq F(u_0, v_0)$ and $v_0 \geq F(v_0, u_0)$*

(f_2) *there exist $\psi \in \Psi, k \in (0, 1)$, for $u, v, x, y \in X$ with $u \geq x$, $v \leq y$,*

$$d(F(u,v), F(x,y)) + d(F(v,u), F(y,x)) \leq \frac{k}{2}[d(u,x) + d(v,y)] \tag{38}$$

(f_3)

(a) *F is continuous or*

(b) *X has the following property:*

(i) *If a nondecreasing sequence $\{u_n\} \longrightarrow u$, then $u_n \leq u, \forall n$*

(ii) *If a nonincreasing sequence $\{v_n\} \longrightarrow v$, then $v_n \geq v, \forall n$*

Then there exist $u, v \in X$ such that

$$u = F(u,v) \quad \text{and } v = F(v,u). \tag{39}$$

Proof. Let $\psi(t) = t/2,\ t \in [0, +\infty)$. The proof is finished by Corollary 4. □

Next, we discuss the uniqueness of the coupled fixed point. Since the contractivity assumption is made only on comparable elements in $X \times X$, Theorem 3 cannot guarantee the uniqueness of the coupled fixed point. Before stating our uniqueness results, we require that the product space $X \times X$ endowed with the partial order mentioned earlier have the following property:

(H_5) *For every $(u, v), (s, w) \in X \times X$, there exists $(x, y) \in X \times X$ which is comparable to (u, v) and (s, w).*

Note that (H_5) is equivalent to (see [42]) the following:

(H_5') *Every pair of elements in $X \times X$ has either a lower bound or an upper bound.*

Next, we state our second main result.

Theorem 6. *Assume (H_1) – (H_4) and (H_5) (or (H_5')) hold. Then the coupled fixed point of F is unique.*

Proof. By (H_1) – (H_4), applying Theorem 3, F has a coupled fixed point. Let $(u, v), (w, s)$ be the coupled fixed points of F, i.e.,

$$\begin{aligned} u &= F(u,v), \\ v &= F(v,u), \\ w &= F(w,s), \\ s &= F(s,w). \end{aligned} \tag{40}$$

To show the uniqueness, we need to prove $u = w, v = s$.

By (H_5), there exists $(x, y) \in X \times X$ which is comparable to (u, v) and (s, w). Let

$$\begin{aligned} x_0 &= x, \\ y_0 &= y, \\ x_{n+1} &= F(x_n, y_n), \\ y_{n+1} &= F(x_n, y_n). \end{aligned} \tag{41}$$

Since (x, y) is comparable to (u, v) with respect to the ordering in $X \times X$, we suppose

$$(x_0, y_0) = (x, y) \leq (u, v). \tag{42}$$

Then, for $n = 1, 2, \ldots$,

$$(x_n, y_n) \leq (u, v). \tag{43}$$

In fact, (42) implies that

$$\begin{aligned} x_0 &= x \leq u, \\ y_0 &= y \geq v. \end{aligned} \tag{44}$$

For $n = 1$, by the mixed monotone property, we have

$$\begin{aligned} x_1 &= F(x_0, y_0) \leq F(u, y_0) \leq F(u, v) = u, \\ y_1 &= F(y_0, x_0) \geq F(v, x_0) \geq F(v, u) = v, \\ &\text{i.e., } (x_1, y_1) \leq (u, v). \end{aligned} \tag{45}$$

For $n = k$, we suppose $(x_k, y_k) \leq (u, v)$. Therefore, for $n = k + 1$, we have

$$\begin{aligned} x_{k+1} &= F(x_k, y_k) \leq F(u, y_k) \leq F(u, v) = u, \\ y_{k+1} &= F(y_k, x_k) \geq F(v, x_k) \geq F(v, u) = v, \\ &\text{i.e., } (x_{k+1}, y_{k+1}) \leq (u, v), \end{aligned} \tag{46}$$

which implies (43) holds.

From (41) (H_2),

$$\begin{aligned} &\varphi(d(u, x_{n+1}) + d(y_{n+1}, v)) \\ &\quad = \varphi(d(F(u,v), F(x_n, y_n)) \\ &\qquad + d(F(y_n, x_n), F(v,u)) \\ &\quad \leq k\varphi(d(u, x_n) + d(v, y_n)) \\ &\qquad - \psi(k[d(u, x_n) + d(v, y_n)]) \\ &\quad \leq \varphi(d(u, x_n) + d(v, y_n)) \end{aligned} \tag{47}$$

which implies

$$d(u, x_{n+1}) + d(y_{n+1}, v) \le d(u, x_n) + d(v, y_n). \quad (48)$$

Set $\xi_n := d(u, x_n) + d(y_n, v)$, $n \in \mathbb{N}$. By (48), we have $0 \le \xi_{n+1} \le \xi_n$. Then, there exists $\xi \in \mathbb{R}^+ = [0, +\infty)$ such that

$$\xi = \lim_{n\longrightarrow\infty} \xi_n = \lim_{n\longrightarrow\infty} [d(u, x_n) + d(y_n, v)]. \quad (49)$$

If $\xi > 0$, by (H_2),

$$\begin{aligned}\varphi(\xi) &= \lim_{n\longrightarrow\infty} \varphi(\xi_n) \le \lim_{n\longrightarrow\infty} (k\varphi(\xi_n)) - \lim_{n\longrightarrow\infty} \psi(k\xi_n) \\ &= k\varphi(\xi) - \lim_{n\longrightarrow\infty} \psi(k\xi_n) < k\varphi(\xi) < \varphi(\xi)\end{aligned} \quad (50)$$

which is a contradiction. So $\xi = 0$. We have

$$0 = \lim_n \xi_n = \lim_{n\longrightarrow\infty} [d(u, x_n) + d(y_n, v)] = 0, \quad (51)$$

and then

$$\lim_{n\longrightarrow\infty} d(u, x_n) = \lim_{n\longrightarrow\infty} d(y_n, v) = 0. \quad (52)$$

Similarly, we also have

$$\lim_{n\longrightarrow\infty} d(w, x_n) = \lim_{n\longrightarrow\infty} d(y_n, s) = 0. \quad (53)$$

It follows from (52) (53) that

$$(u, v) = (w, s). \quad (54)$$

□

Corollary 7. *In addition to the hypothesis of Corollary 4, suppose $(H_5)(or(H_5'))$ holds. Then F has a unique coupled fixed point.*

Next, we state our third main result.

Theorem 8. *In addition to the hypothesis of Theorem 6, suppose one of the followings holds:*

(i) Elements of the coupled fixed point (u, v) are comparable

(ii) u_0 and v_0 are comparable

(iii) Every pair of elements of X has an upper bound or a lower bound in X

Then, we have $u = v$; that is, F has a unique fixed point:

$$F(u, u) = u. \quad (55)$$

Proof. From Theorem 6, F has a unique coupled fixed point (u, v).

Case (i). Since u and v are comparable, we have $u \ge v$ or $u \le v$. Suppose $u \ge v$; then $u = F(u, v)$ and $v = F(v, u)$ are comparable. By (4) in (H_2), one obtains

$$\begin{aligned}\varphi(2d(u, v)) &= \varphi(d(u, v) + d(v, u)) \\ &= \varphi(d(F(u, v), F(v, u)) + d(F(v, u), F(u, v))) \\ &\le k\varphi(d(u, v) + d(v, u)) \\ &\quad - \psi(k[d(u, v) + d(v, u)]) \le k\varphi(2d(u, v)).\end{aligned} \quad (56)$$

Noting $k \in (0, 1)$, it follows from (56) that $d(u, v) = 0$, i.e., $u = v$.

Case (ii). Since u_0 and v_0 are comparable, we have $u_0 \ge v_0$ or $u_0 \le v_0$. Suppose we are in the first case, $u_0 \ge v_0$. Let

$$\begin{aligned}u_{n+1} &= F(u_n, v_n), \\ v_{n+1} &= F(v_n, u_n), \\ &\qquad n = 0, 1, 2 \cdots.\end{aligned} \quad (57)$$

From Theorem 3, the coupled fixed point (u, v) satisfies

$$\begin{aligned}u &= \lim_{n\longrightarrow\infty} u_n = \lim_{n\longrightarrow\infty} F(u_n, v_n), \\ v &= \lim_{n\longrightarrow\infty} v_n = \lim_{n\longrightarrow\infty} F(v_n, u_n).\end{aligned} \quad (58)$$

By $u_0 \ge v_0$ and the mixed monotone property of F, one has

$$u_1 = F(u_0, v_0) \ge F(v_0, u_0) = v_1, \quad (59)$$

and, hence, by induction one obtains

$$u_n \ge v_n, \quad n = 0, 1, 2 \cdots. \quad (60)$$

On the one hand, it follows from the continuity of the distance d that

$$\begin{aligned}d(u, v) &= d\left(\lim_{n\longrightarrow\infty} F(u_n, v_n), \lim_{n\longrightarrow\infty} F(v_n, u_n)\right) \\ &= \lim_{n\longrightarrow\infty} d(F(u_n, v_n), F(v_n, u_n)) \\ &= \lim_{n\longrightarrow\infty} d(u_{n+1}, v_{n+1}).\end{aligned} \quad (61)$$

On the other hand, by (4) in (H_2), one gets

$$\begin{aligned}&\varphi(2d(u_{n+1}, v_{n+1})) = \varphi(d(u_{n+1}, v_{n+1}) \\ &\quad + d(v_{n+1}, u_{n+1})) = \varphi(d(F(u_n, v_n), F(v_n, u_n)) \\ &\quad + d(F(v_n, u_n), F(u_n, v_n))) \le k\varphi(d(u_n, v_n) \\ &\quad + d(v_n, u_n)) - \psi(k(d(u_n, v_n) + d(v_n, u_n))) \\ &\quad \le k\varphi(2d(u_n, v_n)) \le k \cdot k\varphi(2d(u_{n-1}, v_{n-1})) \\ &\quad \le k^n\varphi(2d(u_1, v_1)),\end{aligned} \quad (62)$$

which implies

$$0 = \lim_{n\longrightarrow\infty} \varphi(2d(u_{n+1}, v_{n+1})) = \varphi(2d(u, v)) \quad (63)$$

and therefore $d(u, v) = 0$, which finishes the proof. □

Case (iii). If u, v are comparable, see Case (i). If u, v are not comparable, then there exists $w \in X$ comparable to u and v. Without loss of generality, we suppose $u \le w, v \le w$ (the other case is similar). In view of the order of $X \times X$ (or X^2 for short), one has

$$\begin{aligned}(u, v) &\ge (u, w), \\ (u, w) &\le (w, u), \\ (w, u) &\ge (v, u);\end{aligned} \quad (64)$$

that is, $(u,v),(u,w),(u,w),(w,u),(w,u),(v,u)$ are comparable in X^2.

Inspired by [31], we consider the functional $d_2 : X^2 \times X^2 \longrightarrow \mathbb{R}^+ = [0,+\infty)$ defined by

$$d_2(Y,V) = \frac{1}{2}\left[d(x,u) + d(y,v)\right], \quad \text{for } Y = (x,y),\ V = (u,v) \in X^2. \tag{65}$$

d_2 is a metric on X^2 and, moreover, if (X,d) is complete, then (X^2,d_2) is a complete metric space, too. Define another operator $T : X^2 \longrightarrow X^2$ as follows:

$$T(V) = (F(u,v), F(v,u)) \quad \text{for } V = (u,v) \in X^2. \tag{66}$$

For $Y = (x,y), V = (u,v) \in X^2$, let $\overline{Y} = (y,x), \overline{V} = (v,u)$; one has

$$\begin{aligned} d_2(Y,V) &= \frac{d(x,u) + d(y,v)}{2}, \\ d_2(T(Y),T(V)) &= \frac{d(F(x,y),F(u,v)) + d(F(y,x),F(v,u))}{2} \\ &= \frac{d(F(Y),F(V)) + d\left(F\left(\overline{Y}\right),F\left(\overline{V}\right)\right)}{2}. \end{aligned} \tag{67}$$

By contractive condition (4) in (H_2), we get a Banach type contraction condition:

$$\begin{aligned} &\varphi(2d_2(T(Y),T(V)) \\ &\quad = \varphi\left(d(F(x,y),F(u,v)) + d(F(y,x),F(v,u))\right) \\ &\quad \le k\varphi\left(2d_2(Y,V)\right), \end{aligned} \tag{68}$$

and thus

$$\begin{aligned} &\varphi(2d_2(T(T(Y)),T(T(V))) \\ &\quad \le k\varphi\left(2d_2(T(Y),T(V))\right) \le k^2\varphi\left(2d_2(Y,V)\right), \end{aligned} \tag{69}$$

and, hence, by induction we have

$$\varphi(2d_2(T^n(Y),T^n(V)) \le k^n\varphi\left(2d_2(Y,V)\right), \quad n = 1,2,3,\cdots. \tag{70}$$

Now, applying (70) to the comparable pairs $Y = (u,v), V = (u,w), Y = (u,w), V = (w,u), Y = (w,u), V = (v,u)$, one obtains

$$\begin{aligned} &\varphi(2d_2(T^n((u,v)),T^n((u,w))) \\ &\quad \le k^n\varphi\left(2d_2((u,v),(u,w))\right) = k^n\varphi(d(v,w)), \end{aligned} \tag{71}$$

$$\begin{aligned} &\varphi(2d_2(T^n((u,w)),T^n((w,u))) \\ &\quad \le k^n\varphi\left(2d_2((u,w),(w,u))\right) = k^n\varphi(2d(u,w)), \end{aligned} \tag{72}$$

$$\begin{aligned} &\varphi(2d_2(T^n((w,u)),T^n((v,u))) \\ &\quad \le k^n\varphi\left(2d_2((w,u),(v,u))\right) = k^n\varphi(d(w,v)). \end{aligned} \tag{73}$$

Now, for $Y = (u,v), V = (v,u)$, note that $u = F(u,v) = F(Y), v = F(v,u) = F(V)$; we get

$$\begin{aligned} T(Y) &= (F(u,v),F(v,u)) = (u,v) = Y, \\ d(u,v) &= d(F(Y),F(V)) = d(F(u,v),F(v,u)) \\ &= d\left(F\left(\overline{V}\right),F\left(\overline{Y}\right)\right). \end{aligned} \tag{74}$$

So, using the triangle inequality and (4) (71)-(74), we have

$$\begin{aligned} &\varphi(2d(u,v)) \\ &\quad = \varphi\left(2 \times \frac{d(F(Y),F(V)) + d\left(F\left(\overline{V}\right),F\left(\overline{Y}\right)\right)}{2}\right) \\ &\quad = \varphi\left(2d_2\left(T^n((u,v)),T^n((v,u))\right)\right) \\ &\quad \le \varphi\left(2d_2\left(T^n((u,v)),T^n((u,w))\right)\right) \\ &\qquad + \varphi\left(2d_2\left(T^n((u,w)),T^n((w,u))\right)\right) \\ &\qquad + \varphi\left(2d_2\left(T^n((w,u)),T^n((v,u))\right)\right) \\ &\quad \le k^n\left[\varphi(d(v,w)) + \varphi(2d(u,w)) + \varphi(d(w,v))\right] \\ &\quad \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty), \end{aligned} \tag{75}$$

which implies $d(u,v) = 0$, so $u = v$.

Corollary 9. *In addition to the hypothesis of Corollary 7, suppose one of the followings holds:*

(i) Elements of the coupled fixed point (u,v) are comparable

(ii) u_0 and v_0 are comparable

(iii) Every pair of elements of X has an upper bound or a lower bound in X

Then, we have $u = v$; that is, F has a unique fixed point:

$$F(u,u) = u. \tag{76}$$

Remark 10. Corollary 9 includes theorems in [31].

Example 11. Let $X = \mathbb{R}, d(x,y) = |x-y|$ and $F : X \times X \longrightarrow X$ be defined by

$$F(x,y) = \frac{1}{6}(x-2y), \quad (x,y) \in X^2 = X \times X. \tag{77}$$

Then,

(i) F has a coupled fixed point

(ii) F has a unique coupled fixed point

(iii) F has a unique fixed point

Proof. Let $\varphi(t) = 4t, \psi(t) = t, t \in [0,+\infty)$. Since

$$F(x,y) = \frac{1}{6}(x-2y), \quad (x,y) \in X^2, \tag{78}$$

one gets, for $(u,v),(m,n) \in X^2$,

$$\begin{aligned} d(F(u,v),F(m,n)) &= \frac{1}{6}|u-2v-m+2n| \\ &= \frac{1}{6}|(u-m)+2(n-v)| \\ &\le \frac{1}{6}|u-m| + \frac{1}{3}|n-v|, \end{aligned} \tag{79}$$

$$\begin{aligned} d(F(v,u),F(n,m)) &= \frac{1}{6}|v-2u-n+2m| \\ &= \frac{1}{6}|(v-n)+2(m-u)| \\ &\le \frac{1}{6}|n-v|+\frac{1}{3}|u-m|, \end{aligned} \tag{80}$$

and

$$\begin{aligned} &\varphi(d(F(u,v),F(m,n))+d(F(v,u),F(n,m))) \\ &\quad \le 2|(u-m)|+2|(n-v)|. \end{aligned} \tag{81}$$

Note that

$$d(u,m)+d(v,n)=|u-m|+|n-v|, \tag{82}$$

and, for $k \in [2/3,1)$,

$$\begin{aligned} &k\varphi(d(u,m)+d(v,n))-\psi(k[d(u,m)+d(v,n)]) \\ &\quad = 3k|u-m|+3k|n-v|, \end{aligned} \tag{83}$$

and, hence, (81) (83) imply that

$$\begin{aligned} &\varphi(d(F(u,v),F(m,n))+d(F(v,u),F(n,m))) \\ &\quad \le k\varphi(d(u,m)+d(v,n)) \\ &\qquad -\psi(k[d(u,m)+d(v,n)]). \end{aligned} \tag{84}$$

On the other hand, F is continuous and $u_0=-1$, $v_0=1$ satisfy

$$\begin{aligned} F(u_0,v_0) &= \frac{1}{6}(u_0-2v_0) = -\frac{1}{2} > -1 = u_0, \\ F(v_0,u_0) &= \frac{1}{6}(v_0-2u_0) = \frac{1}{2} < 1 = v_0. \end{aligned} \tag{85}$$

(i) As mentioned above, by Theorem 3, F has a coupled fixed point

(ii) Since X^2 has property (H_5) or (H_5'), by Theorem 6, the uniqueness of the coupled fixed point is obtained

(iii) Noting that u_0 and v_0 are comparable, by Theorem 8 (ii), F has a unique fixed point □

3. Applications to Integral Equations

As an application to the fixed point theorem proved in Section 2, we shall study a class of integral equation

$$\begin{aligned} u(x) &= \int_m^n (q_1(x,y)+q_2(x,y)) \\ &\quad \cdot (f(y,u(y))+g(y,u(y)))\,dy+p(x), \end{aligned} \tag{86}$$

where $x \in I = [m,n]$. We will establish existence and uniqueness results. It is well known that some boundary value problems are equivalent to an integral equation or a system of integral equations.

Let $X = C(I,\mathbb{R})$ be a partially ordered set such that, for $u,v \in X$,

$$\begin{aligned} &u \le v \Longleftrightarrow \\ &u(x) \le v(x), \\ &x \in I. \end{aligned} \tag{87}$$

X is endowed with the sup metric:

$$d(u,v)=\sup_{x\in I}|u(x)-v(x)|, \quad u,v \in X, \tag{88}$$

so (X,d) is a complete metric space. The corresponding metric d_1 on X^2 is defined by

$$\begin{aligned} &d_1((u_1,v_1),(u_2,v_2)) \\ &\quad = \frac{1}{2}\left[\sup_{x\in I}|u_1(x)-u_2(x)|+\sup_{x\in I}|v_1(x)-v_2(x)|\right], \end{aligned} \tag{89}$$

and then consider on X^2 the partial order relation:

$$\begin{aligned} &(u_1,v_1) \le (u_2,v_2) \Longleftrightarrow \\ &u_1(x) \le u_2(x) \\ &\text{and } v_1(x) \ge v_2(x), \\ &x \in I. \end{aligned} \tag{90}$$

A pair $(\alpha,\beta) \in X^2$ is called a *coupled lower – upper* solution of (86) if

$$\begin{aligned} &\alpha(x) \\ &\quad \le \int_m^n q_1(x,y)(f(y,\alpha(y))+g(y,\beta(y)))\,dy \\ &\qquad + \int_m^n q_2(x,y)(f(y,\beta(y))+g(y,\alpha(y)))\,dy \\ &\qquad + p(x), \\ &\beta(x) \\ &\quad \ge \int_m^n q_1(x,y)(f(y,\beta(y))+g(y,\alpha(y)))\,dy \\ &\qquad + \int_m^n q_2(x,y)(f(y,\alpha(y))+g(y,\beta(y)))\,dy \\ &\qquad + p(x). \end{aligned} \tag{91}$$

If $\alpha=\beta$, α will be a solution of (86).

Let $\theta : [0,+\infty) \longrightarrow [0,+\infty)$ be a nondecreasing function such that

$$\theta(u) = \frac{u}{2} - \psi\left(\frac{u}{2}\right), \quad u \in [0,+\infty), \tag{92}$$

where $\psi \in \Psi$ satisfies $\psi(t_1 + t_2) \leq \psi(t_1) + \psi(t_2)$, $t_1, t_2 \in [0,+\infty)$.

We make the following assumptions:

(i) $q_1 \in C(I \times I, [0,+\infty))$, $q_2 \in C(I \times I, (-\infty, 0])$

(ii) $p \in C(I, \mathbb{R})$

(iii) $f, g \in C(I \times \mathbb{R}, \mathbb{R})$

(iv) there exist constants $\lambda, \mu > 0$ such that, for $u, v \in \mathbb{R}, u \geq$

$$\begin{aligned} &0 \leq f(x,u) - f(x,v) \leq \lambda\theta(u-v), \\ &-\mu\theta(u-v) \leq g(x,u) - g(x,v) \leq 0, \\ &\max\{\lambda,\mu\} \sup_{x\in I} \int_m^n (q_1(x,y) - q_2(x,y))\, dy \leq \frac{1}{4}, \end{aligned} \tag{93}$$

(v) $(\alpha, \beta) \in X^2$ is a *coupled lower–upper* solution of (86)

Theorem 12. *Suppose (i) – (v) hold. Then (86) has a unique solution.*

Proof. In order to obtain the (unique) solution of (86), define for $x \in [m,n]$ the operator $F : X \times X \longrightarrow X$ by

$$\begin{aligned} &F(u,v)(x) \\ &\quad = \int_m^n q_1(x,y)(f(y,u(y)) + g(y,v(y)))\, dy \\ &\qquad + \int_m^n q_2(x,y)(f(y,v(y)) + g(y,u(y)))\, dy \\ &\qquad + p(x), \quad \forall x \in [m,n]. \end{aligned} \tag{94}$$

If $(u,v) \in X^2$ is a coupled fixed point of F, then we have

$$\begin{aligned} u(x) &= F(u,v)(x), \\ v(x) &= F(v,u)(x). \end{aligned} \tag{95}$$

It is obvious that the fixed point of F is the solution of (86). In what follows, we will show that F has a unique fixed point.

(1) We will show that the operator F has a unique coupled fixed point in X^2

Firstly, by (v), we have

$$\begin{aligned} \alpha &\leq F(\alpha,\beta), \\ F(\beta,\alpha) &\leq \beta. \end{aligned} \tag{96}$$

Secondly, we check F is mixed monotone for $u_1, u_2 \in X$ such that $u_1 \leq u_2$. From (i) and (iv), we have

$$\begin{aligned} &F(u_1,v)(x) - F(u_2,v)(x) \\ &\quad = \int_m^n q_1(x,y)(f(y,u_1(y)) - (f(y,u_2(y)))\, dy \\ &\qquad + \int_m^n q_2(x,y)(g(y,u_1(y)) - g(y,u_2(y)))\, dy \\ &\quad \leq 0. \end{aligned} \tag{97}$$

Similarly, for $v_1, v_2 \in X$ such that $v_1 \leq v_2$, we have

$$\begin{aligned} &F(u,v_1)(x) - F(u,v_2)(x) \\ &\quad = \int_m^n q_1(x,y)(g(y,v_1(y)) - (g(y,v_2(y)))\, dy \\ &\qquad + \int_m^n q_2(x,y)(f(y,v_1(y)) - f(y,v_2(y)))\, dy \\ &\quad \geq 0, \end{aligned} \tag{98}$$

which yields that F is mixed monotone.

Thirdly, we show that F verifies the contraction condition. Let us consider $u, v, a, b \in X$ with $u \geq a$, $v \leq b$; we have

$$\begin{aligned} d(F(u,v), F(a,b)) &= \sup_{x\in I} \left| \int_m^n q_1(x,y)[(f(y,u(y)) - f(y,a(y)) - (g(y,b(y)) - g(y,v(y))]\, dy \right. \\ &\quad \left. - \int_m^n q_2(x,y)[(f(y,b(y)) - f(y,v(y)) - (g(y,u(y)) - g(y,a(y))]\, dy \right| \\ &\leq \sup_{x\in I} \left\{ \int_m^n q_1(x,y)[\lambda\theta(u(y) - a(y)) + \mu\theta(b(y) - v(y))]\, dy \right. \\ &\quad \left. - \int_m^n q_2(x,y)[\lambda\theta(b(y) - v(y)) + \mu\theta(u(y) - a(y))]\, dy \right\} \leq \max\{\lambda,\mu\} \\ &\quad \cdot \sup_{x\in I} \int_m^n (q_1(x,y) - q_2(x,y))[\theta(u(y) - a(y)) + \theta(b(y) - v(y))]\, dy. \end{aligned} \tag{99}$$

Similarly, one obtains

$$\begin{aligned} d(F(v,u),F(b,a)) &\le \max\{\lambda,\mu\} \\ &\cdot \sup_{x\in I}\int_m^n (q_1(x,y)-q_2(x,y)) \\ &\cdot [\theta(u(y)-a(y))+\theta(b(y)-v(y))]\,dy. \end{aligned} \tag{100}$$

Since

$$\begin{aligned} \theta(u(y)-a(y)) &\le \theta(d(u,a)), \\ \theta(b(y)-v(y)) &\le \theta(d(b,v)), \end{aligned} \tag{101}$$

which together with (iv) imply that

$$\begin{aligned} &d(F(u,v),F(a,b))+d(F(v,u),F(b,a)) = 2 \\ &\cdot \max\{\lambda,\mu\}\sup_{x\in I}\int_m^n (q_1(x,y)-q_2(x,y)) \\ &\cdot [\theta(u(y)-a(y))+\theta(b(y)-v(y))]\,dy \\ &\le \theta(d(u,a))+\theta(d(b,v)) = \frac{1}{2}[d(u,a)+d(b,v)] \\ &-\left[\psi\left(\frac{1}{2}d(u,a)\right)+\psi\left(\frac{1}{2}d(b,v)\right)\right] \\ &\le \frac{d(u,a)+d(b,v)}{2}-\psi\left(\frac{d(u,a)+d(b,v)}{2}\right), \end{aligned} \tag{102}$$

which proves that F verifies contraction condition (36) in Corollary 4.

Next, we consider a monotone nondecreasing sequence $\{u_n\}\subset X$ converging to $u\in X$. Then, for every $x\in I$, the sequence of real numbers

$$u_1(x)\le u_2(x)\le\cdots\le u_n(x)\le\cdots, \tag{103}$$

converges to $u(x)$. So, for all $x\in I, n\in\mathbb{N}, u_n(x)\le u(x)$ Hence, $u_n\le u$, for all n. Similarly, we can verify that the limit $v(x)$ of monotone nonincreasing sequence $\{v_n\}$ in X is a lower bound for all the elements in the sequence. That is, $v\le v_n$ for all n.

Therefore, it follows from Corollary 4 that F has a coupled fixed point $(u_0,v_0)\in X^2$, i.e.,

$$\begin{aligned} u_0 &= F(u_0,v_0) \\ \text{and } v_0 &= F(v_0,u_0). \end{aligned} \tag{104}$$

On the other hand, X has property (H_5') since, for any $g,h\in X$, $M(x)=\max(g(x),h(x))$, $m(x)=\min(g(x),h(x))$, for each $x\in I$, are in X and are the upper and lower bounds of g,h, respectively. Then, by Corollary 7, F has a unique coupled fixed point.

(2) Noting that (α,β) is a coupled *lower – upper* solution of (86), one obtains $\alpha(x)\le\beta(x)$ for all $x\in I$. So, α and β are comparable. By Corollary 9, F has a unique fixed point □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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