Research Article

A Transference Result of the L^p -Continuity of the Jacobi Littlewood-Paley g -Function to the Gaussian and Laguerre Littlewood-Paley g -Function

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We develop a transference method to obtain the L^p -continuity of the Gaussian-Littlewood-Paley g -function and the L^p -continuity of the Laguerre-Littlewood-Paley g -function from the L^p -continuity of the Jacobi-Littlewood-Paley g -function, in dimension one, using the well-known asymptotic relations between Jacobi polynomials and Hermite and Laguerre polynomials.

1. Preliminaries

In the theory of classical orthogonal polynomials, the asymptotic relations between Jacobi polynomials and Hermite and Laguerre polynomials are well known. Using those asymptotic relations we develop a transference method to obtain L^p -continuity for the Gaussian-Littlewood-Paley g -function and the L -continuity of the Laguerre-Littlewood-Paley g -function from the L^p -continuity of the Jacobi-Littlewood-Paley g -function, in dimension one. We are going to use the normalizations given in G. Szegő's book [1], for all classical polynomials.

(i) *Jacobi Polynomials.* For $\alpha, \beta > -1$, the Jacobi polynomials $\{P_n^{(\alpha, \beta)}\}_{n \in \mathbb{N}}$ are defined (up to a multiplicative constant) as the orthogonal polynomials associated with the Jacobi measure $\mu_{\alpha, \beta}$ (or beta measure) in $(-1, 1)$, defined as

$$\begin{aligned} \mu_{\alpha, \beta}(dx) &= \omega_{\alpha, \beta}(x) dx \\ &= \eta_{\alpha, \beta} \chi_{(-1, 1)}(x) (1-x)^\alpha (1+x)^\beta dx, \end{aligned} \quad (1)$$

where $\eta_{\alpha, \beta} = 1/2^{\alpha+\beta+1} B(\alpha+1, \beta+1) = \Gamma(\alpha+\beta+2)/2^{\alpha+\beta+1} \Gamma(\alpha+1) \Gamma(\beta+1)$.

The function $\omega_{\alpha, \beta}$ is called the (normalized) *Jacobi weight*.

The Jacobi polynomials can be obtained (up to a multiplicative constant) from the polynomial canonical basis $\{1, x, x^2, \dots, x^n, \dots\}$ using the Gram-Schmidt orthogonalization process with respect to the inner product in $L^2(\mu_{\alpha, \beta})$. Thus we have the *orthogonality property* of Jacobi polynomials with respect to $\mu_{\alpha, \beta}$,

$$\begin{aligned} \int_{-\infty}^{\infty} P_n^{(\alpha, \beta)}(y) P_m^{(\alpha, \beta)}(y) \mu_{\alpha, \beta}(dy) &= \eta_{\alpha, \beta} h_n^{(\alpha, \beta)} \delta_{n, m} \\ &= \widehat{h}_n^{(\alpha, \beta)} \delta_{n, m}, \end{aligned} \quad (2)$$

$n, m = 0, 1, 2, \dots$, where

$$\begin{aligned} h_n^{(\alpha, \beta)} &= \frac{2^{\alpha+\beta+1}}{(2n+\alpha+\beta+1)} \frac{\Gamma(n+\alpha+1) \Gamma(n+\beta+1)}{\Gamma(n+1) \Gamma(n+\alpha+\beta+1)}, \\ \widehat{h}_n^{(\alpha, \beta)} &= \frac{1}{(2n+\alpha+\beta+1)} \\ &\quad \cdot \frac{\Gamma(\alpha+\beta+2) \Gamma(n+\alpha+1) \Gamma(n+\beta+1)}{\Gamma(\alpha+1) \Gamma(\beta+1) \Gamma(n+1) \Gamma(n+\alpha+\beta+1)} \\ &= \|P_n^{(\alpha, \beta)}\|_{2, (\alpha, \beta)}^2. \end{aligned} \quad (3)$$

We will denote the *normalized Jacobi polynomial* of degree n as

$$p_n^{(\alpha, \beta)}(x) = \frac{P_n^{(\alpha, \beta)}(x)}{P_n^{(\alpha, \beta)}(1)}. \quad (4)$$

On the other hand, the Jacobi polynomial of parameter (α, β) of degree n , $P_n^{(\alpha, \beta)}$, is a polynomial solution of the *Jacobi differential equation*, with parameters α, β, n ,

$$(1-x^2)y'' + [\beta - \alpha - (\alpha + \beta + 2)x]y' + n(n + \alpha + \beta + 1)y = 0; \quad (5)$$

that is, $P_n^{(\alpha, \beta)}$ is an eigenfunction of the (one-dimensional) second-order diffusion operator

$$\mathcal{L}^{\alpha, \beta} = -(1-x^2) \frac{d^2}{dx^2} - (\beta - \alpha - (\alpha + \beta + 2)x) \frac{d}{dx}, \quad (6)$$

associated with the eigenvalue $\lambda_n^{\alpha, \beta} = n(n + \alpha + \beta + 1)$. $\mathcal{L}^{\alpha, \beta}$ is called the *Jacobi differential operator*. Observe that if we choose $\delta_{\alpha, \beta} = \sqrt{1-x^2}(d/dx)$, and consider its formal $L^2(\mu_{\alpha, \beta})$ -adjoint,

$$\delta_{\alpha, \beta}^* = -\sqrt{1-x^2} \frac{d}{dx} + \left[\left(\alpha + \frac{1}{2} \right) \sqrt{\frac{1+x}{1-x}} - \left(\beta + \frac{1}{2} \right) \sqrt{\frac{1-x}{1+x}} \right] I, \quad (7)$$

then $\mathcal{L}^{\alpha, \beta} = \delta_{\alpha, \beta}^* \delta_{\alpha, \beta}$. The differential operator $\delta_{\alpha, \beta}$ is considered the “natural” notion of derivative in the Jacobi case.

The operator semigroup associated with the Jacobi polynomials is defined for positive or bounded measurable Borel functions of $(-1, 1)$, as

$$T_t^{\alpha, \beta} f(x) = \int_{-1}^1 p^{\alpha, \beta}(t, x, y) f(y) \mu_{\alpha, \beta}(dy), \quad (8)$$

where

$$p^{\alpha, \beta}(t, x, y) = \sum_k \frac{e^{-k(k+\alpha+\beta+1)t}}{\widehat{h}_k^{(\alpha, \beta)}} P_k^{(\alpha, \beta)}(x) P_k^{(\alpha, \beta)}(y). \quad (9)$$

No simple explicit representation of $p^{\alpha, \beta}(t, x, y)$ is known, since the eigenvalues λ_n are not linearly distributed; nevertheless there is one obtained by Gasper which is analog of Bailey’s F_4 representation of the kernel of Abel summability for Jacobi series, also called the Jacobi-Poisson integral; see [2]. From that form, taking $x = -y = 1$, it can be proved that $p^{\alpha, \beta}(t, x, y)$ is a positive kernel.

$\{T_t^{\alpha, \beta}\}$ is called the *Jacobi semigroup* and can be proved, that is, a Markov semigroup; for details see [3]. The Jacobi-Poisson semigroup $\{P_t^{\alpha, \beta}\}$ can be defined, using Bochner’s subordination formula,

$$e^{-\lambda^{1/2}t} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-\lambda t^2/4u} du, \quad (10)$$

as the subordinated semigroup of the Jacobi semigroup,

$$P_t^{\alpha, \beta} f(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} T_{t^2/4u}^{\alpha, \beta} f(x) du. \quad (11)$$

For a function $f \in L^2([-1, 1], \mu_{(\alpha, \beta)})$ let us consider its Fourier- Jacobi expansion

$$f = \sum_{k=0}^\infty \frac{\langle f, P_k^{(\alpha, \beta)} \rangle}{\widehat{h}_k^{(\alpha, \beta)}} P_k^{(\alpha, \beta)}, \quad (12)$$

where

$$\langle f, P_k^{(\alpha, \beta)} \rangle = \int_{-1}^1 f(y) P_k^{(\alpha, \beta)}(y) \mu_{\alpha, \beta}(dy). \quad (13)$$

Then, the action of T_t and P_t can be expressed as

$$\begin{aligned} T_t^{\alpha, \beta} f &= \sum_{k=0}^\infty \frac{\langle f, P_k^{(\alpha, \beta)} \rangle}{\widehat{h}_k^{(\alpha, \beta)}} e^{-\lambda_k t} P_k^{(\alpha, \beta)}, \\ P_t^{\alpha, \beta} f &= \sum_{k=0}^\infty \frac{\langle f, P_k^{(\alpha, \beta)} \rangle}{\widehat{h}_k^{(\alpha, \beta)}} e^{-\sqrt{\lambda_k} t} P_k^{(\alpha, \beta)}. \end{aligned} \quad (14)$$

Following the classical case, the Jacobi-Littlewood-Paley g function can be defined as

$$g^{(\alpha, \beta)} f(x) = \left(\int_0^\infty t \left| \nabla_{(\alpha, \beta)} P_t^{\alpha, \beta} f(x) \right|^2 dt \right)^{1/2}, \quad (15)$$

where $\nabla_{(\alpha, \beta)} = (\partial/\partial t, \delta_{\alpha, \beta}) = (\partial/\partial t, \sqrt{1-x^2}(\partial/\partial x))$.

Moreover, observe that $g^{(\alpha, \beta)} f$ can be written as a singular integral with values in the Hilbert space $L^2((0, \infty), dt/t)$

$$g^{(\alpha, \beta)} f(x) = \left\| t \nabla_{(\alpha, \beta)} P_t^{\alpha, \beta} f \right\|_{L^2((0, \infty), dt/t)}. \quad (16)$$

We could also consider the following Jacobi-Littlewood-Paley functions:

$$g_1^{(\alpha, \beta)} f(x) = \left(\int_0^\infty t \left| \frac{\partial P_t^{\alpha, \beta} f}{\partial t}(x) \right|^2 dt \right)^{1/2}, \quad (17)$$

the time Jacobi-Littlewood-Paley function, and

$$\begin{aligned} g_2^{(\alpha, \beta)} f(x) &= \left(\int_0^\infty t \left| \delta_{\alpha, \beta} P_t^{\alpha, \beta} f(x) \right|^2 dt \right)^{1/2} \\ &= \left(\int_0^\infty t \left| \sqrt{1-x^2} \frac{\partial P_t^{\alpha, \beta} f}{\partial x}(x) \right|^2 dt \right)^{1/2}, \end{aligned} \quad (18)$$

the spatial Jacobi-Littlewood-Paley function.

The $L^p(\mu_{\alpha, \beta})$ -boundedness of the Jacobi-Littlewood-Paley $g^{(\alpha, \beta)}$ -function was proved by Nowak and Sjögren in [4].

Theorem 1. Assume that $1 < p < \infty$ and $\alpha, \beta \in [-1/2, \infty)^d$. There exists a constant c_p such that

$$\|g^{(\alpha, \beta)} f\|_{p, (\alpha, \beta)} \leq c_p \|f\|_{p, (\alpha, \beta)}. \quad (19)$$

Observe that, from Theorem 1, we get immediately the $L^p(\mu_{\alpha, \beta})$ -boundedness of $g_1^{(\alpha, \beta)}$ and of $g_2^{(\alpha, \beta)}$.

Now, if we define

$$T^{(\alpha, \beta)} f(x, t) = t \nabla_{(\alpha, \beta)} P_t^{\alpha, \beta} f(x), \quad (x, t) \in (-1, 1) \times (0, \infty), \quad (20)$$

then, the $L^p(\mu_{\alpha, \beta})$ -boundedness of $g^{(\alpha, \beta)} f$ is equivalent to the boundedness of $T^{(\alpha, \beta)}$ from $L^p(\mu_{\alpha, \beta})$ into $L^p_{L^2((0, \infty), dt/t)}(\mu_{\alpha, \beta})$ (for the case of $g_1^{(\alpha, \beta)}$ that could be also linked to $L^2((0, \infty), dt/t)$ -Jacobi multipliers).

(ii) *Hermite Polynomials.* The Hermite polynomials, $\{H_n\}_n$, are defined as the orthogonal polynomials associated with the Gaussian measure in $\mathbb{R}$, $\gamma(dx) = (e^{-x^2}/\sqrt{\pi})dx$, that is,

$$\int_{-\infty}^{\infty} H_n(y) H_m(y) \gamma(dy) = 2^n n! \delta_{n,m}, \quad (21)$$

$n, m = 0, 1, 2, \dots$, with the normalization

$$\begin{aligned} H_{2n+1}(0) &= 0, \\ H_{2n}(0) &= (-1)^n \frac{(2n)!}{n!}. \end{aligned} \quad (22)$$

We have

$$\begin{aligned} H'_n(x) &= 2nH_{n-1}(x), \\ H''_n(x) - 2xH'_n(x) + 2nH_n(x) &= 0; \end{aligned} \quad (23)$$

thus H_n is an eigenfunction of the one-dimensional Ornstein-Uhlenbeck operator (or harmonic oscillator operator),

$$L = -\frac{1}{2} \frac{d^2}{dx^2} + x \frac{d}{dx}, \quad (24)$$

associated with the eigenvalue $\lambda_n = n$. Observe that if we choose $\delta_\gamma = (1/\sqrt{2})(d/dx)$, and consider its formal $L^2(\gamma)$ -adjoint,

$$\delta_\gamma^* = -\frac{1}{\sqrt{2}} \frac{d}{dx} + \sqrt{2}xI, \quad (25)$$

then $L = \delta_\gamma^* \delta_\gamma$. The differential operator δ_γ is considered the “natural” notion of derivative in the Hermite case.

The Gaussian-Littlewood-Paley g^γ function can be defined as

$$g^\gamma f(x) = \left(\int_0^\infty t |\nabla_\gamma P_t^\gamma f(x)|^2 dt \right)^{1/2}, \quad (26)$$

where $\nabla_\gamma = (\partial/\partial t, \delta_\gamma) = (\partial/\partial t, (1/\sqrt{2})(\partial/\partial x))$ and $\{P_t^\gamma\}$ is the Poisson-Hermite semigroup, that is, the subordinated

semigroup to the Ornstein-Uhlenbeck semigroup; for more information see [3].

Moreover, observe that $g^\alpha f$ can be written as a singular integral with values in the Hilbert space $L^2((0, \infty), dt/t)$

$$g^\gamma f(x) = \left\| t \nabla_\gamma P_t^\gamma f \right\|_{L^2((0, \infty), dt/t)}. \quad (27)$$

We could also consider the following Gaussian Littlewood-Paley functions:

$$g_1^\gamma f(x) = \left(\int_0^\infty t \left| \frac{\partial P_t^\gamma f}{\partial t}(x) \right|^2 dt \right)^{1/2}, \quad (28)$$

the time Gaussian-Littlewood-Paley, and

$$g_2^\gamma f(x) = \frac{1}{\sqrt{2}} \left(\int_0^\infty t \left| \frac{\partial P_t^\gamma f}{\partial x}(x) \right|^2 dt \right)^{1/2}, \quad (29)$$

the spatial Gaussian-Littlewood-Paley.

The L^p -continuity of the Gaussian-Littlewood-Paley g^γ function was proved by Gutiérrez in [5].

Theorem 2. Assume that $1 < p < \infty$. There exists a constant c_p such that

$$\|g^\gamma f\|_{p, \gamma} \leq c_p \|f\|_{p, \gamma}. \quad (30)$$

Observe that, from Theorem 2, we get immediately the $L^p(\gamma)$ -boundedness of g_1^γ and of g_2^γ .

Now, if we define

$$T^\gamma f(x, t) = t \nabla_\gamma P_t^\gamma f(x), \quad (x, t) \in \mathbb{R} \times (0, \infty), \quad (31)$$

then, the $L^p(\gamma)$ -boundedness of $g^\gamma f$ is equivalent to the boundedness of T^γ from $L^p(\gamma)$ into $L^p_{L^2((0, \infty), dt/t)}(\gamma)$.

(iii) *Laguerre Polynomials.* For $\alpha > -1$, the Laguerre polynomials $\{L_k^\alpha\}$ are defined as the orthogonal polynomials associated with the Gamma measure on $(0, \infty)$, $\mu_\alpha(dx) = \chi_{(0, \infty)}(x)(x^\alpha e^{-x}/\Gamma(\alpha+1))dx$, that is,

$$\begin{aligned} \int_0^\infty L_n^\alpha(y) L_m^\alpha(y) \mu_\alpha(dy) &= \binom{n+\alpha}{n} \delta_{n,m} \\ &= \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)n!} \delta_{n,m}, \end{aligned} \quad (32)$$

$n, m = 0, 1, 2, \dots$. We have

$$\begin{aligned} (L_k^\alpha(x))' &= -L_{k-1}^{\alpha+1}(x), \\ x(L_k^\alpha(x))'' + (\alpha+1-x)(L_k^\alpha(x))' + kL_k^\alpha(x) &= 0; \end{aligned} \quad (33)$$

thus L_k^α is an eigenfunction of the (one-dimensional) Laguerre differential operator

$$\mathcal{L}^\alpha = -x \frac{d^2}{dx^2} - (\alpha+1-x) \frac{d}{dx}, \quad (34)$$

associated with the eigenvalue $\lambda_k = k$. Observe that if we choose $\delta_\alpha = \sqrt{x}(d/dx)$, and consider its formal $L^2(\alpha)$ -adjoint,

$$\delta_\alpha^* = -\sqrt{x} \frac{d}{dx} + \left[\frac{\alpha + 1/2}{\sqrt{x}} + \sqrt{x} \right] I, \quad (35)$$

then $\mathcal{L}^\alpha = \delta_\alpha^* \delta_\alpha$. The differential operator δ_α is considered the “natural” notion of derivative in the Laguerre case.

The Laguerre-Littlewood-Paley g function can be defined as

$$g^\alpha f(x) = \left(\int_0^\infty t |\nabla_\alpha P_t^\alpha f(x)|^2 dt \right)^{1/2}, \quad (36)$$

where $\nabla_\alpha = (\partial/\partial t, \delta_\alpha) = (\partial/\partial t, \sqrt{x}(\partial/\partial x))$ and $\{P_t^\alpha\}$ is the Poisson-Laguerre semigroup, that is, the subordinated semigroup to the Laguerre semigroup; for more information see [3].

Moreover, observe that $g^\alpha f$ can be written as a singular integral with values in the Hilbert space $L^2((0, \infty), dt/t)$

$$g^\alpha f(x) = \|t \nabla_\alpha P_t^\alpha f\|_{L^2((0, \infty), dt/t)}. \quad (37)$$

We could also consider the following Laguerre Littlewood-Paley functions:

$$g_1^\alpha f(x) = \left(\int_0^\infty t \left| \frac{\partial P_t^\alpha f}{\partial t}(x) \right|^2 dt \right)^{1/2}, \quad (38)$$

the time Laguerre Littlewood-Paley, and

$$\begin{aligned} g_2^\alpha f(x) &= \left(\int_0^\infty t |\delta_\alpha P_t^\alpha f(x)|^2 dt \right)^{1/2} \\ &= \left(\int_0^\infty t \left| \sqrt{x} \frac{\partial P_t^\alpha f}{\partial x}(x) \right|^2 dt \right)^{1/2}, \end{aligned} \quad (39)$$

the spatial Laguerre Littlewood-Paley.

The L^p -continuity of the Laguerre-Littlewood-Paley g function was proved by Nowak in [6].

Theorem 3. Assume that $1 < p < \infty$ and $\alpha \in [1/2, \infty)^d$. There exists a constant c_p such that

$$\|g^\alpha f\|_{p, \alpha} \leq c_p \|f\|_{p, \alpha}. \quad (40)$$

Observe that, from Theorem 3, we get immediately the $L^p(\mu_\alpha)$ -boundedness of g_1^α and of g_2^α .

Now, if we define

$$T^\alpha f(x, t) = t \nabla_\alpha P_t^\alpha f(x), \quad (x, t) \in (0, \infty)^2, \quad (41)$$

then, the $L^p(\mu_\alpha)$ -boundedness of $g^\alpha f$ is equivalent to the boundedness of T^α from $L^p(\mu_\alpha)$ into $L^p_{L^2((0, \infty), dt/t)}(\mu_\alpha)$.

(iv) Finally, let us consider the asymptotic relations between Jacobi polynomials and other classical orthogonal polynomials (see [1], (5.3.4) and (5.6.3)):

(i) For Hermite polynomials,

$$\lim_{\lambda \rightarrow \infty} \lambda^{-n/2} C_n^\lambda \left(\frac{x}{\sqrt{\lambda}} \right) = \frac{H_n(x)}{n!}, \quad (42)$$

where $\{C_n^\lambda(x)\}$ are the Gegenbauer polynomials defined as

$$C_n^\lambda(x) = \frac{\Gamma(\lambda + 1/2) \Gamma(n + 2\lambda)}{\Gamma(2\lambda) \Gamma(n + \lambda + 1/2)} P_n^{(\lambda-1/2, \lambda-1/2)}(x). \quad (43)$$

(ii) For Laguerre polynomials,

$$\lim_{\beta \rightarrow \infty} P_n^{(\alpha, \beta)} \left(1 - \frac{2x}{\beta} \right) = L_n^\alpha(x). \quad (44)$$

Both relations hold uniformly in every closed interval of $\mathbb{R}$.

Actually these relations are expression of deeper relations between the measures and operators involved. As a consequence of those relations, we have the following technical results that were proved in [7] and are needed to prove Theorem 6.

Proposition 4 (norm relations).

(i) Let $f \in L^2(\mathbb{R}, \gamma)$ and define $f_\lambda(x) = f(\sqrt{\lambda}x)\chi_{[-1,1]}(x)$; then $f_\lambda \in L^2([-1, 1], \mu_\lambda)$ and

$$\lim_{\lambda \rightarrow \infty} \|f_\lambda\|_{2, \lambda} = \|f\|_{2, \gamma}. \quad (45)$$

(ii) Let $f \in L^2(\mathbb{R}, \mu_\alpha)$ and define $f_\beta(x) = f((\beta/2)(1 - x))\chi_{[-1,1]}(x)$; then $f_\beta \in L^2([-1, 1], \mu_{(\alpha, \beta)})$ and

$$\lim_{\beta \rightarrow \infty} \|f_\beta\|_{2, (\alpha, \beta)} = \|f\|_{2, \alpha}. \quad (46)$$

Proposition 5 (inner product relations). With the same notation as in Proposition 4,

(i) let $f \in L^2(\mathbb{R}, \gamma)$, and then

$$\lim_{\lambda \rightarrow \infty} \left\langle f_\lambda, \lambda^{-k/2} C_k^\lambda \right\rangle = \left\langle f, \frac{H_k}{k!} \right\rangle, \quad (47)$$

(ii) let $f \in L^2(\mathbb{R}, \mu_\alpha)$, and then

$$\lim_{\beta \rightarrow \infty} \left\langle f_\beta, P_k^{(\alpha, \beta)} \right\rangle = \left\langle f, L_k^\alpha \right\rangle. \quad (48)$$

The results of this paper follow the same scheme of the proof given in [7], where this transference method was used to obtain the L^p -boundedness for the Riesz transform in the Hermite and Laguerre case from the L^p -boundedness for the Riesz transform in the Jacobi case. Unfortunately due to the nonlinearity of the Littlewood-Paley g -function, the computations for the case $p \neq 2$ are more involved.

2. Main Results

We want to obtain the L^p -continuity for the Gaussian-Littlewood-Paley g and the L^p -continuity for the Laguerre-Littlewood-Paley g from the L^p -continuity of the Jacobi-Littlewood-Paley g , in the one-dimensional case ($d = 1$), using a transference method based on the asymptotic relations between Jacobi polynomials and Hermite and Laguerre polynomials. The case of the transference method in higher dimension is still open.

We will start considering the case $p = 2$; more precisely we want to prove the following.

Theorem 6. *The $L^2(\mu_{\alpha,\beta})$ -boundedness for the Jacobi-Littlewood-Paley g*

$$\|g^{(\alpha,\beta)} f\|_{2,(\alpha,\beta)} \leq C_2 \|f\|_{2,(\alpha,\beta)} \quad (49)$$

implies

$$\begin{aligned} \|g^{(\alpha,\beta)} f_\lambda\|_{2,(\alpha,\beta)}^2 &= \int_{-1}^1 \left(\int_0^\infty t \left| \nabla_{\alpha,\beta} P_t^{\alpha,\beta} f_\lambda(x) \right|^2 dt \right) \mu_{\alpha,\beta}(dx) \\ &= \int_0^\infty t \left(\int_{-1}^1 \left(\left(\frac{\partial}{\partial t} P_t^{\alpha,\beta} f_\lambda(x) \right)^2 + (1-x^2) \left(\frac{\partial}{\partial x} P_t^{\alpha,\beta} f_\lambda(x) \right)^2 \right) \mu_{\alpha,\beta}(dx) \right) dt. \end{aligned} \quad (52)$$

Let us study the first term, by Parseval's identity,

$$\begin{aligned} \int_{-1}^1 \left(\frac{\partial}{\partial t} P_t^{\alpha,\beta} f_\lambda(x) \right)^2 \mu_{\alpha,\beta}(dx) \\ = \sum_{k=1}^\infty \left| \left\langle f_\lambda, \frac{P_k^{(\alpha,\beta)}}{\|P_k^{(\alpha,\beta)}\|_{2,(\alpha,\beta)}} \right\rangle \right|^2 \lambda_k e^{-2t\lambda_k^{1/2}}. \end{aligned} \quad (53)$$

Therefore, interchanging the integral with the series, using Lebesgue's dominated convergence theorem, we get

$$\begin{aligned} \int_0^\infty t \left(\int_{-1}^1 \left(\frac{\partial}{\partial t} P_t^{\alpha,\beta} f_\lambda(x) \right)^2 \mu_{\alpha,\beta}(dx) \right) dt \\ = \frac{1}{4} \sum_{k=1}^\infty \left| \left\langle f_\lambda, \frac{P_k^{(\alpha,\beta)}}{\|P_k^{(\alpha,\beta)}\|_{2,(\alpha,\beta)}} \right\rangle \right|^2 = \frac{1}{4} \|f_\lambda\|_{2,(\alpha,\beta)}^2. \end{aligned} \quad (54)$$

In particular, taking $\alpha = \beta = \lambda - 1/2$, we obtain

$$\begin{aligned} \int_0^\infty t \left(\int_{-1}^1 \left(\frac{\partial}{\partial t} P_t^{\lambda-1/2, \lambda-1/2} f_\lambda(x) \right)^2 \mu_\lambda(dx) \right) dt \\ = \frac{1}{4} \int_{-1}^1 |f_\lambda(x)|^2 \mu_\lambda(dx) = \frac{1}{4} \|f_\lambda\|_{2,\lambda}^2. \end{aligned} \quad (55)$$

(i) *the $L^2(\gamma)$ -boundedness for the Gaussian-Littlewood-Paley g*

$$\|g^\gamma f\|_{2,\gamma} \leq C_2 \|f\|_{2,\gamma}, \quad (50)$$

(ii) *the $L^2(\mu_\alpha)$ -boundedness for the Laguerre-Littlewood-Paley g*

$$\|g^\alpha f\|_{2,\alpha} \leq C_2 \|f\|_{2,\alpha}. \quad (51)$$

Observe that this result implies that if (49) holds for $\alpha = \beta$ big enough, then (50) holds and if (49) holds for $\alpha, \beta > -1$ big enough, then (51) holds.

Proof. Even though this is the Hilbertian case, we will give some details of the proof since the exact constants involved in the computations are crucial here.

(i) Let $f \in L^2(\mathbb{R}, \gamma)$ and define $f_\lambda(x) = f(\sqrt{\lambda}x)$; then

Thus, we get using Proposition 4

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} \int_0^\infty t \left(\int_{-1}^1 \left(\frac{\partial}{\partial t} P_t^{\lambda-1/2, \lambda-1/2} f_\lambda(x) \right)^2 \mu_\lambda(dx) \right) dt \\ = \lim_{\lambda \rightarrow \infty} \frac{1}{4} \|f_\lambda\|_{2,\lambda}^2 = \frac{1}{4} \|f\|_{2,\gamma}^2 \\ = \sum_{k=1}^\infty \left| \left\langle f, \frac{H_k}{\|H_k\|_{2,\gamma}} \right\rangle \right|^2 k \int_0^\infty t e^{-2tk^{1/2}} dt \\ = \int_0^\infty t \left(\int_{-\infty}^\infty \left(\frac{\partial}{\partial t} P_t^\gamma f(x) \right)^2 d\gamma(x) \right) dt. \end{aligned} \quad (56)$$

Hence,

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} \int_0^\infty t \left(\int_{-1}^1 \left(\frac{\partial}{\partial t} P_t^{\lambda-1/2, \lambda-1/2} f_\lambda(x) \right)^2 \mu_\lambda(dx) \right) dt \\ = \int_0^\infty t \left(\int_{-\infty}^\infty \left(\frac{\partial}{\partial t} P_t^\gamma f(x) \right)^2 d\gamma(x) \right) dt. \end{aligned} \quad (57)$$

Now, let us consider the second term. First of all, observe that

$$\begin{aligned} \left\| \sqrt{1-x^2} P_{k-1}^{(\alpha+1, \beta+1)} \right\|_{2,(\alpha,\beta)}^2 \\ = \frac{4(\alpha+1)(\beta+1)}{2^{\alpha+\beta+3}(\alpha+\beta+3)(\alpha+\beta+2)B(\alpha+2, \beta+2)} \\ \times \int_{-1}^1 (1-x)^{\alpha+1} (1+x)^{\beta+1} \left[P_{k-1}^{(\alpha+1, \beta+1)}(x) \right]^2 dx \end{aligned}$$

$$\begin{aligned}
&= \frac{4(\alpha+1)(\beta+1)}{(\alpha+\beta+3)(\alpha+\beta+2)} \|P_{k-1}^{(\alpha+1,\beta+1)}\|_{2,(\alpha+1,\beta+1)}^2 \\
&= \frac{4k}{(k+\alpha+\beta+1)} \|P_k^{(\alpha,\beta)}\|_{2,(\alpha,\beta)}^2.
\end{aligned} \tag{58}$$

Then, using Parseval's identity, we get

$$\begin{aligned}
&\int_{-1}^1 \left(\sqrt{1-x^2} \frac{\partial}{\partial x} P_t^{\alpha,\beta} f_\lambda(x) \right)^2 \mu_{\alpha,\beta}(dx) \\
&= \left\| \sqrt{1-x^2} \frac{\partial}{\partial x} P_t^{\alpha,\beta} f_\lambda \right\|_{2,(\alpha,\beta)}^2 \\
&= \sum_{k=1}^{\infty} \left| \left\langle f_\lambda, \frac{P_k^{(\alpha,\beta)}}{\|P_k^{(\alpha,\beta)}\|_{2,(\alpha,\beta)}} \right\rangle \right|^2 e^{-2t\lambda_k^{1/2}} \\
&\quad \cdot \frac{(k+\alpha+\beta+1)^2}{4} \left\| \sqrt{1-x^2} P_{k-1}^{(\alpha+1,\beta+1)} \right\|_{2,(\alpha,\beta)}^2 \\
&= \sum_{k=1}^{\infty} \left| \left\langle f_\lambda, \frac{P_k^{(\alpha,\beta)}}{\|P_k^{(\alpha,\beta)}\|_2} \right\rangle \right|^2 e^{-2t\lambda_k^{1/2}} \lambda_k \|P_k^{(\alpha,\beta)}\|_{2,(\alpha,\beta)}^2 \\
&= \sum_{k=1}^{\infty} \left| \left\langle f_\lambda, \frac{P_k^{(\alpha,\beta)}}{\|P_k^{(\alpha,\beta)}\|_2} \right\rangle \right|^2 \lambda_k e^{-2t\lambda_k^{1/2}}.
\end{aligned} \tag{59}$$

Now as before, using Lebesgue's dominated convergence theorem, interchanging the integral with the series, we get

$$\begin{aligned}
&\int_0^\infty t \left(\int_{-1}^1 \left(\sqrt{1-x^2} \frac{\partial}{\partial x} P_t^{\alpha,\beta} f_\lambda(x) \right)^2 \mu_{\alpha,\beta}(dx) \right) dt \\
&= \int_0^\infty t \left(\sum_{k=1}^{\infty} \left| \left\langle f_\lambda, \frac{P_k^{(\alpha,\beta)}}{\|P_k^{(\alpha,\beta)}\|_{2,(\alpha,\beta)}} \right\rangle \right|^2 \right. \\
&\quad \cdot \lambda_k e^{-2t\lambda_k^{1/2}} \left. \right) dt = \frac{1}{4} \int_{-1}^1 |f_\lambda(x)|^2 \mu_{\alpha,\beta}(dx).
\end{aligned} \tag{60}$$

Then, for the Gegenbauer case, $\alpha = \beta = \lambda - 1/2$, we get

$$\begin{aligned}
&\int_0^\infty t \left(\int_{-1}^1 \left(\sqrt{1-x^2} \frac{\partial}{\partial x} P_t^{\lambda-1/2,\lambda-1/2} f_\lambda(x) \right)^2 \right. \\
&\quad \cdot \mu_\lambda(dx) \left. \right) dt = \frac{1}{4} \|f_\lambda\|_{2,\lambda}^2.
\end{aligned} \tag{61}$$

On the other hand, again using Parseval's identity, we have

$$\begin{aligned}
&\int_{-\infty}^\infty \left(\frac{1}{\sqrt{2}} \frac{\partial}{\partial x} P_t^\gamma f(x) \right)^2 d\gamma(x) \\
&= \sum_{k=1}^\infty \left| \left\langle f, \frac{H_k}{\|H_k\|_{2,\gamma}} \right\rangle \right|^2 2k^2 e^{-2tk^{1/2}} \|H_{k-1}\|_{2,\gamma}^2 \\
&= \sum_{k=1}^\infty \left| \left\langle f, \frac{H_k}{\|H_k\|_{2,\gamma}} \right\rangle \right|^2 2k^2 e^{-2tk^{1/2}} \frac{1}{2k} \|H_k\|_{2,\gamma}^2 \\
&= \sum_{k=1}^\infty \left| \left\langle f, \frac{H_k}{\|H_k\|_{2,\gamma}} \right\rangle \right|^2 k e^{-2tk^{1/2}}.
\end{aligned} \tag{62}$$

Hence,

$$\begin{aligned}
&\int_0^\infty t \left(\int_{-\infty}^\infty \left(\frac{1}{\sqrt{2}} \frac{\partial}{\partial x} P_t^\gamma f(x) \right)^2 d\gamma(x) \right) dt \\
&= \int_0^\infty t \left(\sum_{k=1}^\infty \left| \left\langle f, \frac{H_k}{\|H_k\|_{2,\gamma}} \right\rangle \right|^2 k e^{-2tk^{1/2}} \right) dt \\
&= \frac{1}{4} \|f\|_{2,\gamma}^2.
\end{aligned} \tag{63}$$

Therefore, using Proposition 4,

$$\begin{aligned}
&\lim_{\lambda \rightarrow \infty} \int_0^\infty t \left(\int_{-1}^1 \left(\sqrt{1-x^2} \frac{\partial}{\partial x} P_t^{\lambda-1/2,\lambda-1/2} f_\lambda(x) \right)^2 \right. \\
&\quad \cdot \mu_\lambda(dx) \left. \right) dt = \lim_{\lambda \rightarrow \infty} \frac{1}{4} \|f_\lambda\|_{2,\lambda}^2 = \frac{1}{4} \|f\|_{2,\gamma}^2 \\
&= \int_0^\infty t \left(\int_{-\infty}^\infty \left(\frac{1}{\sqrt{2}} \frac{\partial}{\partial x} P_t^\gamma f(x) \right)^2 d\gamma(x) \right) dt.
\end{aligned} \tag{64}$$

Now, taking $\alpha = \beta = \lambda - 1/2$,

$$\begin{aligned}
&\lim_{\lambda \rightarrow \infty} \|g^{(\lambda-1/2,\lambda-1/2)} f_\lambda\|_{2,\lambda}^2 \\
&= \lim_{\lambda \rightarrow \infty} \int_0^\infty t \left(\int_{-1}^1 \left(\frac{\partial}{\partial t} P_t^{\lambda-1/2,\lambda-1/2} f_\lambda(x) \right)^2 \mu_\lambda(dx) \right) dt \\
&\quad + \lim_{\lambda \rightarrow \infty} \int_0^\infty t \left(\int_{-1}^1 \left(\sqrt{1-x^2} \frac{\partial}{\partial x} P_t^{\lambda-1/2,\lambda-1/2} f_\lambda(x) \right)^2 \right. \\
&\quad \cdot \mu_\lambda(dx) \left. \right) dt \\
&= \int_0^\infty t \left(\int_{-\infty}^\infty \left(\frac{\partial}{\partial t} P_t^\gamma f(x) \right)^2 d\gamma(x) \right) dt
\end{aligned}$$

$$\begin{aligned}
& + \int_0^\infty t \left(\int_{-\infty}^\infty \left(\frac{1}{\sqrt{2}} \frac{\partial}{\partial x} P_t^\gamma f(x) \right)^2 dy(x) \right) dt \\
& = \|g^\gamma f\|_{2,\gamma}^2.
\end{aligned} \tag{65}$$

Now, by the L^2 -continuity of $g^{(\lambda-1/2, \lambda-1/2)}$, we have

$$\|g^{(\lambda-1/2, \lambda-1/2)} f_\lambda\|_{2,\lambda} \leq C_2 \|f_\lambda\|_{2,\lambda}. \tag{66}$$

Finally,

$$\begin{aligned}
\|g^\gamma f\|_{2,\gamma} &= \lim_{\lambda \rightarrow \infty} \|g^{(\lambda-1/2, \lambda-1/2)} f_\lambda\|_{2,\lambda} \\
&\leq C_2 \lim_{\lambda \rightarrow \infty} \|f_\lambda\|_{2,\lambda} = C_2 \|f\|_{2,\gamma}.
\end{aligned} \tag{67}$$

(ii) Let $f \in L^2((0, \infty), \mu_\alpha)$, and define $f_\beta(x) = f((\beta/2)(1-x))$ and then analogously as in the Hermite case

$$\|g^{(\alpha, \beta)} f_\beta\|_{2,(\alpha, \beta)}^2 = \int_0^\infty t \left(\int_{-1}^1 \left(\left(\frac{\partial}{\partial t} P_t^{\alpha, \beta} f_\beta(x) \right)^2 + (1-x^2) \left(\frac{\partial}{\partial x} P_t^{\alpha, \beta} f_\beta(x) \right)^2 \right) \mu_{\alpha, \beta}(dx) \right) dt, \tag{68}$$

where

$$\begin{aligned}
& \int_0^\infty t \left(\int_{-1}^1 \left(\frac{\partial}{\partial t} P_t^{\alpha, \beta} f_\beta(x) \right)^2 \mu_{\alpha, \beta}(dx) \right) dt \\
&= \frac{1}{4} \int_{-1}^1 |f_\beta(x)|^2 \mu_{\alpha, \beta}(dx) = \frac{1}{4} \|f_\beta\|_{2,(\alpha, \beta)}^2, \\
& \int_0^\infty t \left(\int_{-1}^1 \left(\sqrt{1-x^2} \frac{\partial}{\partial x} P_t^{\alpha, \beta} f_\beta(x) \right)^2 \mu_{\alpha, \beta}(dx) \right) dt \\
&= \frac{1}{4} \int_{-1}^1 |f_\beta(x)|^2 \mu_{\alpha, \beta}(dx) = \frac{1}{4} \|f_\beta\|_{2,(\alpha, \beta)}^2.
\end{aligned} \tag{69}$$

Let us study the first term. Using Proposition 4 and Lebesgue's dominated convergence theorem, interchanging the integral with the series, we get

$$\begin{aligned}
& \lim_{\beta \rightarrow \infty} \int_0^\infty t \left(\int_{-1}^1 \left(\frac{\partial}{\partial t} P_t^{\alpha, \beta} f_\beta(x) \right)^2 \mu_{\alpha, \beta}(dx) \right) dt \\
&= \lim_{\beta \rightarrow \infty} \frac{1}{4} \|f_\beta\|_{2,(\alpha, \beta)}^2 = \frac{1}{4} \|f\|_{2,\alpha}^2 \\
&= \int_0^\infty t \left(\sum_{k=1}^\infty \left| \left\langle f, \frac{L_k^\alpha}{\|L_k^\alpha\|_{2,\alpha}} \right\rangle \right|^2 k e^{-2tk^{1/2}} \right) dt \\
&= \int_0^\infty t \left(\int_0^\infty \left(\frac{\partial}{\partial t} P_t^\alpha f(x) \right)^2 \mu_\alpha(dx) \right) dt.
\end{aligned} \tag{70}$$

Let us look now at the second term. First of all, observe that

$$\begin{aligned}
\|\sqrt{x} L_{k-1}^{\alpha+1}\|_{2,\alpha}^2 &= \int_0^\infty |\sqrt{x} L_{k-1}^{\alpha+1}(x)|^2 \frac{x^\alpha e^{-x}}{\Gamma(\alpha+1)} dx \\
&= (\alpha+1) \int_0^\infty |L_{k-1}^{\alpha+1}(x)|^2 \frac{x^{\alpha+1} e^{-x}}{\Gamma(\alpha+2)} dx \\
&= (\alpha+1) \|L_{k-1}^{\alpha+1}\|_{2,(\alpha+1)}^2 \\
&= (\alpha+1) \frac{\Gamma(k-1+\alpha+1+1)}{\Gamma(\alpha+2)(k-1)!} \\
&= k \|L_k^\alpha\|_{2,\alpha}^2;
\end{aligned} \tag{71}$$

then we get, using Parseval's identity,

$$\begin{aligned}
& \int_0^\infty \left(\sqrt{x} \frac{\partial}{\partial x} P_t^\alpha f(x) \right)^2 \mu_\alpha(dx) \\
&= \sum_{k=1}^\infty \left| \left\langle f, \frac{L_k^\alpha}{\|L_k^\alpha\|_{2,\alpha}} \right\rangle \right|^2 e^{-2tk^{1/2}} \|\sqrt{x} L_{k-1}^{\alpha+1}\|_{2,\alpha}^2 \\
&= \sum_{k=1}^\infty \left| \left\langle f, \frac{L_k^\alpha}{\|L_k^\alpha\|_{2,\alpha}} \right\rangle \right|^2 k e^{-2tk^{1/2}}.
\end{aligned} \tag{72}$$

Hence,

$$\begin{aligned}
& \lim_{\beta \rightarrow \infty} \int_0^\infty t \left(\int_{-1}^1 \left(\sqrt{1-x^2} \frac{\partial}{\partial x} P_t^{\alpha, \beta} f_\beta(x) \right)^2 \mu_{\alpha, \beta}(dx) \right) dt \\
&= \lim_{\beta \rightarrow \infty} \frac{1}{4} \|f_\beta\|_{2,(\alpha, \beta)}^2 \\
&= \frac{1}{4} \|f\|_{2,\alpha}^2 = \int_0^\infty t \left(\sum_{k=1}^\infty \left| \left\langle f, \frac{L_k^\alpha}{\|L_k^\alpha\|_{2,\alpha}} \right\rangle \right|^2 \right. \\
&\quad \cdot k e^{-2tk^{1/2}} \left. \right) dt \\
&= \int_0^\infty t \left(\int_0^\infty \left(\sqrt{x} \frac{\partial}{\partial x} P_t^\alpha f(x) \right)^2 \mu_\alpha(dx) \right) dt.
\end{aligned} \tag{73}$$

Therefore,

$$\begin{aligned}
& \lim_{\beta \rightarrow \infty} \|g^{(\alpha, \beta)} f_\beta\|_{2,(\alpha, \beta)}^2 \\
&= \lim_{\beta \rightarrow \infty} \int_0^\infty t \left(\int_{-1}^1 \left(\frac{\partial}{\partial t} P_t^{\alpha, \beta} f_\beta(x) \right)^2 \mu_{\alpha, \beta}(dx) \right) dt \\
&\quad + \lim_{\beta \rightarrow \infty} \int_0^\infty t \left(\int_{-1}^1 \left(\sqrt{1-x^2} \frac{\partial}{\partial x} P_t^{\alpha, \beta} f_\beta(x) \right)^2 \mu_{\alpha, \beta}(dx) \right) dt.
\end{aligned}$$

$$\begin{aligned}
& \cdot \mu_{\alpha,\beta}(dx) \Big) dt = \int_0^\infty t \left(\int_0^\infty \left(\frac{\partial}{\partial t} P_t^\alpha f(x) \right)^2 \right. \\
& \cdot \mu_\alpha(dx) \Big) dt + \int_0^\infty t \left(\int_0^\infty \left(\sqrt{x} \frac{\partial}{\partial x} P_t^\alpha f(x) \right)^2 \right. \\
& \cdot \mu_\alpha(dx) \Big) dt = \|g^\alpha f\|_{2,\alpha}^2
\end{aligned} \tag{74}$$

and now, by the L^2 -continuity of $g^{(\alpha,\beta)}$, we have

$$\|g^{(\alpha,\beta)} f_\beta\|_{2,(\alpha,\beta)} \leq C_2 \|f_\beta\|_{2,(\alpha,\beta)}. \tag{75}$$

Thus, finally,

$$\begin{aligned}
\|g^\alpha f\|_{2,\alpha} &= \lim_{\beta \rightarrow \infty} \|g^{(\alpha,\beta)} f_\beta\|_{2,(\alpha,\beta)} \leq C_2 \lim_{\beta \rightarrow \infty} \|f_\beta\|_{2,(\alpha,\beta)} \\
&= C_2 \|f\|_{2,\alpha}.
\end{aligned} \tag{76}$$

□

Now we are going to consider the general case $p \neq 2$. For the proof, we will follow the argument given by Betancor et al. in [8].

Theorem 7. *Let $\alpha, \beta > -1$ and $1 < p < \infty$; then the $L^p(\mu_{\alpha,\beta})$ -boundedness for the Jacobi-Littlewood-Paley g -function*

$$\|g^{(\alpha,\beta)} f\|_{p,(\alpha,\beta)} \leq C_p \|f\|_{p,(\alpha,\beta)} \tag{77}$$

implies

(i) *the $L^p(\gamma)$ -boundedness for the Gaussian-Littlewood-Paley g -function*

$$\|g^\gamma f\|_{p,\gamma} \leq C_p \|f\|_{p,\gamma}, \tag{78}$$

(ii) *the $L^p(\mu_\alpha)$ -boundedness for the Laguerre-Littlewood-Paley g -function*

$$\|g^\alpha f\|_{p,\alpha} \leq C_p \|f\|_{p,\alpha}. \tag{79}$$

From the vector representations of the Littlewood-Paley g -functions, this result could be interpreted then as transference of vector-valued operators (moreover, for the case of the time Littlewood-Paley g -functions, this can be interpreted as transference of $L^2((0, \infty), dt/t)$ -valued multipliers; see [9]).

Proof. (i) Assume that the operator $g^{(\lambda-1/2, \lambda-1/2)}$ is bounded in $L^p([-1, 1], \mu_\lambda)$. Let $\phi \in C_0^\infty(\mathbb{R})$, and for each $\lambda > 0$ define the function

$$\phi_\lambda(x) = \phi(\sqrt{\lambda}x), \tag{80}$$

$x \in \mathbb{R}$. Let λ big enough such that $\text{supp } \phi_\lambda$ is contained in $[-1, 1]$. In what follows, λ will be taken satisfying that condition.

Now, from the boundedness of $g^{(\lambda-1/2, \lambda-1/2)}$ we have

$$\|g^{(\lambda-1/2, \lambda-1/2)} \phi_\lambda\|_{p,\lambda} \leq C \|\phi_\lambda\|_{p,\lambda}; \tag{81}$$

that is to say,

$$\left\| \left[\int_0^\infty t \left| \nabla_\lambda P_t^{\lambda-1/2, \lambda-1/2} \phi_\lambda \right|^2 dt \right]^{1/2} \right\|_{p,\lambda} \leq C \|\phi_\lambda\|_{p,\lambda}, \tag{82}$$

where $\nabla_\lambda = \nabla_{(\lambda-1/2, \lambda-1/2)}$. Now making the change of variables $x = y/\sqrt{\lambda}$ and taking $Z(\lambda) = (\lambda^{1/2} [\Gamma(\lambda)]^2 2^{2\lambda}) / (2\pi \Gamma(2\lambda))$ we have

$$\begin{aligned}
& \left\{ \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \left[\int_0^\infty t \left| \nabla_\lambda P_t^{\lambda-1/2, \lambda-1/2} \phi_\lambda \left(\frac{y}{\sqrt{\lambda}} \right) \right|^2 dt \right]^{1/2} \right\}^p Z(\lambda) \\
& \cdot \left(1 - \frac{y^2}{\lambda} \right)^{\lambda-1/2} dy \Big\}^{1/p} \leq C \|\phi_\lambda\|_{p,\lambda},
\end{aligned} \tag{83}$$

which implies

$$\begin{aligned}
& \left\{ \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \left[\int_0^\infty t \left| \nabla_\lambda P_t^{\lambda-1/2, \lambda-1/2} \phi_\lambda \left(\frac{y}{\sqrt{\lambda}} \right) \right|^2 dt \right]^{1/2} \right. \\
& \cdot \left(1 - \frac{y^2}{\lambda} \right)^{\lambda/p-1/2p} e^{y^2/p} \left| \frac{e^{-y^2}}{\sqrt{\pi}} dy \right\}^{1/p} \\
& \leq C (Z(\lambda))^{-1/p} \|\phi_\lambda\|_{p,\lambda}.
\end{aligned} \tag{84}$$

On the other hand, analogously we have, from the case $p = 2$,

$$\begin{aligned}
& \left\{ \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \left[\int_0^\infty t \left| \nabla_\lambda P_t^{\lambda-1/2, \lambda-1/2} \phi_\lambda \left(\frac{y}{\sqrt{\lambda}} \right) \right|^2 dt \right]^{1/2} \right. \\
& \cdot \left(1 - \frac{y^2}{\lambda} \right)^{\lambda/2-1/4} e^{y^2/2} \left| \frac{e^{-y^2}}{\sqrt{\pi}} dy \right\}^{1/2} \\
& \leq C (Z(\lambda))^{-1/2} \|\phi_\lambda\|_{2,\lambda}.
\end{aligned} \tag{85}$$

Now, define, for any $K \in \mathbb{N}$ and $\lambda > 0$ such that $\sqrt{\lambda} > K$, the functions

$$F_{\lambda,K}(y) = \begin{cases} \left[\int_0^\infty t \left| \nabla_\lambda P_t^{\lambda-1/2, \lambda-1/2} \phi_\lambda \left(\frac{y}{\sqrt{\lambda}} \right) \right|^2 dt \right]^{1/2} \left(1 - \frac{y^2}{\lambda} \right)^{\lambda/2-1/4} e^{y^2/2} & \text{if } |y| \leq K \\ 0 & \text{if } |y| > K, \end{cases}$$

$$f_{\lambda,K}(y) = \begin{cases} \left[\int_0^\infty t \left| \nabla_\lambda P_t^{\lambda-1/2, \lambda-1/2} \phi_\lambda \left(\frac{y}{\sqrt{\lambda}} \right) \right|^2 dt \right]^{1/2} \left(1 - \frac{y^2}{\lambda} \right)^{\lambda/p-1/2p} e^{y^2/p} & \text{if } |y| \leq K \\ 0 & \text{if } |y| > K. \end{cases} \quad (86)$$

Observe that $F_{\lambda,K} = f_{\lambda,K} \Omega_\lambda$, where

$$\Omega_\lambda(y) = e^{y^2/2-y^2/p} \left(1 - \frac{y^2}{\lambda} \right)^{\lambda/2-\lambda/p-1/4+1/2p}, \quad (87)$$

for all $K \in \mathbb{N}$ and $\sqrt{\lambda} > K$. Moreover,

$$\begin{aligned} |\Omega_\lambda(y)| &\leq e^{y^2/2-y^2/p} e^{-(y^2/\lambda)(\lambda/2-\lambda/p-1/4+1/2p)} \\ &= e^{y^2(1/4\lambda-1/2p\lambda)}. \end{aligned} \quad (88)$$

Therefore, if $p \leq 2$ $|\Omega_\lambda(y)| \leq 1$ and for $p > 2$

$$|\Omega_\lambda(y)| \leq e^{K^2(1/4\lambda-1/2p\lambda)}, \quad (89)$$

hence Ω_λ is bounded in $[-K, K]$. Now

$$\begin{aligned} (Z(\lambda))^{-1/p} \|\phi_\lambda\|_{p,\lambda} &= (Z(\lambda))^{-1/p} \left\{ \int_{-1}^1 |\phi_\lambda(x)|^p \right. \\ &\quad \cdot \frac{\lambda [\Gamma(\lambda)]^2 2^{2\lambda}}{2\pi\Gamma(2\lambda)} (1-x^2)^{\lambda-1/2} \Big\}^{1/p} dx. \end{aligned} \quad (90)$$

Thus, making the change of variables $x = y/\sqrt{\lambda}$, we get

$$\begin{aligned} (Z(\lambda))^{-1/p} \|\phi_\lambda\|_{p,\lambda} &= (Z(\lambda))^{-1/p} \\ &\quad \cdot \left\{ \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \left| \phi_\lambda \left(\frac{y}{\sqrt{\lambda}} \right) \right|^p Z(\lambda) \left(1 - \frac{y^2}{\lambda} \right)^{\lambda-1/2} dy \right\}^{1/p} \\ &= \left\{ \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} |\phi(y)|^p \left(1 - \frac{y^2}{\lambda} \right)^{\lambda-1/2} dy \right\}^{1/p} \\ &\leq C \left\{ \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} |\phi(y)|^p \frac{e^{-y^2}}{\sqrt{\pi}} dy \right\}^{1/p}. \end{aligned} \quad (91)$$

Therefore,

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} (Z(\lambda))^{-1/p} \|\phi_\lambda\|_{p,\lambda} &\leq \lim_{\lambda \rightarrow \infty} C \left\{ \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} |\phi(y)|^p \frac{e^{-y^2}}{\sqrt{\pi}} dy \right\}^{1/p} \\ &= C \left\{ \int_{-\infty}^{\infty} |\phi(y)|^p \frac{e^{-y^2}}{\sqrt{\pi}} dy \right\}^{1/p} = C \|\phi\|_{p,\gamma}, \end{aligned} \quad (92)$$

and moreover,

$$(Z(\lambda))^{-1/p} \|\phi_\lambda\|_{p,\lambda} \leq C \|\phi\|_{p,\gamma}. \quad (93)$$

On the other hand,

$$\begin{aligned} \|F_{\lambda,K}\|_{2,\gamma} &= \left\{ \int_{\mathbb{R}} |F_{\lambda,K}(y)|^2 \frac{e^{-y^2}}{\sqrt{\pi}} dy \right\}^{1/2} \\ &= \left\{ \int_{-K}^K |F_{\lambda,K}(y)|^2 \frac{e^{-y^2}}{\sqrt{\pi}} dy \right\}^{1/2} \\ &\leq \left\{ \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} |F_{\lambda,K}(y)|^2 \frac{e^{-y^2}}{\sqrt{\pi}} dy \right\}^{1/2} \\ &\leq C (Z(\lambda))^{-1/2} \|\phi_\lambda\|_{2,\lambda}. \end{aligned} \quad (94)$$

Then, from (93) and (94), we have

$$\|F_{\lambda,K}\|_{2,\gamma} \leq C \|\phi\|_{2,\gamma}. \quad (95)$$

Analogously,

$$\begin{aligned} \|F_{\lambda,K}\|_{p,\gamma} &= \left\{ \int_{\mathbb{R}} |F_{\lambda,K}(y)|^p \frac{e^{-y^2}}{\sqrt{\pi}} dy \right\}^{1/p} \\ &= \left\{ \int_{-K}^K |f_{\lambda,K} \Omega_\lambda(y)|^p \frac{e^{-y^2}}{\sqrt{\pi}} dy \right\}^{1/p}. \end{aligned} \quad (96)$$

Since Ω_λ is bounded in $[-k, k]$, we get

$$\begin{aligned} \|F_{\lambda,K}\|_{p,\gamma} &\leq C \left\{ \int_{-\infty}^{\infty} |f_{\lambda,K}(y)|^p \frac{e^{-y^2}}{\sqrt{\pi}} dy \right\}^{1/p} \\ &\leq C (Z(\lambda))^{-1/p} \|\phi_\lambda\|_{p,\lambda}. \end{aligned} \quad (97)$$

Then, from (93) and (99) we have

$$\|F_{\lambda,K}\|_{p,\gamma} \leq C \|\phi\|_{p,\gamma} \quad (98)$$

for all $\sqrt{\lambda} > K$. Thus, $F_{\lambda,K}$ is a bounded sequence in $L^2(\mathbb{R}, \gamma)$ and $L^p(\mathbb{R}, \gamma)$. By Bourbaki-Alaoglu's theorem, there exists a subsequence $(\lambda_j)_{j \in \mathbb{N}}$ such that $\lim_{j \rightarrow \infty} \lambda_j = \infty$ and functions $F_K \in L^2(\mathbb{R}, \gamma)$ and $f_K \in L^p(\mathbb{R}, \gamma)$ satisfying

- (a) $F_{\lambda_j,K} \rightarrow F_K$, as $j \rightarrow \infty$, in the weak topology on $L^2(\mathbb{R}, \gamma)$,
- (b) $F_{\lambda_j,K} \rightarrow f_K$, as $j \rightarrow \infty$, in the weak topology on $L^p(\mathbb{R}, \gamma)$.

Moreover, $\text{supp } F_K \cup \text{supp } f_K \subseteq [-K, K]$, and

$$\|F_K\|_{2,\gamma} \leq \lim_{j \rightarrow \infty} \|F_{\lambda_j,K}\|_{2,\gamma} \leq C \|\phi\|_{2,\gamma}. \quad (99)$$

Analogously, one gets

$$\|f_K\|_{p,\gamma} \leq C \|\phi\|_{p,\gamma}. \quad (100)$$

Observe that, for every $k \in \mathbb{N}$,

$$\tau_K(g) = \int_{-\infty}^{\infty} g(x) \chi_{[-K,K]}(x) dx; \quad (101)$$

then, by Cauchy-Schwartz inequality $\tau_K \in (L^2(\mathbb{R}, \gamma))^*$, and therefore,

$$\begin{aligned} \int_{-K}^K F_K(x) dx &= \int_{-\infty}^{\infty} F_K(x) \chi_{[-K,K]}(x) dx \\ &= \tau_K(F_K) = \lim_{j \rightarrow \infty} \tau_K(F_{\lambda_j, K}) \\ &= \lim_{j \rightarrow \infty} \int_{-\infty}^{\infty} F_{\lambda_j, K}(x) \chi_{[-K,K]}(x) dx \\ &= \int_{-K}^K f_K(x) dx; \end{aligned} \quad (102)$$

thus, $F_K = f_K$ a.e. on $[-K, K]$, for all $K \in \mathbb{N}$, so $F_K = f_K$ a.e. on $\mathbb{R}$. Then from (100), we get

$$\|F_K\|_{p,\gamma} \leq C \|\phi\|_{p,\gamma}, \quad (103)$$

and therefore, from (99) and (103), there exists an increasing sequence $\{\lambda_j\}_{j \in \mathbb{N}} \subset (0, \infty)$, with $\lim_{j \rightarrow \infty} \lambda_j = \infty$, and a function $F \in L^p(\mathbb{R}, \gamma) \cap L^2(\mathbb{R}, \gamma)$, such that

- (a) for each $K \in \mathbb{N}$, $F_{\lambda_j, K} \rightarrow F$, as $j \rightarrow \infty$, in the weak topology of $L^2(\mathbb{R}, \gamma)$ and in the weak topology of $L^p(\mathbb{R}, \gamma)$,
- (b) $\|F\|_{p,\gamma} \leq C \|\phi\|_{p,\gamma}$.

On the other hand,

$$\begin{aligned} F_{\lambda, K}(y) &= \chi_{[-K, K]}(y) \\ &\cdot \left(\int_0^{\infty} t \left| \nabla_{\lambda} P_t^{\lambda-1/2, \lambda-1/2} \phi_{\lambda} \left(\frac{y}{\sqrt{\lambda}} \right) \right|^2 \right. \\ &\cdot \left. \left(1 - \frac{y^2}{\lambda} \right)^{\lambda-1/2} e^{y^2} dt \right)^{1/2} = \chi_{[-K, K]}(y) \\ &\cdot \left[\int_0^{\infty} t \left(\frac{\partial}{\partial t} P_t^{\lambda-1/2, \lambda-1/2} \phi_{\lambda} \left(\frac{y}{\sqrt{\lambda}} \right) \right)^2 dt \left(1 - \frac{y^2}{\lambda} \right)^{\lambda-1/2} e^{y^2} \right. \\ &\cdot \left. \left(\frac{\partial}{\partial x} P_t^{\lambda-1/2, \lambda-1/2} \phi_{\lambda} \left(\frac{y}{\sqrt{\lambda}} \right) \right)^2 dt \left(1 - \frac{y^2}{\lambda} \right)^{\lambda-1/2} e^{y^2} \right]^{1/2}. \end{aligned} \quad (104)$$

Now, define

$$\begin{aligned} g_{1,\lambda} \phi(y) &= \chi_{[-K, K]}(y) \\ &\cdot \left(\int_0^{\infty} t \left(\frac{\partial}{\partial t} P_t^{\lambda-1/2, \lambda-1/2} \phi_{\lambda} \left(\frac{y}{\sqrt{\lambda}} \right) \right)^2 dt \right) \left(1 - \frac{y^2}{\lambda} \right)^{\lambda-1/2} e^{y^2}, \\ g_{2,\lambda} \phi(y) &= \chi_{[-K, K]}(y) \left(\int_0^{\infty} t \left(1 - \frac{y^2}{\lambda} \right) \right. \\ &\cdot \left. \left(\frac{\partial}{\partial y} P_t^{\lambda-1/2, \lambda-1/2} \phi_{\lambda} \left(\frac{y}{\sqrt{\lambda}} \right) \right)^2 dt \right) \left(1 - \frac{y^2}{\lambda} \right)^{\lambda-1/2} e^{y^2}. \end{aligned} \quad (105)$$

Let us study first the function $g_{1,\lambda}$. For a function $\phi \in L^2(\mathbb{R}, \gamma)$,

$$\begin{aligned} \sqrt{t} \frac{\partial}{\partial t} P_t^{\alpha, \beta} \phi_{\lambda} \left(\frac{y}{\sqrt{\lambda}} \right) &= - \sum_{k=1}^{\infty} \left\langle \phi_{\lambda}, \frac{P_k^{(\alpha, \beta)}}{\|P_k^{(\alpha, \beta)}\|_{2, (\alpha, \beta)}^2} \right\rangle \\ &\cdot (t \lambda_k)^{1/2} e^{-t \lambda_k^{1/2}} P_k^{(\alpha, \beta)} \left(\frac{y}{\sqrt{\lambda}} \right) \end{aligned} \quad (106)$$

which converges absolutely. Then, taking the Cauchy product, we obtain

$$\begin{aligned} &\left(\sqrt{t} \frac{\partial}{\partial t} P_t^{\alpha, \beta} \phi_{\lambda} \left(\frac{y}{\sqrt{\lambda}} \right) \right)^2 \\ &= \sum_{k=1}^{\infty} \sum_{n=1}^k \left\langle \phi_{\lambda}, \frac{P_n^{(\alpha, \beta)}}{\|P_n^{(\alpha, \beta)}\|_2^2} \right\rangle \left\langle \phi_{\lambda}, \frac{P_{k-n}^{(\alpha, \beta)}}{\|P_{k-n}^{(\alpha, \beta)}\|_2^2} \right\rangle \\ &\cdot t \lambda_n^{1/2} \lambda_{k-n}^{1/2} e^{-t(\lambda_n^{1/2} + \lambda_{k-n}^{1/2})} P_n^{(\alpha, \beta)} \left(\frac{y}{\sqrt{\lambda}} \right) P_{k-n}^{(\alpha, \beta)} \left(\frac{y}{\sqrt{\lambda}} \right). \end{aligned} \quad (107)$$

Then, for the Gegenbauer case, $\alpha = \beta = \lambda - 1/2$, taking $\widehat{\lambda}_k = k(k+2\lambda)$

$$\begin{aligned} &\left(\sqrt{t} \frac{\partial}{\partial t} P_t^{\lambda-1/2, \lambda-1/2} \phi_{\lambda} \left(\frac{y}{\sqrt{\lambda}} \right) \right)^2 = \sum_{k=1}^{\infty} \sum_{n=1}^k \langle \phi_{\lambda}, C_n^{\lambda} \rangle \\ &\cdot \langle \phi_{\lambda}, C_{k-n}^{\lambda} \rangle t \widehat{\lambda}_n^{1/2} \widehat{\lambda}_{k-n}^{1/2} e^{-t(\widehat{\lambda}_n^{1/2} + \widehat{\lambda}_{k-n}^{1/2})} C_n^{\lambda} \left(\frac{y}{\sqrt{\lambda}} \right) \\ &\cdot C_{k-n}^{\lambda} \left(\frac{y}{\sqrt{\lambda}} \right) \times \left[\frac{\Gamma(2\lambda) \Gamma(n+\lambda+1/2)}{\Gamma(\lambda+1/2) \Gamma(n+2\lambda)} \right. \\ &\cdot \left. \frac{\Gamma(2\lambda) \Gamma(k-n+\lambda+1/2)}{\Gamma(\lambda+1/2) \Gamma(k-n+2\lambda)} \right]^2 \\ &\cdot \frac{2(n+\lambda) [\Gamma(\lambda+1/2)]^2 n! \Gamma(n+2\lambda)}{\Gamma(2\lambda+1) \Gamma(n+\lambda+1/2)} \end{aligned}$$

$$\begin{aligned}
& \times \frac{2(k-n+\lambda) [\Gamma(\lambda+1/2)]^2 (k-n)! \Gamma(k-n+2\lambda)}{\Gamma(2\lambda+1) \Gamma(k-n+\lambda+1/2)} \\
& = \sum_{k=1}^{\infty} \sum_{n=1}^k \langle \phi_{\lambda}, C_n^{\lambda} \rangle \langle \phi_{\lambda}, C_{k-n}^{\lambda} \rangle \\
& \quad \cdot t \hat{\lambda}_n^{1/2} \hat{\lambda}_{k-n}^{1/2} e^{-t(\hat{\lambda}_n^{1/2} + \hat{\lambda}_{k-n}^{1/2})} C_n^{\lambda} \left(\frac{y}{\sqrt{\lambda}} \right) C_{k-n}^{\lambda} \left(\frac{y}{\sqrt{\lambda}} \right) \\
& \quad \times \frac{(n+\lambda) n! (k-n)! (k-n+\lambda)}{\lambda^2 (2\lambda)_n (2\lambda)_{k-n}}.
\end{aligned} \tag{108}$$

Thus,

$$\begin{aligned}
& \left(\sqrt{t} \frac{\partial}{\partial t} P_t^{\lambda-1/2, \lambda-1/2} \phi_{\lambda} \left(\frac{y}{\sqrt{\lambda}} \right) \right)^2 \\
& = \sum_{k=1}^{\infty} \sum_{n=1}^k \left\langle \phi_{\lambda}, \frac{C_n^{\lambda}}{\|C_n^{\lambda}\|^2} \right\rangle \left\langle \phi_{\lambda}, \frac{C_{k-n}^{\lambda}}{\|C_{k-n}^{\lambda}\|^2} \right\rangle \\
& \quad \cdot t \hat{\lambda}_n^{1/2} \hat{\lambda}_{k-n}^{1/2} e^{-t(\hat{\lambda}_n^{1/2} + \hat{\lambda}_{k-n}^{1/2})} C_n^{\lambda} \left(\frac{y}{\sqrt{\lambda}} \right) C_{k-n}^{\lambda} \left(\frac{y}{\sqrt{\lambda}} \right).
\end{aligned} \tag{109}$$

Therefore,

$$\begin{aligned}
g_{1,\lambda} \phi(y) & = \chi_{[-K,K]}(y) \int_0^{\infty} \sum_{k=1}^{\infty} \sum_{n=1}^k \left\langle \phi_{\lambda}, \frac{C_n^{\lambda}}{\|C_n^{\lambda}\|^2} \right\rangle \\
& \quad \cdot \left\langle \phi_{\lambda}, \frac{C_{k-n}^{\lambda}}{\|C_{k-n}^{\lambda}\|^2} \right\rangle t \hat{\lambda}_n^{1/2} \hat{\lambda}_{k-n}^{1/2} e^{-t(\hat{\lambda}_n^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \\
& \quad \times C_n^{\lambda} \left(\frac{y}{\sqrt{\lambda}} \right) C_{k-n}^{\lambda} \left(\frac{y}{\sqrt{\lambda}} \right) \left(1 - \frac{y^2}{\lambda} \right)^{\lambda-1/2} e^{y^2} dt.
\end{aligned} \tag{110}$$

Let us take

$$\begin{aligned}
H_{\lambda,K}^{N,1}(t,y) & = \chi_{[-K,K]}(y) \sum_{k=N+1}^{\infty} \sum_{n=1}^k \left\langle \phi_{\lambda}, \frac{C_n^{\lambda}}{\|C_n^{\lambda}\|^2} \right\rangle \\
& \quad \cdot \left\langle \phi_{\lambda}, \frac{C_{k-n}^{\lambda}}{\|C_{k-n}^{\lambda}\|^2} \right\rangle t \hat{\lambda}_n^{1/2} \hat{\lambda}_{k-n}^{1/2} e^{-t(\hat{\lambda}_n^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \\
& \quad \times C_n^{\lambda} \left(\frac{y}{\sqrt{\lambda}} \right) C_{k-n}^{\lambda} \left(\frac{y}{\sqrt{\lambda}} \right) \left(1 - \frac{y^2}{\lambda} \right)^{\lambda/2-1/4} e^{y^2/2}.
\end{aligned} \tag{111}$$

Thus,

$$\begin{aligned}
g_{1,\lambda} \phi(y) & = \chi_{[-K,K]}(y) \int_0^{\infty} \sum_{k=1}^N \sum_{n=1}^k \left\langle \phi_{\lambda}, \frac{C_n^{\lambda}}{\|C_n^{\lambda}\|^2} \right\rangle \\
& \quad \cdot \left\langle \phi_{\lambda}, \frac{C_{k-n}^{\lambda}}{\|C_{k-n}^{\lambda}\|^2} \right\rangle t \hat{\lambda}_n^{1/2} \hat{\lambda}_{k-n}^{1/2} e^{-t(\hat{\lambda}_n^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \\
& \quad \times C_n^{\lambda} \left(\frac{y}{\sqrt{\lambda}} \right) C_{k-n}^{\lambda} \left(\frac{y}{\sqrt{\lambda}} \right) \left(1 - \frac{y^2}{\lambda} \right)^{\lambda-1/2} e^{y^2} dt
\end{aligned}$$

$$\begin{aligned}
& + \chi_{[-K,K]}(y) \int_0^{\infty} \sum_{k=N+1}^{\infty} \sum_{n=1}^k \left\langle \phi_{\lambda}, \frac{C_n^{\lambda}}{\|C_n^{\lambda}\|^2} \right\rangle \\
& \quad \cdot \left\langle \phi_{\lambda}, \frac{C_{k-n}^{\lambda}}{\|C_{k-n}^{\lambda}\|^2} \right\rangle t \hat{\lambda}_n^{1/2} \hat{\lambda}_{k-n}^{1/2} e^{-t(\hat{\lambda}_n^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \\
& \quad \times C_n^{\lambda} \left(\frac{y}{\sqrt{\lambda}} \right) C_{k-n}^{\lambda} \left(\frac{y}{\sqrt{\lambda}} \right) \left(1 - \frac{y^2}{\lambda} \right)^{\lambda-1/2} e^{y^2} dt \\
& = \sum_{k=1}^N \sum_{n=1}^k \left\langle \phi_{\lambda}, \frac{C_n^{\lambda}}{\|C_n^{\lambda}\|^2} \right\rangle \left\langle \phi_{\lambda}, \frac{C_{k-n}^{\lambda}}{\|C_{k-n}^{\lambda}\|^2} \right\rangle \\
& \quad \cdot \frac{\hat{\lambda}_n^{1/2} \hat{\lambda}_{k-n}^{1/2}}{(\hat{\lambda}_n^{1/2} + \hat{\lambda}_{k-n}^{1/2})^2} C_n^{\lambda} \left(\frac{y}{\sqrt{\lambda}} \right) C_{k-n}^{\lambda} \left(\frac{y}{\sqrt{\lambda}} \right) \times \left(1 - \frac{y^2}{\lambda} \right)^{\lambda-1/2} e^{y^2} \\
& \quad + \int_0^{\infty} H_{\lambda,K}^{N,1}(t,y) dt \left(1 - \frac{y^2}{\lambda} \right)^{\lambda/2-1/4} e^{y^2/2}.
\end{aligned} \tag{112}$$

Now, we want to prove that, for $K \in \mathbb{N}$ and $\lambda > 0$ such that $\sqrt{\lambda} > K$,

$$\int_{-\infty}^{\infty} \left| \int_0^{\infty} H_{\lambda,K}^{N,1}(t,y) dt \right|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} \leq \frac{\theta}{e^{(N+1)}}. \tag{113}$$

Indeed, by Minkowski integral inequality we have

$$\begin{aligned}
& \left(\int_{-\infty}^{\infty} \left| \int_0^{\infty} H_{\lambda,K}^{N,1}(t,y) dt \right|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} \right)^{1/2} \\
& \leq \int_0^{\infty} \left(\int_{-\infty}^{\infty} |H_{\lambda,K}^{N,1}(t,y)|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} \right)^{1/2} dt;
\end{aligned} \tag{114}$$

then,

$$\begin{aligned}
& \int_{-\infty}^{\infty} |H_{\lambda,K}^{N,1}(t,y)|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} \leq \frac{1}{\sqrt{\pi}} \\
& \quad \cdot \int_{-1}^1 \left| \sum_{k=N+1}^{\infty} \sum_{n=1}^k \left\langle \phi_{\lambda}, \frac{C_n^{\lambda}}{\|C_n^{\lambda}\|^2} \right\rangle \left\langle \phi_{\lambda}, \frac{C_{k-n}^{\lambda}}{\|C_{k-n}^{\lambda}\|^2} \right\rangle \right. \\
& \quad \cdot t \hat{\lambda}_n^{1/2} \hat{\lambda}_{k-n}^{1/2} e^{-t(\hat{\lambda}_n^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \times C_n^{\lambda}(x) C_{k-n}^{\lambda}(x) \left. \right|^2 \left(1 - x^2 \right)^{\lambda-1/2} dx.
\end{aligned} \tag{115}$$

Using Gasper's linearization of the product of Jacobi polynomials (see [10]),

$$\begin{aligned}
& C_n^{\lambda}(x) C_{k-n}^{\lambda}(x) \\
& = \frac{[\Gamma(\lambda+1/2)]^2 \Gamma(n+2\lambda) \Gamma(k-n+2\lambda)}{[\Gamma(2\lambda)]^2 \Gamma(n+\lambda+1/2) \Gamma(k-n+\lambda+1/2)}
\end{aligned}$$

$$\begin{aligned}
& \times P_n^{(\lambda-1/2, \lambda-1/2)}(1) P_{k-n}^{(\lambda-1/2, \lambda-1/2)}(1) \\
& \cdot P_n^{(\lambda-1/2, \lambda-1/2)}(x) P_{k-n}^{(\lambda-1/2, \lambda-1/2)}(x) \\
& = \frac{[\Gamma(\lambda+1/2)]^2 \Gamma(n+2\lambda) \Gamma(k-n+2\lambda)}{[\Gamma(2\lambda)]^2 \Gamma(n+\lambda+1/2) \Gamma(k-n+\lambda+1/2)} \\
& \times P_n^{(\lambda-1/2, \lambda-1/2)}(1) P_{k-n}^{(\lambda-1/2, \lambda-1/2)}(1) \\
& \cdot \sum_{i=|k-2n|}^k \nu(i, n, k-n) p_i^{(\lambda-1/2, \lambda-1/2)}(x),
\end{aligned} \tag{116}$$

where $p_i^{(\alpha, \beta)}(x)$ is defined in (4) and

$$\begin{aligned}
\nu(i, k-n, n) &= \frac{1}{\|p_i^{(\alpha, \beta)}\|_{2, (\alpha, \beta)}^2} \\
& \cdot \int_{-1}^1 p_i^{(\alpha, \beta)}(x) p_{k-n}^{(\alpha, \beta)}(x) p_n^{(\alpha, \beta)}(x) \mu_{(\alpha, \beta)}(dx).
\end{aligned} \tag{117}$$

Hence,

$$\begin{aligned}
& \int_{-\infty}^{\infty} |H_{\lambda, K}^{N, 1}(t, y)|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} \leq \frac{1}{\sqrt{\pi}} \\
& \cdot \int_{-1}^1 \left| \sum_{k=N+1}^{\infty} \sum_{n=1}^k \sum_{i=|k-2n|}^k \left\langle \phi_{\lambda}, \frac{C_n^{\lambda}}{\|C_n^{\lambda}\|^2} \right\rangle \right. \\
& \cdot \left\langle \phi_{\lambda}, \frac{C_{k-n}^{\lambda}}{\|C_{k-n}^{\lambda}\|^2} \right\rangle t \hat{\lambda}_n^{1/2} \hat{\lambda}_{k-n}^{1/2} e^{-t(\hat{\lambda}_k^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \\
& \times \frac{[\Gamma(\lambda+1/2)]^2 \Gamma(n+2\lambda) \Gamma(k-n+2\lambda)}{[\Gamma(2\lambda)]^2 \Gamma(n+\lambda+1/2) \Gamma(k-n+\lambda+1/2)} \\
& \times P_n^{(\lambda-1/2, \lambda-1/2)}(1) P_{k-n}^{(\lambda-1/2, \lambda-1/2)}(1) \nu(i, n, k-n) \\
& \cdot p_i^{(\lambda-1/2, \lambda-1/2)}(x) \left| (1-x^2)^{\lambda-1/2} dx \right|.
\end{aligned} \tag{118}$$

Then, by Parseval's identity and Cauchy-Schwartz inequality, we have

$$\begin{aligned}
& \int_{-\infty}^{\infty} |H_{\lambda, K}^{N, 1}(t, y)|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} \leq \frac{1}{\sqrt{\pi}} \frac{2\pi\Gamma(2\lambda)}{\lambda [\Gamma(\lambda)]^2 2^{2\lambda}} \\
& \cdot \sum_{k=N+1}^{\infty} \sum_{n=1}^k \sum_{i=|k-2n|}^k \frac{\|\phi_{\lambda}\|^2}{\|C_{k-n}^{\lambda}\|^2 \|C_n^{\lambda}\|^2} \left| \left\langle \phi_{\lambda}, \frac{C_n^{\lambda}}{\|C_n^{\lambda}\|} \right\rangle \right| \\
& \cdot t^2 \hat{\lambda}_n \hat{\lambda}_{k-n} e^{-2t(\hat{\lambda}_k^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \\
& \times \left| \frac{[\Gamma(\lambda+1/2)]^2 \Gamma(n+2\lambda) \Gamma(k-n+2\lambda)}{[\Gamma(2\lambda)]^2 \Gamma(n+\lambda+1/2) \Gamma(k-n+\lambda+1/2)} \right|^2
\end{aligned}$$

$$\begin{aligned}
& \times \left| P_n^{(\lambda-1/2, \lambda-1/2)}(1) P_{k-n}^{(\lambda-1/2, \lambda-1/2)}(1) \right|^2 \\
& \cdot |\nu(i, n, k-n)|^2 \|p_i^{(\lambda-1/2, \lambda-1/2)}\|^2.
\end{aligned} \tag{119}$$

Now,

$$\begin{aligned}
& \left| \nu(i, k-n, n) \|p_i^{(\alpha, \beta)}\|_{2, (\alpha, \beta)} \right| \leq \frac{1}{\|p_i^{(\alpha, \beta)}\|_{2, (\alpha, \beta)}} \\
& \cdot \int_{-1}^1 |p_i^{(\alpha, \beta)}(x) p_{k-n}^{(\alpha, \beta)}(x)| |p_n^{(\alpha, \beta)}(x)| \mu_{(\alpha, \beta)}(dx) \\
& \leq \frac{\binom{n+q}{n}}{\|p_i^{(\alpha, \beta)}\|_{2, (\alpha, \beta)} \|P_n^{(\alpha, \beta)}(1) P_{k-n}^{(\alpha, \beta)}(1)\|} \|p_i^{(\alpha, \beta)}\|_{2, (\alpha, \beta)} \\
& \cdot \|P_{k-n}^{(\alpha, \beta)}\|_{2, (\alpha, \beta)} = \frac{\binom{n+q}{n} \|P_{k-n}^{(\alpha, \beta)}\|_{2, (\alpha, \beta)}}{\|P_n^{(\alpha, \beta)}(1) P_{k-n}^{(\alpha, \beta)}(1)\|},
\end{aligned} \tag{120}$$

where $q = \max(\alpha, \beta)$. Thus

$$\begin{aligned}
& \left| P_n^{(\alpha, \beta)}(1) P_{k-n}^{(\alpha, \beta)}(1) \right|^2 |\nu(i, k-n, n)|^2 \|p_i^{(\alpha, \beta)}\|_{2, (\alpha, \beta)}^2 \\
& \leq \binom{n+q}{n}^2 \|P_{k-n}^{(\alpha, \beta)}\|_{2, (\alpha, \beta)}^2.
\end{aligned} \tag{121}$$

Therefore, for $\alpha = \beta = \lambda - 1/2$, we have

$$\begin{aligned}
& \int_{-\infty}^{\infty} |H_{\lambda, K}^{N, 1}(t, y)|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} \leq \frac{1}{\sqrt{\pi}} \frac{2\pi\Gamma(2\lambda)}{\lambda [\Gamma(\lambda)]^2 2^{2\lambda}} \\
& \cdot \sum_{k=N+1}^{\infty} \sum_{n=1}^k \sum_{i=|k-2n|}^k \frac{\|\phi_{\lambda}\|^2}{\|C_{k-n}^{\lambda}\|^2 \|C_n^{\lambda}\|^2} \left| \left\langle \phi_{\lambda}, \frac{C_n^{\lambda}}{\|C_n^{\lambda}\|} \right\rangle \right|^2 \\
& \cdot t^2 \hat{\lambda}_n \hat{\lambda}_{k-n} e^{-2t(\hat{\lambda}_k^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \\
& \times \left| \frac{[\Gamma(\lambda+1/2)]^2 \Gamma(n+2\lambda) \Gamma(k-n+2\lambda)}{[\Gamma(2\lambda)]^2 \Gamma(n+\lambda+1/2) \Gamma(k-n+\lambda+1/2)} \right|^2 \\
& \cdot \binom{n+\lambda-1/2}{n}^2 \\
& \times \frac{[\Gamma(2\lambda)]^2 [\Gamma(n+\lambda+1/2)]^2}{[\Gamma(\lambda+1/2)]^2 [\Gamma(n+2\lambda)]^2} \|C_{k-n}^{\lambda}\|^2 \\
& \times \frac{[\Gamma(\lambda+1/2)]^2 [\Gamma(k-n+2\lambda)]^2}{[\Gamma(2\lambda)]^2 [\Gamma(k-n+\lambda+1/2)]^2} \\
& \cdot \frac{[\Gamma(n+\lambda+1/2)]^2 \Gamma(2\lambda) (n+\lambda) n!}{[\Gamma(\lambda+1/2)]^2 [n!]^2 \lambda \Gamma(n+2\lambda)} = \frac{1}{\sqrt{\pi}} \\
& \cdot \frac{2\pi\Gamma(2\lambda)}{\lambda [\Gamma(\lambda)]^2 2^{2\lambda}} \sum_{k=N+1}^{\infty} \sum_{n=1}^k \sum_{i=|k-2n|}^k \|\phi_{\lambda}\|^2
\end{aligned}$$

$$\begin{aligned} & \cdot \left| \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|} \right\rangle \right|^2 t^2 \hat{\lambda}_n \hat{\lambda}_{k-n} e^{-2t(\hat{\lambda}_k^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \\ & \times \frac{[\Gamma(k-n+2\lambda)]^2 [\Gamma(n+\lambda+1/2)]^2 (n+\lambda)}{\Gamma(2\lambda) [\Gamma(k-n+\lambda+1/2)]^2 n! \lambda \Gamma(n+2\lambda)}. \end{aligned} \quad (122)$$

On the other hand, using Stirling's approximation formula for the Gamma function,

$$\Gamma(az+b) \sim \sqrt{2\pi e}^{-az} (az)^{az+b-1/2} \quad (|\arg z| < \pi, a > 0), \quad (123)$$

we obtain, for λ big enough, that

$$\begin{aligned} \frac{[\Gamma(k-n+2\lambda)]^2 [\Gamma(n+\lambda+1/2)]^2 (n+\lambda)}{\Gamma(2\lambda) [\Gamma(k-n+\lambda+1/2)]^2 n! \lambda \Gamma(n+2\lambda)} & \sim \frac{2\pi e^{-4\lambda} (2\lambda)^{4\lambda+2k-2n-1} 2\pi e^{-2\lambda} (\lambda)^{2\lambda+2n} (n+\lambda)}{\sqrt{2\pi e}^{-2\lambda} (2\lambda)^{2\lambda-1/2} 2\pi e^{-2\lambda} (\lambda)^{2\lambda+2k-2n} \sqrt{2\pi e}^{-2\lambda} (2\lambda)^{2\lambda+n-1/2} \lambda n!} \\ & = \frac{2^{2k} \lambda^{n-1} (n+\lambda)}{2^{3n} n!}. \end{aligned} \quad (124)$$

Then, for λ big enough

$$\begin{aligned} \int_{-\infty}^{\infty} |H_{\lambda,K}^{N,1}(t,y)|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} & \leq \frac{1}{\sqrt{\pi}} \frac{2\pi\Gamma(2\lambda) \|\phi_\lambda\|^2}{\lambda [\Gamma(\lambda)]^2 2^{2\lambda}} \\ & \cdot \sum_{k=N+1}^{\infty} \sum_{n=1}^k \sum_{i=|k-2n|}^k \left| \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|} \right\rangle \right|^2 t^2 \hat{\lambda}_n \hat{\lambda}_{k-n} \\ & \times e^{-2t(\hat{\lambda}_k^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \frac{2^{2k} \lambda^{n-1} (n+\lambda)}{2^{3n} n!}. \end{aligned} \quad (125)$$

Also,

$$\begin{aligned} & t^2 \hat{\lambda}_n \hat{\lambda}_{k-n} e^{-2t(\hat{\lambda}_k^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \frac{2^{2k} \lambda^{n-1} (n+\lambda)}{2^{3n} n!} \\ & \leq \frac{e^{-t} e^{-t\hat{\lambda}_{k-n}^{1/2}} 2^{2k} \lambda^n (n/\lambda + 1)}{t^2 2^{3n} n!}. \end{aligned} \quad (126)$$

Taking $\lambda = (k-n)(1+2k)^2$, for $t \geq 1$, we get

$$\begin{aligned} \frac{1}{e^{t\hat{\lambda}_{k-n}^{1/2}}} & \leq \frac{1}{e^{(k-n)^{1/2}(k-n+2(k-n)(1+2k)^2)^{1/2}}} \\ & \leq \frac{1}{e^{(k-n)(1+2(1+2k)^2)^{1/2}}} \leq \frac{1}{e^{(k-n)}} \frac{1}{e^k} \frac{1}{e^k}. \end{aligned} \quad (127)$$

Now, if $0 < t < 1$ we have that, for t near 0, there exists $k > 0$ big enough such that $1/k \leq t$; that is, $1/t^2 \leq k^2$. Hence,

$$\begin{aligned} \frac{1}{e^{t\hat{\lambda}_{k-n}^{1/2}}} & \leq \frac{1}{e^{(1/k)(k-n)^{1/2}(k-n+2(k-n)(1+2k)^2)^{1/2}}} \\ & \leq \frac{1}{e^{(k-n)}} \frac{1}{e^k} \frac{1}{e^k}. \end{aligned} \quad (128)$$

Therefore, for λ big enough, there exists $C > 0$ such that

$$t^2 \hat{\lambda}_n \hat{\lambda}_{k-n} e^{-2t(\hat{\lambda}_k^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \frac{2^{2k} \lambda^{n-1} (n+\lambda)}{2^{3n} n!} \leq \frac{e^{-tC}}{e^k}. \quad (129)$$

Then, for λ big enough, we have

$$\begin{aligned} \int_{-\infty}^{\infty} |H_{\lambda,K}^{N,1}(t,y)|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} & \leq \frac{1}{\sqrt{\pi}} \frac{2\pi\Gamma(2\lambda)}{\lambda [\Gamma(\lambda)]^2 2^{2\lambda}} \\ & \cdot \sum_{k=N+1}^{\infty} \sum_{n=1}^k \sum_{i=|k-2n|}^k \left| \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|} \right\rangle \right|^2 t^2 \hat{\lambda}_n \hat{\lambda}_{k-n} \\ & \times e^{-2t(\hat{\lambda}_k^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \frac{2^{2k} \lambda^{n-1} (n+\lambda)}{2^{3n} n!} \leq \frac{e^{-t}}{\sqrt{\pi}} \\ & \cdot \frac{2\pi\Gamma(2\lambda)}{\lambda [\Gamma(\lambda)]^2 2^{2\lambda}} \frac{MC \|\phi_\lambda\|^2}{e^{(N+1)}} \\ & \cdot \sum_{k=N+1}^{\infty} \sum_{n=1}^k \sum_{i=|k-2n|}^k \left| \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|} \right\rangle \right|^2 \\ & \leq \frac{2\pi\Gamma(2\lambda)}{\lambda [\Gamma(\lambda)]^2 2^{2\lambda}} \frac{e^{-t} MC \|\phi_\lambda\|_{2,\lambda}^4}{e^{(N+1)}} \leq \frac{e^{-t} MC \|\phi\|_{2,\gamma}^4}{e^{(N+1)}}. \end{aligned} \quad (130)$$

Hence,

$$\begin{aligned} & \left(\int_{-\infty}^{\infty} \left| \int_0^\infty H_{\lambda,K}^{N,1}(t,y) dt \right|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} \right)^{1/2} \\ & \leq \int_0^\infty \left(\frac{e^{-t} MC \|\phi\|_{2,\gamma}^4}{e^{(N+1)}} \right)^{1/2} dt \\ & = (MC)^{1/2} \frac{\|\phi\|_{2,\gamma}^2}{e^{(N+1)/2}} \int_0^\infty e^{-t/2} dt = \frac{(MC)^{1/2} \|\phi\|_{2,\gamma}^2}{e^{(N+1)/2}}. \end{aligned} \quad (131)$$

Thus, $\{\int_0^\infty H_{\lambda,K}^{N,1}(t,\cdot) dt\}$ is a bounded sequence on $L^2(\mathbb{R}, \gamma)$, so, by Bourbaki-Alaoglu's theorem, there exists a sequence $(\lambda_j)_{j \in \mathbb{N}}$, $\lim_{j \rightarrow \infty} \lambda_j = \infty$, such that, for all $N \in \mathbb{N}$, $\{\int_0^\infty H_{\lambda_j,K}^{N,1}(t,\cdot) dt\}_{j \in \mathbb{N}}$ converges weakly in $L^2(\mathbb{R}, \gamma)$ to a function $H_K^{N,1} \in L^2(\mathbb{R}, \gamma)$. Moreover,

$$\int_{-\infty}^{\infty} |H_K^{N,1}(y)|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} \leq \frac{(MC)^{1/2} \|\phi\|_{2,\gamma}^2}{e^{(N+1)/2}}. \quad (132)$$

Then, there exists a nondecreasing sequence $(N_j)_{j \in \mathbb{N}}$ such that

$$H_K^{N_j,1}(y) \longrightarrow 0, \quad a.e. y \in \mathbb{R}. \quad (133)$$

The study of $g_{2,\lambda}$ is essentially analogous, so fewer details will be given. For $\phi \in L^2(\mathbb{R}, \gamma)$ we have

$$\begin{aligned} \frac{\partial}{\partial y} P_t^{\alpha,\beta} \phi_\lambda \left(\frac{y}{\sqrt{\lambda}} \right) &= - \sum_{k=1}^{\infty} \left\langle \phi_\lambda, \frac{P_k^{(\alpha,\beta)}}{\|P_k^{(\alpha,\beta)}\|_{2,(\alpha,\beta)}^2} \right\rangle \\ &\cdot e^{-t\lambda_k^{1/2}} \frac{(k + \alpha + \beta + 1)}{2} P_{k-1}^{(\alpha+1,\beta+1)} \left(\frac{y}{\sqrt{\lambda}} \right), \end{aligned} \quad (134)$$

which converges absolutely. Then, again taking the Cauchy product, we obtain

$$\begin{aligned} \left(\frac{\partial}{\partial y} P_t^{\lambda-1/2, \lambda-1/2} \phi_\lambda \left(\frac{y}{\sqrt{\lambda}} \right) \right)^2 &= \sum_{k=1}^{\infty} \sum_{n=1}^k \langle \phi_\lambda, C_n^\lambda \rangle \\ &\cdot \langle \phi_\lambda, C_{k-n}^\lambda \rangle e^{-t(\tilde{\lambda}_n^{1/2} + \tilde{\lambda}_{k-n}^{1/2})} C_{n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right) C_{k-n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right) \\ &\cdot \frac{(n+2\lambda)(k-n+2\lambda)}{4} \\ &\times \frac{\Gamma(2\lambda+2)\Gamma(n+\lambda+1/2)}{\Gamma(\lambda+3/2)\Gamma(n+2\lambda+1)} \\ &\cdot \frac{\Gamma(2\lambda+2)\Gamma(k-n+\lambda+1/2)}{\Gamma(\lambda+3/2)\Gamma(k-n+2\lambda+1)} \\ &\cdot \frac{\Gamma(2\lambda)\Gamma(n+\lambda+1/2)}{\Gamma(\lambda+1/2)\Gamma(n+2\lambda)} \\ &\times \frac{\Gamma(2\lambda)\Gamma(k-n+\lambda+1/2)}{\Gamma(\lambda+1/2)\Gamma(k-n+2\lambda)} \\ &\cdot \frac{2(n+\lambda)[\Gamma(\lambda+1/2)]^2 n! \Gamma(n+2\lambda)}{\Gamma(2\lambda+1)[\Gamma(n+\lambda+1/2)]^2} \\ &\times \frac{2(k-n+\lambda)[\Gamma(\lambda+1/2)]^2 (k-n)! \Gamma(k-n+2\lambda)}{\Gamma(2\lambda+1)[\Gamma(k-n+\lambda+1/2)]^2} \\ &= \sum_{k=1}^{\infty} \sum_{n=1}^k \langle \phi_\lambda, C_n^\lambda \rangle \langle \phi_\lambda, C_{k-n}^\lambda \rangle \\ &\cdot e^{-t(\tilde{\lambda}_n^{1/2} + \tilde{\lambda}_{k-n}^{1/2})} C_{n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right) C_{k-n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right) \\ &\times \frac{4(n+\lambda)n!(k-n+\lambda)(k-n)!}{(2\lambda)_n(2\lambda)_{k-n}}; \end{aligned} \quad (135)$$

then

$$\begin{aligned} t \left(1 - \frac{y^2}{\lambda} \right) \left(\frac{\partial}{\partial t} P_t^{\lambda-1/2, \lambda-1/2} \phi_\lambda \left(\frac{y}{\sqrt{\lambda}} \right) \right)^2 \\ = \left(1 - \frac{y^2}{\lambda} \right) \sum_{k=1}^{\infty} \sum_{n=1}^k \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|^2} \right\rangle \end{aligned}$$

$$\begin{aligned} &\cdot \left\langle \phi_\lambda, \frac{C_{k-n}^\lambda}{\|C_{k-n}^\lambda\|^2} \right\rangle 4\lambda^2 t e^{-t(\tilde{\lambda}_n^{1/2} + \tilde{\lambda}_{k-n}^{1/2})} C_{n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right) \\ &\cdot C_{k-n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right). \end{aligned} \quad (136)$$

Hence,

$$\begin{aligned} g_{2,\lambda} \phi(y) &= \chi_{[-K,K]}(y) \int_0^\infty \sum_{k=1}^\infty \sum_{n=1}^k \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|^2} \right\rangle \\ &\cdot \left\langle \phi_\lambda, \frac{C_{k-n}^\lambda}{\|C_{k-n}^\lambda\|^2} \right\rangle 4\lambda^2 t e^{-t(\tilde{\lambda}_n^{1/2} + \tilde{\lambda}_{k-n}^{1/2})} \\ &\times C_{n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right) C_{k-n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right) \left(1 - \frac{y^2}{\lambda} \right)^{\lambda+1/2} \\ &\cdot e^{y^2} dt. \end{aligned} \quad (137)$$

Taking

$$\begin{aligned} H_{\lambda,K}^{N,2}(t, y) &= \chi_{[-K,K]}(y) \sum_{k=N+1}^\infty \sum_{n=1}^k \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|^2} \right\rangle \\ &\cdot \left\langle \phi_\lambda, \frac{C_{k-n}^\lambda}{\|C_{k-n}^\lambda\|^2} \right\rangle 4\lambda^2 t e^{-t(\tilde{\lambda}_n^{1/2} + \tilde{\lambda}_{k-n}^{1/2})} \\ &\times C_{n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right) C_{k-n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right) \\ &\cdot \left(1 - \frac{y^2}{\lambda} \right)^{\lambda/2+1/4} e^{y^2/2}, \end{aligned} \quad (138)$$

we can write

$$\begin{aligned} g_{2,\lambda} \phi(y) &= \chi_{[-K,K]}(y) \int_0^\infty \sum_{k=1}^\infty \sum_{n=1}^k \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|^2} \right\rangle \\ &\cdot \left\langle \phi_\lambda, \frac{C_{k-n}^\lambda}{\|C_{k-n}^\lambda\|^2} \right\rangle 4\lambda^2 t e^{-t(\tilde{\lambda}_n^{1/2} + \tilde{\lambda}_{k-n}^{1/2})} \\ &\times C_{n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right) C_{k-n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right) \left(1 - \frac{y^2}{\lambda} \right)^{\lambda+1/2} e^{y^2} dt \\ &+ \chi_{[-K,K]}(y) \int_0^\infty \sum_{k=N+1}^\infty \sum_{n=1}^k \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|^2} \right\rangle \\ &\cdot \left\langle \phi_\lambda, \frac{C_{k-n}^\lambda}{\|C_{k-n}^\lambda\|^2} \right\rangle 4\lambda^2 t e^{-t(\tilde{\lambda}_n^{1/2} + \tilde{\lambda}_{k-n}^{1/2})} \\ &\times C_{n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right) C_{k-n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right) \left(1 - \frac{y^2}{\lambda} \right)^{\lambda+1/2} \end{aligned}$$

$$\begin{aligned}
& \cdot e^{y^2} dt \left(1 - \frac{y^2}{\lambda}\right)^{\lambda+1/2} e^{y^2} dt \\
& = \sum_{k=1}^N \sum_{n=1}^k \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|^2} \right\rangle \left\langle \phi_\lambda, \frac{C_{k-n}^\lambda}{\|C_{k-n}^\lambda\|^2} \right\rangle \\
& \cdot \frac{4\lambda^2}{(\hat{\lambda}_n^{1/2} + \hat{\lambda}_{k-n}^{1/2})^2} \times C_{n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}}\right) C_{k-n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}}\right) \left(1 - \frac{y^2}{\lambda}\right)^{\lambda+1/2} e^{y^2} + \int_0^\infty H_{\lambda,K}^{N,2}(t, y) dt \left(1 - \frac{y^2}{\lambda}\right)^{\lambda/2+1/4} e^{y^2/2}.
\end{aligned} \quad (139)$$

Similarly to the previous case, we want to prove that, for any $K \in \mathbb{N}$ and $\lambda > 0$ such that $\sqrt{\lambda} > K$,

$$\int_{-\infty}^\infty \left| \int_0^\infty H_{\lambda,K}^{N,2}(t, y) dt \right|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} \leq \frac{\theta}{e^{(N+1)}}. \quad (140)$$

Indeed, by Minkowski integral inequality we have

$$\begin{aligned}
& \left(\int_{-\infty}^\infty \left| \int_0^\infty H_{\lambda,K}^{N,2}(t, y) dt \right|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} \right)^{1/2} \\
& \leq \int_0^\infty \left(\int_{-\infty}^\infty |H_{\lambda,K}^{N,2}(t, y)|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} \right)^{1/2} dt.
\end{aligned} \quad (141)$$

Now,

$$\begin{aligned}
& \int_{-\infty}^\infty |H_{\lambda,K}^{N,2}(t, y)|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} \leq \frac{1}{\sqrt{\pi}} \\
& \cdot \int_{-1}^1 \left| \sum_{k=N+1}^\infty \sum_{n=1}^k \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|^2} \right\rangle \left\langle \phi_\lambda, \frac{C_{k-n}^\lambda}{\|C_{k-n}^\lambda\|^2} \right\rangle \right. \\
& \cdot 4\lambda^2 t e^{-t(\hat{\lambda}_k^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \times C_{n-1}^{\lambda+1}(x) C_{k-n-1}^{\lambda+1}(x) \left. \right|^2 \left(1 - x^2\right)^{\lambda-1/2} dx.
\end{aligned} \quad (142)$$

Again, using Gasper's linearization of the product of Jacobi polynomials [10], we may write

$$\begin{aligned}
& C_{n-1}^{\lambda+1}(x) C_{k-n-1}^{\lambda+1}(x) \\
& = \frac{[\Gamma(\lambda + 3/2)]^2 \Gamma(n + 2\lambda + 1) \Gamma(k - n + 2\lambda + 1)}{[\Gamma(2\lambda + 2)]^2 \Gamma(n + 2\lambda + 3/2) \Gamma(k - n + 2\lambda + 3/2)} \\
& \times P_{n-1}^{(\lambda+1/2, \lambda+1/2)}(1) P_{k-n-1}^{(\lambda+1/2, \lambda+1/2)}(1) P_{n-1}^{(\lambda+1/2, \lambda+1/2)}(x) \\
& \cdot P_{k-n-1}^{(\lambda+1/2, \lambda+1/2)}(x) \\
& = \frac{[\Gamma(\lambda + 3/2)]^2 \Gamma(n + 2\lambda + 1) \Gamma(k - n + 2\lambda + 1)}{[\Gamma(2\lambda + 2)]^2 \Gamma(n + 2\lambda + 3/2) \Gamma(k - n + 2\lambda + 3/2)} \\
& \times P_{n-1}^{(\lambda+1/2, \lambda+1/2)}(1) P_{k-n-1}^{(\lambda+1/2, \lambda+1/2)}(1) \\
& \cdot \sum_{i=|k-2n|}^{k-2} \nu(i, n-1, k-n-1) p_i^{(\lambda+1/2, \lambda+1/2)}(x).
\end{aligned} \quad (143)$$

Thus,

$$\begin{aligned}
& \int_{-\infty}^\infty |H_{\lambda,K}^{N,2}(t, y)|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} \leq \frac{1}{\sqrt{\pi}} \\
& \cdot \int_{-1}^1 \left| \sum_{k=N+1}^\infty \sum_{n=1}^k \sum_{i=|k-2n|}^{k-2} \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|^2} \right\rangle \right. \\
& \cdot \left\langle \phi_\lambda, \frac{C_{k-n}^\lambda}{\|C_{k-n}^\lambda\|^2} \right\rangle 4\lambda^2 t e^{-t(\hat{\lambda}_n^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \\
& \times \frac{[\Gamma(\lambda + 3/2)]^2 \Gamma(n + 2\lambda + 1) \Gamma(k - n + 2\lambda + 1)}{[\Gamma(2\lambda + 2)]^2 \Gamma(n + 2\lambda + 3/2) \Gamma(k - n + 2\lambda + 3/2)} \\
& \cdot P_{n-1}^{(\lambda+1/2, \lambda+1/2)}(1) P_{k-n-1}^{(\lambda+1/2, \lambda+1/2)}(1) \\
& \times \nu(i, n-1, k-n-1) p_i^{(\lambda+1/2, \lambda+1/2)}(x) \left. \right|^2 \left(1 - x^2\right)^{\lambda-1/2} dx.
\end{aligned} \quad (144)$$

Then, using Parseval's identity and Cauchy-Schwartz inequality, we have

$$\begin{aligned}
& \int_{-\infty}^\infty |H_{\lambda,K}^{N,2}(t, y)|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} \leq \frac{1}{\sqrt{\pi}} \frac{2\pi\Gamma(2(\lambda + 1))}{(\lambda + 1) [\Gamma(\lambda + 1)]^2 2^{2(\lambda+1)}} \sum_{k=N+1}^\infty \sum_{n=1}^k \sum_{i=|k-2n|}^k \left| \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|^2} \right\rangle \right|^2 \frac{\|\phi_\lambda\|^2}{\|C_n^\lambda\|^2 \|C_{k-n}^\lambda\|^2} \\
& \times 16\lambda^4 t^2 e^{-2t(\hat{\lambda}_k^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \left| \frac{[\Gamma(\lambda + 3/2)]^2 \Gamma(n + 2\lambda + 1) \Gamma(k - n + 2\lambda + 1)}{[\Gamma(2\lambda + 2)]^2 \Gamma(n + 2\lambda + 3/2) \Gamma(k - n + 2\lambda + 3/2)} \right|^2 \\
& \times \left| P_{n-1}^{(\lambda+1/2, \lambda+1/2)}(1) P_{k-n-1}^{(\lambda+1/2, \lambda+1/2)}(1) \right|^2 |\nu(i, n-1, k-n-1)|^2 \|p_i^{(\lambda+1/2, \lambda+1/2)}\|^2.
\end{aligned} \quad (145)$$

Then, by a similar argument as in the previous case, we get

$$\begin{aligned} \int_{-\infty}^{\infty} |H_{\lambda,K}^{N,2}(t, y)|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} &\leq \frac{1}{\sqrt{\pi}} \frac{2\pi\Gamma(2(\lambda+1))}{(\lambda+1)[\Gamma(\lambda+1)]^2 2^{2(\lambda+1)}} \sum_{k=N+1}^{\infty} \sum_{n=1}^k \sum_{i=|k-2n|}^k \left| \left\langle \phi_{\lambda}, \frac{C_n^{\lambda}}{\|C_n^{\lambda}\|} \right\rangle \right|^2 \frac{\|\phi_{\lambda}\|^2}{\|C_n^{\lambda}\|^2 \|C_{k-n}^{\lambda}\|^2} \\ &\quad \times 16\lambda^4 t^2 e^{-2t(\tilde{\lambda}_k^{1/2} + \tilde{\lambda}_{k-n}^{1/2})} \left| \frac{[\Gamma(\lambda+3/2)]^2 \Gamma(n+2\lambda+1) \Gamma(k-n+2\lambda+1)}{[\Gamma(2\lambda+2)]^2 \Gamma(n+2\lambda+3/2) \Gamma(k-n+2\lambda+3/2)} \right|^2 \\ &\quad \times \binom{n+\lambda-1/2}{n-1}^2 \left[\frac{\Gamma(2\lambda) \Gamma(k-n+\lambda+1/2)}{\Gamma(\lambda+1/2) \Gamma(k-n+2\lambda)} \right]^2 \frac{(2\lambda+2)(2\lambda+1)(k-n)}{(\lambda+1/2)^2 (k-n+2\lambda)} \|C_{k-n}^{\lambda}\|_{\lambda}^2. \end{aligned} \quad (146)$$

On the other hand,

$$\begin{aligned} &\left| \frac{[\Gamma(\lambda+3/2)]^2 \Gamma(n+2\lambda+1) \Gamma(k-n+2\lambda+1)}{[\Gamma(2\lambda+2)]^2 \Gamma(n+2\lambda+3/2) \Gamma(k-n+2\lambda+3/2)} \right|^2 \\ &\quad \times \binom{n+\lambda-1/2}{n-1}^2 \left[\frac{\Gamma(2\lambda) \Gamma(k-n+\lambda+1/2)}{\Gamma(\lambda+1/2) \Gamma(k-n+2\lambda)} \right]^2 \frac{(2\lambda+2)(2\lambda+1)(k-n)}{(\lambda+1/2)^2 (k-n+2\lambda)} \|C_{k-n}^{\lambda}\|_{\lambda}^2 \\ &= \frac{2[\Gamma(n+2\lambda)]^2 [\Gamma(n+\lambda+1/2)]^2 [\Gamma(k-n+\lambda+1/2)]^2 \|C_{k-n}^{\lambda}\|_{\lambda}^2}{(2\lambda)^4 [\Gamma(2\lambda)]^2 [\Gamma(n+2\lambda+1/2)]^2 [\Gamma(k-n+2\lambda+1/2)]^2 [\Gamma(n)]^2}. \end{aligned} \quad (147)$$

Hence,

$$\begin{aligned} \int_{-\infty}^{\infty} |H_{\lambda,K}^{N,2}(t, y)|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} &\leq \frac{1}{\sqrt{\pi}} \frac{2\pi\Gamma(2(\lambda+1))}{(\lambda+1)[\Gamma(\lambda+1)]^2 2^{2(\lambda+1)}} \sum_{k=N+1}^{\infty} \sum_{n=1}^k \sum_{i=|k-2n|}^k \left| \left\langle \phi_{\lambda}, \frac{C_n^{\lambda}}{\|C_n^{\lambda}\|} \right\rangle \right|^2 \frac{\|\phi_{\lambda}\|^2 t^2 e^{-2t(\tilde{\lambda}_k^{1/2} + \tilde{\lambda}_{k-n}^{1/2})}}{\|C_n^{\lambda}\|^2} \\ &\quad \times \frac{2[\Gamma(n+2\lambda)]^2 [\Gamma(n+\lambda+1/2)]^2 [\Gamma(k-n+\lambda+1/2)]^2 \|C_{k-n}^{\lambda}\|_{\lambda}^2}{[\Gamma(2\lambda)]^2 [\Gamma(n+2\lambda+1/2)]^2 [\Gamma(k-n+2\lambda+1/2)]^2 [\Gamma(n)]^2} \\ &= \frac{1}{\sqrt{\pi}} \frac{2\pi\Gamma(2(\lambda+1)) \|\phi_{\lambda}\|^2}{(\lambda+1)[\Gamma(\lambda+1)]^2 2^{2(\lambda+1)}} \sum_{k=N+1}^{\infty} \sum_{n=1}^k \sum_{i=|k-2n|}^k \left| \left\langle \phi_{\lambda}, \frac{C_n^{\lambda}}{\|C_n^{\lambda}\|} \right\rangle \right|^2 t^2 e^{-2t(\tilde{\lambda}_k^{1/2} + \tilde{\lambda}_{k-n}^{1/2})} \\ &\quad \times \frac{2[\Gamma(n+2\lambda)] [\Gamma(n+\lambda+1/2)]^2 [\Gamma(k-n+\lambda+1/2)]^2 (n+\lambda)n!}{\Gamma(2\lambda) [\Gamma(n+2\lambda+1/2)]^2 [\Gamma(k-n+2\lambda+1/2)]^2 [\Gamma(n)]^2 \lambda}. \end{aligned} \quad (148)$$

Then, for λ big enough, we have

$$\begin{aligned} &\frac{2[\Gamma(n+2\lambda)] [\Gamma(n+\lambda+1/2)]^2 [\Gamma(k-n+\lambda+1/2)]^2 (n+\lambda)n!}{\Gamma(2\lambda) [\Gamma(n+2\lambda+1/2)]^2 [\Gamma(k-n+2\lambda+1/2)]^2 [\Gamma(n)]^2 \lambda} \\ &\sim \frac{2\sqrt{2\pi} e^{-2\lambda} (2\lambda)^{2\lambda+n-1/2} 2\pi e^{-2\lambda} (\lambda)^{2\lambda+2n} 2\pi e^{-2\lambda} (\lambda)^{2\lambda+2k-2n} (n+\lambda)n!}{\sqrt{2\pi} e^{-2\lambda} (2\lambda)^{2\lambda-1/2} 2\pi e^{-4\lambda} (2\lambda)^{4\lambda+2n} 2\pi e^{-4\lambda} (2\lambda)^{4\lambda+2k-2n} \lambda [(n-1)!]^2} = \frac{2^{n+1} \lambda^{n-1} (n+\lambda)n!}{2^{2k+8\lambda} \lambda^{4\lambda} e^{-4\lambda} [(n-1)!]^2} \\ &\leq \left(\frac{n}{\lambda} + 1 \right) \frac{\lambda^n n}{\lambda^{4\lambda} (n-1)!}. \end{aligned} \quad (149)$$

Therefore, for λ big enough, we have

$$\begin{aligned} & \int_{-\infty}^{\infty} |H_{\lambda,K}^{N,2}(t,y)|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} \\ & \leq \frac{\|\phi_{\lambda}\|^2}{\sqrt{\pi}} \frac{2\pi\Gamma(2(\lambda+1))}{(\lambda+1)[\Gamma(\lambda+1)]^2 2^{2(\lambda+1)}} \\ & \times \sum_{k=N+1}^{\infty} \sum_{n=1}^k \sum_{i=|k-2n|}^k \left| \left\langle \phi_{\lambda}, \frac{C_n^{\lambda}}{\|C_n^{\lambda}\|} \right\rangle \right|^2 \\ & \cdot t^2 e^{-2t(\hat{\lambda}_k^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \left(\frac{n}{\lambda} + 1 \right) \frac{\lambda^n n}{\lambda^{4\lambda} (n-1)!}. \end{aligned} \quad (150)$$

Similarly to the previous case, taking $\lambda = (k-n)(1+2k)^2$ we have that for k big enough there exists $C > 0$ such that

$$t^2 e^{-2t(\hat{\lambda}_k^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \left(\frac{n}{\lambda} + 1 \right) \frac{\lambda^n n}{\lambda^{4\lambda} (n-1)!} \leq \frac{e^{-t} C}{e^k}. \quad (151)$$

Thus, for λ big enough, we have

$$\begin{aligned} & \int_{-\infty}^{\infty} |H_{\lambda,K}^{N,2}(t,y)|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} \leq \frac{\|\phi_{\lambda}\|^2}{\sqrt{\pi}} \\ & \cdot \frac{2\pi\Gamma(2(\lambda+1))}{(\lambda+1)[\Gamma(\lambda+1)]^2 2^{2(\lambda+1)}} \\ & \times \sum_{k=N+1}^{\infty} \sum_{n=1}^k \sum_{i=|k-2n|}^k \left| \left\langle \phi_{\lambda}, \frac{C_n^{\lambda}}{\|C_n^{\lambda}\|} \right\rangle \right|^2 \\ & \cdot t^2 e^{-2t(\hat{\lambda}_k^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \left(\frac{n}{\lambda} + 1 \right) \frac{\lambda^n n}{\lambda^{4\lambda} (n-1)!} \leq \frac{e^{-t}}{\sqrt{\pi}} \\ & \cdot \frac{2\pi(2\lambda+1)2\lambda\Gamma(2\lambda)}{(\lambda+1)\lambda^2[\Gamma(\lambda)]^2 2^{2\lambda} 2^2} \frac{MC\|\phi_{\lambda}\|^2}{e^{(N+1)}} \\ & \cdot \sum_{k=N+1}^{\infty} \sum_{n=1}^k \sum_{i=|k-2n|}^k \left| \left\langle \phi_{\lambda}, \frac{C_n^{\lambda}}{\|C_n^{\lambda}\|} \right\rangle \right|^2 \\ & \leq \frac{2\pi\Gamma(2\lambda)}{\lambda[\Gamma(\lambda)]^2 2^{2\lambda}} \frac{e^{-t} MC\|\phi_{\lambda}\|_{2,\lambda}^4}{e^{(N+1)}} \leq \frac{e^{-t} MC\|\phi\|_{2,\gamma}^4}{e^{(N+1)}}. \end{aligned} \quad (152)$$

Hence

$$\begin{aligned} & \left(\int_{-\infty}^{\infty} \left| \int_0^{\infty} H_{\lambda,K}^{N,2}(t,y) dt \right|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} \right)^{1/2} \\ & \leq \int_0^{\infty} \left(\frac{e^{-t} MC\|\phi\|_{2,\gamma}^4}{e^{(N+1)}} \right)^{1/2} dt = \frac{(MC)^{1/2} \|\phi\|_{2,\gamma}^2}{e^{(N+1)/2}}. \end{aligned} \quad (153)$$

Then, $\{\int_0^{\infty} H_{\lambda,K}^{N,2}(t,\cdot) dt\}$ is a bounded sequence on $L^2(\mathbb{R}, \gamma)$, so, by Bourbaki-Alaoglu's theorem, there exists a sequence $(\lambda_j)_{j \in \mathbb{N}}$ $\lim_{j \rightarrow \infty} \lambda_j = \infty$ such that, for all $N \in \mathbb{N}$,

$\{\int_0^{\infty} H_{\lambda_j,K}^{N,2}(t,\cdot) dt\}_{j \in \mathbb{N}}$ converges weakly in $L^2(\mathbb{R}, \gamma)$ to a function $H_K^{N,2} \in L^2(\mathbb{R}, \gamma)$. Moreover,

$$\int_{-\infty}^{\infty} |H_K^{N,2}(y)|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} \leq \frac{(MC)^{1/2} \|\phi\|_{2,\gamma}^2}{e^{(N+1)/2}}. \quad (154)$$

Therefore, there exists a nondecreasing sequence $(N_j)_{j \in \mathbb{N}}$ such that

$$H_K^{N_j,2}(y) \rightarrow 0, \quad a.e. \ y \in \mathbb{R}. \quad (155)$$

On the other hand,

$$\begin{aligned} (F_{\lambda,K}(y))^2 &= g_{1,\lambda}\phi(y) + g_{2,\lambda}\phi(y) \\ &= \sum_{k=1}^N \sum_{n=1}^k \left\langle \phi_{\lambda}, \frac{C_n^{\lambda}}{\|C_n^{\lambda}\|} \right\rangle \left\langle \phi_{\lambda}, \frac{C_{k-n}^{\lambda}}{\|C_{k-n}^{\lambda}\|} \right\rangle \\ & \cdot \frac{\hat{\lambda}_n^{1/2} \hat{\lambda}_{k-n}^{1/2}}{(\hat{\lambda}_n^{1/2} + \hat{\lambda}_{k-n}^{1/2})^2} C_n^{\lambda} \left(\frac{y}{\sqrt{\lambda}} \right) C_{k-n}^{\lambda} \left(\frac{y}{\sqrt{\lambda}} \right) \times \left(1 - \frac{y^2}{\lambda} \right)^{\lambda-1/2} e^{y^2} + \int_0^{\infty} H_{\lambda,K}^{N,1}(t,y) dt \left(1 - \frac{y^2}{\lambda} \right)^{\lambda/2-1/4} e^{y^2/2} + \sum_{k=1}^N \sum_{n=1}^k \left\langle \phi_{\lambda}, \frac{C_n^{\lambda}}{\|C_n^{\lambda}\|} \right\rangle \cdot \left\langle \phi_{\lambda}, \frac{C_{k-n}^{\lambda}}{\|C_{k-n}^{\lambda}\|} \right\rangle \frac{4\lambda^2}{(\hat{\lambda}_n^{1/2} + \hat{\lambda}_{k-n}^{1/2})^2} \times C_{n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right) \cdot C_{k-n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right) \left(1 - \frac{y^2}{\lambda} \right)^{\lambda+1/2} e^{y^2} + \int_0^{\infty} H_{\lambda,K}^{N,2}(t,y) dt \left(1 - \frac{y^2}{\lambda} \right)^{\lambda/2+1/4} e^{y^2/2} = F_{\lambda,K}^N(y) + \int_0^{\infty} H_{\lambda,K}^{N,1}(t,y) dt \left(1 - \frac{y^2}{\lambda} \right)^{\lambda/2-1/4} \cdot e^{y^2/2} + \int_0^{\infty} H_{\lambda,K}^{N,2}(t,y) dt \left(1 - \frac{y^2}{\lambda} \right)^{\lambda/2+1/4} e^{y^2/2}, \end{aligned} \quad (156)$$

where

$$\begin{aligned} F_{\lambda,K}^N(y) &= \sum_{k=1}^N \sum_{n=1}^k \left\langle \phi_{\lambda}, \frac{C_n^{\lambda}}{\|C_n^{\lambda}\|} \right\rangle \left\langle \phi_{\lambda}, \frac{C_{k-n}^{\lambda}}{\|C_{k-n}^{\lambda}\|} \right\rangle \\ & \cdot \frac{\hat{\lambda}_n^{1/2} \hat{\lambda}_{k-n}^{1/2}}{(\hat{\lambda}_n^{1/2} + \hat{\lambda}_{k-n}^{1/2})^2} C_n^{\lambda} \left(\frac{y}{\sqrt{\lambda}} \right) C_{k-n}^{\lambda} \left(\frac{y}{\sqrt{\lambda}} \right) \times \left(1 - \frac{y^2}{\lambda} \right)^{\lambda-1/2} e^{y^2} + \int_0^{\infty} H_{\lambda,K}^{N,1}(t,y) dt \left(1 - \frac{y^2}{\lambda} \right)^{\lambda/2-1/4} e^{y^2/2} + \int_0^{\infty} H_{\lambda,K}^{N,2}(t,y) dt \left(1 - \frac{y^2}{\lambda} \right)^{\lambda/2+1/4} e^{y^2/2}, \end{aligned}$$

$$\begin{aligned}
& - \frac{y^2}{\lambda} \Big)^{\lambda-1/2} e^{y^2} + \sum_{k=1}^N \sum_{n=1}^k \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|^2} \right\rangle \\
& \cdot \left\langle \phi_\lambda, \frac{C_{k-n}^\lambda}{\|C_{k-n}^\lambda\|^2} \right\rangle \frac{4\lambda^2}{(\hat{\lambda}_n^{1/2} + \hat{\lambda}_{k-n}^{1/2})^2} \times C_{n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right) \\
& \cdot C_{k-n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right) \left(1 - \frac{y^2}{\lambda} \right)^{\lambda+1/2} e^{y^2}.
\end{aligned} \tag{157}$$

$$\begin{aligned}
F_{\lambda,K}^N &= (F_{\lambda,K})^2 \\
& - \int_0^\infty H_{\lambda,K}^{N,1}(t, y) dt \left(1 - \frac{y^2}{\lambda} \right)^{\lambda/2-1/4} e^{y^2/2} \\
& - \int_0^\infty H_{\lambda,K}^{N,2}(t, y) dt \left(1 - \frac{y^2}{\lambda} \right)^{\lambda/2+1/4} e^{y^2/2},
\end{aligned} \tag{158}$$

we have, for all $m \in \mathbb{N}$,

$$F_{\lambda,K}^{N_m} \longrightarrow F_K^{N_m}, \quad \text{weakly in } L^2(\mathbb{R}, \gamma), \quad \text{as } j \longrightarrow \infty; \tag{159}$$

Defining, for each $m \in \mathbb{N}$, $F_K^{N_m} = F^2 - H_K^{N_m,1} - H_K^{N_m,2}$, then since $F_{\lambda,K} \rightarrow F_K$, as $j \rightarrow \infty$ in the weak topology of $L^2(\mathbb{R}, \gamma)$ and

also,

$$F_K^{N_m}(y) \longrightarrow F^2(y), \quad \text{as } m \longrightarrow \infty \text{ a.e. } y \in \mathbb{R}. \tag{160}$$

Now,

$$\begin{aligned}
F_{\lambda,K}^N(y) &= \sum_{k=1}^N \sum_{n=1}^k \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|^2} \right\rangle \left\langle \phi_\lambda, \frac{C_{k-n}^\lambda}{\|C_{k-n}^\lambda\|^2} \right\rangle \frac{\hat{\lambda}_n^{1/2} \hat{\lambda}_{k-n}^{1/2}}{(\hat{\lambda}_n^{1/2} + \hat{\lambda}_{k-n}^{1/2})^2} C_n^\lambda \left(\frac{y}{\sqrt{\lambda}} \right) C_{k-n}^\lambda \left(\frac{y}{\sqrt{\lambda}} \right) \times \left(1 - \frac{y^2}{\lambda} \right)^{\lambda-1/2} e^{y^2} \\
& + \sum_{k=1}^N \sum_{n=1}^k \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|^2} \right\rangle \left\langle \phi_\lambda, \frac{C_{k-n}^\lambda}{\|C_{k-n}^\lambda\|^2} \right\rangle \frac{4\lambda^2}{(\hat{\lambda}_n^{1/2} + \hat{\lambda}_{k-n}^{1/2})^2} \times C_{n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right) C_{k-n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right) \left(1 - \frac{y^2}{\lambda} \right)^{\lambda+1/2} e^{y^2} \\
& = \sum_{k=1}^N \sum_{n=1}^k \frac{(n+\lambda) n! (k-n)! (k-n+\lambda)}{\lambda^2 (2\lambda)_n (2\lambda)_{k-n}} \frac{\hat{\lambda}_n^{1/2} \hat{\lambda}_{k-n}^{1/2}}{(\hat{\lambda}_n^{1/2} + \hat{\lambda}_{k-n}^{1/2})^2} \langle \phi_\lambda, C_n^\lambda \rangle \langle \phi_\lambda, C_{k-n}^\lambda \rangle \\
& \times C_n^\lambda \left(\frac{y}{\sqrt{\lambda}} \right) C_{k-n}^\lambda \left(\frac{y}{\sqrt{\lambda}} \right) \left(1 - \frac{y^2}{\lambda} \right)^{\lambda-1/2} e^{y^2} + \sum_{k=1}^N \sum_{n=1}^k \langle \phi_\lambda, C_n^\lambda \rangle \langle \phi_\lambda, C_{k-n}^\lambda \rangle C_{n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right) C_{k-n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right) \\
& \times \frac{4(n+\lambda) n! (k-n+\lambda) (k-n)!}{(2\lambda)_n (2\lambda)_{k-n}} \frac{1}{(\hat{\lambda}_n^{1/2} + \hat{\lambda}_{k-n}^{1/2})^2} \left(1 - \frac{y^2}{\lambda} \right)^{\lambda+1/2} e^{y^2} \\
& = \sum_{k=1}^N \sum_{n=1}^k \frac{(n/\lambda + 1) n! (k-n)! ((k-n)/\lambda + 1)}{2(2+1/\lambda) \cdots (2+(n-1)/\lambda) 2(2+1/\lambda) \cdots (2+(k-n-1)/\lambda)} \\
& \times \frac{(n(n/\lambda + 2))^{1/2} ((k-n)((k-n)/\lambda + 2))^{1/2}}{\{(n(n/\lambda + 2))^{1/2} + ((k-n)((k-n)/\lambda + 2))^{1/2}\}^2} \langle \phi_\lambda, \lambda^{-n/2} C_n^\lambda \rangle \langle \phi_\lambda, \lambda^{-(k-n)/2} C_{k-n}^\lambda \rangle \\
& \times \lambda^{-n/2} C_n^\lambda \left(\frac{y}{\sqrt{\lambda}} \right) \lambda^{-(k-n)/2} C_{k-n}^\lambda \left(\frac{y}{\sqrt{\lambda}} \right) \left(1 - y^2/\lambda \right)^{\lambda-1/2} e^{y^2} \\
& + \sum_{k=1}^N \sum_{n=1}^k \langle \phi_\lambda, \lambda^{-n/2} C_n^\lambda \rangle \langle \phi_\lambda, \lambda^{-(k-n)/2} C_{k-n}^\lambda \rangle \lambda^{-(n-1)/2} C_{n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right) \lambda^{-(k-n-1)/2} C_{k-n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right) \\
& \times \frac{4(n/\lambda + 1) n! ((k-n)/\lambda + 1) (k-n)!}{2(2+1/\lambda) \cdots (2+(n-1)/\lambda) 2(2+1/\lambda) \cdots (2+(k-n-1)/\lambda)} \\
& \times \frac{1}{\{(n(n/\lambda + 2))^{1/2} + ((k-n)((k-n)/\lambda + 2))^{1/2}\}^2} \left(1 - \frac{y^2}{\lambda} \right)^{\lambda+1/2} e^{y^2}.
\end{aligned} \tag{161}$$

Then,

$$\begin{aligned}
\lim_{\lambda \rightarrow \infty} F_{\lambda, K}^N(y) &= \lim_{\lambda \rightarrow \infty} \sum_{k=1}^N \sum_{n=1}^k \frac{(n/\lambda + 1) n! (k-n)! ((k-n)/\lambda + 1)}{2(2+1/\lambda) \cdots (2+(n-1)/\lambda) 2(2+1/\lambda) \cdots (2+(k-n-1)/\lambda)} \\
&\quad \times \frac{(n(n/\lambda + 2))^{1/2} ((k-n)((k-n)/\lambda + 2))^{1/2}}{\{(n(n/\lambda + 2))^{1/2} + ((k-n)((k-n)/\lambda + 2))^{1/2}\}^2} \langle \phi_\lambda, \lambda^{-n/2} C_n^\lambda \rangle \langle \phi_\lambda, \lambda^{-(k-n)/2} C_{k-n}^\lambda \rangle \\
&\quad \times \lambda^{-n/2} C_n^\lambda \left(\frac{y}{\sqrt{\lambda}} \right) \lambda^{-(k-n)/2} C_{k-n}^\lambda \left(\frac{y}{\sqrt{\lambda}} \right) \left(1 - \frac{y^2}{\lambda} \right)^{\lambda-1/2} e^{y^2} \\
&\quad + \lim_{\lambda \rightarrow \infty} \sum_{k=1}^N \sum_{n=1}^k \langle \phi_\lambda, \lambda^{-n/2} C_n^\lambda \rangle \langle \phi_\lambda, \lambda^{-(k-n)/2} C_{k-n}^\lambda \rangle \lambda^{-(n-1)/2} C_{n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right) \lambda^{-(k-n-1)/2} C_{k-n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right) \\
&\quad \times \frac{4(n/\lambda + 1) n! ((k-n)/\lambda + 1) (k-n)!}{2(2+1/\lambda) \cdots (2+(n-1)/\lambda) 2(2+1/\lambda) \cdots (2+(k-n-1)/\lambda)} \\
&\quad \times \frac{1}{\{(n(n/\lambda + 2))^{1/2} + ((k-n)((k-n)/\lambda + 2))^{1/2}\}^2} \left(1 - \frac{y^2}{\lambda} \right)^{\lambda+1/2} e^{y^2} \\
&= \sum_{k=1}^N \sum_{n=1}^k \frac{n! (k-n)!}{2^n 2^{k-n}} \frac{n^{1/2} (k-n)^{1/2}}{(n^{1/2} + (k-n)^{1/2})^2} \left\langle \phi, \frac{H_n}{n!} \right\rangle \left\langle \phi, \frac{H_{k-n}}{(k-n)!} \right\rangle \frac{H_n(y)}{n!} \frac{H_{k-n}(y)}{(k-n)!} \\
&\quad + \sum_{k=1}^N \sum_{n=1}^k \frac{n! (k-n)!}{2^n 2^{k-n}} \frac{4}{2(n^{1/2} + (k-n)^{1/2})^2} \times \left\langle \phi, \frac{H_n}{n!} \right\rangle \left\langle \phi, \frac{H_{k-n}}{(k-n)!} \right\rangle \frac{H_{n-1}(y)}{(n-1)!} \frac{H_{k-n-1}(y)}{(k-n-1)!} \\
&= \sum_{k=1}^N \sum_{n=1}^k n^{1/2} (k-n)^{1/2} \left\langle \phi, \frac{H_n}{\|H_n\|^2} \right\rangle \left\langle \phi, \frac{H_{k-n}}{\|H_{k-n}\|^2} \right\rangle H_n(y) H_{k-n}(y) \times \int_0^\infty t e^{-t(n^{1/2} + (k-n)^{1/2})} dt \\
&\quad + \sum_{k=1}^N \sum_{n=1}^k \frac{4n(k-n)}{2} \left\langle \phi, \frac{H_n}{\|H_n\|^2} \right\rangle \left\langle \phi, \frac{H_{k-n}}{\|H_{k-n}\|^2} \right\rangle H_{n-1}(y) H_{k-n-1}(y) \times \int_0^\infty t e^{-t(n^{1/2} + (k-n)^{1/2})} dt \\
&= \int_0^\infty \sum_{k=1}^N \sum_{n=1}^k n^{1/2} (k-n)^{1/2} t e^{-t(n^{1/2} + (k-n)^{1/2})} \times \left\langle \phi, \frac{H_n}{\|H_n\|^2} \right\rangle \left\langle \phi, \frac{H_{k-n}}{\|H_{k-n}\|^2} \right\rangle H_n(y) H_{k-n}(y) dt \\
&\quad + \int_0^\infty \sum_{k=1}^N \sum_{n=1}^k \frac{4n(k-n)}{2} t e^{-t(n^{1/2} + (k-n)^{1/2})} \times \left\langle \phi, \frac{H_n}{\|H_n\|^2} \right\rangle \left\langle \phi, \frac{H_{k-n}}{\|H_{k-n}\|^2} \right\rangle H_{n-1}(y) H_{k-n-1}(y) dt \\
&= \int_0^\infty \left(\sqrt{t} \sum_{k=1}^N k^{1/2} e^{-tk^{1/2}} \left\langle \phi, \frac{H_k}{\|H_k\|^2} \right\rangle H_k(y) \right)^2 dt \\
&\quad + \int_0^\infty t \left(\frac{1}{\sqrt{2}} \sum_{k=1}^N 2k e^{-tk^{1/2}} \left\langle \phi, \frac{H_k}{\|H_k\|^2} \right\rangle H_{k-1}(y) \right)^2 dt,
\end{aligned} \tag{162}$$

so we have

$$\begin{aligned}
\lim_{\lambda \rightarrow \infty} F_{\lambda, K}^N(y) &= \int_0^\infty \left(\sqrt{t} \sum_{k=1}^N k^{1/2} e^{-tk^{1/2}} \left\langle \phi, \frac{H_k}{\|H_k\|^2} \right\rangle \right. \\
&\quad \cdot H_k(y) \Big)^2 dt + \int_0^\infty t \left(\frac{1}{\sqrt{2}} \sum_{k=1}^N 2k e^{-tk^{1/2}} \left\langle \phi, \frac{H_k}{\|H_k\|^2} \right\rangle \right. \\
&\quad \cdot H_{k-1}(y) \Big)^2 dt.
\end{aligned}$$

$$\cdot \sum_{k=1}^N 2k e^{-tk^{1/2}} \left\langle \phi, \frac{H_k}{\|H_k\|^2} \right\rangle H_{k-1}(y) \Big)^2 dt.$$

(163)

Thus,

$$F_K^{N_m}(y) = \int_0^\infty \left(\sqrt{t} \sum_{k=1}^{N_m} k^{1/2} e^{-tk^{1/2}} \left\langle \phi, \frac{H_k}{\|H_k\|^2} \right\rangle \right.$$

$$\begin{aligned} & \cdot H_k(y) \Big)^2 dt + \int_0^\infty t \left(\frac{1}{\sqrt{2}} \right. \\ & \cdot \sum_{k=1}^{N_m} 2ke^{-tk^{1/2}} \left\langle \phi, \frac{H_k}{\|H_k\|^2} \right\rangle H_{k-1}(y) \Big)^2 dt. \end{aligned} \quad (164)$$

On the other hand, given that

$$|H_k(x)| \leq \eta e^{x^2/2} 2^{k/2} \sqrt{k!}, \quad \eta = 1.086435, \quad (165)$$

we get

$$\begin{aligned} & \left| k^{1/2} e^{-tk^{1/2}} \left\langle \phi, \frac{H_k}{\|H_k\|^2} \right\rangle |H_k(y)| \right| \\ & \leq k^{1/2} e^{-tk^{1/2}} \|\phi\| \frac{\eta e^{y^2/2} 2^{k/2} \sqrt{k!}}{\pi^{1/4} 2^{k/2} \sqrt{k!}} \\ & = \frac{\eta \|\phi\| e^{y^2/2}}{\pi^{1/4}} k^{1/2} e^{-tk^{1/2}}; \end{aligned} \quad (166)$$

then,

$$\begin{aligned} & \left| t \left(\sum_{k=1}^{N_m} k^{1/2} e^{-tk^{1/2}} \left\langle \phi, \frac{H_k}{\|H_k\|^2} \right\rangle H_k(y) \right)^2 \right| \\ & \leq \frac{\eta^2 \|\phi\|^2 e^{y^2}}{\pi^{1/2}} t \left(\sum_{k=1}^{N_m} k^{1/2} e^{-tk^{1/2}} \right)^2 \\ & \leq \frac{\eta^2 \|\phi\|^2 e^{y^2}}{\pi^{1/2}} t \left(\sum_{k=1}^{\infty} k^{1/2} e^{-tk^{1/2}} \right)^2. \end{aligned} \quad (167)$$

Now, for $t \geq 1$, we have

$$\sum_{k=1}^{\infty} k^{1/2} e^{-t(k^{1/2}-1/2)} \leq \sum_{k=1}^{\infty} k^{1/2} e^{-k^{1/2}+1/2}, \quad (168)$$

which converges. Therefore

$$\begin{aligned} & \int_1^\infty t \left(\sum_{k=1}^{\infty} k^{1/2} e^{-tk^{1/2}} \right)^2 dt \\ & \leq \int_1^\infty t e^{-t} \left(\sum_{k=1}^{\infty} k^{1/2} e^{-t(k^{1/2}-1/2)} \right)^2 dt \\ & \leq \int_1^\infty t e^{-t} \left(\sum_{k=1}^{\infty} k^{1/2} e^{-k^{1/2}+1/2} \right)^2 dt \leq C \int_1^\infty t e^{-t} dt \\ & < \infty. \end{aligned} \quad (169)$$

Let $0 < t < 1$. For $y \in [-K, K]$, let us define

$$\tau_m(y) = \sum_{k=1}^{N_m} k^{1/2} e^{-tk^{1/2}} \left\langle \phi, \frac{H_k}{\|H_k\|^2} \right\rangle H_k(y). \quad (170)$$

We know that, for $y \in [-K, K]$,

$$\tau_m(y) \longrightarrow \sum_{k=1}^{\infty} k^{1/2} e^{-tk^{1/2}} \left\langle \phi, \frac{H_k}{\|H_k\|^2} \right\rangle H_k(y), \quad (171)$$

if $m \longrightarrow \infty$,

and also

$$\sum_{k=1}^{\infty} k^{1/2} e^{-tk^{1/2}} \left\langle \phi, \frac{H_k}{\|H_k\|^2} \right\rangle H_k(y) = \frac{\partial}{\partial t} P_t^y(\phi(y)). \quad (172)$$

Now, for $\varepsilon = 1$, there exists $M(\varepsilon, y) > 0$ such that

$$\left| \tau_m(y) - \frac{\partial}{\partial t} P_t^y(\phi(y)) \right| < 1, \quad \text{if } m \geq M; \quad (173)$$

then, for $m \geq M$, we get

$$\begin{aligned} |\tau_m(y)| & < 1 + \left| \frac{\partial}{\partial t} P_t^y(\phi(y)) \right| \\ & < t(1 + C(1 + |y|)e^{-t}). \end{aligned} \quad (174)$$

Therefore, for $y \in [-K, K]$, we have

$$\int_0^1 t(1 + C(1 + |y|)e^{-t})^2 dt < \infty. \quad (175)$$

Let us then define

$$\begin{aligned} \psi(t) & = \begin{cases} \frac{\eta^2 \|\phi\|^2 e^{y^2}}{\pi^{1/2}} t \left(\sum_{k=1}^{\infty} k^{1/2} e^{-tk^{1/2}} \right)^2 & \text{if } t \geq 1 \\ t(1 + C(1 + |y|)e^{-t})^2 & \text{if } 0 < t < 1 \end{cases} \end{aligned} \quad (176)$$

which is integrable in $(0, \infty)$; now for all $m > 0$ we have

$$\left| \left(\sqrt{t} \sum_{k=1}^{N_m} k^{1/2} e^{-tk^{1/2}} \left\langle \phi, \frac{H_k}{\|H_k\|^2} \right\rangle H_k(y) \right)^2 \right| \leq \psi(t), \quad (177)$$

for all $t > 0$;

then, by Lebesgue's dominated convergence theorem, we have

$$\begin{aligned} & \lim_{m \rightarrow \infty} \int_0^\infty \left(\sqrt{t} \sum_{k=1}^{N_m} k^{1/2} e^{-tk^{1/2}} \left\langle \phi, \frac{H_k}{\|H_k\|^2} \right\rangle \right. \\ & \cdot H_k(y) \Big)^2 dt \\ & = \int_0^\infty \left(\sqrt{t} \sum_{k=1}^{\infty} k^{1/2} e^{-tk^{1/2}} \left\langle \phi, \frac{H_k}{\|H_k\|^2} \right\rangle \right. \\ & \cdot H_k(y) \Big)^2 dt = \int_0^\infty t \left(\frac{\partial}{\partial t} P_t^y(\phi(y)) \right)^2 dt. \end{aligned} \quad (178)$$

Similarly, we have

$$\begin{aligned} & \lim_{m \rightarrow \infty} \int_0^\infty t \left(\frac{1}{\sqrt{2}} \right. \\ & \quad \cdot \sum_{k=1}^{N_m} 2ke^{-tk^{1/2}} \left\langle \phi, \frac{H_k}{\|H_k\|^2} \right\rangle H_{k-1}(y) \Big)^2 dt \\ & = \int_0^\infty t \left(\frac{1}{\sqrt{2}} \frac{\partial}{\partial y} P_t^\gamma(\phi(y)) \right)^2 dy; \end{aligned} \quad (179)$$

that is,

$$F_K^{N_m}(y) \longrightarrow (g^\gamma \phi(y))^2, \quad \text{as } m \longrightarrow \infty \text{ a.e. } y \in \mathbb{R}; \quad (180)$$

therefore, from (160) and (180), we get

$$F(y) = g^\gamma(\phi(y)). \quad (181)$$

(ii) The proof is essentially analogous to (i), so fewer details will be provided. Assume that the operator $g^{(\alpha, \beta)}$ is bounded in $L^p([-1, 1], \mu_{(\alpha, \beta)})$. Let $\phi \in C_0^\infty(0, \infty)$ and $\beta > 0$, and then we have

$$\|g^{(\alpha, \beta)} \phi_\beta\|_{L^p([-1, 1], \mu_{(\alpha, \beta)})} \leq C \|\phi_\beta\|_{L^p([-1, 1], \mu_{(\alpha, \beta)})}, \quad (182)$$

where $\phi_\beta(x) = \phi((\beta/2)(1-x))$, for $x \in [-1, 1]$; that is,

$$\begin{aligned} & \left\| \left\{ \int_0^\infty t \left| \nabla_{(\alpha, \beta)} P_t^{\alpha, \beta} \phi_\beta \right|^2 dt \right\}^{1/2} \right\|_{L^p([-1, 1], \mu_{(\alpha, \beta)})} \\ & \leq C \|\phi_\beta\|_{L^p([-1, 1], \mu_{(\alpha, \beta)})}. \end{aligned} \quad (183)$$

Now making the change of variable $x = 1 - (2/\beta)y$, we have

$$\begin{aligned} & \left\{ \eta_{\alpha, \beta} \frac{2^{\alpha+\beta+1}}{\beta^{\alpha+1}} \right. \\ & \quad \cdot \int_0^\beta \left| \left\{ \int_0^\infty t \left| \nabla_{(\alpha, \beta)} P_t^{\alpha, \beta} \phi_\beta \left(1 - \frac{2}{\beta}y\right) \right|^2 dt \right\}^{1/2} \right. \\ & \quad \cdot \left(1 - \frac{y}{\beta}\right)^{\beta/p} e^{y/p} \left| y^\alpha e^{-y} dy \right|^p \Big\}^{1/p} \\ & \leq C \|\phi_\beta\|_{L^p([-1, 1], \mu_\lambda)}; \end{aligned} \quad (184)$$

thus,

$$\begin{aligned} & \left\{ \int_0^\beta \left| \left\{ \int_0^\infty t \left| \nabla_{(\alpha, \beta)} P_t^{\alpha, \beta} \phi_\beta \left(1 - \frac{2}{\beta}y\right) \right|^2 dt \right\}^{1/2} \right. \right. \\ & \quad \cdot \left(1 - \frac{y}{\beta}\right)^{\beta/p} e^{y/p} \left| y^\alpha e^{-y} dy \right|^p \Big\}^{1/p} \leq C (Z_{\alpha, \beta})^{-1} \\ & \quad \cdot \|\phi_\beta\|_{L^p([-1, 1], \mu_\lambda)}, \end{aligned} \quad (185)$$

where $Z_{\alpha, \beta} = \Gamma(\alpha + \beta + 2)/\Gamma(\alpha + 1)\beta^{\alpha+2}\Gamma(\beta)$. Analogously, we get

$$\begin{aligned} & \left\{ \int_0^\beta \left| \left\{ \int_0^\infty t \left| \nabla_{(\alpha, \beta)} P_t^{\alpha, \beta} \phi_\beta \left(1 - \frac{2}{\beta}y\right) \right|^2 dt \right\}^{1/2} \right. \right. \\ & \quad \cdot \left(1 - \frac{y}{\beta}\right)^{\beta/2} e^{y/2} \left| y^\alpha e^{-y} dy \right|^2 \Big\}^{1/2} \leq C (Z_{\alpha, \beta})^{-1} \\ & \quad \cdot \|\phi_\beta\|_{L^2([-1, 1], \mu_\lambda)}. \end{aligned} \quad (186)$$

Now, for each $K \in \mathbb{N}$ and $\beta > 0$, such that $\beta > K$, define the functions

$$\begin{aligned} F_{\beta, K}(y) &= \chi_{(0, K)}(y) \\ & \cdot \left\{ \int_0^\infty t \left| \nabla_{(\alpha, \beta)} P_t^{\alpha, \beta} \phi_\beta \left(1 - \frac{2}{\beta}y\right) \right|^2 dt \right\}^{1/2} \\ & \cdot \left(1 - \frac{y}{\beta}\right)^{\beta/2} e^{y/2}, \\ f_{\beta, K}(y) &= \chi_{(0, K)}(y) \\ & \cdot \left\{ \int_0^\infty t \left| \nabla_{(\alpha, \beta)} P_t^{\alpha, \beta} \phi_\beta \left(1 - \frac{2}{\beta}y\right) \right|^2 dt \right\}^{1/2} \\ & \cdot \left(1 - \frac{y}{\beta}\right)^{\beta/p} e^{y/p}. \end{aligned} \quad (187)$$

From the previous inequalities both series converge for all $y \in (0, \beta)$, and $F_{\beta, K} = f_{\beta, K} \Omega_\beta$, where

$$\Omega_\beta(y) = e^{y/2-y/p} \left(1 - \frac{y}{\beta}\right)^{\beta/2-\beta/p} \quad (188)$$

for all $K \in \mathbb{N}$ and $\beta > K$. Now, as

$$\begin{aligned} |\Omega_\beta(y)| &= e^{y/2-y/p} \left(1 - \frac{y}{\beta}\right)^{\beta/2-\beta/p} \\ &\leq e^{(y/2)-(y/p)} (e^{-y/\beta})^{\beta/2-\beta/p} = 1 \end{aligned} \quad (189)$$

we conclude that Ω_β is bounded in $(0, K)$. On the other hand,

$$\begin{aligned} & (Z_{\alpha, \beta})^{-1/p} \|\phi_\beta\|_{L^p([-1, 1], \mu_{(\alpha, \beta)})} = (Z_{\alpha, \beta})^{-1/p} \\ & \cdot \left\{ \eta_{\alpha, \beta} \int_{-1}^1 |\phi_\beta(x)|^p (1-x)^\alpha (1+x)^\beta dx \right\}^{1/p}, \end{aligned} \quad (190)$$

and making the change of variable $x = 1 - (2/\beta)y$ we get

$$\begin{aligned}
 & (Z_{\alpha,\beta})^{-1/p} \\
 & \cdot \left\{ \eta_{\alpha,\beta} \int_{-1}^1 |\phi_\beta(x)|^p (1-x)^\alpha (1+x)^\beta dx \right\}^{1/p} \\
 & = (Z_{\alpha,\beta})^{-1/p} \left\{ \eta_{\alpha,\beta} \frac{2}{\beta} \right. \\
 & \cdot \left. \int_0^\beta \left| \phi_\beta \left(1 - \frac{2}{\beta}y \right) \right|^p \left(\frac{2y}{\beta} \right)^\alpha \left(2 - \frac{2y}{\beta} \right)^\beta dy \right\}^{1/p} \quad (191) \\
 & \leq C \left\{ \int_0^\beta |\phi(y)|^p y^\alpha \left(1 - \frac{y}{\beta} \right)^\beta dy \right\}^{1/p} \\
 & \leq C \left\{ \int_0^\infty |\phi(y)|^p y^\alpha e^{-y} dy \right\}^{1/p} \\
 & = C \|\phi\|_{L^p((0,\infty),\mu_\alpha)}.
 \end{aligned}$$

Then,

$$\begin{aligned}
 & \lim_{\beta \rightarrow \infty} (Z_{\alpha,\beta})^{-1/p} \|\phi_\beta\|_{L^p([-1,1],\mu_{(\alpha,\beta)})} \\
 & \leq \lim_{\beta \rightarrow \infty} C \left\{ \int_0^\infty |\phi(y)|^p y^\alpha e^{-y} dy \right\}^{1/p} \quad (192) \\
 & = C \|\phi\|_{L^p((0,\infty),\mu_\alpha)}.
 \end{aligned}$$

Moreover,

$$(Z_{\alpha,\beta})^{-1/p} \|\phi_\beta\|_{L^p([-1,1],\mu_{(\alpha,\beta)})} \leq C \|\phi\|_{L^p((0,\infty),\mu_\alpha)}. \quad (193)$$

Now,

$$\|F_{\beta,K}\|_{L^2((0,\infty),\mu_\alpha)} \leq C (Z_{\alpha,\beta})^{-1/2} \|\phi_\beta\|_{L^2([-1,1],\mu_{(\alpha,\beta)})} \quad (194)$$

and therefore, from (193) for $p = 2$ and (196), we get

$$\|F_{\beta,K}\|_{L^2((0,\infty),\mu_\alpha)} \leq C \|\phi\|_{L^2((0,\infty),\mu_\alpha)}. \quad (195)$$

Analogously, using the fact that Ω_β is bounded in $(0, K)$,

$$\begin{aligned}
 & \|F_{\beta,K}\|_{L^p((0,\infty),\mu_\alpha)} \\
 & \leq C \left\{ \frac{1}{\Gamma(\alpha+1)} \int_0^K |f_{\beta,K}(y)|^p y^\alpha e^{-y} dy \right\}^{1/p} \quad (196) \\
 & \leq C (Z_{\alpha,\beta})^{-1/p} \|\phi_\beta\|_{L^p([-1,1],\mu_{(\alpha,\beta)})}.
 \end{aligned}$$

Then, from (193) and (196),

$$\|F_{\beta,K}\|_{L^p((0,\infty),\mu_\alpha)} \leq C \|\phi\|_{L^p((0,\infty),\mu_\alpha)}, \quad (197)$$

for all $\beta > K$. Therefore $\{F_{\beta,K}\}$ is a bounded subsequence in $L^2((0,\infty),\mu_\alpha)$ with $L^p((0,\infty),\mu_\alpha)$. Thus by Bourbaki-Alaoglu's theorem, there exists an increasing sequence $\{\beta_j\}_{j \in \mathbb{N}}$ with $\lim_{j \rightarrow \infty} \beta_j = \infty$ and functions $F_K \in L^2((0,\infty),\mu_\alpha)$ and $f_K \in L^p((0,\infty),\mu_\alpha)$ satisfying that

(a) $F_{\beta_j,K} \rightarrow F_K$, as $j \rightarrow \infty$, in the weak topology of $L^2((0,\infty),\mu_\alpha)$,

(b) $F_{\beta_j,K} \rightarrow f_K$, as $j \rightarrow \infty$, in the weak topology of $L^p((0,\infty),\mu_\alpha)$.

Then, as in (i), we can conclude that there exists an increasing sequence $(\beta_j)_{j \in \mathbb{N}} \subset (0, \infty)$ such that $\lim_{j \rightarrow \infty} \beta_j = \infty$, and a function $F \in L^p((0,\infty),\mu_\alpha) \cap L^2((0,\infty),\mu_\alpha)$, such that

(a) for each $K \in \mathbb{N}$, $F_{\beta_j,K} \rightarrow F$, as $j \rightarrow \infty$, in the weak topology on $L^2((0,\infty),\mu_\alpha)$ and in the weak topology on $L^p((0,\infty),\mu_\alpha)$,

(b) $\|F\|_{L^p((0,\infty),\mu_\alpha)} \leq C \|\phi\|_{L^p((0,\infty),\mu_\alpha)}$.

Analogously to the Hermite case, we have

$$\begin{aligned}
 F_{\beta,K}(y) &= \chi_{(0,K)}(y) \\
 & \cdot \left\{ \int_0^\infty t \left(\frac{\partial}{\partial t} P_t^{\alpha,\beta} \phi_\beta \left(1 - \frac{2}{\beta}y \right) \right)^2 dt \left(1 - \frac{y}{\beta} \right)^\beta e^y \right. \\
 & + \left. \int_0^\infty \frac{t}{\beta} \left(\frac{\partial}{\partial y} P_t^{\alpha,\beta} \phi_\beta \left(1 - \frac{2}{\beta}y \right) \right)^2 dt \right. \\
 & \cdot \left. 4y \left(1 - \frac{y}{\beta} \right)^{\beta+1} e^y \right\}^{1/2}. \quad (198)
 \end{aligned}$$

Now, let us define

$$\begin{aligned}
 g_{1,\beta} \phi(y) &= \chi_{(0,K)}(y) \\
 & \cdot \int_0^\infty t \left(\frac{\partial}{\partial t} P_t^{\alpha,\beta} \phi_\beta \left(1 - \frac{2}{\beta}y \right) \right)^2 dt \left(1 - \frac{y}{\beta} \right)^\beta e^y, \\
 g_{2,\beta} \phi(y) &= \chi_{(0,K)}(y) \\
 & \cdot \int_0^\infty \frac{t}{\beta} \left(\frac{\partial}{\partial y} P_t^{\alpha,\beta} \phi_\beta \left(1 - \frac{2}{\beta}y \right) \right)^2 dt \\
 & \cdot 4y \left(1 - \frac{y}{\beta} \right)^{\beta+1} e^y. \quad (199)
 \end{aligned}$$

Let us first consider $g_{1,\beta}$. For a function $\phi \in L^2((0,\infty),\mu_\alpha)$,

$$\begin{aligned}
 & \left(\sqrt{t} \frac{\partial}{\partial t} P_t^{\alpha,\beta} \phi_\beta \left(1 - \frac{2}{\beta}y \right) \right)^2 \\
 & = \sum_{k=1}^\infty \sum_{n=1}^k \left\langle \phi_\beta, \frac{P_n^{(\alpha,\beta)}}{\|P_n^{(\alpha,\beta)}\|_2^2} \right\rangle \left\langle \phi_\beta, \frac{P_{k-n}^{(\alpha,\beta)}}{\|P_{k-n}^{(\alpha,\beta)}\|_2^2} \right\rangle \quad (200) \\
 & \cdot t \lambda_n^{1/2} \lambda_{k-n}^{1/2} \times e^{-t(\lambda_k^{1/2} + \lambda_{k-n}^{1/2})} P_n^{(\alpha,\beta)} \left(1 - \frac{2}{\beta}y \right) \\
 & \cdot P_{k-n}^{(\alpha,\beta)} \left(1 - \frac{2}{\beta}y \right).
 \end{aligned}$$

Therefore, as in the previous case,

$$\begin{aligned}
 g_{1,\beta}\phi(y) &= \sum_{k=1}^N \sum_{n=1}^k \left\langle \phi_\beta, \frac{P_n^{(\alpha,\beta)}}{\|P_n^{(\alpha,\beta)}\|_2^2} \right\rangle \\
 &\cdot \left\langle \phi_\beta, \frac{P_{k-n}^{(\alpha,\beta)}}{\|P_{k-n}^{(\alpha,\beta)}\|_2^2} \right\rangle \frac{\lambda_n^{1/2} \lambda_{k-n}^{1/2}}{(\lambda_k^{1/2} + \lambda_{k-n}^{1/2})^2} \times P_n^{(\alpha,\beta)} \left(1 - \frac{2}{\beta} y \right) P_{k-n}^{(\alpha,\beta)} \left(1 - \frac{2}{\beta} y \right) \left(1 - \frac{y}{\beta} \right)^\beta e^y \\
 &+ \int_0^\infty H_{\beta,K}^{N,1}(t, y) dt \left(1 - \frac{y}{\beta} \right)^{\beta/2} e^{y/2},
 \end{aligned} \quad (201)$$

where

$$\begin{aligned}
 H_{\beta,K}^{N,1}(t, y) &= \chi_{(0,K)}(y) \sum_{k=N+1}^\infty \sum_{n=1}^k \left\langle \phi_\beta, \frac{P_n^{(\alpha,\beta)}}{\|P_n^{(\alpha,\beta)}\|_2^2} \right\rangle \\
 &\cdot \left\langle \phi_\beta, \frac{P_{k-n}^{(\alpha,\beta)}}{\|P_{k-n}^{(\alpha,\beta)}\|_2^2} \right\rangle t \lambda_n^{1/2} \lambda_{k-n}^{1/2} \\
 &\times e^{-t(\lambda_k^{1/2} + \lambda_{k-n}^{1/2})} P_n^{(\alpha,\beta)} \left(1 - \frac{2}{\beta} y \right) P_{k-n}^{(\alpha,\beta)} \left(1 - \frac{2}{\beta} y \right) \\
 &\cdot \left(1 - \frac{y}{\beta} \right)^{\beta/2} e^{y/2}.
 \end{aligned} \quad (202)$$

We want to prove that, for $K \in \mathbb{N}$ and $\beta > K$,

$$\int_0^\infty \left| \int_0^\infty H_{\beta,K}^{N,1}(t, y) dt \right|^2 y^\alpha e^{-y} \frac{dy}{\Gamma(\alpha+1)} \leq \frac{C \|\phi\|_{2,\alpha}^4}{e^{N+1}}. \quad (203)$$

Now,

$$\begin{aligned}
 &\int_0^\infty |H_{\beta,K}^{N,1}(t, y)|^2 y^\alpha e^{-y} \frac{dy}{\Gamma(\alpha+1)} \leq \frac{\beta^\alpha}{2^{\alpha+\beta} \Gamma(\alpha+1)} \\
 &\cdot \int_0^\infty \left| \sum_{k=N+1}^\infty \sum_{n=1}^k \left\langle \phi_\beta, \frac{P_n^{(\alpha,\beta)}}{\|P_n^{(\alpha,\beta)}\|_2^2} \right\rangle \right. \\
 &\cdot \left\langle \phi_\beta, \frac{P_{k-n}^{(\alpha,\beta)}}{\|P_{k-n}^{(\alpha,\beta)}\|_2^2} \right\rangle t \lambda_n^{1/2} \lambda_{k-n}^{1/2} e^{-t(\lambda_k^{1/2} + \lambda_{k-n}^{1/2})} \\
 &\times P_n^{(\alpha,\beta)} \left(1 - \frac{2}{\beta} y \right) P_{k-n}^{(\alpha,\beta)} \left(1 - \frac{2}{\beta} y \right) \left. \right|^2 \left(\frac{2y}{\beta} \right)^\alpha \\
 &\cdot 2^\beta \left(1 - \frac{y}{\beta} \right)^\beta dy = \frac{\beta^{\alpha+1}}{2^{\alpha+\beta+1} \Gamma(\alpha+1)}
 \end{aligned}$$

$$\begin{aligned}
 &\cdot \int_{-1}^1 \left| \sum_{k=N+1}^\infty \sum_{n=1}^k \left\langle \phi_\beta, \frac{P_n^{(\alpha,\beta)}}{\|P_n^{(\alpha,\beta)}\|_2^2} \right\rangle \right. \\
 &\cdot \left\langle \phi_\beta, \frac{P_{k-n}^{(\alpha,\beta)}}{\|P_{k-n}^{(\alpha,\beta)}\|_2^2} \right\rangle t \lambda_n^{1/2} \lambda_{k-n}^{1/2} e^{-t(\lambda_k^{1/2} + \lambda_{k-n}^{1/2})} \\
 &\times P_n^{(\alpha,\beta)}(x) P_{k-n}^{(\alpha,\beta)}(x) \left. \right|^2 (1-x)^\alpha (1+x)^\beta dx.
 \end{aligned} \quad (204)$$

Then, again by Gasper's linearization of the product of Jacobi polynomials [10] and using Parseval's identity, we have

$$\begin{aligned}
 &\int_0^\infty |H_{\beta,K}^{N,1}(t, y)|^2 y^\alpha e^{-y} \frac{dy}{\Gamma(\alpha+1)} \\
 &\leq \frac{\beta^{\alpha+1}}{\eta_{\alpha,\beta} 2^{\alpha+\beta+1} \Gamma(\alpha+1)} \\
 &\cdot \sum_{k=N+1}^\infty \sum_{n=1}^k \sum_{i=k-2n}^k \left| \left\langle \phi_\beta, \frac{P_n^{(\alpha,\beta)}}{\|P_n^{(\alpha,\beta)}\|_2^2} \right\rangle \right|^2 \\
 &\cdot \left| \left\langle \phi_\beta, \frac{P_{k-n}^{(\alpha,\beta)}}{\|P_{k-n}^{(\alpha,\beta)}\|_2^2} \right\rangle \right|^2 t^2 \lambda_n \lambda_{k-n} \times |P_n^{(\alpha,\beta)}(1)| \\
 &\cdot |P_{k-n}^{(\alpha,\beta)}(1)|^2 e^{-2t(\lambda_k^{1/2} + \lambda_{k-n}^{1/2})} |\gamma(i, k-n, n)|^2 \|P_i^{(\alpha,\beta)}\|_2^2.
 \end{aligned} \quad (205)$$

Using analogous boundedness argument as in the Hermite case, we get

$$\begin{aligned}
 &\int_0^\infty |H_{\beta,K}^{N,1}(t, y)|^2 y^\alpha e^{-y} \frac{dy}{\Gamma(\alpha+1)} \\
 &\leq \frac{\beta^{\alpha+1} \|\phi_\beta\|^2}{\eta_{\alpha,\beta} 2^{\alpha+\beta+1} \Gamma(\alpha+1)} \\
 &\cdot \sum_{k=N+1}^\infty \sum_{n=1}^k \left| \left\langle \phi_\beta, \frac{P_n^{(\alpha,\beta)}}{\|P_n^{(\alpha,\beta)}\|_2^2} \right\rangle \right|^2 \\
 &\cdot \frac{t^2 \lambda_n \lambda_{k-n} 2n \binom{n+q}{n}^2 e^{-2t(\lambda_k^{1/2} + \lambda_{k-n}^{1/2})}}{\|P_n^{(\alpha,\beta)}\|_2^2} \\
 &\leq \frac{e^{-t} \beta^{\alpha+1} \|\phi_\beta\|^2}{\eta_{\alpha,\beta} 2^{\alpha+\beta+1} \Gamma(\alpha+1)} \\
 &\cdot \sum_{k=N+1}^\infty \sum_{n=1}^k \left| \left\langle \phi_\beta, \frac{P_n^{(\alpha,\beta)}}{\|P_n^{(\alpha,\beta)}\|_2^2} \right\rangle \right|^2 \frac{2n \binom{n+q}{n}^2 e^{-t\lambda_{k-n}^{1/2}}}{t^2 \|P_n^{(\alpha,\beta)}\|_2^2}.
 \end{aligned} \quad (206)$$

Now, for β big enough,

$$\frac{1}{\|P_n^{(\alpha,\beta)}\|_2^2} = \frac{(2n + \alpha + \beta + 1) \Gamma(\alpha + 1) \Gamma(\beta + 1) \Gamma(n + 1) \Gamma(n + \alpha + \beta + 1)}{\Gamma(\alpha + \beta + 2) \Gamma(n + \alpha + 1) \Gamma(n + \beta + 1)} \leq \frac{Cn!}{(\alpha + 1)_n}, \quad (207)$$

$$\binom{n + q}{n} = \binom{n + \beta}{n} = \frac{\Gamma(n + \beta + 1)}{\Gamma(\beta + 1) n!} \sim \frac{\beta^n}{n!}.$$

Thus, for β big enough,

$$\begin{aligned} & \int_0^\infty |H_{\beta,K}^{N,1}(t, y)|^2 y^\alpha e^{-y} \frac{dy}{\Gamma(\alpha + 1)} \\ & \leq \frac{C e^{-t} \beta^{\alpha+1} \|\phi_\beta\|^2}{\eta_{\alpha,\beta} 2^{\alpha+\beta+1} \Gamma(\alpha + 1)} \\ & \cdot \sum_{k=N+1}^\infty \sum_{n=1}^k \left| \left\langle \phi_\beta, \frac{P_n^{(\alpha,\beta)}}{\|P_n^{(\alpha,\beta)}\|_2} \right\rangle \right|^2 \frac{2n\beta^{2n} e^{-t\lambda_{k-n}^{1/2}}}{t^2 (\alpha + 1)_n}. \end{aligned} \quad (208)$$

Taking $\beta = (k - n)(1 + 2k^2)^2$, then, k is big enough if β is big enough and we have two cases.

(i) Case $t \geq 1$. In this case, we have

$$\frac{1}{e^{t\lambda_{k-n}^{1/2}}} \leq \frac{1}{e^{(k-n)^{1/2}(k-n+(k-n)(1+2k^2)^2+\alpha+1)^{1/2}}} \leq \frac{1}{e^{(k-n)}} \frac{1}{e^k}. \quad (209)$$

Therefore, for k big enough, there exists $C > 0$ such that

$$\begin{aligned} & \sum_{k=N+1}^\infty \sum_{n=1}^k \left| \left\langle \phi_\beta, \frac{P_n^{(\alpha,\beta)}}{\|P_n^{(\alpha,\beta)}\|_2} \right\rangle \right|^2 \frac{2n\beta^{2n} e^{-t\lambda_{k-n}^{1/2}}}{t^2 (\alpha + 1)_n} \\ & \leq \frac{C}{e^N} \sum_{k=N+1}^\infty \sum_{n=1}^k \left| \left\langle \phi_\beta, \frac{P_n^{(\alpha,\beta)}}{\|P_n^{(\alpha,\beta)}\|_2} \right\rangle \right|^2; \end{aligned} \quad (210)$$

thus,

$$\int_0^\infty |H_{\beta,K}^{N,1}(t, y)|^2 y^\alpha e^{-y} \frac{dy}{\Gamma(\alpha + 1)} \leq \frac{C e^{-t} \|\phi\|_{2,\alpha}^4}{e^{N+1}}. \quad (211)$$

(ii) Case $0 < t < 1$. For t near 0, there exists $k > 0$ big enough such that $1/k \leq t$; that is, $1/t^2 \leq k^2$. Then, analogously to the Hermite case,

$$\frac{1}{e^{t\lambda_{k-n}^{1/2}}} \leq \frac{1}{e^{(k-n)}} \frac{1}{e^k} \frac{1}{e^k}. \quad (212)$$

Hence,

$$\int_0^\infty |H_{\beta,K}^{N,1}(t, y)|^2 y^\alpha e^{-y} \frac{dy}{\Gamma(\alpha + 1)} \leq \frac{C e^{-t} \|\phi\|_{2,\alpha}^4}{e^{N+1}}. \quad (213)$$

Therefore,

$$\begin{aligned} & \left(\int_0^\infty \left| \int_0^\infty H_{\beta,K}^{N,1}(t, y) dt \right|^2 y^\alpha e^{-y} \frac{dy}{\Gamma(\alpha + 1)} \right)^{1/2} \\ & \leq \int_0^\infty \left(\frac{e^{-t} C \|\phi\|_{2,\alpha}^4}{e^{(N+1)}} \right)^{1/2} dt = \frac{(C)^{1/2} \|\phi\|_{2,\alpha}^2}{e^{(N+1)/2}}. \end{aligned} \quad (214)$$

Thus, $\{\int_0^\infty H_{\beta,K}^{N,1}(t, \cdot) dt\}$ is a bounded sequence on $L^2((0, \infty), \mu_\alpha)$, so, by Bourbaki-Alaoglu's theorem, there exists a sequence $(\lambda_j)_{j \in \mathbb{N}}$ $\lim_{j \rightarrow \infty} \lambda_j = \infty$ such that, for all $N \in \mathbb{N}$, $\{\int_0^\infty H_{\beta,K}^{N,1}(t, \cdot) dt\}_{j \in \mathbb{N}}$ converges weakly in $L^2((0, \infty), \mu_\alpha)$ to a function $H_K^{N,1} \in L^2((0, \infty), \mu_\alpha)$. Moreover,

$$\int_0^\infty |H_K^{N,1}(y)|^2 y^\alpha e^{-y} \frac{dy}{\Gamma(\alpha + 1)} \leq \frac{C}{e^{(N+1)/2}}. \quad (215)$$

Then, there exists a nondecreasing sequence $(N_j)_{j \in \mathbb{N}}$, such that

$$H_K^{N_j,1}(y) \rightarrow 0, \quad a.e. \ y \in \mathbb{R}. \quad (216)$$

For the function $g_{2,\beta}$, we get similar estimates. Given $\phi \in L^2((0, \infty), \mu_\alpha)$,

$$\begin{aligned} & \left(\frac{\sqrt{t}}{\sqrt{\beta}} \frac{\partial}{\partial y} P_t^{\alpha,\beta} \phi_\beta \left(1 - \frac{2}{\beta} y \right) \right)^2 \\ & = \sum_{k=1}^\infty \sum_{n=1}^k \left\langle \phi_\beta, \frac{P_n^{(\alpha,\beta)}}{\|P_n^{(\alpha,\beta)}\|_2} \right\rangle \left\langle \phi_\beta, \frac{P_{k-n}^{(\alpha,\beta)}}{\|P_{k-n}^{(\alpha,\beta)}\|_2} \right\rangle \\ & \cdot \frac{\lambda_n \lambda_{k-n}}{4n(k-n)} \times P_{n-1}^{(\alpha+1,\beta+1)} \left(1 - \frac{2}{\beta} y \right) P_{k-n-1}^{(\alpha+1,\beta+1)} \left(1 - \frac{2}{\beta} y \right) \\ & \cdot \frac{t}{\beta} e^{-t(\lambda_k^{1/2} + \lambda_{k-n}^{1/2})}. \end{aligned} \quad (217)$$

Thus,

$$\begin{aligned}
 g_{2,\beta}\phi(y) &= \sum_{k=1}^N \sum_{n=1}^k \left\langle \phi_\beta, \frac{P_n^{(\alpha,\beta)}}{\|P_n^{(\alpha,\beta)}\|_2^2} \right\rangle \left\langle \phi_\beta, \frac{P_{k-n}^{(\alpha,\beta)}}{\|P_{k-n}^{(\alpha,\beta)}\|_2^2} \right\rangle \\
 &\cdot P_{n-1}^{(\alpha+1,\beta+1)} \left(1 - \frac{2}{\beta}y\right) P_{k-n-1}^{(\alpha+1,\beta+1)} \left(1 - \frac{2}{\beta}y\right) \\
 &\times \frac{y}{\beta n(k-n)} \frac{\lambda_n \lambda_{k-n}}{(\lambda_k^{1/2} + \lambda_{k-n}^{1/2})^2} \left(1 - \frac{y}{\beta}\right)^{\beta+1} e^y \\
 &+ \int_0^\infty H_{\beta,K}^{N,2}(t, \\
 &y) dt y^{1/2} \left(1 - \frac{y}{\beta}\right)^{(\beta+1)/2} e^{y/2},
 \end{aligned} \quad (218)$$

where

$$\begin{aligned}
 H_{\beta,K}^{N,2}(t, y) &= \chi_{(0,K)}(y) \sum_{k=N+1}^\infty \sum_{n=1}^k \left\langle \phi_\beta, \frac{P_n^{(\alpha,\beta)}}{\|P_n^{(\alpha,\beta)}\|_2^2} \right\rangle \\
 &\cdot \left\langle \phi_\beta, \frac{P_{k-n}^{(\alpha,\beta)}}{\|P_{k-n}^{(\alpha,\beta)}\|_2^2} \right\rangle \frac{t}{\beta} e^{-t(\lambda_k^{1/2} + \lambda_{k-n}^{1/2})} \frac{\lambda_n \lambda_{k-n}}{n(k-n)} \\
 &\times P_{n-1}^{(\alpha+1,\beta+1)} \left(1 - \frac{2}{\beta}y\right) P_{k-n-1}^{(\alpha+1,\beta+1)} \left(1 - \frac{2}{\beta}y\right) \\
 &\cdot y^{1/2} \left(1 - \frac{y}{\beta}\right)^{(\beta+1)/2} e^{y/2}.
 \end{aligned} \quad (219)$$

Again, we want to prove that, for $K \in \mathbb{N}$ and $\beta > K$,

$$\begin{aligned}
 \int_0^\infty \left| \int_0^\infty H_{\beta,K}^{N,2}(t, y) dt \right|^2 y^\alpha e^{-y} \frac{dy}{\Gamma(\alpha+1)} \\
 \leq \frac{(C)^{1/2} \|\phi\|_{2,\alpha}^2}{e^{(N+1)/2}}.
 \end{aligned} \quad (220)$$

Let us see the following:

$$\begin{aligned}
 \int_0^\infty |H_{\beta,K}^{N,2}(t, y)|^2 y^\alpha e^{-y} \frac{dy}{\Gamma(\alpha+1)} &\leq \frac{\beta^{\alpha+2}}{2^{\alpha+\beta+3} \Gamma(\alpha+1)} \\
 &\cdot \int_{-1}^1 \left| \sum_{k=N+1}^\infty \sum_{n=1}^k \sum_{i=k-2n}^{k-2} \left\langle \phi_\beta, \frac{P_n^{(\alpha,\beta)}}{\|P_n^{(\alpha,\beta)}\|_2^2} \right\rangle \right. \\
 &\cdot \left\langle \phi_\beta, \frac{P_{k-n}^{(\alpha,\beta)}}{\|P_{k-n}^{(\alpha,\beta)}\|_2^2} \right\rangle \frac{t}{\beta} \times e^{-t(\lambda_k^{1/2} + \lambda_{k-n}^{1/2})} \frac{\lambda_n \lambda_{k-n}}{n(k-n)} \\
 &\cdot P_{n-1}^{(\alpha+1,\beta+1)}(1) P_{k-n-1}^{(\alpha+1,\beta+1)}(1) \nu(i, k-n-1, n-1) \\
 &\cdot p_i^{(\alpha+1,\beta+1)}(x) \left. \right|^2 \times (1-x)^{\alpha+1} (1+x)^{\beta+1} dx.
 \end{aligned} \quad (221)$$

Thus, using Parseval's identity,

$$\begin{aligned}
 \int_0^\infty |H_{\beta,K}^{N,2}(t, y)|^2 y^\alpha e^{-y} \frac{dy}{\Gamma(\alpha+1)} \\
 \leq \frac{\beta^{\alpha+2}}{\eta_{\alpha+1,\beta+1} 2^{\alpha+\beta+3} \Gamma(\alpha+1)} \\
 \cdot \sum_{k=N+1}^\infty \sum_{n=1}^k \sum_{i=k-2n}^{k-2} \left| \left\langle \phi_\beta, \frac{P_n^{(\alpha,\beta)}}{\|P_n^{(\alpha,\beta)}\|_2^2} \right\rangle \right|^2 \\
 \cdot \left| \left\langle \phi_\beta, \frac{P_{k-n}^{(\alpha,\beta)}}{\|P_{k-n}^{(\alpha,\beta)}\|_2^2} \right\rangle \right|^2 \left| \frac{t}{\beta} \right|^2 \\
 \times e^{-2t(\lambda_k^{1/2} + \lambda_{k-n}^{1/2})} \left| \frac{\lambda_n \lambda_{k-n}}{n(k-n)} P_{n-1}^{(\alpha+1,\beta+1)}(1) \right. \\
 \cdot P_{k-n-1}^{(\alpha+1,\beta+1)}(1) \times |\nu(i, k-n-1, n-1)|^2 \\
 \cdot \left. \|P_i^{(\alpha+1,\beta+1)}\|_{(\alpha+1,\beta+1)}^2 \right|.
 \end{aligned} \quad (222)$$

Analogously to the previous case,

$$\begin{aligned}
 \int_0^\infty |H_{\beta,K}^{N,2}(t, y)|^2 y^\alpha e^{-y} \frac{dy}{\Gamma(\alpha+1)} \\
 \leq \frac{\beta^{\alpha+2} \|\phi_\beta\|^2}{\eta_{\alpha+1,\beta+1} 2^{\alpha+\beta+3} \Gamma(\alpha+1)} \\
 \cdot \sum_{k=N+1}^\infty \sum_{n=1}^k \left| \left\langle \phi_\beta, \frac{P_n^{(\alpha,\beta)}}{\|P_n^{(\alpha,\beta)}\|_2^2} \right\rangle \right|^2 \\
 \cdot \frac{t^2 2(n-1)}{\beta^2 \|P_n^{(\alpha,\beta)}\|_2^2 \|P_{k-n}^{(\alpha,\beta)}\|_2^2} \\
 \times e^{-2t(\lambda_k^{1/2} + \lambda_{k-n}^{1/2})} \left| \frac{\lambda_n \lambda_{k-n}}{n(k-n)} \right|^2 \cdot \left(\frac{n-1+q}{n-1} \right)^2 \\
 \cdot \left. \|P_{k-n-1}^{(\alpha+1,\beta+1)}\|_{(\alpha+1,\beta+1)}^2 \right|.
 \end{aligned} \quad (223)$$

Also, given that

$$\begin{aligned}
 &\|P_{k-n-1}^{(\alpha+1,\beta+1)}\|_{(\alpha+1,\beta+1)}^2 \\
 &= \frac{(\alpha+\beta+3)(\alpha+\beta+2)(k-n)}{(\alpha+1)(\beta+1)(k-n+\alpha+\beta+1)} \|P_{k-n}^{(\alpha,\beta)}\|_{(\alpha,\beta)}^2,
 \end{aligned} \quad (224)$$

we get

$$\begin{aligned}
& \int_0^\infty |H_{\beta,K}^{N,2}(t, y)|^2 y^\alpha e^{-y} \frac{dy}{\Gamma(\alpha+1)} \\
& \leq \frac{\beta^{\alpha+2} \|\phi_\beta\|^2}{\eta_{\alpha+1, \beta+1} 2^{\alpha+\beta+3} \Gamma(\alpha+1)} \\
& \cdot \sum_{k=N+1}^\infty \sum_{n=1}^k \left| \left\langle \phi_\beta, \frac{P_n^{(\alpha, \beta)}}{\|P_n^{(\alpha, \beta)}\|_2} \right\rangle \right|^2 \\
& \cdot \frac{t^2 2e^{-2t(\lambda_k^{1/2} + \lambda_{k-n}^{1/2})} \lambda_n^2 \lambda_{k-n}}{\beta^2 n} \\
& \times \frac{\binom{n-1+q}{n-1}^2 (\alpha + \beta + 3)(\alpha + \beta + 2)}{(\alpha + 1)(\beta + 1) \|P_n^{(\alpha, \beta)}\|_2^2} \\
& \leq \frac{e^{-t} \beta^{\alpha+1} \|\phi_\beta\|^2}{\eta_{\alpha, \beta} 2^{\alpha+\beta+1} \Gamma(\alpha+1)} \\
& \cdot \sum_{k=N+1}^\infty \sum_{n=1}^k \left| \left\langle \phi_\beta, \frac{P_n^{(\alpha, \beta)}}{\|P_n^{(\alpha, \beta)}\|_2} \right\rangle \right|^2 \\
& \cdot \frac{2e^{-t\lambda_{k-n}^{1/2}} (n + \alpha + \beta + 1) \binom{n-1+q}{n-1}^2}{t^2 \beta \|P_n^{(\alpha, \beta)}\|_2^2}.
\end{aligned} \tag{225}$$

Then we have for $\beta > 0$ big enough

$$\frac{\binom{n-1+q}{n-1}^2 (n + \alpha + \beta + 1)}{\beta \|P_n^{(\alpha, \beta)}\|_2^2} \leq \frac{Cn\beta^{2(n-1)}}{(\alpha + 1)_n} \leq \frac{Cn\beta^{2n}}{(\alpha + 1)_n}. \tag{226}$$

Hence, for $\beta > 0$ big enough,

$$\begin{aligned}
& \int_0^\infty |H_{\beta,K}^{N,2}(t, y)|^2 y^\alpha e^{-y} \frac{dy}{\Gamma(\alpha+1)} \\
& \leq \frac{Ce^{-t} \beta^{\alpha+1} \|\phi_\beta\|^2}{\eta_{\alpha, \beta} 2^{\alpha+\beta+1} \Gamma(\alpha+1)} \\
& \cdot \sum_{k=N+1}^\infty \sum_{n=1}^k \left| \left\langle \phi_\beta, \frac{P_n^{(\alpha, \beta)}}{\|P_n^{(\alpha, \beta)}\|_2} \right\rangle \right|^2 \frac{2n\beta^{2n} e^{-t\lambda_{k-n}^{1/2}}}{t^2 (\alpha + 1)_n};
\end{aligned} \tag{227}$$

then, analogous to (211), we have

$$\int_0^\infty |H_{\beta,K}^{N,2}(t, y)|^2 y^\alpha e^{-y} \frac{dy}{\Gamma(\alpha+1)} \leq \frac{Ce^{-t} \|\phi\|_{2, \alpha}^4}{e^{N+1}}; \tag{228}$$

therefore,

$$\begin{aligned}
& \left(\int_0^\infty \left| \int_0^\infty H_{\beta,K}^{N,2}(t, y) dt \right|^2 y^\alpha e^{-y} \frac{dy}{\Gamma(\alpha+1)} \right)^{1/2} \\
& \leq \frac{(C)^{1/2} \|\phi\|_{2, \alpha}^2}{e^{(N+1)/2}}.
\end{aligned} \tag{229}$$

Thus, $\{\int_0^\infty H_{\beta,K}^{N,2}(t, \cdot) dt\}$ is a bounded sequence on $L^2((0, \infty), \mu_\alpha)$, so, by Bourbaki-Alaoglu's theorem, there exists a sequence $(\lambda_j)_{j \in \mathbb{N}}$ $\lim_{j \rightarrow \infty} \lambda_j = \infty$ such that, for all $N \in \mathbb{N}$, $\{\int_0^\infty H_{\beta,K}^{N,2}(t, \cdot) dt\}_{j \in \mathbb{N}}$ converges weakly in $L^2((0, \infty), \mu_\alpha)$ to a function $H_K^{N,2} \in L^2((0, \infty), \mu_\alpha)$. Moreover,

$$\int_0^\infty |H_K^{N,2}(y)|^2 y^\alpha e^{-y} \frac{dy}{\Gamma(\alpha+1)} \leq \frac{C}{e^{(N+1)/2}}. \tag{230}$$

Then, there exists a nondecreasing sequence $(N_j)_{j \in \mathbb{N}}$, such that

$$H_K^{N_j,2}(y) \longrightarrow 0, \quad a.e. y \in (0, \infty). \tag{231}$$

Therefore, similarly to (156),

$$\begin{aligned}
(F_{\beta,K}(y))^2 &= g_{1,\beta} \phi(y) + g_{2,\beta} \phi(y) \\
&= F_{\beta,K}^N(y) + \int_0^\infty H_{\beta,K}^{N,1}(t, y) dt \left(1 - \frac{y}{\beta}\right)^{\beta/2} e^{y/2} \\
&\quad + \int_0^\infty H_{\beta,K}^{N,2}(t, y) dt y^{1/2} \left(1 - \frac{y}{\beta}\right)^{(\beta+1)/2} e^{y/2},
\end{aligned} \tag{232}$$

where

$$\begin{aligned}
F_{\beta,K}^N(y) &= \sum_{k=1}^N \sum_{n=1}^k \left\langle \phi_\beta, \frac{P_n^{(\alpha, \beta)}}{\|P_n^{(\alpha, \beta)}\|_2} \right\rangle \\
&\cdot \left\langle \phi_\beta, \frac{P_{k-n}^{(\alpha, \beta)}}{\|P_{k-n}^{(\alpha, \beta)}\|_2} \right\rangle \frac{\lambda_n^{1/2} \lambda_{k-n}^{1/2}}{(\lambda_k^{1/2} + \lambda_{k-n}^{1/2})^2} \times P_n^{(\alpha, \beta)} \left(1 - \frac{2}{\beta} y\right) P_{k-n}^{(\alpha, \beta)} \left(1 - \frac{2}{\beta} y\right) \left(1 - \frac{y}{\beta}\right)^\beta e^y \\
&\quad + \sum_{k=1}^N \sum_{n=1}^k \left\langle \phi_\beta, \frac{P_n^{(\alpha, \beta)}}{\|P_n^{(\alpha, \beta)}\|_2} \right\rangle \left\langle \phi_\beta, \frac{P_{k-n}^{(\alpha, \beta)}}{\|P_{k-n}^{(\alpha, \beta)}\|_2} \right\rangle \\
&\cdot \frac{\lambda_n \lambda_{k-n}}{(\lambda_k^{1/2} + \lambda_{k-n}^{1/2})^2} \times \frac{y}{\beta n (k-n)} P_{n-1}^{(\alpha+1, \beta+1)} \left(1 - \frac{2}{\beta} y\right) P_{k-n-1}^{(\alpha+1, \beta+1)} \left(1 - \frac{2}{\beta} y\right) \left(1 - \frac{y}{\beta}\right)^{\beta+1} e^y.
\end{aligned} \tag{233}$$

Defining, for each $m \in \mathbb{N}$, $F_K^{N_m} = F^2 - H_K^{N_m,1} - H_K^{N_m,2}$, then since $F_{\beta,K} \rightarrow F_K$, as $j \rightarrow \infty$ in the weak topology of $L^2((0, \infty), \mu_\alpha)$ and

$$F_{\beta,K}^N = (F_{\beta,K})^2 - \int_0^\infty H_{\beta,K}^{N,1}(t, y) dt \left(1 - \frac{y}{\beta}\right)^{\beta/2} e^{y/2} - \int_0^\infty H_{\beta,K}^{N,2}(t, y) dt y^{1/2} \left(1 - \frac{y}{\beta}\right)^{(\beta+1)/2} e^{y/2}, \quad (234)$$

we have, for all $m \in \mathbb{N}$,

$$F_{\beta,j,K}^{N_m} \longrightarrow F_K^{N_m}, \quad (235)$$

weakly in $L^2((0, \infty), \mu_\alpha)$, as $j \rightarrow \infty$,

and also

$$F_K^{N_m}(y) \longrightarrow F^2(y), \quad (236)$$

as $m \rightarrow \infty$ a.e. $y \in (0, \infty)$.

Now, given that

$$\begin{aligned} & \lim_{\beta \rightarrow \infty} P_{n-1}^{(\alpha+1, \beta+1)} \left(1 - \frac{2}{\beta} y\right) \\ &= \lim_{\beta \rightarrow \infty} P_{n-1}^{(\alpha+1, \beta+1)} \left(1 - \frac{2y}{\beta+1} \frac{\beta+1}{\beta}\right) = L_{n-1}^{\alpha+1}(y), \\ & \lim_{\beta \rightarrow \infty} \frac{1}{\|P_n^{(\alpha, \beta)}\|_{(\alpha, \beta)}^2} = \lim_{\beta \rightarrow \infty} \left(\frac{2n}{\beta} + \frac{\alpha}{\beta} + 1 + \frac{1}{\beta}\right) \\ & \cdot n! \frac{\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)} \frac{\beta^{\alpha+1} \Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} \frac{\Gamma(\beta+n+\alpha+1)}{\beta^\alpha \Gamma(n+\beta+1)} \\ &= \frac{n! \Gamma(\alpha+1)}{\Gamma(n+\alpha+1)} = \frac{1}{\|L_n^\alpha\|_\alpha^2}, \end{aligned} \quad (237)$$

we get

$$\begin{aligned} \lim_{\beta \rightarrow \infty} F_{\beta,K}^N(y) &= \lim_{\beta \rightarrow \infty} \sum_{k=1}^N \sum_{n=1}^k \left\langle \phi_\beta, \frac{P_n^{(\alpha, \beta)}}{\|P_n^{(\alpha, \beta)}\|_2^2} \right\rangle \\ & \cdot \left\langle \phi_\beta, \frac{P_{k-n}^{(\alpha, \beta)}}{\|P_{k-n}^{(\alpha, \beta)}\|_2^2} \right\rangle \frac{\lambda_n^{1/2} \lambda_{k-n}^{1/2}}{(\lambda_k^{1/2} + \lambda_{k-n}^{1/2})^2} \times P_n^{(\alpha, \beta)} \left(1 - \frac{2}{\beta} y\right) P_{k-n}^{(\alpha, \beta)} \left(1 - \frac{2}{\beta} y\right) \left(1 - \frac{y}{\beta}\right)^\beta e^y \\ &+ \lim_{\beta \rightarrow \infty} \sum_{k=1}^N \sum_{n=1}^k \left\langle \phi_\beta, \frac{P_n^{(\alpha, \beta)}}{\|P_n^{(\alpha, \beta)}\|_2^2} \right\rangle \\ & \cdot \left\langle \phi_\beta, \frac{P_{k-n}^{(\alpha, \beta)}}{\|P_{k-n}^{(\alpha, \beta)}\|_2^2} \right\rangle \frac{\lambda_n \lambda_{k-n}}{(\lambda_k^{1/2} + \lambda_{k-n}^{1/2})^2} \end{aligned}$$

$$\begin{aligned} & \times \frac{y}{\beta n (k-n)} P_{n-1}^{(\alpha+1, \beta+1)} \left(1 - \frac{2}{\beta} y\right) P_{k-n-1}^{(\alpha+1, \beta+1)} \left(1 - \frac{2}{\beta} y\right) \left(1 - \frac{y}{\beta}\right)^{\beta+1} e^y = \sum_{k=1}^N \sum_{n=1}^k \left\langle \phi, \frac{L_n^\alpha}{\|L_n^\alpha\|_\alpha^2} \right\rangle \\ & \cdot \left\langle \phi, \frac{L_{k-n}^\alpha}{\|L_{k-n}^\alpha\|_\alpha^2} \right\rangle \frac{n^{1/2} (k-n)^{1/2}}{(n^{1/2} + (k-n)^{1/2})^2} L_n^\alpha(y) \\ & \cdot L_{k-n}^\alpha(y) + \sum_{k=1}^N \sum_{n=1}^k \left\langle \phi, \frac{L_n^\alpha}{\|L_n^\alpha\|_\alpha^2} \right\rangle \\ & \cdot \left\langle \phi, \frac{L_{k-n}^\alpha}{\|L_{k-n}^\alpha\|_\alpha^2} \right\rangle \frac{y}{(n^{1/2} + (k-n)^{1/2})^2} L_{n-1}^{\alpha+1}(y) \\ & \cdot L_{k-n-1}^{\alpha+1}(y) = \int_0^\infty t \left(\sum_{k=1}^N \left\langle \phi, \frac{L_n^\alpha}{\|L_n^\alpha\|_\alpha^2} \right\rangle \right. \\ & \cdot k^{1/2} e^{-tk^{1/2}} L_n^\alpha(y) \left. \right)^2 dt \\ & + \int_0^\infty t \left(\sum_{k=1}^N \left\langle \phi, \frac{L_n^\alpha}{\|L_n^\alpha\|_\alpha^2} \right\rangle \right. \\ & \cdot \sqrt{y} e^{-tk^{1/2}} L_{n-1}^{\alpha+1}(y) \left. \right)^2 dt. \end{aligned} \quad (238)$$

Thus,

$$\begin{aligned} F_K^{N_m}(y) &= \int_0^\infty t \left(\sum_{k=1}^{N_m} \left\langle \phi, \frac{L_n^\alpha}{\|L_n^\alpha\|_\alpha^2} \right\rangle \right. \\ & \cdot k^{1/2} e^{-tk^{1/2}} L_n^\alpha(y) \left. \right)^2 dt \\ & + \int_0^\infty t \left(\sum_{k=1}^{N_m} \left\langle \phi, \frac{L_n^\alpha}{\|L_n^\alpha\|_\alpha^2} \right\rangle \right. \\ & \cdot \sqrt{y} e^{-tk^{1/2}} L_{n-1}^{\alpha+1}(y) \left. \right)^2 dt. \end{aligned} \quad (239)$$

We want to prove that

$$F_K^{N_m}(y) \longrightarrow (g^\alpha \phi(y))^2, \quad \text{as } m \rightarrow \infty \text{ a.e. } y \in \mathbb{R}. \quad (240)$$

Indeed, initially we have

$$\begin{aligned} & \left| \sum_{k=1}^{N_m} \left\langle \phi, \frac{L_k^\alpha}{\|L_k^\alpha\|_\alpha^2} \right\rangle k^{1/2} e^{-tk^{1/2}} L_k^\alpha(x) \right| \\ & \leq \|\phi\| \sum_{k=1}^\infty k^{1/2} e^{-tk^{1/2}} \frac{|L_k^\alpha(x)|}{\|L_k^\alpha\|_\alpha}. \end{aligned} \quad (241)$$

From (8.22.1) of [1], for $\alpha, x > 0$, we have

$$\begin{aligned} L_k^\alpha(x) &= \pi^{-1/2} e^{x/2} x^{-\alpha/2-1/4} \cos \left\{ 2(kx)^{1/2} - \alpha\pi/2 - \pi/4 \right\} \\ &\quad + O(k^{\alpha/2-3/4}). \end{aligned} \quad (242)$$

Thus, there exist $M > 0$ such that

$$\begin{aligned} \left| L_k^\alpha(x) - \pi^{-1/2} e^{x/2} x^{-\alpha/2-1/4} k^{\alpha/2-1/4} \cos \left\{ 2(kx)^{1/2} \right. \right. \\ \left. \left. - \alpha\pi/2 - \pi/4 \right\} \right| \leq M k^{\alpha/2-3/4}, \end{aligned} \quad (243)$$

hence,

$$|L_k^\alpha(x)| \leq k^{\alpha/2-1/4} (M + \pi^{-1/2} e^{x/2} x^{-\alpha/2-1/4}). \quad (244)$$

On the other hand, since for k big enough $\|L_k^\alpha\|_\alpha^2 \approx k^\alpha/(\Gamma(\alpha+1))$, then, there exist $N_1 > 0$ such that

$$\begin{aligned} \frac{|L_k^\alpha(x)|}{\|L_k^\alpha\|_\alpha} &\leq \frac{(M + \pi^{-1/2} e^{x/2} x^{-\alpha/2-1/4}) [\Gamma(k+1)]^{1/2}}{k^{1/4}}, \\ &\text{for any } k \geq N_1. \end{aligned} \quad (245)$$

Thus,

$$\begin{aligned} \sum_{k=1}^{\infty} k^{1/2} e^{-tk^{1/2}} \frac{|L_k^\alpha(x)|}{\|L_k^\alpha\|_\alpha} &\leq (M + \pi^{-1/2} e^{x/2} x^{-\alpha/2-1/4}) \\ &\quad \cdot [\Gamma(\alpha+1)]^{1/2} \sum_{k=1}^{N_1} k^{1/4} e^{-tk^{1/2}} \frac{k^{\alpha/2}}{\|L_k^\alpha\|_\alpha} \\ &\quad + (M + \pi^{-1/2} e^{x/2} x^{-\alpha/2-1/4}) [\Gamma(\alpha+1)]^{1/2} \\ &\quad \cdot \sum_{k=1}^{\infty} k^{1/2} e^{-tk^{1/2}}. \end{aligned} \quad (246)$$

Then, for $t \geq 1$, we have

$$\begin{aligned} \left| t \left(\sum_{k=1}^{N_m} \left\langle \phi, \frac{L_k^\alpha}{\|L_k^\alpha\|_\alpha^2} \right\rangle k^{1/2} e^{-tk^{1/2}} L_k^\alpha(x) \right)^2 \right| \\ \leq (M + \pi^{-1/2} e^{x/2} x^{-\alpha/2-1/4})^2 \Gamma(\alpha+1) \\ \cdot \left(\sum_{k=1}^{N_1} k^{1/4} e^{-tk^{1/2}} \frac{k^{\alpha/2}}{\|L_k^\alpha\|_\alpha} \right)^2 t \\ + (M + \pi^{-1/2} e^{x/2} x^{-\alpha/2-1/4})^2 \Gamma(\alpha+1) \\ \cdot \left(\sum_{k=1}^{N_1} k^{1/4} e^{-tk^{1/2}} \frac{k^{\alpha/2}}{\|L_k^\alpha\|_\alpha} \right) t \left(\sum_{k=1}^{\infty} k^{1/2} e^{-tk^{1/2}} \right) \\ + (M + \pi^{-1/2} e^{x/2} x^{-\alpha/2-1/4})^2 \Gamma(\alpha+1) \\ \cdot t \left(\sum_{k=1}^{\infty} k^{1/2} e^{-tk^{1/2}} \right)^2. \end{aligned} \quad (247)$$

Now, analogously to the Hermite case, we have

$$\begin{aligned} \int_1^\infty t \left\{ \left(M + \pi^{-1/2} e^{x/2} x^{-\alpha/2-1/4} \right)^2 \Gamma(\alpha+1) \right. \\ \cdot \left(\sum_{k=1}^{N_1} k^{1/4} e^{-tk^{1/2}} \frac{k^{\alpha/2}}{\|L_k^\alpha\|_\alpha} \right)^2 \\ + \left(M + \pi^{-1/2} e^{x/2} x^{-\alpha/2-1/4} \right)^2 \Gamma(\alpha+1) \\ \cdot \left(\sum_{k=1}^{N_1} k^{1/4} e^{-tk^{1/2}} \frac{k^{\alpha/2}}{\|L_k^\alpha\|_\alpha} \right) \left(\sum_{k=1}^{\infty} k^{1/2} e^{-tk^{1/2}} \right) \\ + \left(M + \pi^{-1/2} e^{x/2} x^{-\alpha/2-1/4} \right)^2 \Gamma(\alpha+1) \\ \cdot \left. \left(\sum_{k=1}^{\infty} k^{1/2} e^{-tk^{1/2}} \right)^2 \right\} dt \leq (M \\ + \pi^{-1/2} e^{x/2} x^{-\alpha/2-1/4})^2 \Gamma(\alpha+1) \\ \cdot \left\{ \left(\sum_{k=1}^{N_1} \frac{e^{-(k^{1/2}-1/2)} k^{\alpha/2+1/4}}{\|L_k^\alpha\|_\alpha} \right)^2 \right. \\ + \left(\sum_{k=1}^{N_1} \frac{e^{-k^{1/2}} k^{\alpha/2+1/4}}{\|L_k^\alpha\|_\alpha} \right) \left(\sum_{k=1}^{\infty} k^{1/2} e^{-(k^{1/2}-1)} \right) \\ + \left. \left(\sum_{k=1}^{\infty} k^{1/2} e^{-(k^{1/2}-1/2)} \right)^2 \right\} \int_1^\infty t e^{-t} dt < \infty. \end{aligned} \quad (248)$$

For $0 < t < 1$, given that $|(\partial/\partial t) P_t^\alpha(\phi(y))| \leq C(1+|y|)e^{-t}$, we get for $y \in (0, K)$

$$\begin{aligned} t \left| \sum_{k=1}^{N_m} \left\langle \phi, \frac{L_k^\alpha}{\|L_k^\alpha\|_\alpha^2} \right\rangle k^{1/2} e^{-tk^{1/2}} L_k^\alpha(x) \right|^2 \\ < t \left(1 + \left| \frac{\partial}{\partial t} P_t^\alpha(\phi(y)) \right| \right)^2 \\ < t (1 + C(1+|y|)e^{-t})^2, \end{aligned} \quad (249)$$

where

$$\int_0^1 t (1 + C(1+|y|)e^{-t})^2 dt < \infty; \quad (250)$$

then, by Lebesgue's dominated convergence theorem, we have

$$\begin{aligned} \lim_{m \rightarrow \infty} \int_0^\infty t \left(\sum_{k=1}^{N_m} \left\langle \phi, \frac{L_k^\alpha}{\|L_k^\alpha\|_\alpha^2} \right\rangle k^{1/2} e^{-tk^{1/2}} L_k^\alpha(x) \right)^2 dt \\ = \int_0^\infty t \left(\sum_{k=1}^\infty \left\langle \phi, \frac{L_k^\alpha}{\|L_k^\alpha\|_\alpha^2} \right\rangle k^{1/2} e^{-tk^{1/2}} L_k^\alpha(x) \right)^2 dt \end{aligned}$$

$$= \int_0^\infty t \left(\frac{\partial}{\partial t} P_t^\alpha (\phi(y)) \right)^2 dt. \quad (251)$$

Similarly, as before,

$$\begin{aligned} \lim_{m \rightarrow \infty} \int_0^\infty t \left(\sum_{k=1}^{N_m} \left\langle \phi, \frac{L_n^\alpha}{\|L_n^\alpha\|_\alpha^2} \right\rangle \right. \\ \cdot \sqrt{y} e^{-tk^{1/2}} L_{n-1}^{\alpha+1}(y) \Big)^2 dt = \int_0^\infty t \left(\sqrt{y} \frac{\partial}{\partial y} \right. \\ \cdot P_t^\alpha (\phi(y)) \Big)^2 dt. \end{aligned} \quad (252)$$

Thus,

$$F_K^{N_m}(y) \longrightarrow (g^\alpha \phi(y))^2, \quad \text{as } m \longrightarrow \infty \text{ a.e. } y \in \mathbb{R}; \quad (253)$$

hence, from (236) and (240), we have $F(y) = g^\alpha(\phi(y))$. $\square$

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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