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Research Article

Integration in Orlicz-Bochner Spaces

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Let (Ω, Σ, μ) be a complete σ-finite measure space, φ be a Young function, and X and Y be Banach spaces. Let $L^\varphi(X)$ denote the Orlicz-Bochner space, and $\mathcal{T}_\varphi^\wedge$ denote the finest Lebesgue topology on $L^\varphi(X)$. We study the problem of integral representation of $(\mathcal{T}_\varphi^\wedge, \|\cdot\|_Y)$-continuous linear operators $T : L^\varphi(X) \to Y$ with respect to the representing operator-valued measures. The relationships between $(\mathcal{T}_\varphi^\wedge, \|\cdot\|_Y)$-continuous linear operators $T : L^\varphi(X) \to Y$ and the topological properties of their representing operator measures are established.

1. Introduction and Preliminaries

Throughout the paper, $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ denote real Banach spaces and X^* and Y^* denote their Banach duals, respectively. By B_X and B_{Y^*} we denote the closed unit ball in X and in Y^*. Let $\mathcal{L}(X, Y)$ stand for the space of all bounded operators from X and Y, equipped with the uniform operator norm $\|\cdot\|$.

We assume that (Ω, Σ, μ) is a complete σ-finite measure space. Denote by $\Sigma_f(\mu)$ the δ-ring of sets $A \in \Sigma$ with $\mu(A) < \infty$. By $L^0(X)$ we denote the linear space of μ-equivalence classes of all strongly Σ-measurable functions $f : \Omega \to X$, equipped with the topology $\mathcal{T}_0$ of convergence in measure on sets of finite measure.

Now we recall the basic concepts and properties of Orlicz-Bochner spaces (see [1–6] for more details).

By a *Young function* we mean here a continuous convex mapping $\varphi : [0, \infty) \to [0, \infty)$ that vanishes only at 0 and $\varphi(t)/t \to 0$ as $t \to 0$ and $\varphi(t)/t \to \infty$ as $t \to \infty$. Let φ^* stand for the complementary Young function of φ in the sense of Young.

Let $L^\varphi(X)$ (resp., L^φ) denote the *Orlicz-Bochner space* (resp., *Orlicz space*) defined by a Young function φ; that is,

$$L^\varphi(X) := \left\{ f \in L^0(X) : \int_\Omega \varphi(\lambda \|f(\omega)\|_X) \, d\mu < \infty \text{ for some } \lambda > 0 \right\} = \left\{ f \in L^0(X) : \|f(\cdot)\|_X \in L^\varphi \right\}. \tag{1}$$

Then $L^\varphi(X)$, equipped with the topology $\mathcal{T}_\varphi$ of the norm

$$\|f\|_\varphi := \inf \left\{ \lambda > 0 : \int_\Omega \varphi\left(\frac{\|f(\omega)\|_X}{\lambda} \right) d\mu \le 1 \right\}, \tag{2}$$

is a Banach space. For a sequence (f_n) in $L^\varphi(X)$, $\|f_n\|_\varphi \to 0$ if and only if $\int_\Omega \varphi(\lambda \|f_n(\omega)\|_X) d\mu \to 0$ for all $\lambda > 0$. Let

$$B_{L^\varphi(X)} := \left\{ f \in L^\varphi(X) : \|f\|_\varphi \le 1 \right\}. \tag{3}$$

Let

$$E^\varphi(X) = \left\{ f \in L^0(X) : \int_\Omega \varphi(\lambda \|f(\omega)\|_X) \, d\mu < \infty \ \forall \lambda > 0 \right\}. \tag{4}$$

Then $E^\varphi(X)$ is a $\|\cdot\|_\varphi$-closed subspace of $L^\varphi(X)$.

Recall that a subset H of $L^{\varphi}(X)$ is said to be *solid* whenever $\|f_1(\omega)\|_X \leq \|f_2(\omega)\|_X$ μ-a.e. and $f_1 \in L^{\varphi}(X)$, $f_2 \in H$ imply $f_1 \in H$. A linear topology ξ on $L^{\varphi}(X)$ is said to be *locally solid* if it has a local basis at 0 consisting of solid sets (see [4]).

According to [7, Definition 2.2] and [6] we have the following definition.

Definition 1. A locally solid topology ξ on $L^{\varphi}(X)$ is said to be a *Lebesgue topology* if for a net (f_α) in $L^{\varphi}(X)$, $\|f_\alpha(\cdot)\|_X \xrightarrow{(o)} 0$ in the Banach lattice L^{φ} implies $f_\alpha \to 0$ in ξ.

In view of the super Dedekind completeness of L^{φ} one can restrict in the above definition to usual sequences (f_n) in $L^{\varphi}(X)$ (see [7, Definition 2.2, p. 173]).

Note that, for a sequence (f_n) in $L^{\varphi}(X)$, $\|f_n(\cdot)\|_X \xrightarrow{(o)} 0$ in L^{φ} if and only if $\|f_n(\omega)\|_X \to 0$ μ-a.e. and $\|f_n(\omega)\|_X \leq u(\omega)$ μ-a.e. for some $0 \leq u \in L^{\varphi}$.

For $\varepsilon > 0$ let $U_\varphi(\varepsilon) = \{f \in L^{\varphi}(X) : \int_\Omega \varphi(\|f(\omega)\|_X) d\mu \leq \varepsilon\}$. Then the family of all sets of the form:

$$\bigcup_{n=1}^{\infty}\left(\sum_{i=1}^{n} U_\varphi(\varepsilon_i)\right), \qquad (*)$$

where (ε_n) is a sequence of positive numbers and is a local basis at 0 for a linear topology $\mathcal{T}_\varphi^\wedge$ on $L^{\varphi}(X)$ (see [4, 6] for more details). Using [4, Lemma 1.1] one can show that the sets of the form $(*)$ are convex and solid, so $\mathcal{T}_\varphi^\wedge$ is a locally convex-solid topology.

We now recall terminology and basic facts concerning the spaces of weak*-measurable functions $g : \Omega \to X^*$ (see [8, 9]). Given a function $g : \Omega \to X^*$ and $x \in X$, let $g_x(\omega) = g(\omega)(x)$ for $\omega \in \Omega$. By $L^0(X^*, X)$ we denote the linear space of the weak*-equivalence classes of all weak*-measurable functions $g : \Omega \to X^*$. In view of the super Dedekind completeness of L^0 the set $\{|g_x| : x \in B_X\}$ is order bounded in L^0 for each $g \in L^0(X^*, X)$. Thus one can define the so-called *abstract norm* $\vartheta : L^0(X^*, X) \to L^0$ by

$$\vartheta(g) := \sup\{|g_x| : x \in B_X\} \quad \text{in } L^0. \qquad (5)$$

One can easy check that the following properties of ϑ hold:

$\vartheta(g) = 0$ if and only if $g = 0$ and $g \in L^0(X^*, X)$,

$\vartheta(\lambda g) = |\lambda|\varphi(g)$ for $\lambda \in \mathbb{R}$ and $g \in L^0(X^*, X)$,

$\vartheta(g_1 + g_2) \leq \vartheta(g_1) + \vartheta(g_2)$ if $g_1, g_2 \in L^0(X^*, X)$,

$\vartheta(\mathbb{1}_A g) = \mathbb{1}_A \vartheta(g)$ for $A \in \Sigma$ and $g \in L^0(X^*, X)$.

It is known that, for $f \in L^0(X)$, $g \in L^0(X^*, X)$, the function $\langle f, g\rangle : \Omega \to \mathbb{R}$ defined by $\langle f, g\rangle(\omega) = \langle f(\omega), g(\omega)\rangle$ is measurable and

$$|\langle f(\omega), g(\omega)\rangle| \leq \|f(\omega)\|_X \vartheta(g)(\omega) \quad \mu\text{-a.e.} \qquad (6)$$

Moreover, $\vartheta(g) = \|g(\cdot)\|_{X^*}$ for $g \in L^0(X^*)$. Let

$$L^{\varphi^*}(X^*, X) := \left\{g \in L^0(X^*, X) : \vartheta(g) \in L^{\varphi^*}\right\}. \qquad (7)$$

Clearly $L^{\varphi^*}(X^*) \subset L^{\varphi^*}(X^*, X)$. If, in particular, X^* has the Radon-Nikodym property (i.e., X is an *Asplund space*; see [10, p. 213]), then $L^{\varphi^*}(X^*, X) = L^{\varphi^*}(X^*)$.

Let $L^{\varphi}(X)^*$ stand for the Banach dual of $L^{\varphi}(X)$, equipped with the conjugate norm $\|\cdot\|_\varphi^*$.

Recall that a Young function φ satisfies the Δ_2-*condition* if $\varphi(2t) \leq d\varphi(t)$ for some $d > 1$ and all $t \geq 0$. We shall say that a Young function ψ is *completely weaker* than another φ (in symbols, $\psi \lhd \varphi$) if for an arbitrary $c > 1$ there exists $d > 1$ such that $\psi(ct) \leq d\varphi(t)$ for all $t \geq 0$. Note that a Young function φ satisfies the Δ_2-condition if and only if $\varphi \lhd \varphi$. If $\psi \lhd \varphi$, then $L^{\varphi} \subset E^{\psi}$ and it follows that $L^{\varphi}(X) \subset E^{\psi}(X)$.

Now we present basic properties of the topology $\mathcal{T}_\varphi^\wedge$ on $L^{\varphi}(X)$.

Theorem 2. *Let φ be a Young function. Then the following statements hold:*

(i) *$\mathcal{T}_\varphi^\wedge \subset \mathcal{T}_\varphi$ and $\mathcal{T}_\varphi^\wedge = \mathcal{T}_\varphi$ if φ satisfies the Δ_2-condition.*

(ii) *$\mathcal{T}_\varphi^\wedge$ is the finest Lebesgue topology on $L^{\varphi}(X)$.*

(iii) *$\mathcal{T}_\varphi^\wedge$ is generated by the family of norms $\{\|\cdot\|_\psi|_{L^{\varphi}(X)} : \psi \lhd \varphi\}$.*

(iv) *$(L^{\varphi}(X), \mathcal{T}_\varphi^\wedge)^* = \{F_g : g \in L^{\varphi^*}(X^*, X)\}$, where for $g \in L^{\varphi^*}(X^*, X)$,*

$$F_g(f) = \int_\Omega \langle f(\omega), g(\omega)\rangle d\mu \quad \text{for } f \in L^{\varphi}(X),$$
$$\|F_g\|_\varphi^* = \sup\left\{\int_\Omega \|f(\omega)\|_X \vartheta(g)(\omega) d\mu : f \in B_{L^{\varphi}(X)}\right\} = \|\vartheta(g)\|_{\varphi^*}. \qquad (8)$$

(v) *$(L^{\varphi}(X), \mathcal{T}_\varphi^\wedge)$ is a closed subset of the Banach space $(L^{\varphi}(X), \|\cdot\|_\varphi^*)$.*

(vi) *If X^* has the Radon-Nikodym property, then the space $(L^{\varphi}(X), \mathcal{T}_\varphi^\wedge)$ is strongly Mackey; hence $\mathcal{T}_\varphi^\wedge$ coincides with the Mackey topology $\tau(L^{\varphi}(X), L^{\varphi^*}(X^*))$.*

Proof. (i)–(iii) See [4, Theorems 6.1, 6.3 and 6.5].

(iv) In view of [6, Corollary 4.4 and Theorem 1.2], we get $(L^{\varphi}(X), \mathcal{T}_\varphi^\wedge)^* = L^{\varphi}(X)_n^\sim$, where $L^{\varphi}(X)_n^\sim$ stands for the order continuous dual of $L^{\varphi}(X)$ (see [7, 8, 11] for more details). According to [8, Theorem 4.1] $L^{\varphi}(X)_n^\sim = \{F_g : g \in L^{\varphi^*}(X^*, X)\}$.

Using [11, Theorem 1.3] for $g \in L^{\varphi^*}(X^*, X)$ we have

$$\|F_g\|_\varphi^* := \sup\left\{\left|\int_\Omega \langle f(\omega), g(\omega)\rangle d\mu\right| : f \in B_{L^{\varphi}(X)}\right\}$$
$$= \sup\left\{\int_\Omega \|f(\omega)\|_X \vartheta(g)(\omega) d\mu : f \in B_{L^{\varphi}(X)}\right\} = \|\vartheta(g)\|_{\varphi^*}. \qquad (9)$$

(v) See [12, § 3, Theorem 2].

(vi) See [6, Theorem 4.5]. □

Let $\gamma_\varphi[\mathcal{T}_\varphi, \mathcal{T}_0]$ (briefly γ_φ) denote the natural *mixed topology* on $L^\varphi(X)$; that is, γ_φ is the finest linear topology that agrees with $\mathcal{T}_0$ on $\|\cdot\|_\varphi$-bounded sets in $L^\varphi(X)$ (see [5, 13, 14] for more details). Then γ_φ is a locally convex-solid Hausdorff topology (see [14, Theorem 3.2]) and γ_φ and $\mathcal{T}_\varphi$ have the same bounded sets. This means that $(L^\varphi(X), \gamma_\varphi)$ is a generalized DF-space (see [15]) and its follows that $(L^\varphi(X), \gamma_\varphi)$ is quasinormable (see [15, p. 422]). Moreover, for a sequence (f_n) in $L^\varphi(X)$, $f_n \to 0$ in γ_φ if and only if $f_n \to 0$ in $\mathcal{T}_0$ and $\sup_n \|f_n\|_\varphi < \infty$ (see [14, Theorem 3.1]).

We say that a Young function φ *increases essentially more rapidly* than another ψ (in symbols, $\psi \ll \varphi$) if for arbitrary $c > 0$, $\psi(ct)/\varphi(t) \to 0$ as $t \to 0$ and $t \to \infty$.

Theorem 3. *Let φ be a Young function. Then the mixed topology γ_φ on $L^\varphi(X)$ is generated by the family of norms $\{\|\cdot\|_\psi|_{L^\varphi(X)} : \psi \ll \varphi\}$.*

Proof. It is known that the mixed topology γ_φ on L^φ is generated by the family of norms $\{\|\cdot\|_\psi|_{L^\varphi} : \psi \ll \varphi\}$ (see [16, Theorem 2.1]). Since $\|f\|_\psi = \|\|f(\cdot)\|_X\|_\psi$ for $f \in L^\varphi(X)$, by [14, (54), p. 97], the mixed topology γ_φ on $L^\varphi(X)$ is generated by the family of norms $\{\|\cdot\|_\psi|_{L^\varphi(X)} : \psi \ll \varphi\}$. □

Since $\psi \ll \varphi$ implies $\psi \lhd \varphi$, in view of Theorems 2 and 3, we get

$$\gamma_\varphi \subset \mathcal{T}_\varphi^\wedge. \tag{10}$$

The problem of integral representation of bounded linear operators on Banach function spaces of vector-valued functions to Banach spaces in terms of the corresponding operator-valued measures has been the object of much study (see [5, 17–24]). In particular, Dinculeanu (see [19, § 13, Sect. 3], [20], [21, § 8, Sect. B]) studied the problem of integral representation of bounded linear operators from $L^p(X)$ to a Banach space Y. It is known that if $1 \le p < \infty$, $\mu(\Omega) < \infty$ and an operator measure $m : \Sigma \to \mathcal{L}(X, Y)$ vanishes on μ-null sets and has the finite q-semivariation $\widetilde{m}_q(\Omega)$ ($1 < q \le \infty$, $1/p + 1/q = 1$), then one can define the integral $\int_\Omega f\, dm$ for all $f \in L^p(X)$. Moreover, if $T : L^p(X) \to Y$ is a bounded linear operator, then the associated operator measure $m : \Sigma \to \mathcal{L}(X, Y)$ has the finite q-semivariation $\widetilde{m}_q(\Omega)$ and $T(f) = \int_\Omega f\, dm$ for all $f \in L^p(X)$ (see [19, § 13, Theorem 1 p. 259], [20, Theorem 4]). The relationships of the q-semivariation $\widetilde{m}_q$ to the properties of operators from $L^p(X)$ to Y were studied in [22]. Diestel [23] found the integral representation of bounded linear operators from an Orlicz-Bochner space $L^\varphi(X)$ to a Banach spaces if $\mu(\Omega) < \infty$ and a Young φ satisfies the Δ_2-condition.

The present paper is a continuation of [5], where we establish integral representation of $(\gamma_\varphi, \|\cdot\|_Y)$-continuous linear operators $T : L^\varphi(X) \to Y$. We study the problem of integration of functions in $L^\varphi(X)$ with respect to the representing operator measures of $(\mathcal{T}_\varphi^\wedge, \|\cdot\|_Y)$-continuous linear operators $T : L^\varphi(X) \to Y$. An integral representation theorem for $(\mathcal{T}_\varphi^\wedge, \|\cdot\|_Y)$-continuous linear operators $T : L^\varphi(X) \to Y$ is established (see Theorem 9 below). We study the relationships between $(\mathcal{T}_\varphi^\wedge, \|\cdot\|_Y)$-continuous operators $T : L^\varphi(X) \to Y$ and the properties of their representing measures $m : \Sigma_f(\mu) \to \mathcal{L}(X, Y)$.

2. φ^*-Semivariation of Operator Measures

Assume that $m : \Sigma_f(\mu) \to \mathcal{L}(X, Y)$ is an additive measure such that $m \ll \mu$; that is, $m(A) = 0$ if $\mu(A) = 0$.

Let $\mathcal{S}(\Sigma_f(\mu), X)$ denote the space of all X-valued $\Sigma_f(\mu)$-simple functions on Ω. Then $s \in \mathcal{S}(\Sigma_f(\mu), X)$ if $s = \sum(\mathbb{1}_{A_i} \otimes x_i)$, where (A_i) is a finite pairwise disjoint sequence in $\Sigma_f(\mu)$ and $x_i \in X$. For $s = \sum_{i=1}^n (\mathbb{1}_{A_i} \otimes x_i) \in \mathcal{S}(\Sigma_f(\mu), X)$ and $A \in \Sigma$, we can define the *integral* $\int_A s\, dm$ by

$$\int_A s\, dm := \sum_{i=1}^n m(A_i \cap A)(x_i). \tag{11}$$

Note that

$$\int_A s\, dm = \int_\Omega \mathbb{1}_A s\, dm. \tag{12}$$

For $y^* \in Y^*$, we define a measure $m_{y^*} : \Sigma_f(\mu) \to X^*$ by the equality

$$m_{y^*}(A)(x) := y^*(m(A)(x)) \quad \text{for } A \in \Sigma_f(\mu),\ x \in X. \tag{13}$$

For $s = \sum_{i=1}^n (\mathbb{1}_{A_i} \otimes x_i) \in \mathcal{S}(\Sigma_f(\mu), X)$ and $A \in \Sigma$, we define the integral $\int_A s\, dm_{y^*}$ by the equality:

$$\int_A s\, dm_{y^*} := \sum_{i=1}^n m_{y^*}(A_i \cap A)(x_i). \tag{14}$$

Then

$$y^*\left(\int_A s\, dm\right) = \int_A s\, dm_{y^*}. \tag{15}$$

Following [23], [19, § 13] one can define the φ^*-*semivariation* $\widetilde{m}_{\varphi^*}(A)$ of m on $A \in \Sigma$ by

$$\widetilde{m}_{\varphi^*}(A) := \sup \left\| \sum_{i=1}^n m(A \cap A_i)(x_i) \right\|_Y, \tag{16}$$

where the supremum is taken over all finite pairwise disjoint sets $\{A_1, \ldots, A_n\}$ in $\Sigma_f(\mu)$ and $x_i \in X$ for $i = 1, \ldots, n$ such that $\|\sum_{i=1}^n (\mathbb{1}_{A_i} \otimes x_i)\|_\varphi \le 1$.

One can observe that

$$\widetilde{m}_{\varphi^*}(A) = \sup \left\{ \left\| \int_A s\, dm \right\|_Y : s \in \mathcal{S}(\Sigma_f(\mu), X), \|s\|_\varphi \le 1 \right\}. \tag{17}$$

Note that

$$\begin{gathered} \widetilde{m}_{\varphi^*}(A) \le \widetilde{m}_{\varphi^*}(B) \quad \text{if } A, B \in \Sigma \text{ with } A \subset B, \\ \widetilde{m}_{\varphi^*}(A \cup B) \le \widetilde{m}_{\varphi^*}(A) + \widetilde{m}_{\varphi^*}(B) \quad \text{for } A, B \in \Sigma. \end{gathered} \tag{18}$$

Let $(\widetilde{m_{y^*}})_{\varphi^*}(A)$ stand for the φ^**-semivariation* of m_{y^*} on $A \in \Sigma$; that is,

$$\begin{aligned}&\left(\widetilde{m_{y^*}}\right)_{\varphi^*}(A)\\&\quad=\sup\left\{\left|\int_A s\,dm_{y^*}\right| : s\in\mathcal{S}\left(\Sigma_f(\mu),X\right),\|s\|_\varphi\le 1\right\}.\end{aligned}\tag{19}$$

The following lemma will be useful.

Lemma 4. *Let φ be a Young function and $m : \Sigma_f(\mu) \to \mathscr{L}(X,Y)$ be a measure with $m \ll \mu$ and $\widetilde{m}_{\varphi^*}(\Omega) < \infty$. Then the following statements hold:*

(i) *If $f \in E^\varphi(X)$, then there exists a $\|\cdot\|_\varphi$-Cauchy sequence (s_n) in $\mathcal{S}(\Sigma_f(\mu),X)$ such that $\|s_n(\omega)-f(\omega)\|_X \to 0$ μ-a.e.*

(ii) *If (s_n) is a $\|\cdot\|_\varphi$-Cauchy sequence in $\mathcal{S}(\Sigma_f(\mu),X)$, then for $A\in\Sigma$, $(\int_A s_n dm)$ is a Cauchy sequence in a Banach space Y and for every $y^*\in Y^*$, $(\int_A s_n dm_{y^*})$ is a Cauchy sequence in $\mathbb{R}$.*

(iii) *If $f \in E^\varphi(X)$ and (s_n') and (s_n'') are $\|\cdot\|_\varphi$-Cauchy sequence in $\mathcal{S}(\Sigma_f(\mu),X)$ such that $\|s_n'(\omega)-f(\omega)\|_X \to 0$ μ-a.e. and $\|s_n''(\omega)-f(\omega)\|_X \to 0$ μ-a.e., then for $A\in\Sigma$, one has*

$$\lim\int_A s_n'dm = \lim\int_A s_n''dm,\tag{20}$$

and for every $y^*\in Y^*$, one has

$$\lim\int_A s_n'dm_{y^*} = \lim\int_A s_n''dm_{y^*}.\tag{21}$$

Proof. (i) Let $f \in E^\varphi(X)$. Then there exists a sequence (s_n) in $\mathcal{S}(\Sigma_f(\mu),X)$ such that $\|s_n(\omega)-f(\omega)\|_X \to 0$ μ-a.e. and $\|s_n(\omega)\|_X \le \|f(\omega)\|_X$ μ-a.e. for all $n\in\mathbb{N}$ (see [21, Theorem 6, p. 4]). Using the Lebesgue dominated convergence theorem, we obtain that $\int_\Omega \varphi(\lambda(\|s_n(\omega)-f(\omega)\|_X)d\mu \to 0$ for all $\lambda>0$, so $\|s_n-f\|_\varphi \to 0$. Hence (s_n) is a $\|\cdot\|_\varphi$-Cauchy sequence.

(ii) Assume that (s_n) is a $\|\cdot\|_\varphi$-Cauchy sequence in $\mathcal{S}(\Sigma_f(\mu),X)$. Hence for $n,k\in\mathbb{N}$, we have

$$\begin{aligned}\left\|\int_A s_n dm - \int_A s_k dm\right\|_Y &= \left\|\int_A (s_n-s_k)\,dm\right\|_Y\\&\le \left\|s_n-s_k\right\|_\varphi \widetilde{m}_{\varphi^*}(A) \le \left\|s_n-s_k\right\|_\varphi \widetilde{m}_{\varphi^*}(\Omega).\end{aligned}\tag{22}$$

It follows that $(\int_A s_n dm)$ is a Cauchy sequence in Y. Hence in view of (15), for $y^*\in Y^*$, $(\int_A s_n dm_{y^*})$ is a Cauchy sequence in $\mathbb{R}$.

(iii) Note that $(s_n'-s_n'')$ is a $\|\cdot\|_\varphi$-Cauchy sequence and $\|s_n'(\omega)-s_n''(\omega)\|_X \to 0$ μ-a.e. Hence there exists $h\in E^\varphi(X)$ such that $\|(s_n'-s_n'')-h\|_\varphi \to 0$. Note that $\mathscr{T}_0|_{E^\varphi(X)} \subset \mathscr{T}_\varphi|_{E^\varphi(X)}$. Hence $(s_n'-s_n'')-h \to 0$ in $\mathscr{T}_0$ and it follows that there exists a subsequence $(s_{k_n}'-s_{k_n}'')$ of $(s_n'-s_n'')$ such that $\|(s_{k_n}'(\omega)-s_{k_n}''(\omega))-h(\omega)\|_X \to 0$ μ-a.e. Then $h(\omega)=0$ μ-a.e., so $\|s_n'-s_n''\|_\varphi \to 0$ and for $A\in\Sigma$, we get

$$\begin{aligned}\left\|\int_A s_n'dm - \int_A s_n''dm\right\|_Y &= \left\|\int_A \left(s_n'-s_n''\right)dm\right\|_Y\\&\le \left\|s_n'-s_n''\right\|_\varphi \widetilde{m}_{\varphi^*}(A).\end{aligned}\tag{23}$$

It follows that

$$\lim\int_A s_n'dm = \lim\int_A s_n''dm\tag{24}$$

and hence, in view of (15) for every $y^*\in Y^*$, we have

$$\lim\int_A s_n'dm_{y^*} = \lim\int_A s_n''dm_{y^*}.\tag{25}$$

□

Following [21, § 13, Definition 1, p. 254], in view of Lemma 4 we have the following.

Definition 5. Let φ be a Young function and $m : \Sigma_f(\mu) \to \mathscr{L}(X,Y)$ be an additive measure such that $m \ll \mu$ and $\widetilde{m}_{\varphi^*}(\Omega) < \infty$. Then for every $f\in E^\varphi(X)$ and $A\in\Sigma$, we can define the *integral* $\int_A f\,dm$ by the equality

$$\int_A f\,dm := \lim\int_A s_n dm\tag{26}$$

and for $y^*\in Y^*$, we can define the *integral* $\int_A f\,dm_{y^*}$ by the equality

$$\int_A f\,dm_{y^*} := \lim\int_A s_n dm_{y^*},\tag{27}$$

where (s_n) is an arbitrary $\|\cdot\|_\varphi$-Cauchy sequence in $\mathcal{S}(\Sigma_f(\mu),X)$ such that $\|s_n(\omega)-f(\omega)\|_X \to 0$ μ-a.e.

3. Integral Representation of Continuous Operators on Orlicz-Bochner Spaces

For a bounded linear operator $T : L^\varphi(X) \to Y$ let

$$\|T\|_\varphi := \sup\left\{\|T(f)\|_Y : f\in B_{L^\varphi(X)}\right\}.\tag{28}$$

Proposition 6. *Let $T : L^\varphi(X) \to Y$ be a bounded linear operator and*

$$m(A)(x) := T\left(\mathbb{1}_A\otimes x\right)\quad \text{for } A\in\Sigma_f(\mu),\ x\in X.\tag{29}$$

Then the following statements hold:

(i) *For $A\in\Sigma_f(\mu) m(A)\in\mathscr{L}(X,Y)$ and $\|m(A)\|\le\|T\|_\varphi\cdot\|\mathbb{1}_A\|_\varphi$.*

(ii) *$m\ll\mu$.*

(iii) *$\|m(A_n)\|\to 0$ if $A_n\downarrow\emptyset$ with $A_n\in\Sigma_f(\mu)$.*

(iv) *$m : \Sigma_f(\mu)\to\mathscr{L}(X,Y)$ is countably additive; that is, $m(\bigcup_{n=1}^\infty B_n)=\sum_{n=1}^\infty m(B_n)$ if (B_n) is a pairwise disjoint sequence in $\Sigma_f(\mu)$ with $\bigcup_{n=1}^\infty B_n\in\Sigma_f(\mu)$.*

(v) *$\widetilde{m}_{\varphi^*}(\Omega)\le\|T\|_\varphi$.*

Proof. (i) Let $A \in \Sigma_f(\mu)$. Then for $x \in B_X$, we have $\|\mathbb{1}_A \otimes x\|_\varphi \le \|\mathbb{1}_A\|_\varphi$ and hence

$$\begin{aligned}\|m(A)(x)\|_Y &= \|T(\mathbb{1}_A \otimes x)\|_Y \le \|T\|_\varphi \cdot \|\mathbb{1}_A \otimes x\|_\varphi \\ &\le \|T\|_\varphi \|\mathbb{1}_A\|_\varphi,\end{aligned} \tag{30}$$

so $\|m(A)\| \le \|T\|_\varphi \cdot \|\mathbb{1}_A\|_\varphi$.

(ii) This follows from (i) because $\|\mathbb{1}_A\|_\varphi = 0$ if $\mu(A) = 0$.

(iii) Assume that $A_n \downarrow \emptyset$ with $A_n \in \Sigma_f(\mu)$. Then $\mathbb{1}_{A_1}(\omega) \ge \mathbb{1}_{A_n}(\omega) \downarrow 0$ for $\omega \in \Omega$. By the Lebesgue dominated convergence theorem, we obtain that $\int_\Omega \varphi(\lambda \mathbb{1}_{A_n}(\omega))d\mu \to 0$ for every $\lambda > 0$. This means that $\|\mathbb{1}_{A_n}\|_\varphi \to 0$ and by (i), $\|m(A_n)\| \to 0$.

(iv) Assume that (B_n) is a pairwise disjoint sequence in $\Sigma_f(\mu)$ with $B = \bigcup_{n=1}^\infty B_n \in \Sigma_f(\mu)$. Let $A_n = B \setminus \bigcup_{i=1}^n B_i$ for $n \in \mathbb{N}$. Then $A_n \in \Sigma_f(\mu)$ and $A_n \downarrow \emptyset$. Hence by (iii) $\|m(B) - \sum_{i=1}^n m(B_i)\| = \|m(B) - m(\bigcup_{i=1}^n B_i)\| = \|m(A_n)\| \to 0$.

Statement (v) is obvious. □

Definition 7. Let $T : L^\varphi(X) \to Y$ be a bounded linear operator and

$$m(A)(x) := T(\mathbb{1}_A \otimes x) \quad \text{for } A \in \Sigma_f(\mu),\ x \in X. \tag{31}$$

Then the measure $m : \Sigma_f(\mu) \to \mathscr{L}(X, Y)$ will be called a *representing measure* of T.

Proposition 8. *Let $T : L^\varphi(X) \to Y$ be a $(\mathscr{T}_\varphi^\wedge, \|\cdot\|_Y)$-continuous linear operator and $m : \Sigma_f(\mu) \to \mathscr{L}(X, Y)$ be its representing measure. Then there exists a Young function ψ such that $\psi \lhd \varphi$ and $\widetilde{m}_{\psi^*}(\Omega) < \infty$.*

Proof. According to Theorem 2 there exist a finite set $\{\psi_i : i = 1, \ldots, n\}$ of Young functions with $\psi_i \lhd \varphi$ for $i = 1, \ldots, n$ and $a > 0$ such that

$$\|T(f)\|_Y \le a \max_{1\le i\le n} \|f\|_{\psi_i} \quad \forall f \in L^\varphi(X). \tag{32}$$

Let $\psi(t) = \max_{1\le i\le n}\psi_i(t)$ for $t \ge 0$. Then ψ is a Young function with $\psi \lhd \varphi$ and

$$\|T(f)\|_Y \le a\|f\|_\psi \quad \forall f \in L^\varphi(X). \tag{33}$$

Hence

$$\begin{aligned}&\widetilde{m}_{\psi^*}(\Omega) \\ &\quad = \sup\left\{\|T(s)\|_Y : s \in \mathcal{S}\left(\Sigma_f(\mu), X\right),\ \|s\|_\psi \le 1\right\} \\ &\quad \le a < \infty.\end{aligned} \tag{34}$$

□

For a linear operator $T : L^\varphi(X) \to Y$ and $A \in \Sigma$, let

$$T_A(f) := T(\mathbb{1}_A f) \quad \text{for } f \in L^\varphi(X). \tag{35}$$

Now we can state our main result that extends the classical results concerning the integral representation of operators on Lebesgue-Bochner spaces $L^p(X)$ $(1 \le p < \infty)$ (see [19, § 13, Theorem 1, pp. 259–261]) to operators on Orlicz-Bochner spaces $L^\varphi(X)$.

Theorem 9. *Let $T : L^\varphi(X) \to Y$ be a $(\mathscr{T}_\varphi^\wedge, \|\cdot\|_Y)$-continuous linear operator and $m : \Sigma_f(\mu) \to \mathscr{L}(X, Y)$ be its representing measure. Then for $A \in \Sigma$ the following statements hold:*

(i) *$T_A : L^\varphi(X) \to Y$ is a $(\mathscr{T}_\varphi^\wedge, \|\cdot\|_Y)$-continuous linear operator.*

(ii) *For $f \in L^\varphi(X)$, one has*

$$T_A(f) = \int_A f\, dm \tag{36}$$

and for $y^* \in Y^*$, one has

$$y^*(T_A(f)) = \int_A f\, dm_{y^*}. \tag{37}$$

(iii) *For $f \in L^\varphi(X)$, the measure $m_f : \Sigma \to Y$ defined by the equality*

$$m_f(A) := \int_A f\, dm \quad \text{for } A \in \Sigma \tag{38}$$

is countably additive.

(iv) $\|T_A\|_\varphi = \widetilde{m}_{\varphi^*}(A)$

and for $y^* \in Y^*$, $\|y^* \circ T_A\|_\varphi^* = \|(y^* \circ T)_A\|_\varphi^* = (\widetilde{m_{y^*}})_{\varphi^*}(A)$.

(v) $\widetilde{m}_{\varphi^*}(A) = \sup\{(\widetilde{m_{y^*}})_{\varphi^*}(A) : y^* \in B_{Y^*}\}$.

(vi) *For $f \in L^\varphi(X)$, one has*

$$\left\|\int_A f\, dm\right\|_Y \le \widetilde{m}_{\varphi^*}(A)\|f\|_\varphi \tag{39}$$

and for $y^* \in Y^*$, one has

$$\left|\int_A f\, dm_{y^*}\right| \le \left(\widetilde{m_{y^*}}\right)_{\varphi^*}(A)\|f\|_\varphi. \tag{40}$$

Proof. (i) Assume that (f_α) is a net in $L^\varphi(X)$ such that $f_\alpha \to 0$ in $\mathscr{T}_\varphi^\wedge$. Since $\mathscr{T}_\varphi^\wedge$ is a locally solid topology on $L^\varphi(X)$, we get $\mathbb{1}_A f_\alpha \to 0$ in $\mathscr{T}_\varphi^\wedge$. Hence

$$\|T_A(f_\alpha)\|_Y = \|T(\mathbb{1}_A f_\alpha)\|_Y \longrightarrow 0. \tag{41}$$

(ii) In view of Proposition 8 there exists a Young function ψ such that $\psi \lhd \varphi$ and $\widetilde{m}_{\psi^*}(\Omega) < \infty$. Then $L^\varphi(X) \subset E^\psi(X)$. Let $f \in L^\varphi(X)$. Then there exists a sequence (s_n) in $\mathcal{S}(\Sigma_f(\mu), X)$ such that $\|s_n(\omega) - f(\omega)\|_X \to 0$ μ-a.e. and $\|s_n(\omega)\|_X \le \|f(\omega)\|_X$ μ-a.e. for all $n \in \mathbb{N}$ (see [21, Theorem 6, p. 4]). Then $s_n \to f$ in $\mathscr{T}_\varphi^\wedge$ because $\mathscr{T}_\varphi^\wedge$ is a Lebesgue topology. Hence $\|s_n - f\|_\psi \to 0$. In view of Lemma 4 we can define the integral $\int_A f\, dm$ by the equality

$$\int_A f\, dm := \lim \int_A s_n dm. \tag{42}$$

Since $T_A(s_n) = \int_A s_n dm$ and by (i), T_A is $(\mathcal{T}_\varphi^\wedge, \|\cdot\|_Y)$-continuous, we get

$$T_A(f) = \lim \int_A s_n dm. \quad (43)$$

Hence

$$T_A(f) = \int_A f\, dm \quad (44)$$

and for $y^* \in Y^*$, we have

$$\begin{aligned} y^*(T_A(f)) &= \lim y^*\left(\int_A s_n dm\right) = \lim \int_A s_n dm_{y^*} \\ &= \int_A f\, dm_{y^*}. \end{aligned} \quad (45)$$

(iii) Let $f \in L^\varphi(X)$ and (A_n) be a sequence in Σ such that $A_n \downarrow \emptyset$. Then $\mathbb{1}_{A_n}(\omega) \downarrow 0$ for $\omega \in \Omega$, and hence $\|\mathbb{1}_{A_n}(\omega)f(\omega)\|_X \to 0$ μ-a.e. and $\|\mathbb{1}_{A_n}(\omega)f(\omega)\|_X \le \|f(\omega)\|_X$ μ-a.e. Hence $\mathbb{1}_{A_n}f \to 0$ in $\mathcal{T}_\varphi^\wedge$ because $\mathcal{T}_\varphi^\wedge$ is a Lebesgue topology, and by (i) we get

$$\|m_f(A_n)\|_Y = \left\|\int_{A_n} f\, dm\right\|_Y = \left\|T\left(\mathbb{1}_{A_n}f\right)\right\|_Y \longrightarrow 0. \quad (46)$$

(iv) Note that $\widetilde{m}_{\varphi^*}(A) \le \|T_A\|_\varphi$. To show that $\|T_A\|_\varphi \le \widetilde{m}_{\varphi^*}(A)$, assume that $f \in B_{L^\varphi(X)}$. Choose a sequence (s_n) in $\mathcal{S}(\Sigma_f(\mu), X)$ such that $\|s_n(\omega) - f(\omega)\|_X \to 0$ μ-a.e. and $\|s_n(\omega)\|_X \le \|f(\omega)\|_X$ μ-a.e. for all $n \in \mathbb{N}$. Since $\mathcal{T}_\varphi^\wedge$ is a Lebesgue topology, we have $s_n \to f$ in $\mathcal{T}_\varphi^\wedge$ and hence $\|T_A(s_n) - T_A(f)\|_Y \to 0$. Note that $T_A(s_n) = \int_A s_n dm$.

Let $\varepsilon > 0$ be given. Choose $n_0 \in \mathbb{N}$ such that $\|T_A(f) - \int_A s_{n_0} dm\|_Y \le \varepsilon$. Then

$$\begin{aligned} \|T_A(f)\|_Y &\le \left\|T_A(f) - \int_A s_{n_0} dm\right\|_Y + \left\|\int_A s_{n_0} dm\right\|_Y \\ &\le \varepsilon + \widetilde{m}_{\varphi^*}(A). \end{aligned} \quad (47)$$

It follows that $\|T_A\|_\varphi \le \widetilde{m}_{\varphi^*}(A)$, so $\widetilde{m}_{\varphi^*}(A) = \|T_A\|_\varphi$. Hence for $y^* \in Y^*$, we easily get

$$\|(y^* \circ T)_A\|_\varphi^* = \|y^* \circ T_A\|_\varphi^* = \left(\widetilde{m_{y^*}}\right)_{\varphi^*}(A). \quad (48)$$

(v) Using (iv) we have

$$\begin{aligned} \widetilde{m}_{\varphi^*}(A) &= \|T_A\|_\varphi \\ &= \sup\left\{\|T_A(f)\|_Y : f \in L^\varphi(X), \|f\|_\varphi \le 1\right\} \\ &= \sup_{y^* \in B_{Y^*}} \left\{\left|(y^* \circ T_A)(f)\right| : f \in L^\varphi(X), \|f\|_\varphi \le 1\right\} \\ &= \sup_{y^* \in B_{Y^*}} \|y^* \circ T_A\|_\varphi^* = \sup_{y^* \in B_{Y^*}} \left(\widetilde{m_{y^*}}\right)_{\varphi^*}(A). \end{aligned} \quad (49)$$

(vi) This follows from (ii) and (iv). □

For a sequence (A_n) in Σ, we will write $A_n \searrow_\mu \emptyset$ if $A_n \downarrow$ and $\mu(A_n \cap A) \to 0$ for every $A \in \Sigma_f(\mu)$.

Definition 10. A measure $m : \Sigma_f(\mu) \to \mathcal{L}(X, Y)$ with $m \ll \mu$ and $\widetilde{m}_{\varphi^*}(\Omega) < \infty$ is said to be φ^**-semivariationally* μ*-continuous* if $\widetilde{m}_{\varphi^*}(A_n) \to 0$ whenever $A_n \searrow_\mu \emptyset$, $(A_n) \subset \Sigma$.

Using a standard argument we can show the following.

Proposition 11. *Let $m : \Sigma \to \mathcal{L}(X, Y)$ be an additive measure such that $m \ll \mu$ and $\widetilde{m}_\varphi(\Omega) < \infty$. Then the following statements are equivalent:*

(i) *m is φ^*-semivariationally μ-continuous.*

(ii) *The following two conditions hold simultaneously:*

(a) *For every $\varepsilon > 0$ there exists $\delta > 0$ such that $\widetilde{m}_{\varphi^*}(A) \le \varepsilon$ whenever $\mu(A) \le \delta$, $A \in \Sigma$.*

(b) *For every $\varepsilon > 0$ there exists $A_0 \in \Sigma_f(\mu)$ such that $\widetilde{m}_{\varphi^*}(\Omega \setminus A_0) \le \varepsilon$.*

The following theorem characterizes φ^*-semivariationally μ-continuous representing measures.

Theorem 12. *Let $T : L^\varphi(X) \to Y$ be a $(\mathcal{T}_\varphi^\wedge, \|\cdot\|_Y)$-continuous linear operator and $m : \Sigma_f(\mu) \to \mathcal{L}(X, Y)$ be its representing measure. Then the following statements are equivalent:*

(i) *m is φ^*-semivariationally μ-continuous.*

(ii) *T is $(\gamma_\varphi, \|\cdot\|_Y)$-continuous.*

(iii) *$\|T(f_n)\|_Y \to 0$ if $f_n \to 0$ in $\mathcal{T}_0$ and $\sup_n \|f_n\|_\varphi < \infty$.*

(iv) *$\|T_{A_n}\|_\varphi \to 0$ if $A_n \searrow_\mu \emptyset$, $(A_n) \subset \Sigma$.*

Proof. (i) ⇔ (ii) ⇔ (iii) See [5, Corollary 2.8 and Proposition 1.1].

(i) ⇔ (iv) This follows from Theorem 9. □

Now assume that Ω is a completely regular Hausdorff space. Let $\mathcal{B}a$ denote the σ-algebra of Baire sets in Ω, which is the σ-algebra generated by the class $\mathcal{Z}$ of all zero sets of bounded continuous positive functions on ω. By $\mathcal{P}$ we denote the family of all cozero (=positive) in Ω (see [25, p. 108]).

Let $\mu : \mathcal{B}a \to [0, \infty)$ be a countably additive measure. Then μ is zero-set regular; that is, for every $A \in \mathcal{B}a$ and $\varepsilon > 0$ there exists $Z \in \mathcal{Z}$ with $Z \subset A$ such that $\mu(A \setminus Z) \le \varepsilon$ (see [25, p. 118]). It follows that for every $A \in \mathcal{B}a$ and $\varepsilon > 0$ there exist $U \in \mathcal{P}$, $U \supset A$ such that $\mu(U \setminus A) \le \varepsilon$.

We can assume that μ to be complete (if necessary we can take the completion $(\Omega, \overline{\mathcal{B}a}, \overline{\mu})$ of the measure space $(\Omega, \mathcal{B}a, \mu)$).

Proposition 13. *Assume that Ω is a completely regular Hausdorff space and $(\Omega, \mathcal{B}a, \mu)$ is a complete finite measure space. Let $T : L^\varphi(X) \to Y$ be a $(\mathcal{T}_\varphi^\wedge, \|\cdot\|_Y)$-continuous linear operator and $m : \mathcal{B}a \to \mathcal{L}(X, Y)$ be its representing measure. Then the following statements are equivalent:*

(i) *m is φ^*-semivariationally μ-continuous.*

(ii) *For every sequence* (A_n) *in* $\mathscr{B}a$ *such that* $A_n \downarrow$ *and* $\mu(A_n) \to 0$ *there exists a sequence* (U_n) *in* $\mathscr{P}$ *with* $A_n \subset U_n \downarrow$ *such that* $\widetilde{m}_{\varphi^*}(U_n) \to 0$.

(iii) *For every sequence* (A_n) *in* $\mathscr{B}a$ *such that* $A_n \downarrow$ *and* $\mu(A_n) \to 0$ *there exists a sequence* (U_n) *in* $\mathscr{P}$ *with* $A_n \subset U_n \downarrow$ *such that*

$$\sup\left\{\|T(f)\|_Y : f \in B_{L^\varphi(X)}, \operatorname{supp} f \subset U_n\right\} \longrightarrow 0. \tag{50}$$

Proof. (i) $\Rightarrow$ (ii) Assume that (i) holds and (A_n) is a sequence in $\mathscr{B}a$ such that $A_n \downarrow$ and $\mu(A_n) \to 0$. Then there exists a sequence (U_n) in $\mathscr{P}$ such that $A_n \subset U_n \downarrow$ and $\mu(U_n \backslash A_n) \le 1/n$ for $n \in \mathbb{N}$.

Let $\varepsilon > 0$ be given. Then in view of Proposition 11 there exists $\delta > 0$ such that $\widetilde{m}_{\varphi^*}(A) \le \varepsilon/2$ if $\mu(A) \le \delta$ with $A \in \mathscr{B}a$. Choose $n_1 \in \mathbb{N}$ such that $\mu(U_n \setminus A_n) \le \delta$ for $n \ge n_2$. Then $\widetilde{m}_{\varphi^*}(U_n \setminus A_n) \le \varepsilon/2$ for $n \ge n_1$. Since $\widetilde{m}_{\varphi^*}(A_n) \to 0$, we can choose $n_2 \in \mathbb{N}$ such that $\widetilde{m}_{\varphi^*}(A_n) \le \varepsilon/2$ for $n \ge n_2$. Then for $n \ge n_0 = \max(n_1, n_2)$, we get

$$\widetilde{m}_{\varphi^*}(U_n) \le \widetilde{m}_{\varphi^*}(U_n \setminus A_n) + \widetilde{m}_{\varphi^*}(A_n) \le \varepsilon; \tag{51}$$

that is, (ii) holds.

(ii) $\Rightarrow$ (iii) Assume that (ii) holds and (A_n) is a sequence in $\mathscr{B}o$ such that $A_n \downarrow$ and $\mu(A_n) \to 0$. Then there exists a sequence (U_n) in $\mathscr{P}$ with $A_n \subset U_n \downarrow$ such that $\widetilde{m}_{\varphi^*}(U_n) \to 0$. Note that, for $f \in B_{L^\varphi(X)}$ with $\operatorname{supp} f \subset U_n$ for $n \in \mathbb{N}$, by Theorem 9 we have

$$\|T(f)\|_Y = \left\|\int_\Omega f\, dm\right\|_Y = \left\|\int_{U_n} f\, dm\right\|_Y \le \widetilde{m}_{\varphi^*}(U_n). \tag{52}$$

It follows that (iii) holds.

(iii) $\Rightarrow$ (i) Assume that (iii) holds and $A_n \downarrow$ with $\mu(A_n) \to 0$. Then there exists a sequence (U_n) in $\mathscr{P}$ with $A_n \subset U_n \downarrow$ such that

$$\sup\left\{\|T(f)\|_Y : f \in B_{L^\varphi(X)}, \operatorname{supp} f \subset U_n\right\} \longrightarrow 0. \tag{53}$$

Assume on the contrary that (i) fails to hold. Then without loss of generality we can assume that

$$\widetilde{m}_{\varphi^*}(A_n) > \varepsilon_0 \quad \text{for some } \varepsilon_0 > 0, \text{ all } n \in \mathbb{N}. \tag{54}$$

Choose $n_0 \in \mathbb{N}$ such that

$$\sup\left\{\|T(f)\|_Y : f \in B_{L^\varphi(X)}, \operatorname{supp} f \subset U_{n_0}\right\} < \frac{\varepsilon_0}{2}. \tag{55}$$

In view of (54) there exists a pairwise disjoint set $\{B_1, \ldots, B_k\}$ in $\mathscr{B}a$, $x_i \in X$ for $i = 1, \ldots, k$ and $y^* \in B_{Y^*}$ such that $\|\sum_{i=1}^k (\mathbb{1}_{B_i} \otimes x_i)\|_\varphi \le 1$ and

$$\left| y^* \left(\sum_{i=1}^k m\left(A_{n_0} \cap B_i\right)(x_i) \right) \right| \ge \varepsilon_0. \tag{56}$$

Let $s_0 = \sum_{i=1}^k (\mathbb{1}_{A_{n_0} \cap B_i} \otimes x_i)$. Then $\|s_0\|_\varphi \le 1$ and $\operatorname{supp} s_0 \subset A_{n_0} \subset U_{n_0}$. Then by (55) we get $\|T(s_0)\|_Y < \varepsilon_0/2$.

On the other hand, in view of (56) we have $\|T(s_0)\|_Y \ge \varepsilon_0$. This contradiction establishes that (i) holds. □

Corollary 14. *Assume that* Ω *is a completely regular Hausdorff space and* $(\Omega, \mathscr{B}a, \mu)$ *is complete finite measure space. Let* $T : L^\varphi(X) \to Y$ *be a* $(\gamma_\varphi, \|\cdot\|_Y)$*-continuous linear operator and* $m : \mathscr{B}a \to \mathscr{L}(X, Y)$ *be its representing measure. Then* $\widetilde{m}_{\varphi^*}$ *is regular; that is, for every* $A \in \mathscr{B}a$ *and* $\varepsilon > 0$ *there exist* $Z \in \mathscr{Z}$ *and* $U \in \mathscr{P}$ *with* $Z \subset A \subset U$ *such that* $\widetilde{m}_{\varphi^*}(U \setminus Z) \le \varepsilon$.

Proof. In view of Theorem 12 m is φ^*-semivariationally μ-continuous. Let $A \in \mathscr{B}a$ and $\varepsilon > 0$ be given. Then by Proposition 11 there exists $\delta > 0$ such that $\widetilde{m}_{\varphi^*}(B) \le \varepsilon$ whenever $B \in \mathscr{B}a$ and $\mu(B) \le \delta$. By the regularity of μ one can choose $Z \in \mathscr{Z}$ and $U \in \mathscr{P}$ with $Z \subset A \subset U$ such that $\mu(U \setminus Z) \le \delta$. Hence $\widetilde{m}_{\varphi^*}(U \setminus Z) \le \varepsilon$, as desired. □

4. Compact Operators on Orlicz-Bochner Spaces

The following theorem presents necessary conditions for a $(\mathscr{T}_\varphi^\wedge, \|\cdot\|_Y)$-continuous operator $T : L^\varphi(X) \to Y$ to be compact.

Theorem 15. *Assume that a Young function* φ *such that* φ^* *satisfies the* Δ_2*-condition. Let* $T : L^\varphi(X) \to Y$ *be a* $(\mathscr{T}_\varphi^\wedge, \|\cdot\|_Y)$*-continuous linear operator and* $m : \Sigma_f(\mu) \to \mathscr{L}(X, Y)$ *be its representing measure. If* T *is compact, then* m *is* φ^**-semivariationally* μ*-continuous.*

Proof. Assume that T is compact and m fails to be φ^*-semivariationally μ-continuous. Then there exist $\varepsilon > 0$ and a sequence (A_n) in Σ with $A_n \searrow_\mu \emptyset$ such that $\|T_{A_n}\| = \widetilde{m_\varphi^*}(A_n) > \varepsilon$ for $n \in \mathbb{N}$ (see Theorem 9). Hence one can choose a sequence (y_n^*) in B_{Y^*} such that

$$\left\| y_n^* \circ T_{A_n} \right\|_\varphi^* \ge \varepsilon \quad \forall n \in \mathbb{N}. \tag{57}$$

By Schauder's theorem the conjugate mapping $T^* : Y^* \to L^\varphi(X)^*$ is compact. Note that $T^*(y_n^*) = y_n^* \circ T \in L^\varphi(X)_n^\sim$ for all $n \in \mathbb{N}$, where $L^\varphi(X)_n^\sim$ is a closed subspace of the Banach space $(L^\varphi(X)^*, \|\cdot\|_\varphi^*)$ (see Theorem 2). Then for every $n \in \mathbb{N}$ there exists $g_n \in L^{\varphi^*}(X^*, X)$ such that

$$\begin{aligned} (y_n^* \circ T)(f) &= \int_\Omega \langle f(\omega), g_n(\omega) \rangle \, d\mu \\ &\qquad\qquad \text{for } f \in L^\varphi(X), \\ \left\| y_n^* \circ T \right\|_\varphi^* & \\ &= \sup\left\{ \int_\Omega \|f(\omega)\|_X \vartheta(g_n)(\omega) \, d\mu : f \in B_{L^\varphi(X)} \right\} \\ &= \left\|\vartheta(g_n)\right\|_{\varphi^*}. \end{aligned} \tag{58}$$

Hence we obtain that, for each $n \in \mathbb{N}$,

$$\left\| y_n^* \circ T_{A_n} \right\|_\varphi^* = \left\| \mathbb{1}_{A_n} \vartheta(g_n) \right\|_{\varphi^*} = \left\| \vartheta\left(\mathbb{1}_{A_n} g_n\right) \right\|_{\varphi^*}. \tag{59}$$

Since $T^*(B_{Y^*})$ is a relatively sequentially compact subset of $((L^\varphi(X))_n^\sim, \|\cdot\|_\varphi^*)$, there exist a subsequence (g_{k_n}) of (g_n) and $g \in L^{\varphi^*}(X^*, X)$ such that

$$\left\|F_{g_n} - F_g\right\|_\varphi^* = \left\|\vartheta\left(g_{k_n} - g\right)\right\|_{\varphi^*} \longrightarrow 0. \tag{60}$$

Choose $n_\varepsilon \in \mathbb{N}$ such that $\|\vartheta(g_{k_n} - g)\|_{\varphi^*} \le \varepsilon/2$ for $n \ge n_\varepsilon$. Hence for $n \ge n_\varepsilon$,

$$\begin{aligned}\Big|\left\|\vartheta\left(\mathbb{1}_{A_{k_n}} g\right)\right\|_{\varphi^*} &- \left\|\vartheta\left(\mathbb{1}_{A_{k_n}} g_{k_n}\right)\right\|_{\varphi^*}\Big| \\ &\le \left\|\vartheta\left(\mathbb{1}_{A_{k_n}}\left(g_{k_n} - g\right)\right)\right\|_{\varphi^*} = \left\|\mathbb{1}_{A_{k_n}}\vartheta\left(g_{k_n} - g\right)\right\|_{\varphi^*} \\ &\le \left\|\vartheta\left(g_{k_n} - g\right)\right\|_{\varphi^*} \le \frac{\varepsilon}{2}.\end{aligned} \tag{61}$$

Using (57) and (61), for $n \ge n_\varepsilon$, we get

$$\begin{aligned}\varepsilon &\le \left\|y^* \circ T_{A_{k_n}}\right\|_\varphi^* = \left\|\vartheta\left(\mathbb{1}_{A_{k_n}} g_{k_n}\right)\right\|_{\varphi^*} \\ &\le \frac{\varepsilon}{2} + \left\|\vartheta\left(\mathbb{1}_{A_{k_n}} g\right)\right\|_{\varphi^*}\end{aligned} \tag{62}$$

and hence

$$\left\|\mathbb{1}_{A_{k_n}}\vartheta(g)\right\|_{\varphi^*} = \left\|\vartheta\left(\mathbb{1}_{A_{k_n}} g\right)\right\|_{\varphi^*} \ge \frac{\varepsilon}{2}. \tag{63}$$

On the other hand, since φ^* is supposed to satisfy the Δ_2-condition, we have that $\|\mathbb{1}_{A_{k_n}}\vartheta(g)\|_{\varphi^*} \to 0$ (see [26, Theorem 3, pp. 58-59]). This contradiction establishes that m is φ^*-semivariationally μ-continuous. □

Corollary 16. *Assume that φ is a Young function such that φ^* satisfies the Δ_2-condition. Let $T : L^\varphi(X) \to Y$ be a $(\mathcal{T}_\varphi^\wedge, \|\cdot\|_Y)$-continuous linear operator. Then the following statements are equivalent:*

(i) *T is compact.*

(ii) *T is $(\gamma_\varphi, \|\cdot\|_Y)$-compact; that is, there exists a γ_φ-neighborhood V of 0 in $L^\varphi(X)$ such that $T(V)$ is a relatively norm compact set in Y.*

(iii) *There exists a Young function ψ with $\psi \ll \varphi$ such that $\{\int_\Omega f\,dm : f \in L^\varphi(X), \|f\|_\psi \le 1\}$ is a relatively norm compact set in Y.*

Proof. (i) ⇒ (ii) Assume that (i) holds. Then by Theorems 12 and 15 T is $(\gamma_\varphi, \|\cdot\|_Y)$-continuous. Since the space $(L^\varphi(X), \gamma_\varphi)$ is quasinormable, by Grothendieck's classical result (see [15, p. 429]), we obtain that T is $(\gamma_\varphi, \|\cdot\|_Y)$-compact.

(ii) ⇒ (i) The implication is obvious.

(ii) ⇔ (iii) This follows from Theorem 3. □

5. Topology Associated with the φ^*-Semivariation of a Representing Measure

Assume that $T : L^\varphi(X) \to Y$ be a $(\mathcal{T}_\varphi^\wedge, \|\cdot\|_Y)$-continuous linear operator. Let $m : \Sigma_f(\mu) \to \mathcal{L}(X, Y)$ be its representing measure. Let us put

$$p_m\left(y^*\right) := \left(\widetilde{m_{y^*}}\right)_{\varphi^*}(\Omega) \quad \text{for } y^* \in Y^*. \tag{64}$$

Note that p_m is a seminorm on Y^*. Following [22, 27] let δ_{m,φ^*} stand for the topology on B_{Y^*} defined by the seminorm p_m restricted to B_{Y^*}.

The following theorem characterizes $(\mathcal{T}_\varphi^\wedge, \|\cdot\|_Y)$-continuous compact operators $T : L^\varphi(X) \to Y$ in terms of the topological properties of the space $(B_{Y^*}, \delta_{m,\varphi^*})$ (see [22, Theorem 3]).

Theorem 17. *Let $T : L^\varphi(X) \to Y$ be a $(\mathcal{T}_\varphi^\wedge, \|\cdot\|_Y)$-continuous linear operator and $m : \Sigma_f(\mu) \to \mathcal{L}(X, Y)$ be its representing measure. Then the following statements are equivalent:*

(i) *The space $(B_{Y^*}, \delta_{m,\varphi^*})$ is compact.*

(ii) *T is compact.*

Proof. (i) ⇒ (ii) Assume that $(B_{Y^*}, \delta_{m,\varphi^*})$ is compact. Let (y_n^*) be a sequence in B_{Y^*}. Without loss of generality we can assume that $y_n^* \to y_0^*$ in δ_{m,φ^*} for some $y^* \in B_{Y^*}$. Then using Theorem 9 for $f \in L^\varphi(X)$, we have

$$\begin{aligned}\left|\left(T^*\left(y_n^*\right) - T^*\left(y_0^*\right)\right)(f)\right| &= \left|\left(y_n^* - y_0^*\right)(T(f))\right| \\ &= \left|\int_\Omega f\,dm_{y_n^* - y_0^*}\right| \le \left(\widetilde{m_{y_n^* - y_0^*}}\right)_{\varphi^*}(\Omega)\,\|f\|_\varphi.\end{aligned} \tag{65}$$

It follows that $\|T^*(y_n^*) - T^*(y_0^*)\|_\varphi^* \le (\widetilde{m_{y_n^* - y_0^*}})_{\varphi^*}(\Omega)$, where $p_m(y_n^* - y_0^*) = (\widetilde{m_{y_n^* - y_0^*}})_{\varphi^*}(\Omega) \underset{n}{\to} 0$. This means that T^* is compact and hence T is compact.

(ii) ⇒ (i) Assume that T is compact and (y_α^*) is a net in B_{Y^*}. Since B_{Y^*} is $\sigma(Y^*, Y)$-compact, without loss of generality we can assume that $y_\alpha^* \underset{\alpha}{\to} y_0^*$ in $\sigma(Y^*, Y)$ for some $y_0^* \in B_{Y^*}$. In view of the compactness of the conjugate operator $T^* : Y^* \to L^\varphi(X)^*$, there exists a subset (y_β^*) of (y_α^*) and $\Phi_0 \in L^\varphi(X)^*$ such that $\|T^*(y_\beta^*) - \Phi_0\|_\varphi^* \underset{\beta}{\to} 0$. On the other hand, since T^* is $(\sigma(Y^*, Y), \sigma(L^\varphi(X)^*, L^\varphi(X)))$-continuous, we get $T^*(y_\beta^*) \underset{\beta}{\to} T^*(y_0^*)$ in $\sigma(L^\varphi(X)^*, L^\varphi(X))$. Hence $\Phi_0 = T^*(y_0^*)$; that is, $\|T^*(y_\beta^*) - T^*(y_0^*)\|_\varphi^* \underset{\beta}{\to} 0$.

Let $\varepsilon > 0$ be given. Then there exist a pairwise disjoint set $\{A_1, \ldots, A_n\}$ in $\Sigma_f(\mu)$ and $x_i \in X$ for $i = 1, \ldots, n$ such that $\|\sum_{i=1}^n(\mathbb{1}_{A_i} \otimes x_i)\|_\varphi \le 1$ and

$$\left(\widetilde{m_{y_\beta^* - y_0^*}}\right)_{\varphi^*}(\Omega) \le \left|\sum_{i=1}^n\left(y_\beta^* - y_0^*\right)\left(m\left(A_i\right)\left(x_i\right)\right)\right| + \varepsilon. \tag{66}$$

Hence

$$\begin{aligned}\left(\widetilde{m_{y_\beta^* - y_0^*}}\right)_{\varphi^*}(\Omega) &\le \left|\sum_{i=1}^n\left(y_\beta^* - y_0^*\right)\left(T\left(\mathbb{1}_{A_i} \otimes x_i\right)\right)\right| + \varepsilon \\ &\le \left|\left(y_\beta^* - y_0^*\right)T\left(\sum_{i=1}^n\left(\mathbb{1}_{A_i} \otimes x_i\right)\right)\right| \\ &\quad + \varepsilon\end{aligned}$$

$$= \left| T^* \left(y^*_\beta - y^*_0 \right) \left(\sum_{i=1}^{n} \left(\mathbb{1}_{A_i} \otimes x_i \right) \right) \right|$$
$$+ \varepsilon$$
$$\leq \left\| T^* \left(y^*_\beta - y^*_0 \right) \right\|^*_\varphi \left\| \sum_{i=1}^{n} \left(\mathbb{1}_{A_i} \otimes x_i \right) \right\|_\varphi$$
$$+ \varepsilon \leq \left\| T^* \left(y^*_\beta \right) - T \left(y^*_0 \right) \right\|^*_\varphi + \varepsilon. \quad (67)$$

Hence $p_m(y^*_\beta - y^*_0) = (\widetilde{m_{y^*_\beta - y^*_0}})_{\varphi^*}(\Omega) \underset{\beta}{\to} 0$, and this means that the space $(B_{Y^*}, \delta_{m,\varphi^*})$ is compact. □

As a consequence of Theorems 17 and 15, we have the following.

Corollary 18. *Assume that φ is a Young function such that φ^* satisfies the Δ_2-condition. Let $T : L^\varphi(X) \to Y$ be a $(\mathcal{T}^\wedge_\varphi, \|\cdot\|_Y)$-continuous linear operator and $m : \Sigma_f(\mu) \to \mathscr{L}(X, Y)$ be its representing measure. If the space $(B_{Y^*}, \delta_{m,\varphi^*})$ is compact, then m is φ^*-semivariationally μ-continuous.*

Conflicts of Interest

The author declares that there are no conflicts of interest.

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