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Research Article

Weak Convergence Theorems on the Split Common Fixed Point Problem for Demicontractive Continuous Mappings

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We are concerned with the split common fixed point problem in Hilbert spaces. We propose a new method for solving this problem and establish a weak convergence theorem whenever the involved mappings are demicontractive and Lipschitz continuous. As an application, we also obtain a new method for solving the split equality problem in Hilbert spaces.

1. Introduction

The split common fixed point problem (SCFP) is an inverse problem that aims to find an element in a fixed point set such that its image under a linear transformation belongs to another fixed point set. More specifically, given two Hilbert spaces H_1 and H_2, the SCFP consists in finding $x \in H_1$ such that

$$\begin{aligned} x &\in F(U), \\ Ax &\in F(T), \end{aligned} \tag{P1}$$

where $A : H_1 \to H_2$ is a bounded linear mapping and $F(U)$ and $F(T)$ are, respectively, the fixed point sets of nonlinear mappings $U : H_1 \to H_1$ and $T : H_2 \to H_2$. Particularly, if U and T are both metric projections, the SCFP is reduced to the well-known split feasibility problem (SFP). Actually, the SFP can be formulated as the problem of finding a point $x \in H_1$ such that

$$\begin{aligned} x &\in C, \\ Ax &\in Q, \end{aligned} \tag{1}$$

where $C \subseteq H_1$ and $Q \subseteq H_2$ are nonempty closed convex sets, and mapping A is as above. These two problems have been extensively investigated since they play an important role in various areas including signal processing and image reconstruction [1–5].

We assume throughout the paper that problem (P1) is consistent, which means that its solution set, denoted by S, is nonempty. Censor and Segal [6] studied the SCFP when U and T are firmly quasi-nonexpansive mappings, and proposed the following method:

$$x_{n+1} = U\left[x_n - \tau_n A^* (I - T) Ax_n\right], \tag{2}$$

where τ_n is a properly chosen stepsize. It is shown that if τ_n is chosen in $(0, 2/\|A\|^2)$, then the sequence generated by method (2) converges weakly to a solution of problem (P1). Subsequently, this result was extended to quasi-nonexpansive operators [7], demicontractive operators [8, 9], two groups of finitely many firmly quasi-nonexpansive mappings [10, 11], and the more general common null point problem [12]. Also, some variants of method (2) have been considered in [13–16].

Since the choice of the stepsize is related to $\|A\|$, thus, to implement method (3), one has to compute (or at least estimate) the norm $\|A\|$, which is generally not easy in practice. A way to avoid this is to adopt variable stepsize which ultimately has no relation with $\|A\|$ [8, 10, 17–19]. Wang [18] recently proposed a new method for solving the SCFP:

$$x_{n+1} = x_n - \rho_n \left[(I - U) x_n + A^* (I - T) Ax_n\right], \tag{3}$$

where $\{\rho_n\} \subset (0, \infty)$ is chosen such that

$$\rho_n = \frac{\|(I - U) x_n\|^2 + \|(I - T) Ax_n\|^2}{\|(I - U) x_n + A^* (I - T) Ax_n\|^2}. \tag{4}$$

Wang proved that if mappings U and T are firmly quasi-nonexpansive, then the sequence $\{x_n\}$ generated by (3)-(4) converges weakly to a solution of problem (P1). Wang and Xu [19] recently proposed another choice of the stepsize:

$$\rho_n = \frac{\tau_n}{\left\|(I-U)\,x_n + A^*\,(I-T)\,Ax_n\right\|}, \tag{5}$$

where $\{\tau_n\} \subset (0,\infty)$ is chosen such that

$$\begin{aligned} &\sum_{n=0}^{\infty} \tau_n = \infty, \\ &\sum_{n=0}^{\infty} \tau_n^2 < \infty. \end{aligned} \tag{6}$$

They proved that if mappings U and T are nonexpansive, then the sequence $\{x_n\}$ generated by (3) and (5)-(6) converges weakly to a solution of problem (P1). It is clear that these choices of the stepsize do not rely on the norm $\|A\|$.

In this paper, we first extend the above result for method (3) from nonexpansive mappings to demicontractive continuous mappings. By using properties of product spaces, we change the split equality problem into a special split common fixed point problem. As a result, based on our extension, we obtain a new method for solving the split equality problem in Hilbert spaces.

2. Preliminaries

Throughout the paper, H_i, $i = 1,2,3$, are Hilbert spaces, I is the identity operator, "$\rightarrow$" stands for strong convergence, and "$\rightharpoonup$" stands for weak convergence. For a mapping $W : H_1 \rightarrow H_1$, $F(W)$ is the set of the fixed points of W, $W^{-1}(0) = \{x \in H_1 : Wx = 0\}$, and $W^c := I - W$.

Definition 1. Let $W : H_1 \rightarrow H_1$ be a nonlinear mapping.

(i) W is called *firmly nonexpansive*, if

$$\begin{aligned} \left\|Wx - Wy\right\|^2 \le \left\|x-y\right\|^2 - \left\|W^c x - W^c y\right\|^2, \\ \forall x, y \in H_1. \end{aligned} \tag{7}$$

(ii) W is called *nonexpansive*, if

$$\left\|Wx - Wy\right\| \le \left\|x-y\right\|, \quad \forall x, y \in H_1. \tag{8}$$

(iii) W is called *strictly pseudo-contractive*, if there exists $k < 1$ such that

$$\begin{aligned} \left\|Wx - Wy\right\|^2 \le \left\|x-y\right\|^2 + k\left\|W^c x - W^c y\right\|^2, \\ \forall x, y \in H_1. \end{aligned} \tag{9}$$

(iv) W is called *L-Lipschitz continuous*, if there exists $L > 0$ such that

$$\left\|Wx - Wy\right\| \le L\left\|x-y\right\|, \quad \forall x, y \in H_1. \tag{10}$$

Definition 2. Let $W : H_1 \rightarrow H_1$ be a nonlinear mapping with $F(W) \neq \emptyset$.

(i) W is called *firmly quasi-nonexpansive*, if

$$\begin{aligned} \|Wx - z\|^2 \le \|x - z\|^2 - \|(I-W)\,x\|^2, \\ \forall\,(x,z) \in H_1 \times F\,(W). \end{aligned} \tag{11}$$

(ii) W is called *quasi-nonexpansive*, if

$$\|Wx - z\| \le \|x - z\|, \quad \forall\,(x,z) \in H_1 \times F\,(W). \tag{12}$$

(iii) W is called *k-demicontractive*, if there exists $k < 1$ such that

$$\begin{aligned} \|Wx - z\|^2 \le \|x - z\|^2 + k\,\|(I-W)\,x\|^2, \\ \forall\,(x,z) \in H_1 \times F\,(W). \end{aligned} \tag{13}$$

Note that the class of strictly pseudo-contractive mappings properly includes the class of nonexpansive mappings, while the class of nonexpansive mappings properly includes the class of firmly nonexpansive mappings. And the class of demicontractive mappings properly includes the class of quasi-nonexpansive mappings, while the class of quasi-nonexpansive mappings properly includes the class of firmly quasi-nonexpansive mappings.

Definition 3 (demiclosedness property). Let $\{x_n\}$ be a sequence in H_1 and $W : H_1 \rightarrow H_1$ be a mapping. Then W is said to have demiclosedness property if the following implication holds:

$$\left.\begin{aligned} (I-W)\,x_n \longrightarrow 0 \\ x_n \rightharpoonup x \end{aligned}\right] \Longrightarrow x \in F\,(W). \tag{14}$$

It is known that strictly pseudo-contractive mappings possess the demiclosedness property [20]. In particular, both nonexpansive and firmly nonexpansive mappings possess such a property.

Lemma 4 (see [20]). *Let $W : H_1 \rightarrow H_1$ be a k-strictly pseudo-contractive mapping. Then W is demicontractive and Lipschitz continuous and moreover has the demiclosedness property.*

The metric projection P_C from H_1 onto a nonempty closed convex subset $C \subseteq H_1$ is defined by

$$P_C x := \operatorname*{argmin}_{y \in C} \left\|x - y\right\|, \tag{15}$$

which is characterized by

$$\left\langle x - P_C x, z - P_C x\right\rangle \le 0, \quad \forall z \in C. \tag{16}$$

It is well known that the metric projection is firmly nonexpansive.

Definition 5. Let C be a nonempty closed convex subset in H_1.

(i) A sequence $\{x_n\}$ in H_1 is Fejér-monotone with respect to C if

$$\|x_{n+1} - z\| \leq \|x_n - z\|, \quad \forall n \geq 0,\ \forall z \in C. \tag{17}$$

(ii) A sequence $\{x_n\}$ in H_1 is quasi Fejér-monotone with respect to C if

$$\|x_{n+1} - z\|^2 \leq \|x_n - z\|^2 + \varepsilon_n, \quad \forall n \geq 0,\ \forall z \in C, \tag{18}$$

where $\{\varepsilon_n\} \subseteq (0, +\infty)$ satisfies $\sum_{n=1}^{\infty} \varepsilon_n < \infty$.

Lemma 6 (see [21]). *A quasi Fejér-monotone sequence $\{x_n\}$ (with respect to C) is weakly convergent to $z \in C$ if and only if every weak cluster point of $\{x_n\}$ belongs to C.*

Lemma 7 (see [22]). *Let $\{\varepsilon_n\}$ and $\{s_n\}$ be positive real sequences such that $\sum_{n=0}^{\infty} \varepsilon_n < \infty$. If $s_{n+1} \leq (1 + \varepsilon_n)s_n$, or $s_{n+1} \leq s_n + \varepsilon_n$, then the limit of the sequence $\{s_n\}$ exists.*

3. The Case for Demicontractive Continuous Mappings

In this section, we consider the SCFP (P1) for demicontractive continuous mappings. Under this situation, we shall prove that the sequence $\{x_n\}$ generated by (3) and (5)-(6) still converges weakly to a solution of problem (P1).

Lemma 8. *Let $k_1, k_2 \in (-\infty, 1)$, $L_1, L_2 \in (0, +\infty)$, and $W = (I - U) + A^*(I - T)A$, where U, T, and A are mappings defined in* (P1). *Assume that U is k_1-demicontractive and L_1-Lipschitz continuous, T is k_2-demicontractive and L_2-Lipschitz continuous, and $I - U$ and $I - T$ are demiclosed at the origin. For any $(x, z) \in H_1 \times S$, we have the following:*

(i) $S = W^{-1}(0)$.

(ii) $2\langle Wx, x - z\rangle \geq \min((1 - k_1)/2, (1 - k_2)/2\|A\|^2)\|Wx\|^2$.

(iii) *W is L-Lipschitz continuous with $L = \max((L_1 + 1), (L_2 + 1)\|A\|^2)$.*

(iv) *If $\|Wx_n\| \to 0$ and $x_n \rightharpoonup x$ as $n \to \infty$, then $x \in S$.*

Proof. (i) It is readily seen that $S \subseteq W^{-1}(0)$. To see the converse, let $x \in W^{-1}(0)$ and fix any $z \in S$. Since U and T are demicontractive, we have

$$\begin{aligned} \frac{1 - k_1}{2} \|(I - U)x\|^2 &\leq \langle (I - U)x, x - z\rangle, \\ \frac{1 - k_2}{2} \|(I - T)Ax\|^2 &\leq \langle (I - T)Ax, Ax - Az\rangle. \end{aligned} \tag{19}$$

Adding up these two inequalities, we have

$$\begin{aligned} \langle Wx, x - z\rangle \geq{}& \frac{1 - k_1}{2} \|(I - U)x\|^2 \\ &+ \frac{1 - k_2}{2} \|(I - T)Ax\|^2, \end{aligned} \tag{20}$$

which yields $\|(I - U)x\| = \|(I - T)Ax\| = 0$, that is, $x \in S$. This implies $S \supseteq W^{-1}(0)$.

(ii) Let $(x, z) \in H_1 \times S$. It follows from (20) that

$$\begin{aligned} \langle Wx, x - z\rangle &\geq \frac{1 - k_1}{2} \|(I - U)x\|^2 + \frac{1 - k_2}{2} \|(I - T) \\ &\cdot Ax\|^2 = \frac{1 - k_1}{2} \|(I - U)x\|^2 + \frac{1 - k_2}{2\|A\|^2} \|A^*\|^2 \\ &\cdot \|(I - T)Ax\|^2 \geq \frac{1 - k_1}{2} \|(I - U)x\|^2 \\ &+ \frac{1 - k_2}{2\|A\|^2} \|A^*(I - T)Ax\|^2 \\ &\geq \min\left(\frac{1 - k_1}{2}, \frac{1 - k_2}{2\|A\|^2}\right) \\ &\cdot \left(\|(I - U)x\|^2 + \|A^*(I - T)Ax\|^2\right) \\ &\geq \min\left(\frac{1 - k_1}{4}, \frac{1 - k_2}{4\|A\|^2}\right) \\ &\cdot \|((I - U) + A^*(I - T)A)x\|^2, \end{aligned} \tag{21}$$

which yields the desired inequality.

(iii) Let $x, y \in H_1$. We have

$$\begin{aligned} &\|Wx - Wy\| \\ &\quad \leq \|(I - U)(x - y)\| + \|A^*(I - T)A(x - y)\| \\ &\quad \leq (L_1 + 1)\|x - y\| + \|A\| \|(I - T)A(x - y)\| \\ &\quad \leq (L_1 + 1)\|x - y\| + (L_2 + 1)\|A\|^2 \|x - y\| \\ &\quad \leq \max\left((L_1 + 1), (L_2 + 1)\|A\|^2\right) \|x - y\|. \end{aligned} \tag{22}$$

(iv) We note that $\{x_n\}$ is bounded by its weak convergence. By inequality (20), we have

$$\begin{aligned} &\frac{1 - k_1}{2} \|(I - U)x_n\|^2 + \frac{1 - k_2}{2} \|(I - T)Ax_n\|^2 \\ &\quad \leq \langle Wx_n, x_n - z\rangle \leq \|Wx_n\| \|x_n - z\| \longrightarrow \\ &\quad 0, \end{aligned} \tag{23}$$

which implies that

$$\lim_{n\to\infty} \|(I - U)x_n\| = \lim_{n\to\infty} \|(I - T)Ax_n\| = 0. \tag{24}$$

Since $x_n \rightharpoonup x$, this by the demiclosedness property implies $x \in F(U)$. On the other hand, for any $y \in H_2$, we have

$$\begin{aligned} \lim_{n\to\infty} \langle Ax_n, y\rangle &= \lim_{n\to\infty} \langle x_n, A^* y\rangle = \langle x, A^* y\rangle \\ &= \langle Ax, y\rangle. \end{aligned} \tag{25}$$

Hence $Ax_n \rightharpoonup Ax$, which yields $Ax \in F(T)$. Altogether, $x \in S$. □

Theorem 9. *Let $k_1, k_2 \in (-\infty, 1), L_1, L_2 \in (0, +\infty)$. Assume that U is k_1-demicontractive and L_1-Lipschitz continuous, T is k_2-demicontractive and L_2-Lipschitz continuous, and $I-U$ and $I-T$ are demiclosed at the origin. If condition (6) holds, then the sequence $\{x_n\}$, generated by (3) and (5), converges weakly to a solution of problem* (P1).

Proof. Let $z \in S$, $\tau = \min((1-k_1)/2, (1-k_2)/2\|A\|^2)$, and $L = \max((L_1+1), (L_2+1)\|A\|^2)$. It then follows from Lemma 8 that

$$\begin{aligned}\|x_{n+1}-z\|^2 &= \|x_n - z\|^2 - 2\rho_n \langle Wx_n, x_n - z\rangle \\ &\quad + \rho_n^2 \|Wx_n\|^2 \\ &\le \|x_n - z\|^2 - \tau\rho_n \|Wx_n\|^2 + \rho_n^2 \|Wx_n\|^2 .\end{aligned} \tag{26}$$

By (5), we have

$$\|x_{n+1}-z\|^2 \le \|x_n - z\|^2 - \tau\tau_n \|Wx_n\| + \tau_n^2; \tag{27}$$

in particular,

$$\|x_{n+1}-z\|^2 \le \|x_n - z\|^2 + \tau_n^2. \tag{28}$$

By our hypothesis (6), this implies that $\{x_n\}$ is quasi Fejér-monotone with respect to S.

Next, we deduce from (27) and the boundedness of $\{x_n\}$ (guaranteed by the quasi-Fejér-monotonicity) that

$$\tau_n \|Wx_n\| \le \frac{1}{\tau}\left(\|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \tau_n^2\right). \tag{29}$$

Thus, we have

$$\sum_{n=0}^{\infty} \tau_n \|Wx_n\| < \infty. \tag{30}$$

In view of (6), this implies

$$\varliminf_{n\to\infty} \|Wx_n\| = 0. \tag{31}$$

On the other hand, since

$$\|Wx_{n+1} - Wx_n\| \le L\|x_n - x_{n+1}\| = L\rho_n \|Wx_n\|, \tag{32}$$

then we have

$$\begin{aligned}\|Wx_{n+1}\|^2 &= \|Wx_n\|^2 + \|Wx_{n+1} - Wx_n\|^2 \\ &\quad + 2\langle Wx_n, Wx_{n+1} - Wx_n\rangle \\ &\le \|Wx_n\|^2 + \|Wx_{n+1} - Wx_n\|^2 \\ &\quad + 2\|Wx_n\| \|Wx_{n+1} - Wx_n\| \\ &\le \|Wx_n\|^2 + L^2\rho_n^2 \|Wx_n\|^2 + 2L\rho_n \|Wx_n\|^2 \\ &= \|Wx_n\|^2 + L^2\tau_n^2 + 2L\tau_n \|Wx_n\| .\end{aligned} \tag{33}$$

In light of (30) and (6), we have $\sum_n (L^2\tau_n^2 + 2L\tau_n\|Wx_n\|) < \infty$. By Lemma 7, $\lim_n \|Wx_n\|$ exists, and further we have $\lim_n \|Wx_n\| = 0$ by (31). Hence, by Lemma 8, we conclude that every weak cluster point of $\{x_n\}$ belongs to S.

Finally, we deduce from Lemma 6 that $\{x_n\}$ converges weakly to a solution of problem (P1). □

Corollary 10. *Assume that U and T are both strictly pseudo-contractive mappings. If condition (6) holds, then the sequence $\{x_n\}$, generated by (3) and (5), converges weakly to a solution of problem* (P1).

Proof. It follows from Lemma 4 and Theorem 9. □

Remark 11. It is readily seen that the above result holds true for nonexpansive and firmly nonexpansive mappings. As a result, it extends the results in [19] from nonexpansive mappings to demicontractive continuous mappings.

4. New Methods for the Split Equality Problem

The split equality problem (SEP) is an inverse problem that requests finding

$$(x, y) \in F(U_1) \times F(U_2) \quad \text{s.t. } A_1 x = A_2 y, \tag{P2}$$

where $A_1 : H_1 \to H_3$ and $A_2 : H_2 \to H_3$ are two bounded linear mappings, while $U_1 : H_1 \to H_1$ and $U_2 : H_2 \to H_2$ are two nonlinear mappings. The SEP was first introduced by Moudafi and Al-Shemas [23], and they proposed the following iterative method:

$$\begin{aligned} x_{n+1} &= U_1\left[x_n - \tau_n A^*(Ax_n - By_n)\right], \\ y_{n+1} &= U_2\left[y_n + \tau_n B^*(Ax_n - By_n)\right]. \end{aligned} \tag{34}$$

Under some certain conditions, they proved the weak convergence of the iterative sequence generated by method (34).

Our method is actually motivated by (3), since problem (P2) can be regarded as a special SCFP: find $\mathbf{x} = (x_1, x_2) \in H_1 \times H_2$ such that

$$\begin{aligned} \mathbf{x} &\in F(\mathbf{U}), \\ \mathbf{A}\mathbf{x} &\in F(\mathbf{T}), \end{aligned} \tag{35}$$

where $\mathbf{U}\mathbf{x} = (U_1x_1, U_2x_2)$, $\mathbf{A}\mathbf{x} = A_1x_1 - A_2x_2$, and $\mathbf{T}$ is the projection onto the set $\{0\}$. Motivated by (3), we now propose a new method for solving problem (P2). For an arbitrary initial guess (x_0, y_0), define (x_n, y_n) recursively by

$$\begin{aligned} x_{n+1} &= x_n - \rho_n\left[(I - U_1)x_n + A_1^*(A_1x_n - A_2y_n)\right], \\ y_{n+1} &= y_n - \rho_n\left[(I - U_2)y_n - A_2^*(A_1x_n - A_2y_n)\right], \end{aligned} \tag{36}$$

where $\{\rho_n\}$ is chosen as

$$\rho_n = \frac{\tau_n}{\left(\left\|(I-U_1)x_n + A_1^*(A_1x_n - A_2y_n)\right\|^2 + \left\|(I-U_2)y_n - A_2^*(A_1x_n - A_2y_n)\right\|^2\right)^{1/2}}. \quad (37)$$

In what follows, we will show the SEP amounts to problem (35). Now consider the product space $H_1 \times H_2$, in which the inner product and the norm are, respectively, defined by

$$\langle \mathbf{x}, \mathbf{y}\rangle = \langle x_1, y_1\rangle + \langle x_2, y_2\rangle,$$
$$\|\mathbf{x}\| = \left(\|x_1\|^2 + \|x_2\|^2\right)^{1/2}, \quad (38)$$

where $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$ with $x_1, y_1 \in H_1$, $x_2, y_2 \in H_2$.

Lemma 12. *Let $x_1 \in H_1$, $x_2 \in H_2$, and $A_1 : H_1 \to H_3$ and $A_2 : H_2 \to H_3$ be as in problem* (P2). *Define a mapping $\mathbf{A} : H_1 \times H_2 \to H_3$ by*

$$\mathbf{A}\mathbf{x} = \mathbf{A}(x_1, x_2) = A_1x_1 - A_2x_2. \quad (39)$$

Then we have the following:

(i) *$\mathbf{A}$ is a bounded linear mapping.*

(ii) $\mathbf{A}^*\mathbf{A}\mathbf{x} = (A_1^*(A_1x_1 - A_2x_2), -A_2^*(A_1x_1 - A_2x_2))$.

Proof. (i) Let $\alpha, \beta \in \mathbb{R}$. Since A_1 and A_2 are both linear, we have

$$\begin{aligned}\mathbf{A}(\alpha\mathbf{x} + \beta\mathbf{y}) &= \mathbf{A}(\alpha(x_1, x_2) + \beta(y_1, y_2))\\ &= \mathbf{A}((\alpha x_1 + \beta y_1), (\alpha x_2 + \beta y_2))\\ &= A_1(\alpha x_1 + \beta y_1) - A_2(\alpha x_2 + \beta y_2)\\ &= \alpha(A_1x_1 - A_2x_2) + \beta(A_1y_1 - A_2y_2)\\ &= \alpha\mathbf{A}(x_1, x_2) + \beta\mathbf{A}(y_1, y_2)\\ &= \alpha\mathbf{A}\mathbf{x} + \beta\mathbf{A}\mathbf{y};\end{aligned} \quad (40)$$

on the other hand, we have

$$\begin{aligned}\|\mathbf{A}\mathbf{x}\| &= \|A_1x_1 - A_2x_2\| \le \|A_1\|\|x_1\| + \|A_2\|\|x_2\|\\ &\le \max(\|A_1\|, \|A_2\|)(\|x_1\| + \|x_2\|)\\ &\le \sqrt{2}\max(\|A_1\|, \|A_2\|)\sqrt{\|x_1\|^2 + \|x_2\|^2}\\ &= \sqrt{2}\max(\|A_1\|, \|A_2\|)\|\mathbf{x}\|,\end{aligned} \quad (41)$$

which implies $\|\mathbf{A}\| \le \sqrt{2}\max(\|A_1\|, \|A_2\|)$. Thus $\mathbf{A}$ is linear and bounded.

(ii) For $w \in H_3$, we have

$$\begin{aligned}\langle \mathbf{A}\mathbf{x}, w\rangle &= \langle \mathbf{A}(x_1, x_2), w\rangle = \langle A_1x_1 - A_2x_2, w\rangle\\ &= \langle x_1, A_1^*w\rangle + \langle x_2, -A_2^*w\rangle\\ &= \langle (x_1, x_2), (A_1^*w, -A_2^*w)\rangle\\ &= \langle \mathbf{x}, (A_1^*w, -A_2^*w)\rangle.\end{aligned} \quad (42)$$

This gives $\mathbf{A}^*w = (A_1^*w, -A_2^*w)$, which implies that

$$\begin{aligned}\mathbf{A}^*\mathbf{A}\mathbf{x} &= \mathbf{A}^*(A_1x_1 - A_2x_2)\\ &= (A_1^*(A_1x_1 - A_2x_2), -A_2^*(A_1x_1 - A_2x_2)).\end{aligned} \quad (43)$$

Hence the lemma is proved. □

Lemma 13. *Assume that $U_1 : H_1 \to H_1$ is k_1-demicontractive and L_1-Lipschitz continuous, $U_2 : H_2 \to H_2$ is k_2-demicontractive and L_2-Lipschitz continuous, and $I - U_1$ and $I - U_2$ are demiclosed at the origin. Define a mapping $\mathbf{U} : H_1 \times H_2 \to H_1 \times H_2$ by*

$$\mathbf{U}\mathbf{x} = (U_1x_1, U_2x_2), \quad (44)$$

where $\mathbf{x} = (x_1, x_2)$ is in $H_1 \times H_2$ with $x_1 \in H_1$, $x_2 \in H_2$. Then

(i) *$F(\mathbf{U}) = F(U_1) \times F(U_2)$;*

(ii) *$\mathbf{U}$ is demicontractive and Lipschitz continuous;*

(iii) *$\mathbf{I} - \mathbf{U}$ is demiclosed at the origin.*

Proof. It is easy to check (i). For (ii), fix $z = (z_1, z_2) \in F(\mathbf{U})$. It follows that

$$\begin{aligned}\|\mathbf{U}\mathbf{x} - \mathbf{z}\|^2 &= \|U_1x_1 - z_1\|^2 + \|U_2x_2 - z_2\|^2\\ &= \|x_1 - z_1\|^2 + \|x_2 - z_2\|^2 + k_1\|U_1x_1 - x_1\|^2\\ &\quad + k_2\|U_2x_2 - x_2\|^2\\ &\le \|x_1 - z_1\|^2 + \|x_2 - z_2\|^2\\ &\quad + \max(k_1, k_2)\left(\|U_1x_1 - x_1\|^2 + \|U_2x_2 - x_2\|^2\right)\\ &= \|(x_1, x_2) - (z_1, z_2)\|^2\\ &\quad + \max(k_1, k_2)\|(\mathbf{I} - \mathbf{U})(x_1, x_2)\|^2\\ &= \|\mathbf{x} - \mathbf{z}\|^2 + \max(k_1, k_2)\|(\mathbf{I} - \mathbf{U})\mathbf{x}\|^2,\end{aligned} \quad (45)$$

which implies that $\mathbf{U}$ is demicontractive. On the other hand, we have

$$\begin{aligned}\|\mathbf{U}\mathbf{x} - \mathbf{U}\mathbf{y}\| &= \|(U_1x_1 - U_1y_1, U_2x_2 - U_2y_2)\|\\ &= \left(\|U_1x_1 - U_1y_1\|^2 + \|U_2x_2 - U_2y_2\|^2\right)^{1/2}\\ &\le \left(L_1^2\|x_1 - y_1\|^2 + L_2^2\|x_2 - y_2\|^2\right)^{1/2}\\ &\le \max(L_1, L_2)\left(\|x_1 - y_1\|^2 + \|x_2 - y_2\|^2\right)^{1/2}\\ &= \max(L_1, L_2)\|(x_1, y_1) - (x_2, y_1)\|\\ &= \max(L_1, L_2)\|\mathbf{x} - \mathbf{y}\|,\end{aligned} \quad (46)$$

where $\mathbf{y} = (y_1, y_2)$ is in $H_1 \times H_2$ with $y_1 \in H_1$, $y_2 \in H_2$. This implies that $\mathbf{U}$ is Lipschitz continuous.

To show (iii), let $\{(x_n, y_n)\}$ be a sequence such that it converges weakly to $\{(x, y)\}$ and $(\mathbf{I} - \mathbf{U})(x_n, y_n)$ converges strongly to 0. This implies that $x_n \rightharpoonup x$ and $(I - U_1)x_n \to 0$, which, by the demiclosedness of $I - U_1$, gives $x \in F(U_1)$. Similarly, we have $y \in F(U_2)$, so that $(x, y) \in F(U_1) \times F(U_2) = F(\mathbf{U})$. So the lemma is proved. □

Theorem 14. *Assume that $U_1 : H_1 \to H_1$ and $U_2 : H_2 \to H_2$ are two demicontractive and Lipschitz continuous mappings such that $I - U_1$ and $I - U_2$ are demiclosed at the origin. If condition (6) is fulfilled, then the sequence $\{(x_n, y_n)\}$ generated by (36), (37), and (48) converges weakly to a solution of problem* (P2).

Proof. Let $\mathbf{z}_n = (x_n, y_n)$, $\mathbf{U}$, $\mathbf{A}$ be defined as in the previous lemmas, and $\mathbf{T}$ be the projection onto the set $\{0\}$. Then method (36) can be rewritten as

$$\mathbf{z}_{n+1} = \mathbf{z}_n - \rho_n \left[(\mathbf{I} - \mathbf{U}) \mathbf{z}_n + \mathbf{A}^* (\mathbf{I} - \mathbf{T}) \mathbf{A} \mathbf{z}_n \right], \qquad (47)$$

where

$$\rho_n = \frac{\tau_n}{\left\| (\mathbf{I} - \mathbf{U}) \mathbf{z}_n + \mathbf{A}^* (\mathbf{I} - \mathbf{T}) \mathbf{A} \mathbf{z}_n \right\|}. \qquad (48)$$

Note that $\mathbf{T}$ is firmly nonexpansive. By Lemma 13, all assumptions in Theorem 9 are fulfilled. Hence, by applying Theorem 9, we conclude that $\{\mathbf{z}_n\}$ converges weakly to some $\mathbf{z} = (x, y)$ such that $\mathbf{z} \in F(\mathbf{U}) = F(U_1) \times F(U_2)$ and $\mathbf{A}\mathbf{z} \in F(\mathbf{T}) = \{0\}$, which clearly yields $x \in F(U_1)$, $y \in F(U_2)$, and $A_1 x = A_2 y$. Hence the theorem is proved. □

5. Conclusions

We studied the split common fixed point problem in Hilbert spaces. We proposed a new method for solving such a problem and established a weak convergence theorem whenever the involved mappings are demicontractive and continuous. We also obtained a new method for solving the split equality problem in Hilbert spaces.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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