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Research Article

On the Non-Newtonian Fluid Equation with a Source Term and a Damping Term

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A kind of non-Newtonian fluid equation with a damping term and a source term is considered. After giving a result of the existence, if the diffusion coefficient is degenerate on the boundary, the local stability of the weak solutions is established without any boundary condition. If the diffusion coefficient is degenerate on a part of the boundary, by imposing the homogeneous value condition on the other part of the boundary, the local stability of the weak solutions is proved. Moreover, if the equation is with a damping term, other than the finite propagation property, the results of this paper reveal the essential differences between the non-Newtonian fluid equation and the heat conduction equation in a new way.

1. Introduction

Consider the parabolic equation

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(a(x)\left|\nabla u\right|^{p-2}\nabla u\right) - b(x,t)\left|\nabla u\right|^{2} + f(u,x,t), \quad (x,t)\in\Omega\times(0,T), \tag{1}$$

related to the p–Laplacian, with the initial value

$$u(x,0) = u_0(x), \quad x\in\Omega, \tag{2}$$

and the usual boundary value

$$u(x,t) = 0, \quad (x,t)\in\partial\Omega\times(0,T), \tag{3}$$

where $a(x)\in C(\overline{\Omega})$ and $a(x)\geq 0$, $b(x,t)\in C(\overline{Q_T})$, $f(s,x,t)$ is a continuous function, $\Omega\subset\mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$, $p>1$, $q>0$. The equation comes from a host of applied fields such as the theory of non-Newtonian fluid, the water infiltration through porous media, and the oil combustion process; one can refer to [1–4] and the references therein. For the evolutionary p–Laplacian equation

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(\left|\nabla u\right|^{p-2}\nabla u\right), \quad (x,t)\in\Omega\times(0,T), \tag{4}$$

and with the initial-boundary value conditions (2) and (3), the weak solution is unique and has finite propagation property [3]. However, the damping term and the source term in (1) may change the situation.

Bertsh et. al. [5] and Zhou et. al. [6] had discussed the existence and the properties of the viscosity solutions for the equation

$$u_t = u\triangle u - \gamma\left|\nabla u\right|^{2}, \tag{5}$$

and shown that the uniqueness of the weak solution is not true, where γ is a positive constant. Zhang et. al. [7] had discussed the existence and the properties of the viscosity solution for the equation

$$u_t = \triangle u - g(x)\left|u\right|^{q-1}\left|\nabla u\right|^{2}, \tag{6}$$

and shown that the uniqueness of the weak solution is not true, where $g(x)\geq 0$ and at least there exists a point $x_0\in\Omega$ such that $g(x_0)>0$, $q\geq 1$.

Meanwhile, Jiří Benedikt et. al. [8, 9] had shown that the uniqueness of the solution of the following equation is true

$$u_t = \operatorname{div}\left(\left|\nabla u\right|^{p-2}\nabla u\right) + q(x)\left|u\right|^{\alpha-1}u, \quad (x,t)\in(x,t)\in\Omega\times(0,T), \tag{7}$$

is not true provided that $0 < \alpha < 1$, $q(x) \geq 0$ and at least there exists a point $x_0 \in \Omega$ such that $q(x_0) > 0$.

In this paper, we first assume that

$$a(x) = 0, \quad x \in \partial\Omega, \qquad a(x) > 0, \quad x \in \Omega, \tag{8}$$

and then (1) is degenerate on the boundary. Such a degeneracy may have a substantial influence on the solutions. If one considers the well-posedness of (1), one expects that such a degeneracy may counteract the effects from the damp term and the source term. A typical example is the equation

$$u_t = \operatorname{div}\left(d^{\alpha}(x)|\nabla u|^{p-2}\nabla u\right), \quad (x,t) \in \Omega \times (0,T), \tag{9}$$

which was studied by Yin-Wang [10, 11]. Here, $d(x) = \operatorname{dist}(x, \partial\Omega)$ is the distance function from the boundary and satisfies (8). Yin-Wang showed that, if $\alpha > p-1$, although the weak solution may lack the regularity to be defined and the trace on the boundary and the boundary value condition (3) cannot be imposed in the trace sense, the uniqueness of the weak solution is still true. Moreover, the author had studied the equation

$$u_t = \operatorname{div}\left(a(x)|\nabla u|^{p-2}\nabla u\right), \quad (x,t) \in \Omega \times (0,T), \tag{10}$$

and shown that condition (8) may act as the role as the boundary value condition (3) and ensure the well-posedness of the solutions [12–15].

Coming back to (1). On the one hand, based on the knowledge of (5) and (6), if $a(x) = 1$, $p = 2$, and $b(x) \geq \gamma > 0$ in (1), then the uniqueness of the solution is not true. Accordingly, in this paper, we consider the well-posedness of (1) whether $p > 2$ or $b(x) = 0$ on the boundary $\partial\Omega$. On the other hand, based on the knowledge of [10–15], when $p > 2$ and $a(x)$ satisfies (8), we can expect that the uniqueness of the weak solution to (1) is still true, even if $f(u,x,t) = q(x)|u|^{\alpha-1}u$ as (6).

Definition 1. A function $u(x,t)$ is said to be a weak solution of (1) with the initial value (2), if

$$u \in L^{\infty}(Q_T), \qquad u_t \in L^2(Q_T), \qquad a(x)|\nabla u|^p \in L^1(Q_T), \tag{11}$$

and for any function $\varphi \in C_0^1(Q_T)$,

$$\iint_{Q_T} \left[u_t\varphi(x) + a(x)|\nabla u|^{p-2}\nabla u \cdot \nabla\varphi + b(x,t)|\nabla u|^2\varphi(x) - f(u,x,t)\varphi(x)\right] dxdt = 0. \tag{12}$$

The initial value is satisfied in the sense

$$\lim_{t\longrightarrow 0}\int_{\Omega}\left|u(x,t) - u_0(x)\right| dx = 0. \tag{13}$$

Definition 2. The function $u(x,t)$ is said to be the weak solution of (1) with the initial value (2) and the boundary value condition (3), if u satisfies Definition 1, and the boundary value condition (3) is satisfied in the sense of trace.

Theorem 3. *Let $a(x) \in C(\overline{\Omega})$ satisfy (8), $0 \leq b(x,t) \in C(\overline{Q_T})$, $f(s,x,t)$ be a Lipschitz function,*

$$f(s,x.t) > 0, \quad if\ s < 0. \tag{14}$$

If $p > 4$,

$$0 \leq u_0(x) \in L^{\infty}(\Omega), \qquad a(x)\left|\nabla u_0\right|^p \in L^1(\Omega), \tag{15}$$

and

$$\int_{\Omega}[b(x,t)]^{2p/(p-4)}[a(x)]^{-4/(p-4)}dx \leq c, \tag{16}$$

then (1) with initial value (2) has a nonnegative weak solution. Moreover, if

$$\int_{\Omega}[a(x)]^{-1/(p-1)}dx \leq c, \tag{17}$$

then the initial-boundary value problem (1), (2), (3) has a nonnegative solution in the sense of Definition 2.

Since $a(x) = 0$ when $x \in \partial\Omega$, condition (16) implies that $b(x,t)|_{x\in\partial\Omega} = 0$; hereafter, the constants c may depend on T. We think the existence of the weak solutions can be proved only if $p > 2$, and the condition $p > 4$ is just a makeshift. Also condition (16) may not be necessary, but we are not ready to pay so much attentions to the existence. We will focus on the uniqueness of the weak solution.

Theorem 4. *Let $u(x,t), v(x,t)$ be two weak solutions of (1) with the initial values $u_0(x), v_0(x)$, respectively. If $a(x) \in C(\overline{\Omega})$ satisfies (8), $|b(x,t)| \leq ca(x)$, $f(s,x,t)$ is a continuous function, $p > 2$, then there is a constant $\beta \geq 4$ such that*

$$\int_{\Omega} a(x)^{\beta}|u(x,t) - v(x,t)|dx \leq \int_{\Omega} a(x)^{\beta}\left|u_0(x) - v_0(x)\right|dx, \quad a.e.\ t \in [0,T). \tag{18}$$

Since $a(x)$ satisfies (8), Theorem 4 implies the uniqueness of the weak solution to (1) is true even without the boundary value condition. Moreover, we have the following two simple comments.

(1) Theorem 4 includes the case of $f(u,x,t) = q(x)|u|^{\alpha-1}u$ and $b(x,t) \equiv 0$; in other words, the uniqueness of the weak solution to the following equation

$$u_t = \operatorname{div}\left(a(x)|\nabla u|^{p-2}\nabla u\right) + q(x)|u|^{\alpha-1}u \tag{19}$$

is true, where $p > 2$ and $a(x)|_{x\in\partial\Omega} = 0$.

(2) Theorem 4 includes the case of $f(u,x,t) \equiv 0$ and $b(x,t) \equiv g(x) \geq 0$ and at least there exists a point $x_0 \in \Omega$

such that $g(x_0) > 0$. In other words, the uniqueness of the weak solution to the following equation

$$u_t = \operatorname{div}\left(a(x)|\nabla u|^{p-2}\nabla u\right) - g(x)|\nabla u|^2, \tag{20}$$

is true, where $p > 2$ and $a(x)|_{x\in\partial\Omega} = 0$.

Compared with (6) and (7), Theorem 4 reveals that the degeneracy of $a(x)$ brings the new change about the property of the solutions.

In order to illustrate the problem more clearly, secondly, we assume that

$$\begin{gathered}\partial\Omega = \Sigma_p \cup \Sigma_p', \\ \overline{\Sigma_p} \cap \overline{\Sigma_p'} = \emptyset,\end{gathered} \tag{21}$$

$$a(x) \geq c > 0, \quad x \in \Sigma_p, \tag{22}$$

$$a(x) = 0, \quad x \in \Sigma_p'. \tag{23}$$

In this case, we consider the uniqueness of weak solution to (1) under a partial boundary value condition. This is the following theorem.

Theorem 5. *Let $\partial\Omega$ satisfy (21) and $u(x,t)$, $v(x,t)$ be two weak solutions of (1) with the initial values $u_0(x)$, $v_0(x)$, respectively, with the same partial boundary value condition*

$$u(x,t) = v(x,t) = 0, \quad x \in \Sigma_p. \tag{24}$$

If $p > 2$, $b(x,t) \in C(\overline{Q_T})$, $a(x) \in C(\overline{\Omega})$ satisfies (17), (22), and (23), and $f(s,x,t)$ is a continuous function, then there is a constant $\beta > 1$ such that the local stability is true in the sense of (18).

If we notice that $a(x)$ satisfies (22), according to [5–7], the uniqueness of the solution to the equation

$$u_t = \Delta u - g(x)|\nabla u|^2 \tag{25}$$

is not true when $x \in \Omega$ is near to Σ_p, while Theorem 5 implies that the uniqueness of the solution to the equation

$$u_t = \operatorname{div}\left(a(x)|\nabla u|^{p-2}\nabla u\right) - g(x)|\nabla u|^2, \tag{26}$$

is true provided that $p > 2$. This fact shows the differences between the heat conduction equation ($p = 2$) and the non-Newtonian fluid equation ($p > 2$) again. It is well-known that the heat conduction equation has the infinite propagation property, while the non-Newtonian fluid equation has the finite propagation property.

2. The Weak Solutions Depend on the Initial Value

It is supposed that u_0 satisfies

$$\begin{gathered}0 \leq u_0(x) \in L^\infty(\Omega), \\ a(x)|\nabla u_0|^p \in L^1(\Omega).\end{gathered} \tag{27}$$

Let $u_{\varepsilon,0}(x) \in C_0^\infty(\Omega)$ and $a(x)|\nabla u_{\varepsilon,0}(x)|^p \in L^1(\Omega)$ be uniformly bounded, and let $a(x)u_{\varepsilon,0}(x)$ converge to $a(x)u_0(x)$ in $W_0^{1,p}(\Omega)$. For simplicity, we may assume that $f(s,x,t)$ is a C^1 function without loss the generality.

We now consider the following regularized problem

$$\begin{aligned}&u_{\varepsilon t} - \operatorname{div}\left((a(x)+\varepsilon)\left(|\nabla u_\varepsilon|^2+\varepsilon\right)^{(p-2)/2}\nabla u_\varepsilon\right) \\ &\quad + b(x,t)|\nabla u_\varepsilon|^2 = f(u_\varepsilon, x, t), \quad (x,t) \in Q_T,\end{aligned} \tag{28}$$

$$u_\varepsilon(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{29}$$

$$u_\varepsilon(x,0) = u_{\varepsilon,0}(x), \quad x \in \Omega. \tag{30}$$

Since $f(s,x,t)$ satisfies (4), it is well-known that the above problem has a unique nonnegative classical solution [3, 16].

By the maximum principle, we have

$$0 \leq u_\varepsilon \leqslant c. \tag{31}$$

Multiplying (28) by u_ε and integrating it over Q_T, we get

$$\begin{aligned}&\frac{1}{2}\int_\Omega u_\varepsilon^2 \mathrm{d}x \\ &\quad + \iint_{Q_T}(a(x)+\varepsilon)\left(|\nabla u_\varepsilon|^2+\varepsilon\right)^{(p-2)/2}|\nabla u_\varepsilon|^2 \mathrm{d}x\mathrm{d}t \\ &\quad + \iint_{Q_T} b(x,t)|\nabla u_\varepsilon|^2 u_\varepsilon dxdt \\ &= \frac{1}{2}\int_\Omega u_0^2 \mathrm{d}x + \iint_{Q_T} f(u_\varepsilon, x, t)u_\varepsilon.\end{aligned} \tag{32}$$

Since $b(x,t) \geq 0$,

$$\iint_{Q_T} b(x,t)|\nabla u_\varepsilon|^2 u_\varepsilon dxdt \geq 0, \tag{33}$$

and by $|f(u_\varepsilon, x, t)u_\varepsilon| \leq c$, we have

$$\begin{aligned}&\frac{1}{2}\int_\Omega u_\varepsilon^2 \mathrm{d}x \\ &\quad + \iint_{Q_T}(a(x)+\varepsilon)\left(|\nabla u_\varepsilon|^2+\varepsilon\right)^{(p-2)/2}|\nabla u_\varepsilon|^2 \mathrm{d}x\mathrm{d}t \\ &\leqslant c,\end{aligned} \tag{34}$$

and

$$\begin{aligned}&\iint_{Q_T} a(x)|\nabla u_\varepsilon|^p dxdt \\ &\quad \leqslant c\iint_{Q_T}(a(x)+\varepsilon)|\nabla u_\varepsilon|^p dxdt \leqslant c.\end{aligned} \tag{35}$$

Multiplying (28) by $\partial u_\varepsilon/\partial t$, integrating it over Q_T,

$$\begin{aligned}&\iint_{Q_T} u_{\varepsilon t}\frac{\partial u_\varepsilon}{\partial t}dxdt\\&=\iint_{Q_T}\operatorname{div}\left((a(x)+\varepsilon)\left(|\nabla u|^2+\varepsilon\right)^{(p-2)/2}\nabla u_\varepsilon\right)\\&\cdot\frac{\partial u_\varepsilon}{\partial t}dxdt-\iint_{Q_T} b(x,t)\left|\nabla u_\varepsilon\right|^2\frac{\partial u_\varepsilon}{\partial t}dxdt\\&+\iint_{Q_t}\frac{\partial u_\varepsilon}{\partial t}f(u_\varepsilon,x,t)\,dxdt.\end{aligned}\tag{36}$$

Noticing that

$$\begin{aligned}&\left(\left|\nabla u_\varepsilon\right|^2+\varepsilon\right)^{(p-2)/2}\nabla u_\varepsilon\cdot\nabla\frac{\partial u_\varepsilon}{\partial t}\\&=\frac{1}{2}\frac{d}{dt}\int_0^{|\nabla u_\varepsilon(x,t)|^2+\varepsilon}s^{(p-2)/2}ds,\end{aligned}\tag{37}$$

we have

$$\begin{aligned}&\iint_{Q_T}\operatorname{div}\left((a(x)+\varepsilon)\left(\left|\nabla u_\varepsilon\right|^2+\varepsilon\right)^{(p-2)/2}\nabla u_\varepsilon\frac{\partial u_\varepsilon}{\partial t}\right)dxdt\\&=-\iint_{Q_T}(a(x)+\varepsilon)\left(\left|\nabla u_\varepsilon\right|^2+\varepsilon\right)^{(p-2)/2}\nabla u_\varepsilon\nabla\frac{\partial u_\varepsilon}{\partial t}dxdt\\&=-\frac{1}{2}\iint_{Q_T}(a(x)+\varepsilon)\frac{d}{dt}\int_0^{|\nabla u_\varepsilon(x,t)|^2+\varepsilon}s^{(p-2)/2}dsdxdt\\&=-\frac{1}{2}\int_\Omega(a(x)+\varepsilon)\int_0^{|\nabla u_\varepsilon(x,t)|^2+\varepsilon}s^{(p-2)/2}dsdx\\&+\frac{1}{2}\int_\Omega(a(x)+\varepsilon)\int_0^{|\nabla u_\varepsilon(x,0)|^2+\varepsilon}s^{(p-2)/2}dsdx.\end{aligned}\tag{38}$$

Moreover, by the Young inequality and the Hölder inequality,

$$\begin{aligned}&\iint_{Q_T}\frac{\partial u_\varepsilon}{\partial t}b(x,t)\left|\nabla u_\varepsilon\right|^2dxdt\le\frac{3}{4}\\&\cdot\iint_{Q_T}\left[b(x,t)\left|\nabla u_\varepsilon\right|^2\right]^2dxdt+\frac{1}{4}\iint_{Q_T}\left|\frac{\partial u_\varepsilon}{\partial t}\right|^2dxdt\\&\le\frac{1}{4}\iint_{Q_T}\left|\frac{\partial u_\varepsilon}{\partial t}\right|^2dxdt\\&+c\left(\iint_{Q_T}\left[b(x,t)^2a(x)^{-4/p}\right]^{p/(p-4)}dxdt\right)^{(p-4)/p}\\&\cdot\left(\iint_{Q_T}a(x)\left|\nabla u_\varepsilon\right|^pdxdt\right)^{4/p}\le\frac{1}{4}\\&\cdot\iint_{Q_T}\left|\frac{\partial u_\varepsilon}{\partial t}\right|^2dxdt+c.\end{aligned}\tag{39}$$

$$\left|\iint_{Q_T}\frac{\partial u_\varepsilon}{\partial t}f(u_\varepsilon,x,t)\,dxdt\right|\le c+\frac{1}{4}\iint_{Q_T}\left|\frac{\partial u_\varepsilon}{\partial t}\right|^2dxdt.\tag{40}$$

By (36), (38), (39), and (40),

$$\iint_{Q_T}\left|\frac{\partial u_\varepsilon}{\partial t}\right|^2dxdt\le c.\tag{41}$$

By (35), (41), we know u_ε can be embedded into $L^2_{loc}(Q_T)$ compactly. Then $u_\varepsilon\longrightarrow u$ a.e. in Q_T. At the same time, since $p>4$,

$$\begin{aligned}&\iint_{Q_T}|b(x,t)|\left|\nabla u_\varepsilon\right|^2dxdt\le c\left(\iint_{Q_T}a(x)\left|\nabla u_\varepsilon\right|^pdxdt\right)^{2/p}\\&\cdot\left(\iint_{Q_T}\left|b(x,t)a(x)^{-2/p}\right|^{p/(p-2)}dxdt\right)^{(p-2)/p}\\&\le c\left(\iint_{Q_T}\left|b(x,t)a(x)^{-2/p}\right|^{p/(p-2)}dxdt\right)^{(p-2)/p}\\&=c\left(\iint_{Q_T}|b(x,t)|^{p/(p-2)}a(x)^{-2/(p-2)}dxdt\right)^{(p-2)/p}\\&\le c\left(\iint_{Q_T}\left|b(x,t)^{p/(p-2)}\right.\right.\\&\left.\left.\cdot a(x)^{-2/(p-2)}\right|^{((p-2)/2)(4/(p-4))}dxdt\right)^{((p-2)/p)(2(p-4)/4(p-2))}\\&=c\left(\iint_{Q_T}\left|b(x,t)^{2p/(p-4)}a(x)^{-4/(p-4)}\right|dxdt\right)^{(p-4)/2p}\le c.\end{aligned}\tag{42}$$

Hence, by (31), (35), (40), (41), (42) there exists a function u, n–dimensional vector function $\vec{\zeta}=(\zeta_1,\ldots,\zeta_n)$, and a function $\nu\in L^2(Q_T)$ such that

$$\begin{aligned}&u\in L^\infty(Q_T),\\&\left|\vec{\zeta}\right|\in L^{p/(p-1)}(Q_T),\end{aligned}\tag{43}$$

and

$$\begin{aligned}&u_\varepsilon\rightharpoonup *u,\quad\text{in } L^\infty(Q_T),\\&u_\varepsilon\longrightarrow u,\quad\text{a.e. in } Q_T.\\&a(x)\left|\nabla u_\varepsilon\right|^{p-2}\nabla u_\varepsilon\rightharpoonup\vec{\zeta}\quad\text{in } L^{p/(p-1)}(Q_T).\\&b(x,t)\left|\nabla u_\varepsilon\right|^2\rightharpoonup\nu,\quad\text{in } \mathfrak{M}(Q_T).\end{aligned}\tag{44}$$

Here, $\mathfrak{M}(Q_T)$ is the signed Radon measures on Q_T. In order to prove that u satisfies (1), we notice that for any function $\varphi\in C_0^1(Q_T)$,

$$\begin{aligned}&\iint_{Q_T}\left[\frac{\partial u_\varepsilon}{\partial t}\varphi+(a(x)+\varepsilon)\left(\left|\nabla u_\varepsilon\right|^2+\varepsilon\right)^{(p-2)/2}\nabla u_\varepsilon\cdot\nabla\varphi\right.\\&\left.+b(x,t)\left|\nabla u_\varepsilon\right|^2\right]\mathrm{d}x\mathrm{d}t\\&=\iint_{Q_T}f(u_\varepsilon,x,t)\,\varphi dxdt.\end{aligned}\tag{45}$$

Now, in the first place, we can prove

$$\begin{aligned}
&\lim_{\varepsilon\longrightarrow 0}(a(x)+\varepsilon)\left(\left|\nabla u_\varepsilon\right|^2+\varepsilon\right)^{(p-2)/2}\nabla u_\varepsilon\cdot\nabla\varphi dxdt\\
&\quad=\lim_{\varepsilon\longrightarrow 0}\iint_{Q_T}a(x)\left|\nabla u_\varepsilon\right|^{p-2}\nabla u_\varepsilon\cdot\nabla\varphi dxdt\\
&\quad=\iint_{Q_T}a(x)\left|\nabla u\right|^{p-2}\nabla u\cdot\nabla\varphi \mathrm{d}x\mathrm{d}t\\
&\quad=\iint_{Q_T}\overrightarrow{\zeta}\cdot\nabla\varphi \mathrm{d}x\mathrm{d}t,
\end{aligned}\tag{46}$$

in a similar way to that of the usual p–Laplacian equation. Then, letting $\varepsilon\longrightarrow 0$ in (45),

$$\begin{aligned}
&\iint_{Q_T}\left(\frac{\partial u}{\partial t}\varphi+a(x)\left|\nabla u\right|^{p-2}\nabla u\cdot\nabla\varphi+\nu\varphi\right)\mathrm{d}x\mathrm{d}t\\
&\quad=\iint_{Q_T}f(u,x,t)\varphi dxdt.
\end{aligned}\tag{47}$$

What is more, by the weak convergent theorem, for any $\varphi\in C_0^1(Q_T)$,

$$\begin{aligned}
&\lim_{\varepsilon\longrightarrow 0}\iint_{Q_T}a(x)\left|\nabla u_\varepsilon\right|^{p-2}\nabla u_\varepsilon\cdot\nabla u\varphi dxdt\\
&\quad=\iint_{Q_T}a(x)\left|\nabla u\right|^{p}\varphi dxdt.
\end{aligned}\tag{48}$$

For any $0\le\varphi\in C_0^1(Q_T)$,

$$\begin{aligned}
&\lim_{\varepsilon\longrightarrow 0}\iint_{Q_T}a(x)\left(\left|\nabla u_\varepsilon\right|^{p-2}\nabla u_\varepsilon-\left|\nabla u\right|^{p-2}\nabla u\right)\\
&\quad\cdot\nabla u_\varepsilon\varphi dxdt=\lim_{\varepsilon\longrightarrow 0}\iint_{Q_T}a(x)\\
&\quad\cdot\left(\left|\nabla u_\varepsilon\right|^{p-2}\nabla u_\varepsilon-\left|\nabla u\right|^{p-2}\nabla u\right)(\nabla u_\varepsilon-\nabla u)\varphi dxdt\\
&\quad+\lim_{\varepsilon\longrightarrow 0}\iint_{Q_T}a(x)\\
&\quad\cdot\left(\left|\nabla u_\varepsilon\right|^{p-2}\nabla u_\varepsilon-\left|\nabla u\right|^{p-2}\nabla u\right)\nabla u\varphi dxdt\\
&\quad\ge\lim_{\varepsilon\longrightarrow 0}\iint_{Q_T}a(x)\\
&\quad\cdot\left(\left|\nabla u_\varepsilon\right|^{p-2}\nabla u_\varepsilon-\left|\nabla u\right|^{p-2}\nabla u\right)\nabla u\varphi dxdt=0.
\end{aligned}\tag{49}$$

By (48),

$$\begin{aligned}
&\lim_{\varepsilon\longrightarrow 0}\iint_{Q_T}a(x)\left(\left|\nabla u\right|^{p-2}\nabla u-\left|\nabla u_\varepsilon\right|^{p-2}\nabla u_\varepsilon\right)\\
&\quad\cdot\nabla u_\varepsilon\varphi dxdt=\lim_{\varepsilon\longrightarrow 0}\iint_{Q_T}a(x)\\
&\quad\cdot\left(\left|\nabla u\right|^{p-2}\nabla u-\left|\nabla u_\varepsilon\right|^{p-2}\nabla u_\varepsilon\right)(\nabla u-\nabla u_\varepsilon)\varphi dxdt\\
&\quad+\lim_{\varepsilon\longrightarrow 0}\iint_{Q_T}a(x)\\
&\quad\cdot\left(\left|\nabla u\right|^{p-2}\nabla u-\left|\nabla u_\varepsilon\right|^{p-2}\nabla u_\varepsilon\right)\nabla u\varphi dxdt\\
&\quad\ge\lim_{\varepsilon\longrightarrow 0}\iint_{Q_T}a(x)\\
&\quad\cdot\left(\left|\nabla u\right|^{p-2}\nabla u-\left|\nabla u_\varepsilon\right|^{p-2}\nabla u_\varepsilon\right)\nabla u\varphi dxdt=0.
\end{aligned}\tag{50}$$

Combining (49) with (50), we have

$$\begin{aligned}
&\lim_{\varepsilon\longrightarrow 0}\iint_{Q_T}a(x)\left(\left|\nabla u_\varepsilon\right|^{p-2}\nabla u_\varepsilon-\left|\nabla u\right|^{p-2}\nabla u\right)\\
&\quad\cdot\nabla u_\varepsilon\varphi dxdt=0,
\end{aligned}\tag{51}$$

for any $0\le\varphi\in C_0^1(Q_T)$. Clearly, for any $\varphi\in C_0^1(Q_T)$, (50) is still true.

Then, by (48) and (51),

$$\begin{aligned}
&\lim_{\varepsilon\longrightarrow 0}\iint_{Q_T}a(x)\left(\left|\nabla u_\varepsilon\right|^{p}-\left|\nabla u\right|^{p}\right)\varphi dxdt\\
&\quad=\lim_{\varepsilon\longrightarrow 0}\iint_{Q_T}a(x)\\
&\quad\cdot\left(\left|\nabla u_\varepsilon\right|^{p-2}\nabla u_\varepsilon-\left|\nabla u\right|^{p-2}\nabla u\right)\nabla u_\varepsilon\varphi dxdt\\
&\quad+\lim_{\varepsilon\longrightarrow 0}\iint_{Q_T}a(x)\left|\nabla u\right|^{p-2}\nabla u\cdot\nabla u_\varepsilon\varphi dxdt\\
&\quad-\iint_{Q_T}a(x)\left|\nabla u\right|^{p}\varphi dxdt=0.
\end{aligned}\tag{52}$$

By (52), by the arbitrary of $0\le\varphi\in C_0^1(Q_T)$, we know that

$$\lim_{\varepsilon\longrightarrow 0}\iint_{Q_T}a(x)\left|\nabla u_\varepsilon\right|^{p}dxdt=\iint_{Q_T}a(x)\left|\nabla u\right|^{p}dxdt.\tag{53}$$

Thus $\nabla u_\varepsilon\longrightarrow\nabla u$ *a.e.* in Q_T.

Since $p>4$, by (53), for any function $\varphi\in C_0^1(Q_T)$

$$\lim_{\varepsilon\longrightarrow 0}\iint_{Q_T}b(x,t)\left(\left|\nabla u_\varepsilon\right|^{2}-\left|\nabla u\right|^{2}\right)\varphi dxdt=0\tag{54}$$

is clearly. Then

$$\iint_{Q_T}b(x,t)\left|\nabla u\right|^{2}\varphi dxdt=\iint_{Q_T}\nu\varphi dxdt,\tag{55}$$

for any function $\varphi\in C_0^1(Q_T)$. Combining (46) with (55), u satisfies (7).

At last, we are able to prove (13) as in [17]; thus we have Theorem 3.

3. The Proof of Theorem 4

For small $\eta > 0$, let

$$S_\eta(s) = \int_0^s h_\eta(\tau)\,d\tau, \qquad h_\eta(s) = \frac{2}{\eta}\left(1 - \frac{|s|}{\eta}\right)_+. \tag{56}$$

Obviously $h_\eta(s) \in C(\mathbb{R})$, and

$$\begin{gathered} h_\eta(s) \geq 0, \\ \left|s h_\eta(s)\right| \leq 1, \\ \left|S_\eta(s)\right| \leq 1; \\ \lim_{\eta\longrightarrow 0} S_\eta(s) = \operatorname{sgn} s, \\ \lim_{\eta\longrightarrow 0} s S_\eta'(s) = 0. \end{gathered} \tag{57}$$

Theorem 6. *Let $u(x,t), v(x,t)$ be two weak solutions of (1) with the initial values $u_0(x), v_0(x)$, respectively. If $a(x) \in C(\overline{\Omega})$ satisfies (8), $b(x,t) \in C(\overline{Q_T})$, there is a constant $\beta > 1$ such that one of the following conditions is true:*

(i) $p > 4$;

$$\int_\Omega [a(x)]^{2p(\beta-4)/(p-4)} |b(x,t)|^{2p/(p-4)}\,dx \leq c, \tag{58}$$

(ii) $2 < p \leq 4$,

$$[a(x)]^{2(\beta-1)/(p-2)} |b(x,t)|^{p/(p-2)} \leq c, \tag{59}$$

(ii) $1 < p \leq 2$, *and*

$$\begin{gathered} \left|[a(x)]^{\beta/2} b(x,t)\right| |\nabla u|^2\,dx \leq c, \\ \left|[a(x)]^{\beta/2} b(x,t)\right| |\nabla v|^2\,dx \leq c; \end{gathered} \tag{60}$$

then

$$\begin{aligned} &\int_\Omega a(x)^\beta |u(x,t) - v(x,t)|^2\,dx \\ &\quad \leq \int_\Omega a(x)^\beta \left|u_0(x) - v_0(x)\right|^2 dx, \quad a.e.\ t \in [0,T). \end{aligned} \tag{61}$$

Proof. Let $u(x,t), v(x,t)$ be two solutions of (1) with the initial values $u_0(x), v_0(x)$. We can choose $a(x)^\beta(u-v)$ as the test function. Then

$$\begin{aligned} &\int_\Omega a(x)^\beta (u-v) \frac{\partial(u-v)}{\partial t} dx \\ &\quad + \int_\Omega a(x)^{\beta+1} \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right) \\ &\quad \cdot \nabla(u-v)\,dx \\ &\quad + \int_\Omega a(x) \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right) \\ &\quad \cdot \nabla a^\beta (u-v)\,dx \\ &\quad + \int_\Omega b(x,t)\left(|\nabla u|^2 - |\nabla v|^2\right) a(x)^\beta (u-v)\,dx \\ &\quad = \int_\Omega \left[f(u,x,t) - f(v,x,t)\right] a(x)^\beta (u-v)\,dx. \end{aligned} \tag{62}$$

Thus

$$\begin{aligned} &\int_\Omega a(x)^\beta (u-v) \frac{\partial(u-v)}{\partial t} dx \\ &\quad = \frac{1}{2}\frac{d}{dt}\int_\Omega a(x)^\beta |u(x,t) - v(x,t)|^2\,dx, \end{aligned} \tag{63}$$

$$\begin{aligned} &\int_\Omega a(x)^{\beta+1} \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right) \cdot \nabla(u-v)\,dx \\ &\quad \geq 0. \end{aligned} \tag{64}$$

By that $|\nabla a(x)| \leq c$ in Ω, we have

$$\begin{aligned} &\left|\int_\Omega a(x)\left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right) \cdot \nabla a^\beta (u-v)\,dx\right| \\ &\quad \leq \beta\left(\int_\Omega \left|a(x)^{1/p+\beta-1}\nabla a(u-v)\right|^p dx\right)^{1/p} \\ &\quad \cdot \left(\int_\Omega a(x)\left(|\nabla u|^p + |\nabla v|^p\right) dx\right)^{(p-1)/p} \\ &\quad \leq c\left(\int_\Omega a(x)^{p(\beta-1)+1} |u-v|^p\,dx\right)^{1/p}. \end{aligned} \tag{65}$$

If $p > 2$, since $\beta > 1$, $p(\beta-1) + 1 - \beta > 0$, we have

$$\begin{aligned} &\left(\int_\Omega a(x)^{p(\beta-1)+1} |u-v|^p\,dx\right)^{1/p} \\ &\quad = \left(\int_\Omega a(x)^{p(\beta-1)+1-\beta} |u-v|^{p-2} a(x)^\beta \right. \\ &\quad \left. \cdot |u-v|^2\,dx\right)^{1/p} \leq c\left(\int_\Omega a(x)^\beta |u-v|^2\,dx\right)^{1/p}. \end{aligned} \tag{66}$$

If $p = 2$, since $\beta > 1$, $p(\beta-1) + 1 - \beta > 0$, we have

$$\begin{aligned} &\left(\int_\Omega a(x)^{p(\beta-1)+1} |u-v|^p\,dx\right)^{1/p} \\ &\quad \leq c\left(\int_\Omega a(x)^\beta |u-v|^2\,dx\right)^{1/2}. \end{aligned} \tag{67}$$

If $1 < p < 2$

$$\begin{aligned}&\left(\int_\Omega a(x)^{p(\beta-1)+1}|u-v|^p\,dx\right)^{1/p}\\&\quad\le\left(\int_\Omega a(x)^{[p(\beta-1)+1-\beta p/2](2/(2-p))}\,dx\right)^{(2-p)/2p}\\&\quad\cdot\left(\int_\Omega a(x)^\beta|u-v|^2\,dx\right)^{1/2}\\&\quad\le c\left(\int_\Omega a(x)^\beta|u-v|^2\,dx\right)^{1/2},\end{aligned}\tag{68}$$

only if

$$\int_\Omega a(x)^{[p(\beta-1)+1-\beta p/2](2/(2-p))}\,dx<\infty.\tag{69}$$

However, this inequality is natural since $\beta > 1$.

By (65)-(68), we have

$$\begin{aligned}&\left|\int_\Omega a(x)\left(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v\right)\cdot\nabla a^\beta(u-v)\,dx\right|\\&\quad\le c\left(\int_\Omega a(x)^\beta|u-v|^2\,dx\right)^q,\end{aligned}\tag{70}$$

where $q < 1$.

If $4 \ge p > 2$, then $p/(p-2) \ge 2$. Since

$$\begin{aligned}&[a(x)]^{2(\beta-1)/(p-2)}|b(x,t)|^{p/(p-2)}\\&\quad=[a(x)]^{(p\beta-2)/(p-2)-\beta}|b(x,t)|^{p/(p-2)}\le c,\end{aligned}\tag{71}$$

we have

$$\begin{aligned}&\left|\int_\Omega b(x,t)\left(|\nabla u|^2-|\nabla v|^2\right)a(x)^\beta(u-v)\,dx\right|\\&\quad\le\left(\int_\Omega a(x)\left(|\nabla u|^2+|\nabla v|^2\right)^{p/2}dx\right)^{2/p}\\&\quad\cdot\left(\int_\Omega\left|[a(x)]^{\beta-2/p}b(x,t)\right.\right.\\&\quad\left.\left.\cdot(u-v)\right|^{p/(p-2)}dx\right)^{(p-2)/p}\\&\quad\le c\left(\int_\Omega[a(x)]^{(p\beta-2)/(p-2)}|b(x,t)|^{p/(p-2)}|u\right.\\&\quad\left.-v|^{p/(p-2)}dx\right)^{(p-2)/p}\\&\quad\le c\left(\int_\Omega[a(x)]^{(p\beta-2)/(p-2)-(\beta/2)(p/(p-2))}|b(x,t)|^{p/(p-2)}\right.\\&\quad\left.\cdot[a(x)]^{(\beta/2)(p/(p-2))}|u-v|^{p/(p-2)}dx\right)^{(p-2)/p}\\&\quad\le\left(\int_\Omega\left([a(x)]^{(p\beta-2)/(p-2)-(\beta/2)(p/(p-2))}\right.\right.\\&\quad\left.\left.\cdot|b(x,t)|^{p/(p-2)}\right)^{2(p-2)/(p-4)}dx\right)^{(p-4)/2(p-2)}\\&\quad\cdot\left(\int_\Omega[a(x)]^\beta|u-v|^2\,dx\right)^{1/2}\\&\quad\le\left(\int_\Omega\left([a(x)]^{p(\beta-4)/(p-2)}\right.\right.\\&\quad\left.\left.\cdot|b(x,t)|^{p/(p-2)}\right)^{2(p-2)/(p-4)}dx\right)^{(p-4)/2(p-2)}\\&\quad\cdot\left(\int_\Omega[a(x)]^\beta|u-v|^2\,dx\right)^{1/2}\\&\quad\le\left(\int_\Omega[a(x)]^{2p(\beta-4)/(p-4)}\right.\\&\quad\left.\cdot|b(x,t)|^{2p/(p-4)}dx\right)^{(p-4)/2(p-2)}\cdot\left(\int_\Omega[a(x)]^\beta\right.\\&\quad\left.\cdot|u-v|^2\,dx\right)^{1/2}\le c\left(\int_\Omega[a(x)]^\beta|u\right.\\&\quad\left.-v|^2\,dx\right)^{1/2}.\end{aligned}\tag{72}$$

If $p > 4$, then $p/(p-2) < 2$. Since

$$\int_\Omega[a(x)]^{2p(\beta-4)/(p-4)}|b(x,t)|^{2p/(p-4)}\,dx\le c,\tag{73}$$

and we have

$$\begin{aligned}&\left|\int_\Omega b(x,t)\left(|\nabla u|^2-|\nabla v|^2\right)a(x)^\beta(u-v)\,dx\right|\\&\quad\le\left(\int_\Omega a(x)\left(|\nabla u|^2+|\nabla v|^2\right)^{p/2}dx\right)^{2/p}\\&\quad\cdot\left(\int_\Omega\left|[a(x)]^{\beta-2/p}b(x,t)(u-v)\right|^{p/(p-2)}dx\right)^{(p-2)/p}\\&\quad\le c\left(\int_\Omega[a(x)]^{(\beta-2)p/(p-2)}|b(x,t)|^{p/(p-2)}|u\right.\\&\quad\left.-v|^{p/(p-2)}dx\right)^{(p-2)/p}\\&\quad\le c\left(\int_\Omega[a(x)]^{(\beta-2)p/(p-2)-(\beta/2)(p/(p-2))}|b(x,t)|^{p/(p-2)}\right.\\&\quad\left.\cdot[a(x)]^{(\beta/2)(p/(p-2))}|u-v|^{p/(p-2)}dx\right)^{(p-2)/p}\\&\quad\le\left(\int_\Omega\left([a(x)]^{(\beta-2)p/(p-2)-(\beta/2)(p/(p-2))}\right.\right.\\&\quad\left.\left.\cdot|b(x,t)|^{p/(p-2)}\right)^{2(p-2)/(p-4)}dx\right)^{(p-4)/2(p-2)}\\&\quad\cdot\left(\int_\Omega[a(x)]^\beta|u-v|^2\,dx\right)^{1/2}\\&\quad\le\left(\int_\Omega\left([a(x)]^{p(\beta-4)/(p-2)}\right.\right.\\&\quad\left.\left.\cdot|b(x,t)|^{p/(p-2)}\right)^{2(p-2)/(p-4)}dx\right)^{(p-4)/2(p-2)}\\&\quad\cdot\left(\int_\Omega[a(x)]^\beta|u-v|^2\,dx\right)^{1/2}\\&\quad\le\left(\int_\Omega[a(x)]^{2p(\beta-4)/(p-4)}\right.\\&\quad\left.\cdot|b(x,t)|^{2p/(p-4)}dx\right)^{(p-4)/2(p-2)}\cdot\left(\int_\Omega[a(x)]^\beta\right.\\&\quad\left.\cdot|u-v|^2\,dx\right)^{1/2}\le c\left(\int_\Omega[a(x)]^\beta|u\right.\\&\quad\left.-v|^2\,dx\right)^{1/2}.\end{aligned}\tag{74}$$

If $1 < p \le 2$, by (60), we have

$$\begin{aligned}&\left|\int_\Omega b(x,t)\left(|\nabla u|^2-|\nabla v|^2\right)[a(x)]^\beta(u-v)\,dx\right|\\&\quad\le c\int_\Omega[a(x)]^{\beta/2}|u-v|\,dx\\&\quad\le c\left(\int_\Omega[a(x)]^\beta|u-v|^2\,dx\right)^{1/2}.\end{aligned}\tag{75}$$

Moreover, since $\|u(x,t)\|_{L^\infty(Q_T)} \le c$, $\|u(x,t)\|_{L^\infty(Q_T)} \le c$, $f(s,x,t)$ is a continuous function,

$$\begin{aligned}&\left|\int_\Omega [f(u,x,t) - f(v,x,t)]\, a(x)^\beta \right.\\ &\left. \cdot [u(x,t) - v(x,t)]\, dxdt\right| \le c\int_\Omega a(x)^\beta \\ &\cdot |u(x,t) - v(x,t)|\, dxdt.\end{aligned} \tag{76}$$

Now, let $\eta \longrightarrow 0$ in (62). Then

$$\begin{aligned}&\frac{d}{dt}\int_\Omega a(x)^\beta |u(x,t) - v(x,t)|^2\, dx \\ &\leqslant \left(\int_\Omega a(x)^\beta |u(x,t) - v(x,t)|^2\, dx\right)^q,\end{aligned} \tag{77}$$

where $q \le 1$. This inequality implies that

$$\begin{aligned}&\int_\Omega a(x)^\beta |u(x,t) - v(x,t)|^2\, dx \\ &\leqslant c\int_\Omega a(x)^\beta |u_0(x) - v_0(x)|^2\, dx.\end{aligned} \tag{78}$$

□

Proof of Theorem 4. Since $|b(x,t)| \le ca(x)$, $\beta \ge 4$, and $p > 2$, conditions (58) and (59) are true naturally, by Theorem 6, we have the conclusion. □

4. The Proof of Theorem 5

Lemma 7. *If $\int_\Omega a(x)^{-1/(p-1)}dx < \infty$, u is a weak solution of (7) with the initial condition (2). Then the trace of u on the boundary $\partial\Omega$ can be defined in the traditional way.*

This lemma can be found in [14]. Recall that we have assumed (21)-(23), i.e.,

$$\begin{aligned}&\partial\Omega = \Sigma_p \cup \Sigma_p', \\ &\overline{\Sigma_p} \cap \overline{\Sigma_p'} = \emptyset, \\ &a(x) \ge c > 0, \quad x \in \Sigma_p, \\ &a(x) = 0, \quad x \in \Sigma_p'.\end{aligned} \tag{79}$$

Let $\varphi(x)$ be a $C^1(\overline{\Omega})$ function satisfying that

$$\begin{aligned}&\varphi(x)|_{x\in\Sigma_p'} = 0, \\ &\varphi(x)|_{x\in\overline{\Omega}\setminus\Sigma_p'} > 0,\end{aligned} \tag{80}$$

and

$$\Omega_\eta = \{x \in \Omega : \varphi(x) > \eta\}. \tag{81}$$

Let

$$\varphi_\eta(x) = \begin{cases} 1, & \text{if } x \in \Omega_\eta, \\ \dfrac{1}{\eta}\varphi(x), & \text{if } x \in \Omega \setminus \Omega_\eta. \end{cases} \tag{82}$$

Then $\varphi_\eta(x)|_{x\in\Sigma_p'} = 0$.

Theorem 8. *Let $u(x,t), v(x,t)$ be two weak solutions of (1) with the initial values $u_0(x), v_0(x)$, respectively, with the same partial boundary value condition*

$$u(x,t) = v(x,t), \quad x \in \Sigma_p. \tag{83}$$

If $b(x,t) \in C(\overline{Q_T})$, $a(x) \in C(\overline{\Omega})$ satisfies (17), (22)-(23), and there is a constant $\beta > 1$ such that one of the following conditions is true,

$$\begin{aligned}&\int_\Omega a(x)^{-1/(p-1)}\, dx \le c, \\ &\frac{a(x)^{p\beta}|\nabla\varphi|^p}{\eta^{p-1}} \le c,\end{aligned} \tag{84}$$

(i) $p > 2$;
(ii) $1 < p \le 2$, *and (60) is true;*
then the local stability (18) is true.

Proof. Since $\int_\Omega a(x)^{-1/(p-1)}dx \le c$, then $\int_\Omega |\nabla u|dx < \infty$; accordingly, we can choose $\varphi_\eta(x)a(x)^\beta(u - v)$ as the test function. Then

$$\begin{aligned}&\int_\Omega \varphi_\eta(x)\, a(x)^\beta (u - v)\frac{\partial(u - v)}{\partial t}dx + \int_\Omega a(x)^{\beta+1} \\ &\cdot \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right)\cdot\nabla(u - v) \\ &\cdot \varphi_\eta(x)\, dx + \int_\Omega a(x)^{\beta+1}\varphi_\eta(x) \\ &\cdot \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right)\cdot\nabla\varphi_\eta(x)(u - v)\, dx \\ &+ \int_\Omega a(x)\left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right)\cdot\nabla a^\beta(u - v) \\ &\cdot \varphi_\eta(x)\, dx + \int_\Omega b(x,t)\left(|\nabla u|^2 - |\nabla v|^2\right)a(x)^\beta \\ &\cdot (u - v)\varphi_\eta(x)\, dx = \int_\Omega [f(u,x,t) - f(v,x,t)] \\ &\cdot a(x)^\beta (u - v)\varphi_\eta(x)\, dx.\end{aligned} \tag{85}$$

At first

$$\begin{aligned}&\lim_{\eta\longrightarrow 0}\int_\Omega \varphi_\eta(x)\, a(x)^\beta (u - v)\frac{\partial(u - v)}{\partial t}dx = \frac{1}{2}\frac{d}{dt} \\ &\cdot \int_\Omega a(x)|u(x,t) - v(x,t)|^2\, dx,\end{aligned} \tag{86}$$

$$\begin{aligned}&\int_\Omega a(x)^{\beta+1}\left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right)\cdot\nabla(u - v) \\ &\cdot \varphi_\eta(x)\, dx \ge 0.\end{aligned} \tag{87}$$

In the second place, by the fact that $|\nabla a(x)| \le c$ in Ω, we have

$$\begin{aligned}
&\lim_{\eta\longrightarrow 0}\left|\int_\Omega \varphi_\eta(x)\, a(x)\left(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v\right)\right.\\
&\quad\left.\cdot\nabla a^\beta(u-v)\,dx\right|\\
&\quad\cdot\left|\int_\Omega a(x)\left(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v\right)\right.\\
&\quad\left.\cdot\nabla a^\beta(u-v)\,dx\right|\\
&\quad\le\beta\left(\int_\Omega\left|a(x)^{1/p}a(x)^{\beta-1}\nabla a(u-v)\right|^p dx\right)^{1/p}\\
&\quad\cdot\left(\int_\Omega a(x)\left(|\nabla u|^p+|\nabla v|^p\right)dx\right)^{(p-1)/p}\\
&\quad\le c\left(\int_\Omega a(x)^{p(\beta-1)+1}|u-v|^p\,dx\right)^{1/p}\\
&\quad\le c\left(\int_\Omega a(x)^\beta|u-v|^2\,dx\right)^q.
\end{aligned}\tag{88}$$

The last inequality is obtained similar to (66)-(68), where $q<1$. Thirdly,

$$\begin{aligned}
&\lim_{\eta\longrightarrow 0}\left|\int_\Omega a(x)^{\beta+1}\left(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v\right)\cdot\nabla\varphi_\eta(x)\right.\\
&\quad\left.\cdot(u-v)\,dx\right|=\left|\lim_{\eta\longrightarrow 0}\frac{1}{\eta}\int_{\Omega\setminus\Omega_\eta}a(x)^{\beta+1}\right.\\
&\quad\left.\cdot\left(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v\right)\cdot\nabla\varphi(x)(u-v)\,dx\right|\\
&\quad\le\lim_{\eta\longrightarrow 0}\frac{1}{\eta}\left(\int_{\Omega\setminus\Omega_\eta}a(x)^{p\beta+1}\right.\\
&\quad\left.\cdot\left|\nabla\varphi(x)(u-v)\right|^p dx\right)^{1/p}\cdot\left(\int_{\Omega\setminus\Omega_\eta}a(x)\left(|\nabla u|^p\right.\right.\\
&\quad\left.\left.+|\nabla v|^p\right)dx\right)^{(p-1)/p}.
\end{aligned}\tag{89}$$

Since $u=v=0$ when $x\in\Sigma_p$, and $a(x)$ satisfies

$$\begin{aligned}
&\frac{a(x)^{p\beta}|\nabla\varphi|^p}{\eta^{p-1}}\le c,\\
&a(x)|_{x\in\Sigma_p'}=0,
\end{aligned}\tag{90}$$

we have

$$\begin{aligned}
&\lim_{\eta\longrightarrow 0}\frac{1}{\eta}\left(\int_{\Omega\setminus\Omega_\eta}a(x)^{p\beta+1}\left|\nabla\varphi(x)(u-v)\right|^p dx\right)^{1/p}\\
&\quad\le c\lim_{\eta\longrightarrow 0}\left(\frac{1}{\eta}\int_{\Omega\setminus\Omega_\eta}\frac{a(x)^{p\beta+1}|\nabla\varphi|^p}{\eta^{p-1}}|u-v|^p\,dx\right)^{1/p}\\
&\quad\le c\lim_{\eta\longrightarrow 0}\left(\frac{1}{\eta}\int_{\Omega\setminus\Omega_\eta}a(x)|u-v|^p\,dx\right)^{1/p}\\
&\quad=c\left(\int_{\Sigma_p}a(x)|u-v|^p\,dx+\int_{\Sigma_p'}a(x)|u-v|\,dx\right)^{1/p}\\
&\quad=0.
\end{aligned}\tag{91}$$

Fourthly, if $p>2$,

$$\begin{aligned}
&\lim_{\eta\longrightarrow 0}\left|\int_\Omega b(x,t)\,\varphi_\eta(x)\left(|\nabla u|^2-|\nabla v|^2\right)a(x)^\beta\right.\\
&\quad\left.\cdot(u-v)\,dx\right|\le\lim_{\eta\longrightarrow 0}\left(\int_\Omega b(x,t)\right.\\
&\quad\left.\cdot\left(|\nabla u|^2+|\nabla v|^2\right)^{2/p}dx\right)^{2/p}\left(\int_\Omega b(x,t)\right.\\
&\quad\left.\cdot\left|\varphi_\eta(x)\,a(x)^\beta(u-v)\right|^{p/(p-2)}dx\right)^{(p-2)/p}\\
&\quad\le c\lim_{\eta\longrightarrow 0}\left(\int_\Omega b(x,t)\left|\varphi_\eta(x)\,a^\beta(u-v)\right|^{p/(p-2)}dx\right)^{(p-2)/p}\\
&\quad\le c\left(\int_\Omega a(x)^\beta|u-v|^2\,dx\right)^q,
\end{aligned}\tag{92}$$

where $q<1$.

If $1<p\le 2$, by (60), we have

$$\begin{aligned}
&\lim_{\eta\longrightarrow 0}\left|\int_\Omega b(x,t)\,\varphi_\eta(x)\left(|\nabla u|^2-|\nabla v|^2\right)a(x)^\beta\right.\\
&\quad\left.\cdot(u-v)\,dx\right|=\left|\int_\Omega b(x,t)\left(|\nabla u|^2-|\nabla v|^2\right)a(x)^\beta\right.\\
&\quad\left.\cdot(u-v)\,dx\right|\le c\int_\Omega[a(x)]^{\beta/2}|u-v|\,dx\\
&\quad\le c\left(\int_\Omega[a(x)]^\beta|u-v|^2\,dx\right)^{1/2}.
\end{aligned}\tag{93}$$

Last but not least,

$$\begin{aligned}
&\lim_{\eta\longrightarrow 0}\left|\int_\Omega\varphi_\eta(x)\left[f(u,x,t)-f(v,x,t)\right]a(x)^\beta\right.\\
&\quad\left.\cdot[u(x,t)-v(x,t)]\,dx\right|\\
&\quad=\left|\int_\Omega\left[f(u,x,t)-f(v,x,t)\right]a(x)^\beta\right.\\
&\quad\left.\cdot[u(x,t)-v(x,t)]\,dx\right|\le c\int_\Omega a(x)^\beta|u(x,t)\\
&\quad-v(x,t)|\,dx.
\end{aligned}\tag{94}$$

Now, let $\eta\longrightarrow 0$ in (85). Then

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega a(x)^\beta|u(x,t)-v(x,t)|^2\,dx\\
&\quad\leqslant\left(\int_\Omega a(x)^\beta|u(x,t)-v(x,t)|^2\,dx\right)^q,
\end{aligned}\tag{95}$$

where $q \leq 1$. This inequality implies that

$$\int_{\Omega} a(x)^{\beta} |u(x,t) - v(x,t)|^2 dx \leqslant c \int_{\Omega} a(x)^{\beta} |u_0(x) - v_0(x)|^2 dx. \quad (96)$$

□

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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