Research Article

Some Results on N-Tupled Coincidence and Fixed Points of Graphs on Metric Spaces and an Application to Integral Equations

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In this work, we establish some N-tupled common coincidence and N-tupled common fixed points for the mappings satisfying a $(\varphi - \psi)$ -type contractive condition in a complete metric space endowed with a directed graph (for short digraph). Also, we apply our theoretical results to study the existence and uniqueness of solutions for systems of integral equations.

1. Introduction

The theory of fixed point has become very important and useful to solve many mathematical problems in different subjects of sciences such as dynamical programming [1], optimization theory [2], signal processing [3], and iterative process [4], and other times it is used to prove the existence of solution of differential and integral equations [5, 6].

Coupled fixed point was started by Guo and Lakshmikantham [7] in 1987 which is additionally trailed by Bhasker and Lakshmikantham [8] wherein authors gave the notion of mixed monotone property for a generalized contraction mapping $F : X^2 \rightarrow X$ (where X is an ordered metric space) and utilized the same to introduce some results on existence of coupled fixed points which are seen as coupled definition of specific results of Nieto and Lopez [9]. In 2009, Lakshmikantham and Ćirić [10] extended these results for nonlinear contraction mappings by giving the notions of mixed g-monotone property and coupled coincidence point. Recently, the idea of coupled fixed point is reached out to higher dimensions by numerous authors. In fact, in 2010, Samet and Vetro [11] defined the notion of fixed point of n-tupled fixed point, $n \in \mathbb{N}$ and $n \geq 2$, as

natural extension of coupled fixed point and proved some theorems for n-tupled fixed point in complete metric spaces. In 2013, Imdad et al. [12] extended the notion of n-tupled fixed point for a pair of mappings F and g by defining n-tupled coincidence point and also introduced the general concept of mixed g-monotone property on X^n for merely even n . Also, they employed the same to prove an even-tupled coincidence theorem for nonlinear contractions satisfying mixed g-monotone property. In fact, this result is true for only even n but not for odd ones. Recently, Kir et al. [13] considered the existence of a coupled fixed point of (φ, ψ) -contractive mappings endowed with a digraph and applied the obtained results to the system of two integral equations. There already exists an extensive literature on this topic. Keeping in view the relevance of this paper, we refer to [14–21].

In this paper, we will obtain more general results of Kir et al. [13], which are common N-tupled coincidence points and common N-tupled fixed points of operators satisfying (φ, ψ) -contractive condition in metric spaces endowed with a digraph.

Our paper is organized as follows. In Section 2, some basic definitions are introduced. Section 3 contains some results on N-tupled coincidence points. In Section 4, the existence of a common N-tupled fixed point of operators

satisfying (φ, ψ) -contractive conditions in metric spaces endowed with a digraph is considered. Section 5 gives an application to integral equations to support availability of the obtained results.

2. Preliminaries

Through all of this paper let (X, d) be a complete metric space. We recall the following definition which is given in [22].

Definition 1. An element $(x^1, x^2, \dots, x^N) \in X^N$ is said to be an N -tupled fixed point of the function $F : X^N \rightarrow X$ if

$$\begin{aligned} F(x^1, x^2, \dots, x^N) &= x^1, \\ F(x^2, x^3, \dots, x^1) &= x^2, \\ &\vdots \\ F(x^N, x^2, \dots, x^{N-1}) &= x^N. \end{aligned} \quad (1)$$

Therefore, an element $(x^1, x^2, \dots, x^N) \in X^N$ is said to be a common N -tupled coincidence point of the functions $F : X^N \rightarrow X$ and $g : X \rightarrow X$ if

$$\begin{aligned} F(x^1, x^2, \dots, x^N) &= gx^1, \\ F(x^2, x^3, \dots, x^1) &= gx^2, \\ &\vdots \\ F(x^N, x^2, \dots, x^{N-1}) &= gx^N. \end{aligned} \quad (2)$$

Also, an element $(x^1, x^2, \dots, x^N) \in X^N$ is said to be a common N -tupled fixed point of the functions $F : X^N \rightarrow X$ and $g : X \rightarrow X$ if $(x^1, x^2, \dots, x^N)$ is N -tupled fixed point for both F and g .

The following definition of commutative mappings in X^N is the generalization of the definition of Lakshmikantham and Ćirić [10].

Definition 2. The two mappings $F : X^N \rightarrow X$ and $g : X \rightarrow X$ are said to be commutative if

$$gF(x^1, x^2, \dots, x^N) = F(gx^1, gx^2, \dots, gx^N) \quad (3)$$

for all $(x^1, x^2, \dots, x^N) \in X^N$.

Inspired by the definition of compatible mappings in X^2 of Choudhury and Kundu [23], we introduce the following definition.

Definition 3. The mappings $F : X^N \rightarrow X$ and $g : X \rightarrow X$ are called compatible if

$$\begin{aligned} &\lim_{n \rightarrow \infty} d(g(F(x_n^1, x_n^2, \dots, x_n^N))), \\ &F(gx_n^1, gx_n^2, \dots, gx_n^N) = 0, \\ &\lim_{n \rightarrow \infty} d(g(F(x_n^2, x_n^3, \dots, x_n^N, x_n^1))), \\ &F(gx_n^2, gx_n^3, \dots, gx_n^N, gx_n^1) = 0, \\ &\vdots \\ &\lim_{n \rightarrow \infty} d(g(F(x_n^N, x_n^1, \dots, x_n^{N-1}))), \\ &F(gx_n^N, gx_n^1, \dots, gx_n^{N-1}) = 0, \end{aligned} \quad (4)$$

where $\{x_n^1\}, \{x_n^2\}, \dots, \{x_n^N\}$ are sequences in X such that

$$\begin{aligned} \lim_{n \rightarrow \infty} F(x_n^1, x_n^2, \dots, x_n^N) &= \lim_{n \rightarrow \infty} g(x_n^1) = x^1, \\ &\vdots \\ \lim_{n \rightarrow \infty} F(x_n^N, x_n^1, \dots, x_n^{N-1}) &= \lim_{n \rightarrow \infty} g(x_n^N) = x^N, \end{aligned} \quad (5)$$

where $(x^1, x^2, \dots, x^N) \in X^N$.

Definition 4 (see [24]). Let Φ be the set of all functions $\varphi : [0, \infty) \rightarrow [0, \infty)$ such that, for any $\varphi \in \Phi$, all following conditions are satisfied:

- (1) φ is lower semi-continuous and (strictly) increasing;
- (2) $\varphi(t) < t$ for all $t > 0$;
- (3) $\varphi(t + s) \leq \varphi(t) + \varphi(s)$ for all $t, s \in [0, \infty)$;
one can note that $\lim_{n \rightarrow \infty} \varphi(t_n) = 0$ if and only if $\lim_{n \rightarrow \infty} t_n = 0$ for $t_n \in [0, \infty)$;
- (4) for $\varphi \in \Phi$, Ψ_φ denote all functions $\psi : [0, \infty) \rightarrow [0, \infty)$ which satisfy the following:

$$(4A) \quad \limsup_{n \rightarrow \infty} \psi(t_n) < \varphi(r) \text{ if } \lim_{n \rightarrow \infty} t_n = r > 0;$$

$$(4B) \quad \lim_{n \rightarrow \infty} \psi(t_n) = 0 \text{ if } \lim_{n \rightarrow \infty} t_n = 0 \text{ for a sequence } \{t_n\} \in [0, \infty).$$

Now, let (X, d) be a metric space. Define $\Delta = \{(x, x) : x \in X\}$ to be a diagonal of X^2 and $G = (V(G), E(G))$ is digraph which does not have parallel edges, where $V(G)$ is the set of vertices of G and $E(G)$ is the set of edges of G .

Through all of this paper, let $V(G)$ coincide with X and $\Delta \in E(G)$. Chifu and Petrusel [25] considered the concept of G -continuous mappings which is extended in X^N by the following definition.

Definition 5. Let $F : X^N \rightarrow X$ and $g : X \rightarrow X$ be two mappings. Then (I) F is said to be G – N –continuous if for all $(x^{1*}, x^{2*}, \dots, x^{N*}) \in X^N$ and any positive sequence $\{n_i\}_i \in \mathbb{N}$

$$\begin{aligned} F(x_{n_i}^1, x_{n_i}^2, \dots, x_{n_i}^N) &\rightarrow x^{1*}, \\ F(x_{n_i}^2, x_{n_i}^3, \dots, x_{n_i}^1) &\rightarrow x^{2*}, \\ &\vdots \\ F(x_{n_i}^N, x_{n_i}^1, \dots, x_{n_i}^{N-1}) &\rightarrow x^{N*}, \end{aligned} \quad (6)$$

as $i \rightarrow \infty$ and

$$\begin{aligned} (F(x_{n_i}^1, x_{n_i}^2, \dots, x_{n_i}^N), F(x_{n_{i+1}}^1, x_{n_{i+1}}^2, \dots, x_{n_{i+1}}^N)) \\ \in E(G); \\ (F(x_{n_i}^2, x_{n_i}^3, \dots, x_{n_i}^1), F(x_{n_{i+1}}^2, x_{n_{i+1}}^3, \dots, x_{n_{i+1}}^1)) \\ \in E(G), \\ &\vdots \\ (F(x_{n_i}^N, x_{n_i}^1, \dots, x_{n_i}^{N-1}), F(x_{n_{i+1}}^N, x_{n_{i+1}}^1, \dots, x_{n_{i+1}}^{N-1})) \\ \in E(G). \end{aligned} \quad (7)$$

We have

$$\begin{aligned} F(F(x_{n_i}^1, x_{n_i}^2, \dots, x_{n_i}^N), F(x_{n_i}^2, x_{n_i}^3, \dots, x_{n_i}^1), \dots, \\ F(x_{n_i}^N, x_{n_i}^1, \dots, x_{n_i}^{N-1})) = F(x^{1*}, x^{2*}, \dots, x^{N*}), \\ F(F(x_{n_i}^2, x_{n_i}^3, \dots, x_{n_i}^1), F(x_{n_i}^3, x_{n_i}^4, \dots, x_{n_i}^{(2)}), \dots, \\ F(x_{n_i}^1, x_{n_i}^2, \dots, x_{n_i}^N)) = F(x^{2*}, x^{3*}, \dots, x^{1*}), \\ &\vdots \\ F(F(x_{n_i}^N, x_{n_i}^1, \dots, x_{n_i}^{N-1}), F(x_{n_i}^1, x_{n_i}^2, \dots, x_{n_i}^N), \dots, \\ F(x_{n_i}^{N-1}, x_{n_i}^N, \dots, x_{n_i}^{(1)})) = F(x^{N*}, x^{1*}, \dots, x^{N-1*}). \end{aligned} \quad (8)$$

(II) (X, d, G) has property A if any sequence $\{x_n\} \subseteq X$ with $x_n \rightarrow x$ as $n \rightarrow \infty$ and $(x_n, x_{n+1}) \in E(G)$, then $(x_n, x) \in E(G)$.

Consider the set $NTCP(F, g)$ of all N -tupled coincidence points of the mappings $F : X^N \rightarrow X$ and $g : X \rightarrow X$, i.e.,

$$\begin{aligned} NTCP(F, g) &= \{(x^1, x^2, \dots, x^N) \\ &\in X^N : F(x^1, x^2, \dots, x^N) = gx^1, F(x^2, x^3, \dots, x^1) \\ &= gx^2, \dots, F(x^N, x^1, \dots, x^{N-1}) = gx^N\}. \end{aligned} \quad (9)$$

Therefore, $(X^N)_g^F$ is defined as

$$\begin{aligned} (X^N)_g^F &= \{(x^1, x^2, \dots, x^N) \\ &\in X^N : (gx^1, F(x^1, x^2, \dots, x^N)) \\ &\in E(G), \dots, (gx^N, F(x^N, x^1, \dots, x^{N-1})) \in E(G)\}. \end{aligned} \quad (10)$$

In 2015, Suantai et al. [26] considered the definition of G –edge preserving which is extended by the following definition.

Definition 6. The mappings $F : X^N \rightarrow X$ and $g : X \rightarrow X$ are said to be N – G –edge preserving if

$$[(gx^1, gu^1), \dots, (gx^N, gu^N)] \in E(G) \quad (11)$$

implies that

$$\begin{aligned} [(F(x^1, \dots, x^N), F(u^1, \dots, u^N)), \dots, \\ (F(x^N, \dots, x^{N-1}), F(u^N, \dots, u^{N-1}))] \in E(G). \end{aligned} \quad (12)$$

Definition 7 (see [26]). $E(G)$ satisfies the transitivity property in X if and only if $\forall x, y, t \in X$, where $(x, y), (y, t) \in E(G) \rightarrow (x, t) \in E(G)$.

3. An N -Tupled Coincidence Point Theorem

Firstly, we give the following definition which will be used in our main results.

Definition 8. Let (X, d) be a complete metric space endowed with digraph G . The mappings $F : X^N \rightarrow X$ and $g : X \rightarrow X$ are called (ϕ, ψ) –contractive if

- (i) F and f are N – G –edge preserving,
 - (ii) there exist $\phi \in \Phi$ and $\psi \in \Psi_\phi$ such that, for all $(x^1, \dots, x^N)$ and $(u^1, \dots, u^N) \in X^N$,
- $$(gx^1, gu^1), \dots, (gx^N, gu^N) \in E(G), \quad (13)$$

then

$$\begin{aligned} \phi(2^{-1} \{d(F(x^1, \dots, x^N), F(u^1, \dots, u^N)) \\ + d(F(x^2, \dots, x^1), F(u^2, \dots, u^1)) + \dots \\ + d(F(x^N, \dots, x^{N-1}), F(u^N, \dots, u^{N-1}))\}) \\ \leq \psi(d(gx^1, gu^1) + d(gx^2, gu^2) + \dots \\ + d(gx^N, gu^N)). \end{aligned} \quad (14)$$

We study the existence of N -tupled coincidence point of the mappings $F : X^N \rightarrow X$ and $g : X \rightarrow X$ under the following hypotheses:

- (H1) g is continuous and $g(X)$ is closed;
- (H2) $F(X^N) \subseteq g(X)$, and (F, g) is compatible;
- (H3) F is N – G –continuous;
- (H4) the tripled (X, d, G) has a property A .

Now, we are ready to introduce our N -tupled coincidence point theorem.

Theorem 9. Let $F : X^N \longrightarrow X$ and $g : X \longrightarrow X$ be (φ, ψ) -contractive mappings, where (X, d) is a complete metric space and G is the digraph endowed by X . If the hypotheses (H1), (H2), and (H3) hold, then $NTCP(F, g) \neq \emptyset$ if and only if $(X^N)_g^F \neq \emptyset$.

Proof. Consider $(x_0^1, x_0^2, \dots, x_0^N), (y_0^1, y_0^2, \dots, y_0^N) \in X^N$. Since $F(X^N) \subseteq g(X)$, then there exist $x_1^1, x_1^2, \dots, x_1^N, y_1^1, y_1^2, \dots, y_1^N \in X$ such that

$$\begin{aligned} F(x_0^1, x_0^2, \dots, x_0^N) &= gx_1^1 \\ F(x_0^2, x_0^3, \dots, x_0^N) &= gx_1^2 \\ &\vdots \\ F(x_0^N, x_0^1, \dots, x_0^{N-1}) &= gx_1^N \end{aligned} \quad (15)$$

Also, we have that

$$\begin{aligned} F(y_0^1, y_0^2, \dots, y_0^N) &= gy_1^1 \\ F(y_0^2, y_0^3, \dots, y_0^N) &= gy_1^2 \\ &\vdots \\ F(y_0^N, y_0^1, \dots, y_0^{N-1}) &= gy_1^N. \end{aligned} \quad (16)$$

By doing this process, then we get the sequences $\{x_n^1\}, \{x_n^2\}, \dots, \{x_n^N\}, \{y_n^1\}, \{y_n^2\}, \dots, \{y_n^N\}$ such that

$$\begin{aligned} F(x_n^1, x_n^2, \dots, x_n^N) &= gx_{n+1}^1; \\ F(x_n^2, x_n^3, \dots, x_n^N) &= gx_{n+1}^2; \\ &\vdots \\ F(x_n^N, x_n^1, \dots, x_n^{N-1}) &= gx_{n+1}^N. \end{aligned} \quad (17)$$

Also, we have that

$$\begin{aligned} F(y_n^1, y_n^2, \dots, y_n^N) &= gy_{n+1}^1; \\ F(y_n^2, y_n^3, \dots, y_n^N) &= gy_{n+1}^2; \\ &\vdots \\ F(y_n^N, y_n^1, \dots, y_n^{N-1}) &= gy_{n+1}^N. \end{aligned} \quad (18)$$

Now, suppose that $NTCP(F, g) \neq \emptyset$. Let $(u^1, u^2, \dots, u^N) \in NTCP(F, g)$. Thus, we have that

$$\begin{aligned} F(u^1, u^2, \dots, u^N) &= gu^1 \\ F(u^2, u^3, \dots, u^1) &= gu^2 \\ &\vdots \\ F(u^N, u^1, \dots, u^{N-1}) &= gu^N. \end{aligned} \quad (19)$$

Hence, we get that

$$\begin{aligned} (gu^1, F(u^1, u^2, \dots, u^N)) &= (gu^1, gu^1) \in E(G); \\ (gu^2, F(u^2, u^3, \dots, u^1)) &= (gu^2, gu^2) \in E(G); \\ &\vdots \\ (gu^N, F(u^N, u^1, \dots, u^{N-1})) &= (gu^N, gu^N) \in E(G). \end{aligned} \quad (20)$$

Hence, $(u^1, u^2, \dots, u^N) \in (X^N)_g^F$. Then, we obtain $(X^N)_g^F \neq \emptyset$.

Let $(x_0^1, \dots, x_0^N) \in (X^N)_g^F$, then we have that

$$\begin{aligned} (gx_0^1, F(x_0^1, x_0^2, \dots, x_0^N)) &\in E(G); \\ (gx_0^2, F(x_0^2, x_0^3, \dots, x_0^1)) &\in E(G); \\ &\vdots \\ (gx_0^N, F(x_0^N, x_0^1, \dots, x_0^{N-1})) &\in E(G). \end{aligned} \quad (21)$$

By using (15), we get that

$$\begin{aligned} (gx_0^1, F(x_0^1, x_0^2, \dots, x_0^N)) &= (gx_0^1, gx_1^1) \in E(G); \\ (gx_0^2, F(x_0^2, x_0^3, \dots, x_0^1)) &= (gx_0^2, gx_1^2) \in E(G); \\ &\vdots \\ (gx_0^N, F(x_0^N, x_0^1, \dots, x_0^{N-1})) &= (gx_0^N, gx_1^N) \in E(G). \end{aligned} \quad (22)$$

Since F and g are $N - G$ -edge preserving, we have that

$$\begin{aligned} (F(x_0^1, x_0^2, \dots, x_0^N), F(x_1^1, x_1^2, \dots, x_1^N)) \\ = (gx_1^1, gx_2^1) \in E(G); \\ (F(x_0^2, x_0^3, \dots, x_0^1), F(x_1^2, x_1^3, \dots, x_1^1)) &= (gx_1^2, gx_2^2) \\ \in E(G); \\ &\vdots \\ (F(x_0^N, x_0^1, \dots, x_0^{N-1}), F(x_1^N, x_1^1, \dots, x_1^{N-1})) \\ = (gx_1^N, gx_2^N) \in E(G). \end{aligned} \quad (23)$$

By doing this process, we have

$$\begin{aligned} [(gx_n^1, gx_{n+1}^1), (gx_n^2, gx_{n+1}^2), \dots, (gx_n^N, gx_{n+1}^N)] \\ \in E(G). \end{aligned} \quad (24)$$

Let $r_n = ((d(gx_{n+1}^1, gx_{n+2}^1) + d(gx_{n+1}^2, gx_{n+2}^2) + \dots + d(gx_{n+1}^N, gx_{n+2}^N))/2)$. We get that

$$\begin{aligned}
& \varphi \left(\frac{d(gx_{n+1}^1, gx_{n+2}^1) + d(gx_{n+1}^2, gx_{n+2}^2) + \cdots + d(gx_{n+1}^N, gx_{n+2}^N)}{2} \right) \\
&= \varphi \left(\frac{d(F(x_n^1, \dots, x_n^N), F(x_{n+1}^1, \dots, x_{n+1}^N)) + d(F(x_n^2, \dots, x_n^1), F(x_{n+1}^2, \dots, x_{n+1}^1)) + \cdots + d(F(x_n^N, \dots, x_n^{N-1}), F(x_{n+1}^N, \dots, x_{n+1}^{N-1}))}{2} \right) \quad (25) \\
&\leq \psi \left(\frac{d(gx_n^1, gx_{n+1}^1) + d(gx_n^2, gx_{n+1}^2) + \cdots + d(gx_n^N, gx_{n+1}^N)}{2} \right),
\end{aligned}$$

for all $n \in \mathbb{N}$, then we have that

$$\varphi(r_{n+1}) \leq \psi(r_n) < \varphi(r_n) \quad (26)$$

By using (26) and because φ is monotonic function, we obtain $\{r_n\}$ is a positive decreasing sequence. Then, $r_n \rightarrow r$ as $n \rightarrow \infty$ for some $r \in [0, \infty)$. Consider $r > 0$. In (26), taking the limit as $n \rightarrow \infty$ and from the properties of φ, ψ , we have that

∞ for some $r \in [0, \infty)$. Consider $r > 0$. In (26), taking the limit as $n \rightarrow \infty$ and from the properties of φ, ψ , we have that

$$\varphi(r) \leq \limsup_{n \rightarrow \infty} \varphi(r_{n+1}) \leq \limsup_{n \rightarrow \infty} \psi(r_n) < \varphi(r) \quad (27)$$

We obtain contradiction. Thus $r = 0$ and then we get

$$\lim_{n \rightarrow \infty} r_n = \lim_{n \rightarrow \infty} \left(\frac{d(gx_n^1, gx_{n+1}^1) + d(gx_n^2, gx_{n+1}^2) + \cdots + d(gx_n^N, gx_{n+1}^N)}{2} \right) = 0 \quad (28)$$

Now, we will show that $\{gx_n^1\}, \{gx_n^2\}, \dots, \{gx_n^N\}$ are Cauchy sequences. Let $\{gx_n^1\}, \{gx_n^2\}, \dots, \{gx_n^N\}$ be not Cauchy sequences. Then there exist $\epsilon > 0$,

positive integers $n(k) > m(k) \geq k$ and we can find $\{gx_{n(k)}^1\}, \{gx_{m(k)}^1\}, \{gx_{n(k)}^2\}, \{gx_{m(k)}^2\}, \dots, \{gx_{n(k)}^N\}, \{gx_{m(k)}^N\}$ such that

$$\eta_k = \frac{d(gx_{n(k)}^1, gx_{m(k)}^1) + d(gx_{n(k)}^2, gx_{m(k)}^2) + \cdots + d(gx_{n(k)}^N, gx_{m(k)}^N)}{2} \geq \epsilon. \quad (29)$$

Therefore, we can find $n(k)$ is the smallest integer for which (29) holds, and we have that

$$\frac{d(gx_{n(k)-1}^1, gx_{m(k)}^1) + d(gx_{n(k)-1}^2, gx_{m(k)}^2) + \cdots + d(gx_{n(k)-1}^N, gx_{m(k)}^N)}{2} \geq \epsilon. \quad (30)$$

Using (29) and (30), we get

$$\epsilon \leq \eta_k < \epsilon + \frac{d(gx_{n(k)}^1, gx_{n(k)-1}^1) + d(gx_{n(k)}^2, gx_{n(k)-1}^2) + \cdots + d(gx_{n(k)}^N, gx_{n(k)-1}^N)}{2}. \quad (31)$$

From (27) and (31), we get that

$$\eta_k = \frac{d(gx_{n(k)}^1, gx_{m(k)}^1) + d(gx_{n(k)}^2, gx_{m(k)}^2) + \cdots + d(gx_{n(k)}^N, gx_{m(k)}^N)}{2} \rightarrow 0, \text{ as } k \rightarrow \infty. \quad (32)$$

Therefore, we obtain that

$$\begin{aligned}\eta_k &= \frac{(d(gx_{n(k)}^1, gx_{m(k)}^1) + d(gx_{n(k)}^2, gx_{m(k)}^2) + \cdots + d(gx_{n(k)}^N, gx_{m(k)}^N))}{2} \\ &\leq r_{n(k)} + r_{m(k)} + \frac{(d(gx_{n(k)+1}^1, gx_{m(k)+1}^1) + d(gx_{n(k)+1}^2, gx_{m(k)+1}^2) + \cdots + d(gx_{n(k)+1}^N, gx_{m(k)+1}^N))}{2}.\end{aligned}\quad (33)$$

Using the properties of φ , we get that

$$\begin{aligned}\varphi(\eta_k) &\leq \varphi(r_{n(k)}) + \varphi(r_{m(k)}) + \varphi\left(\frac{(d(gx_{n(k)+1}^1, gx_{m(k)+1}^1) + d(gx_{n(k)+1}^2, gx_{m(k)+1}^2) + \cdots + d(gx_{n(k)+1}^N, gx_{m(k)+1}^N))}{2}\right) \\ &\leq \varphi(r_{n(k)}) + \varphi(r_{m(k)}) \\ &\quad + \varphi\left(\frac{(d(F(x_{n(k)}^1, \dots, x_{n(k)}^N), F(x_{m(k)}^1, \dots, x_{m(k)}^N)) + \cdots + d(F(x_{n(k)}^N, \dots, x_{n(k)}^{N-1}), F(x_{m(k)}^N, \dots, x_{m(k)}^{N-1})))}{2}\right) \\ &\leq \varphi(r_{n(k)}) + \varphi(r_{m(k)}) + \psi\left(\frac{(d(gx_{n(k)}^1, gx_{m(k)}^1) + \cdots + d(gx_{n(k)}^N, gx_{m(k)}^N))}{2}\right) \leq \varphi(r_{n(k)}) + \varphi(r_{m(k)}) + \varphi(\eta_k).\end{aligned}\quad (34)$$

Again from the properties of φ which is lower semi-continuous, if we take the limit as $k \rightarrow \infty$, we have that

$$\begin{aligned}\varphi(\epsilon) &\leq \limsup_{k \rightarrow \infty} \varphi(\eta_k) \\ &\leq \lim_{k \rightarrow \infty} \varphi(r_{n(k)}) + \lim_{k \rightarrow \infty} \varphi(r_{m(k)}) \\ &\quad + \limsup_{k \rightarrow \infty} \psi(\eta_k) < \varphi(\epsilon).\end{aligned}\quad (35)$$

That is a contradiction. Hence, $\{gx_n^1\}, \{gx_n^2\}, \dots, \{gx_n^N\}$ are Cauchy sequences. Then, there exist $u^1, u^2, \dots, u^N \in g(X)$ such that

$$\begin{aligned}\lim_{n \rightarrow \infty} F(x_n^1, x_n^2, \dots, x_n^N) &= \lim_{n \rightarrow \infty} gx_n^1 = u^1; \\ \lim_{n \rightarrow \infty} F(x_n^2, x_n^3, \dots, x_n^1) &= \lim_{n \rightarrow \infty} gx_n^2 = u^2; \\ &\vdots \\ \lim_{n \rightarrow \infty} F(x_n^N, x_n^1, \dots, x_n^{N-1}) &= \lim_{n \rightarrow \infty} gx_n^N = u^N.\end{aligned}\quad (36)$$

Since (F, g) is compatible, we get

$$\begin{aligned}\lim_{n \rightarrow \infty} d(g(F(x_n^1, x_n^2, \dots, x_n^N)), \\ F(gx_n^1, gx_n^2, \dots, gx_n^N)) &= 0; \\ \lim_{n \rightarrow \infty} d(g(F(x_n^2, x_n^3, \dots, x_n^1)), F(gx_n^2, gx_n^3, \dots, gx_n^1)) &\end{aligned}$$

= 0;

$$\begin{aligned}&\vdots \\ \lim_{n \rightarrow \infty} d(g(F(x_n^N, x_n^1, \dots, x_n^{N-1})), \\ F(gx_n^N, gx_n^1, \dots, gx_n^{N-1})) &= 0.\end{aligned}\quad (37)$$

Since F is $N - G$ -continuous, we get, for all $n \geq 0$,

$$\begin{aligned}&d(F(gx_n^1, gx_n^2, \dots, gx_n^N), gu^1) \\ &\leq d(F(gx_n^1, gx_n^2, \dots, gx_n^N), gF(x_n^1, x_n^2, \dots, x_n^N)) \\ &\quad + d(gF(x_n^1, x_n^2, \dots, x_n^N), gu^1). \\ &d(F(gx_n^2, gx_n^3, \dots, gx_n^1), gu^2) \\ &\leq d(F(gx_n^2, gx_n^3, \dots, gx_n^1), gF(x_n^2, x_n^3, \dots, x_n^1)) \\ &\quad + d(gF(x_n^2, x_n^3, \dots, x_n^1), gu^2).\end{aligned}\quad (38)$$

$$\begin{aligned}&\vdots \\ &d(F(gx_n^N, gx_n^1, \dots, gx_n^{N-1}), gu^N) \\ &\leq d(F(gx_n^N, gx_n^1, \dots, gx_n^{N-1}), \\ &\quad gF(x_n^N, gx_n^1, \dots, gx_n^{N-1})) \\ &\quad + d(gF(x_n^N, gx_n^1, \dots, gx_n^{N-1}), gu^N).\end{aligned}$$

Taking the limit as $n \rightarrow \infty$ and using (H1) and (H2), we obtain that

$$d(F(u^1, u^2, \dots, u^N), gu^1) = 0, d(F(u^2, u^3, \dots, u^1), gu^2) = 0, \dots, d(F(u^N, u^1, \dots, u^{N-1}), gu^N) = 0. \quad (39)$$

Thus, we have that

$$F(u^1, u^2, \dots, u^N) = gu^1, F(u^2, u^3, \dots, u^1) = gu^2, \dots, F(u^N, u^1, \dots, u^{N-1}) = gu^N. \quad (40)$$

Hence, $NTCP(F, g) \neq \emptyset$. $\square$

Theorem 10. Let $F : X^N \rightarrow X$ and $g : X \rightarrow X$ be (φ, ψ) -contractive mappings, where (X, d) is a complete metric space and G is the digraph endowed by X . If the hypotheses (H1), (H2), and (H4) hold, then $NTCP(F, g) \neq \emptyset$ if and only if $(X^N)_g^F \neq \emptyset$.

Proof. Let $(u^1, \dots, u^N), (x^1, \dots, x^N) \in X^N$ such that $u^1 = gx^1, \dots, u^N = gx^N$. Following the same argument as in the proof of Theorem 9, we get sequences $\{x_n^1\}, \dots, \{x_n^N\}$ such that

$$[(gx_n^1, x^1), \dots, (gx_n^N, x^N)] \in E(G). \quad (41)$$

Thus, we have that

$$\begin{aligned} d(gx^1, F(x^1, \dots, x^N)) &\leq d(gx^1, gx_{n+1}^1) \\ &+ d(gx_{n+1}^1, F(x^1, \dots, x^N)) = d(gx^1, gx_{n+1}^1) \\ &+ d(F(x_n^1, \dots, x_n^N), F(x^1, \dots, x^N)); \\ d(gx^2, F(x^2, \dots, x^1)) &\leq d(gx^2, gx_{n+1}^2) \\ &+ d(gx_{n+1}^2, F(x^2, \dots, x^1)) = d(gx^2, gx_{n+1}^2) \end{aligned}$$

$$+ d(F(x_n^2, \dots, x_n^1), F(x^2, \dots, x^1));$$

$\vdots$

$$\begin{aligned} d(gx^N, F(x^N, \dots, x^{N-1})) &\leq d(gx^N, gx_{n+1}^N) \\ &+ d(gx_{n+1}^N, F(x^N, \dots, x^{N-1})) = d(gx^N, gx_{n+1}^N) \\ &+ d(F(x_n^N, \dots, x_n^{N-1}), F(x^N, \dots, x^{N-1})). \end{aligned} \quad (42)$$

Thus, we obtain that

$$\begin{aligned} d(gx^1, F(x^1, \dots, x^N)) - d(gx^1, gx_{n+1}^1) \\ \leq d(F(x_n^1, \dots, x_n^N), F(x^1, \dots, x^N)); \\ d(gx^2, F(x^2, \dots, x^1)) - d(gx^2, gx_{n+1}^2) \\ \leq d(F(x_n^2, \dots, x_n^1), F(x^2, \dots, x^1)); \\ \vdots \\ d(gx^N, F(x^N, \dots, x^{N-1})) - d(gx^N, gx_{n+1}^N) \\ \leq d(F(x_n^N, \dots, x_n^{N-1}), F(x^N, \dots, x^{N-1})), \end{aligned} \quad (43)$$

which implies that

$$\begin{aligned} &\frac{d(gx^1, F(x^1, \dots, x^N)) - d(gx^1, gx_{n+1}^1) + \dots + d(gx^N, F(x^N, \dots, x^{N-1})) - d(gx^N, gx_{n+1}^N)}{2} \\ &\leq \frac{d(F(x_n^1, \dots, x_n^N), F(x^1, \dots, x^N)) + \dots + d(F(x_n^N, \dots, x_n^{N-1}), F(x^N, \dots, x^{N-1}))}{2}. \end{aligned} \quad (44)$$

Again using the monotonicity property of φ , we get

$$\begin{aligned} & \varphi \left(\frac{d(gx^1, F(x^1, \dots, x^N)) - d(gx^1, gx_{n+1}^1) + \dots + d(gx^N, F(x^N, \dots, x^{N-1})) - d(gx^N, gx_{n+1}^N)}{2} \right) \\ & \leq \varphi \left(\frac{d(F(x_n^1, \dots, x_n^N), F(x^1, \dots, x^N)) + \dots + d(F(x_n^N, \dots, x_n^{N-1}), F(x^N, \dots, x^{N-1}))}{2} \right) \\ & \leq \psi \left(\frac{d(gx_n^1, gx^1) + \dots + d(gx_n^N, gx^N)}{2} \right). \end{aligned} \quad (45)$$

By taking the limit as $n \rightarrow \infty$, we obtain that

$$gx^1 = F(x^1, \dots, x^N), \dots, gx^N = F(x^N, \dots, x^1). \quad (46)$$

Hence, $NTCP(F, g) \neq \emptyset$. $\square$

4. The Existence and Uniqueness of N-Tupled Fixed Points

Let the set of all common N-tupled fixed points of the mappings $F : X^N \rightarrow X$ and $g : X \rightarrow X$ denoted by $NTF(F, g)$ and defined as

$$\begin{aligned} NTF(F, g) &= \{(x^1, \dots, x^N) \in X^N : F(x^1, \dots, x^N) \\ &= gx^1 = x^1, \dots, F(x^N, \dots, x^{N-1}) = gx^N = x^N\}. \end{aligned} \quad (47)$$

Now, we consider the following fixed point property:

(FP) Whenever $F : X^N \rightarrow X$ and $g : X \rightarrow X$ are two mappings, then F, g have a unique common N-tupled fixed point.

Theorem 11. Let $F : X^N \rightarrow X$ and $g : X \rightarrow X$ be commutative (φ, ψ) -contractive mappings, where (X, d) is a complete metric space and G is the digraph endowed by X . If the hypotheses (H1), (H2), and (H3) hold and for any two elements $(x^1, \dots, x^N), (y^1, \dots, y^N) \in X^N$ there exists $(z^1, \dots, z^N) \in X^N$ such that $[(gx^1, gz^1), (gy^1, gz^1), \dots, (gx^N, gz^N), (gy^N, gz^N)] \in E(G)$. Then $(X^N)_g^F \neq \emptyset$ if and only if F, g have the fixed point property (FP).

Proof. By doing the same argument as in the proof of Theorem 9, there exists $(x^1, x^2, \dots, x^N) \in X^N$ such that

$$\begin{aligned} F(x^1, x^2, \dots, x^N) &= gx^1; \\ F(x^2, x^3, \dots, x^1) &= gx^2; \\ &\vdots \\ F(x^N, x^1, \dots, x^{N-1}) &= gx^N. \end{aligned} \quad (48)$$

Assume there exists another element $(y^1, y^2, \dots, y^N) \in X^N$ such that

$$\begin{aligned} F(y^1, y^2, \dots, y^N) &= gy^1; \\ F(y^2, y^3, \dots, y^1) &= gy^2; \\ &\vdots \\ F(y^N, y^1, \dots, y^{N-1}) &= gy^N. \end{aligned} \quad (49)$$

We will show that $gx^1 = gy^1, gx^2 = gy^2, \dots, gx^N = gy^N$. We can find $(z^1, z^2, \dots, z^N) \in X^N$ such that

$$\begin{aligned} [(gx^1, gz^1), (gy^1, gz^1), \dots, (gx^N, gz^N), (gy^N, gz^N)] \\ \in E(G). \end{aligned} \quad (50)$$

By doing the same process as in (17) and (18), we get the sequences $\{x_n^1, \dots, x_n^N\}, \{y_n^1, \dots, y_n^N\}, \{z_n^1, \dots, z_n^N\}$ such that

$$\begin{aligned} x^1 &= x_0^1, x^2 = x_0^2, \dots, x^N = x_0^N; \\ y^1 &= y_0^1, y^2 = y_0^2, \dots, y^N = y_0^N; \end{aligned} \quad (51)$$

and

$$z^1 = z_0^1, z^2 = z_0^2, \dots, z^N = z_0^N. \quad (52)$$

Therefore, we obtain that

$$\begin{aligned} F(x_n^1, x_n^2, \dots, x_n^N) &= gx_{n+1}^1; \\ F(x_n^2, x_n^3, \dots, x_n^1) &= gx_{n+1}^2; \\ &\vdots \\ F(x_n^N, x_n^1, \dots, x_n^{N-1}) &= gx_{n+1}^N; \\ F(y_n^1, y_n^2, \dots, y_n^N) &= gy_{n+1}^1; \end{aligned}$$

$$\begin{aligned}
F(y_n^2, y_n^3, \dots, y_n^1) &= gy_{n+1}^2; \\
&\vdots \\
F(y_n^N, y_n^1, \dots, y_n^{N-1}) &= gy_{n+1}^N; \\
F(z_n^1, z_n^2, \dots, z_n^N) &= gz_{n+1}^1; \\
F(z_n^2, z_n^3, \dots, z_n^1) &= gz_{n+1}^2; \\
&\vdots \\
F(z_n^N, z_n^1, \dots, z_n^{N-1}) &= gz_{n+1}^N;
\end{aligned} \tag{53}$$

and we obtain that

$$\begin{aligned}
F(x^1, x^2, \dots, x^N) &= gx_n^1; \\
F(x^2, x^3, \dots, x^1) &= gx_n^2; \\
&\vdots \\
F(x^N, x^1, \dots, x^{N-1}) &= gx_n^N; \\
F(y^1, y^2, \dots, y^N) &= gy_n^1; \\
F(y^2, y^3, \dots, y^1) &= gy_n^2; \\
&\vdots \\
F(y^N, y^1, \dots, y^{N-1}) &= gy_n^N
\end{aligned} \tag{54}$$

for all $n \in \mathbb{N}$. Then we have that

$$\begin{aligned}
(gx^1, gz^1), (gy^1, gz^1) &\in E(G) \implies \\
(gx^1, gz_0^1) &\in E(G); \\
(gx^2, gz^2), (gy^2, gz^2) &\in E(G) \implies \\
(gx^2, gz_0^2) &\in E(G);
\end{aligned} \tag{55}$$

$$\begin{aligned}
&\vdots \\
(gx^N, gz^N), (gy^N, gz^N) &\in E(G) \implies \\
(gx^N, gz_0^N) &\in E(G);
\end{aligned} \tag{56}$$

therefore, we get that

$$\begin{aligned}
(F(x^1, x^2, \dots, x^N), F(z_0^1, z_0^2, \dots, z_0^N)) &= (gx^1, gz_1^1) \\
&\in E(G); \\
(F(x^2, x^3, \dots, x^1), F(z_0^3, z_0^2, \dots, z_0^1)) &= (gx^2, gz_1^2) \\
&\in E(G); \\
&\vdots \\
(F(x^N, x^1, \dots, x^{N-1}), F(z_0^N, z_0^1, \dots, z_0^{N-1})) &= (gx^N, gz_1^N) \in E(G).
\end{aligned} \tag{57}$$

Similarly, one can get that $[(gx^1, gz_n^1), \dots, (gx^N, gz_n^N)] \in E(G)$. Using the properties of φ, ψ , we get that

$$\begin{aligned}
&\varphi\left(\frac{d(gz_{n+1}^1, gx^1) + \dots + d(gz_{n+1}^N, gx^N)}{2}\right) \\
&\leq \varphi\left(\frac{d(F(z_n^1, \dots, z_n^N), F(x^1, \dots, x^N)) + \dots + d(F(z_n^N, \dots, z_n^{N-1}), F(x^N, \dots, x^{N-1}))}{2}\right) \\
&\leq \psi\left(\frac{d(gz_n^1, gx^1) + \dots + d(gz_n^N, gx^N)}{2}\right).
\end{aligned} \tag{58}$$

We obtain that

$$\varphi(\xi_{n+1}) \leq \psi(\xi_n), \tag{59}$$

where

$$\xi_n = \frac{d(gz_n^1, gx^1) + \dots + d(gz_n^N, gx^N)}{2}. \tag{60}$$

The sequence $\{\xi_n\}$ is monotonic decreasing sequence of positive real numbers; then $\exists \xi \geq 0$ such that

$$\xi_n \rightarrow \xi \text{ as } n \rightarrow \infty. \tag{61}$$

Let $\xi > 0$. Take the limit as $n \rightarrow \infty$ for (59),

$$\varphi(\xi) \leq \limsup_{n \rightarrow \infty} \varphi(\xi_{n+1}) < \varphi(\xi). \tag{62}$$

This is a contradiction; hence $\xi = 0$. Therefore,

$$\lim_{n \rightarrow \infty} \frac{d(gz_n^1, gx^1) + \dots + d(gz_n^N, gx^N)}{2} = 0, \tag{63}$$

which gives that

$$\begin{aligned}\lim_{n \rightarrow \infty} d(gz_n^1, gx^1) &= \lim_{n \rightarrow \infty} d(gz_n^2, gx^2) = \dots \\ &= \lim_{n \rightarrow \infty} d(gz_n^N, gx^N) = 0;\end{aligned}\quad (64)$$

and according to $[(gz^1, gy^1), \dots, (gz^N, gy^N)] \in E(G)$, then we get that

$$\begin{aligned}\lim_{n \rightarrow \infty} d(gz_n^1, gy^1) &= \lim_{n \rightarrow \infty} d(gz_n^2, gy^2) = \dots \\ &= \lim_{n \rightarrow \infty} d(gz_n^N, gy^N) = 0.\end{aligned}\quad (65)$$

Since

$$d(gy^1, gx^1) \leq d(gy^1, gz_n^1) + d(gz_n^1, gx^1) \quad (66)$$

for all $n \in \mathbb{N}$, then by taking the limit as $n \rightarrow \infty$, we have that

$$d(gy^1, gx^1) = d(gy^2, x^2) = \dots = d(gy^N, x^N) = 0. \quad (67)$$

Thus, $gx^1 = gy^1$, $gx^1 = gy^2, \dots, gx^N = gy^N$. Hence the uniqueness of N-tupled common coincidence point for F and g is obtained. Let $(w^1, w^2, \dots, w^N) \in X^N$ be the unique N-tupled common coincidence point for F and g , then it is clear that

$$w^1 = F(w^1, w^2, \dots, w^N), w^2 = F(w^2, w^3, \dots, w^1), \dots, w^N = F(w^N, w^1, \dots, w^{N-1}). \quad (68)$$

Since (F, g) are commutative, we have

$$gw^1 = gF(w^1, w^2, \dots, w^N) = F(gw^1, gw^2, \dots, gw^N), gw^2 = gF(w^2, w^3, \dots, w^1) = F(gw^2, gw^3, \dots, gw^1), \dots, gw^N = gF(w^N, w^1, \dots, w^{N-1}) = F(gw^N, gw^1, \dots, gw^{N-1}). \quad (69)$$

Hence, $(gw^1, gw^2, \dots, gw^N)$ is another N-tupled common coincidence point for F and g . Then, we obtain that

$$w^1 = gw^1 = F(w^1, w^2, \dots, w^N), w^2 = gw^2 = F(w^2, w^3, \dots, w^1), \dots, w^N = gw^N = F(w^N, w^1, \dots, w^{N-1}). \quad (70)$$

To prove the uniqueness of N-tupled fixed point, let $(\widehat{w}^1, \widehat{w}^2, \dots, \widehat{w}^N) \in X^N$ be another fixed point. Then there exists $(u^1, u^2, \dots, u^N) \in X^N$ such that

$$u_0^1 = u^1, u_0^2 = u^2, \dots, u_0^N = u^N, \quad (71)$$

and we obtain that

$$\begin{aligned}F(u_n^1, u_n^2, \dots, u_n^N) &= gu_{n+1}^1; \\ F(u_n^2, u_n^3, \dots, u_n^1) &= gu_{n+1}^2;\end{aligned}$$

$\vdots$

$$F(u_n^N, u_n^1, \dots, u_n^{N-1}) = gu_{n+1}^N \quad (72)$$

for all $n \in \mathbb{N}$. Then we have that

$$[(gw^1, gu_n^1), (gw^2, gu_n^2), \dots, (gw^N, u_n^N)] \in E(G). \quad (73)$$

Then we have that

$$\begin{aligned}&\varphi\left(\frac{d(w^1, gu_{n+1}^1) + \dots + d(w^2, gu_{n+1}^N)}{2}\right) \\ &= \varphi\left(\frac{d(F(w^1, \dots, w^N), F(u_{n+1}^1, \dots, u_{n+1}^N)) + \dots + d(F(w^N, \dots, w^{N-1}), F(u_{n+1}^N, \dots, u_{n+1}^1))}{2}\right) \\ &\leq \psi\left(\frac{d(w_n^1, gu_n^1) + \dots + d(w_n^N, gu_n^N)}{2}\right).\end{aligned}\quad (74)$$

Therefore, by putting

$$\rho_n = \frac{d(w_n^1, gu_n^1) + \dots + d(w_n^N, gu_n^N)}{2}, \quad (75)$$

we obtain

$$\varphi(\rho_{n+1}) \leq \psi(\rho_n) \quad (76)$$

By doing the same steps above, we can show that there exist $\rho \geq 0$ such that $\rho_n \rightarrow \rho$ as $n \rightarrow \infty$. If $\rho > 0$,

$$\varphi(\rho) \leq \limsup_{n \rightarrow \infty} \varphi(\rho_{n+1}) \leq \lim_{n \rightarrow \infty} \psi(\rho_n) < \varphi(\rho), \quad (77)$$

which is a contraction. Hence it must be $\rho = 0$. Then we have that

$$\begin{aligned} \lim_{n \rightarrow \infty} d(w^1, u_n^1) &= \lim_{n \rightarrow \infty} d(w^2, u_n^2) = \dots \\ &= \lim_{n \rightarrow \infty} d(w^N, u_n^N) = 0. \end{aligned} \quad (78)$$

Similarly, we can prove

$$\begin{aligned} \lim_{n \rightarrow \infty} d(\widehat{w^1}, u_n^1) &= \lim_{n \rightarrow \infty} d(\widehat{w^2}, u_n^2) = \dots \\ &= \lim_{n \rightarrow \infty} d(\widehat{w^N}, u_n^N) = 0. \end{aligned} \quad (79)$$

Hence, we get $\widehat{w^1} = w^1, \widehat{w^2} = w^2, \dots, \widehat{w^N} = w^N$. $\square$

Theorem 12. Let $F : X^N \rightarrow X$ and $g : X \rightarrow X$ be commutative (φ, ψ) -contractive mappings, where (X, d) is a complete metric space and G is the digraph endowed by X . If the hypotheses (H1), (H2), and (H4) hold and for any two elements $(x^1, \dots, x^N), (y^1, \dots, y^N) \in X^N$ there exist $(z^1, \dots, z^N) \in X^N$ such that $[(gx^1, gz^1), (gy^1, gz^1), \dots, (gx^N, gz^N), (gy^N, gz^N)] \in E(G)$, then $(X^N)_g^F \neq \emptyset$ if and only if F, g have the fixed point property (FP).

5. An application

Now, we apply the N-tupled fixed point theorem to study the existence and uniqueness of the solution of the system of nonlinear integral equations. This system can be written as

$$\begin{aligned} x_1(t) &= A(t) \\ &+ \int_0^a B(t, s) K(s, x_1(s), x_2(s), \dots, x_N(s)) ds; \\ x_2(t) &= A(t) \\ &+ \int_0^a B(t, s) K(s, x_2(s), x_3(s), \dots, x_1(s)) ds; \\ &\vdots \\ x_N(t) &= A(t) \\ &+ \int_0^a B(t, s) K(s, x_N(s), x_1(s), \dots, x_{N-1}(s)) ds, \end{aligned} \quad (80)$$

where $a > 0, t \in [0, a]$. Define the space $X = C([0, a], \mathfrak{R}^n) = \{x : [0, a] \rightarrow \mathfrak{R}^n : x \text{ is continuous}\}$, where, for all $x \in X$, $\|x\| = \max_{t \in [0, a]} |x(t)|$. Therefore, $d(x, y) = \|x - y\|$ and (X, d) is a complete metric space. Let G be a digraph which $V(G)$ coincides with X and $E(G)$ can be obtained by the the partial order relation

$$(x, y) \in E(G) \iff x \leq y \iff x(t) \leq y(t) \quad \text{for all } t \in [0, a]. \quad (81)$$

Thus (X, d) is endowed with a digraph G . It is clear that the transitivity property is satisfied on $E(G)$ and (x, d, G) has property A. To prove the existence of unique solution of (80), we consider

$$(C1) \ K : [0, a] \times \underbrace{\mathfrak{R}^N \times \mathfrak{R}^N \dots \times \mathfrak{R}^N}_{N\text{-times}} \rightarrow \mathfrak{R}^N \text{ and } A :$$

$[0, a] \rightarrow \mathfrak{R}^N$ are continuous,

(C2) there exists a $0 \leq \theta \leq 1$ such that

$$\begin{aligned} |K(s, x_1, x_2, \dots, x_N) - K(s, y_1, y_2, \dots, y_N)| \\ \leq \theta (|x_1 - y_1| + |x_2 - y_2| + \dots + |x_N - y_N|), \end{aligned} \quad (82)$$

(C3) $\forall s, t \in [0, a]$, there exists a continuous mapping $B : [0, a] \times \mathfrak{R} \rightarrow \mathfrak{R}$ such that

$$\sup_{t \in [0, a]} \int_0^a B(t, s) ds < 1, \quad (83)$$

(C4) there exists $(x_1^0, x_2^0, \dots, x_N^0) \in X^N$ such that

$$\begin{aligned} x_1^0(t) &\leq A(t) \\ &+ \int_0^a B(t, s) K(s, x_1^0(s), x_2^0(s), \dots, x_N^0(s)) ds; \\ x_2^0(t) &\leq A(t) \\ &+ \int_0^a B(t, s) K(s, x_2^0(s), x_3^0(s), \dots, x_1^0(s)) ds; \\ &\vdots \end{aligned} \quad (84)$$

$$x_N^0(t) \leq A(t)$$

$$+ \int_0^a B(t, s) K(s, x_N^0(s), x_1^0(s), \dots, x_{N-1}^0(s)) ds,$$

for all $t \in [0, a]$.

Theorem 13. If conditions (C1), (C2), (C3), and (C4) are satisfied, then system (80) has a unique solution in $C([0, a], \mathfrak{R}^n)$.

Proof. Define $F : X^N \rightarrow X$, where

$$\begin{aligned} F(x_1, x_2, \dots, x_N)(t) \\ = A(t) \\ + \int_0^a B(t, s) K(s, x_1(s), x_2(s), \dots, x_N(s)) ds; \end{aligned} \quad (85)$$

$t \in [0, a].$

Define $g : X \rightarrow X$ as $gx(t) = 2x(t)$. System (80) can be written as

$$x_1 = F(x_1, x_2, \dots, x_N), x_2 = F(x_2, x_3, \dots, x_1), \dots, x_N = F(x_N, x_1, \dots, x_{N-1}). \quad (86)$$

Let $x_1, x_2, \dots, x_N, u_1, u_2, \dots, u_N \in X$ such that

$$gx_1 \leq gu_1, gx_2 \leq gu_2, \dots, gx_N \leq gu_N. \quad (87)$$

Then we get that $x_1 \leq u_1, x_2 \leq u_2, \dots, x_N \leq u_N$, and

$$\begin{aligned} F(x_1, x_2, \dots, x_N)(t) &= A(t) \\ &+ \int_0^a B(t, s) K(s, x_1(s), x_2(s), \dots, x_N(s)) ds \\ &\leq A(t) \\ &+ \int_0^a B(t, s) K(s, u_1(s), u_2(s), \dots, u_N(s)) ds \\ &= F(u_1, u_2, \dots, u_N)(t); \\ F(x_2, x_3, \dots, x_1)(t) &= A(t) \\ &+ \int_0^a B(t, s) K(s, x_2(s), x_3(s), \dots, x_1(s)) ds \end{aligned}$$

$$\begin{aligned} &\leq A(t) \\ &+ \int_0^a B(t, s) K(s, u_2(s), u_3(s), \dots, u_1(s)) ds \\ &= F(u_2, u_3, \dots, u_1)(t); \\ &\vdots \\ &F(x_N, x_1, \dots, x_{N-1})(t) = A(t) \\ &+ \int_0^a B(t, s) K(s, x_N(s), x_1(s), \dots, x_{N-1}(s)) ds \\ &\leq A(t) \\ &+ \int_0^a B(t, s) K(s, u_N(s), u_1(s), \dots, u_{N-1}(s)) ds \\ &= F(u_N, u_1, \dots, u_{N-1})(t); \end{aligned}$$

for all $t \in [0, a]$

(88)

Hence, (F, g) are N-G-edge preserving. By using conditions (C2) and (C3), and for all $t \in [0, a]$, we get that

$$\begin{aligned} &\frac{|F(x_1, \dots, x_N)(t) - F(u_1, \dots, u_N)(t)| + \dots + |F(x_N, \dots, x_{N-1})(t) - F(u_N, \dots, u_{N-1})(t)|}{2} \\ &= \frac{\left| \int_0^a B(t, s) [K(s, x_1(s), x_2(s), \dots, x_N(s)) - K(s, u_1(s), u_2(s), \dots, u_N(s))] ds \right|}{2} + \dots \\ &+ \frac{\left| \int_0^a B(t, s) [K(s, x_N(s), x_1(s), \dots, x_{N-1}(s)) - K(s, u_N(s), u_1(s), \dots, u_{N-1}(s))] ds \right|}{2} \\ &\leq \frac{|F(x_1, \dots, x_N)(t) - F(u_1, \dots, u_N)(t)| + \dots + |F(x_N, \dots, x_{N-1})(t) - F(u_N, \dots, u_{N-1})(t)|}{2} \\ &= \frac{\int_0^a B(t, s) |K(s, x_1(s), x_2(s), \dots, x_N(s)) - K(s, u_1(s), u_2(s), \dots, u_N(s))| ds}{2} + \dots \\ &+ \frac{\int_0^a B(t, s) |K(s, x_N(s), x_1(s), \dots, x_{N-1}(s)) - K(s, u_N(s), u_1(s), \dots, u_{N-1}(s))| ds}{2} \\ &\leq \frac{\theta(2|x_1(s) - u_1(s)| + \dots + 2|x_N(s) - u_N(s)|)}{2} \leq \frac{\theta(|gx_1(s) - gu_1(s)| + \dots + |gx_N(s) - gu_N(s)|)}{2} \\ &= \theta \left(\frac{d(gx_1, gu_1) + \dots + d(gx_N, gu_N)}{2} \right) \end{aligned} \quad (89)$$

Take $\varphi(t) = t$ and $\psi(t) = \theta t$. We get

$$\begin{aligned} & \varphi \left(\frac{d(F(x_1, \dots, x_N), F(u_1, \dots, u_N)) + \dots + d(F(x_N, \dots, x_{N-1}), F(u_N, \dots, u_{N-1}))}{2} \right) \\ & \leq \psi \left(\frac{d(gx_1, gu_1) + \dots + d(gx_N, gu_N)}{2} \right) \end{aligned} \quad (90)$$

Thus F and g are $(\varphi - \psi)$ -contractive. Therefore by using (C4) there exists $(x_1^0, x_2^0, \dots, x_N^0) \in X^N$ such that

$$\begin{aligned} gx_1^0 & \leq F(x_1^0, x_2^0, \dots, x_N^0), gx_2^0 \\ & \leq F(x_2^0, x_3^0, \dots, x_1^0), \dots, gx_N^0 \\ & \leq F(x_N^0, x_1^0, \dots, x_{N-1}^0). \end{aligned} \quad (91)$$

Hence $(X^N)^F_g \neq \emptyset$. Hence there exists a unique N -tupled common fixed point $(x_1^*, x_2^*, \dots, x_N^*) \in X^N$ of F and g is the unique solution of system (80). $\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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