Hindawi
Journal of Function Spaces
Volume 2019, Article ID 1078103, 6 pages
https://doi.org/10.1155/2019/1078103

Research Article

On the Generalized Reflective Function of the Three-Degree Polynomial Differential System

Jian Zhou and Shiyin Zhao

School of Literature and Science, Suqian College, Suqian 223800, China

Correspondence should be addressed to Shiyin Zhao; 52660078@qq.com

Received 8 February 2019; Accepted 2 April 2019; Published 2 May 2019

Academic Editor: Gestur Ólafsson

The structure of the generalized reflective function of three-degree polynomial differential systems is considered in this paper. The generated results are used for discussing the existence of periodic solutions of these systems.

1. Introduction

As we know, it is very important to study the properties of the solution of differential system

$$x' = X(t, x) \tag{1}$$

for both the theory and application of an ordinary differential equation.

If $X(t + 2\omega, x) = X(t, x)$ (ω is a positive constant), we can use the Poincaré mapping introduced in [1] to study the behavior of the solutions of (1). But it is very difficult to find the Poincaré mapping for many systems which cannot be integrated in quadratures. In the 1980s, the Russian mathematician Mironenko first established the theory of reflective functions (*RF*) in [2, 3]. Since then a quite new method to study (1) has been found. In recent years, more and more experts and scholars have achieved many good results; refer to [4–9] in this direction.

The aim of this article is to use the theory of the generalized reflective function to study the behavior of the solutions of differential systems. The obtained results will provide a new theoretical basis and criterion to further explain the laws of the movement of objects.

In the present section, we first recall some basic notions and results of the generalized reflective function (*GRF*), which will be used throughout the rest of this article.

Now consider system (1) with a continuous differentiable right-hand side and with a general solution $\varphi(t; t_0, x_0)$. For each such system, the *GRF* of system (1) is defined as $F(t, x) := \varphi(\alpha(t); t, x)$, $(t, x) \in D \subset R \times R^n$, where $\alpha(t)$ is a continuous differentiable function such that $\alpha(\alpha(t)) = t$, $\alpha(0) = 0$. Then for any solution $x(t)$ of (1), we have $F(t, x(t)) = x(\alpha(t))$ and $F(\alpha(t), F(t, x)) \equiv F(0, x) \equiv x$. By the definition in [3], a continuous differentiable vector function $F(t, x)$ on $R \times R^n$ is called *GRF* if and only if it is a solution of the Cauchy problem

$$\begin{gathered} F_t(t, x) + F_x(t, x) X(t, x) = \alpha'(t) X(\alpha(t), F(t, x)), \\ F(0, x) = x. \end{gathered} \tag{2}$$

Relation (2) is called a basic relation (*BR*).

Besides this, suppose that system (1) is 2ω−periodic with respect to the variable t, and F is its *GRF*; if there exists a number η on R such that $\alpha(\eta) = 2\omega + \eta$, then $T(x) = F(\eta, x) = \varphi(\alpha(\eta); \eta, x)$ is the Poincaré mapping of (1) over the period $[\eta, \eta + 2\omega]$. So, for any solution $x(t)$ of (1) defined on $[\eta, \eta + 2\omega]$, it will be 2ω−periodic if and only if $F(\eta, x) = x$ which is called a basic lemma (*BL*).

Now, we consider the higher dimensional polynomial differential system

$$\begin{aligned} x' &= a_1 + a_2 y + a_3 z \triangleq A(t, x, y, z) \\ y' &= b_1 + b_2 y + b_3 z + b_4 y^2 + b_5 yz + b_6 z^2 \\ &\triangleq B(t, x, y, z) \end{aligned}$$

$$z' = c_1 + c_2 y + c_3 z + c_4 y^2 + c_5 yz + c_6 z^2 \triangleq C(t, x, y, z), \tag{3}$$

where $a_i = a_i(t, x)$, $b_j = b_j(t, x)$, and $c_j = c_j(t, x)$ ($i = 1, 2, 3$; $j = 1, 2, ..., 6$) are continuously differentiable functions in R^2, ${a_2}^2 + {a_3}^2 \neq 0$ (in some deleted neighborhood of $t = 0$ and $|t|$ being small enough but different from zero), and there exists a unique solution for the initial value problem of (3). And suppose that

$$F(t, x, y, z) = (F_1(t, x, y, z), F_2(t, x, y, z), F_3(t, x, y, z))^T \tag{4}$$

is the *GRF* of (3).

In this article, we discuss the structure of $F_i(t, x, y, z)$ ($i = 2, 3$) when $F_1(t, x, y, z) = f(t, x)$ and obtain the good results $F_i(t, x, y, z) = f_{i1}(t, x) + f_{i2}(t, x)y + f_{i3}(t, x)z$ ($i = 2, 3$) which are useful for the research of the existence of periodic solutions and establishing the sufficient conditions of system (3) with the form of *GRF*.

In the following, we denote

$$\begin{aligned} \bar{a}_i &= a_i(\alpha(t), x); \\ \bar{b}_j &= b_j(\alpha(t), x); \\ \bar{c}_j &= c_j(\alpha(t), x); \\ \alpha &= \alpha(t); \\ F_i &= F_i(t, x, y, z) \end{aligned} \tag{5}$$

($i = 1, 2, 3$; $j = 1, 2, ..., 6$). The notation "$a_i(t, x) \neq 0$" means that, in some deleted neighborhood of $t = 0$, $a_i(t, x)$ is small enough but different from zero.

2. Main Results

Without loss of generality, we suppose that $f(t, x) = x$. Otherwise, we can take the transformation $\xi = f(t, x)$, $\eta = y$, $\zeta = z$.

Now, we consider system (3).

Lemma 1. *For system (3), let $F_1 = x$, $\alpha'(0) = -1$; then*

$$a_i(0, x) = 0, \quad i = 1, 2, 3. \tag{6}$$

Proof. Using relation (2), we can get

$$A(t, x, y, z) - \alpha'(t) A(\alpha, x, F_2, F_3) = 0; \tag{7}$$

i.e.,

$$a_1 - \alpha' \bar{a}_1 + a_2 y + a_3 z - \alpha' \bar{a}_2 F_2 - \alpha' \bar{a}_3 F_3 = 0. \tag{8}$$

Putting $t = 0$, we have $a_1(0, x) + a_2(0, x)y + a_3(0, x)z \equiv 0$, $\forall x, y, z$, which implies that relation (6) is valid. □

In the following discussion, we always assume (6) holds without further mentioning.

Case I ($a_3 \neq 0$). From relation (8), we get

$$F_3 = l_1 + l_2 F_2, \tag{9}$$

where

$$\begin{aligned} l_1 &= l_{11} + l_{12} y + l_{13} z, \\ l_{11} &= \frac{a_1 - \alpha' \bar{a}_1}{\alpha' \bar{a}_3}, \\ l_{12} &= \frac{a_2}{\alpha' \bar{a}_3}, \\ l_{13} &= \frac{a_3}{\alpha' \bar{a}_3}, \\ l_2 &= -\frac{\bar{a}_2}{\bar{a}_3}. \end{aligned} \tag{10}$$

Differentiating relation (9) with respect to t implies

$$\sum_{i=0}^{2} P_i {F_2}^i = 0, \tag{11}$$

where

$$\begin{aligned} P_0 &= p_{01} + p_{02} y + p_{03} z + p_{04} y^2 + p_{05} yz + p_{06} z^2, \\ p_{01} &= \alpha' \left[\bar{c}_1 - l_2 \bar{b}_1 + l_{11}\left(\bar{c}_3 - l_2 \bar{b}_3\right) + {l_{11}}^2 \left(\bar{c}_6 - l_2 \bar{b}_6\right)\right] \\ &\quad - \left({l_{11t}}' + {l_{11x}}' a_1 + l_{12} b_1 + l_{13} c_1\right), \\ p_{02} &= \alpha' \left[l_{12}\left(\bar{c}_3 - l_2 \bar{b}_3\right) + 2 l_{11} l_{12}\left(\bar{c}_6 - l_2 \bar{b}_6\right)\right] - \left({l_{12t}}' \right. \\ &\quad \left. + {l_{11x}}' a_2 + {l_{12x}}' a_1 + l_{12} b_2 + l_{13} c_2\right), \\ p_{03} &= \alpha' \left[l_{13}\left(\bar{c}_3 - l_2 \bar{b}_3\right) + 2 l_{11} l_{13}\left(\bar{c}_6 - l_2 \bar{b}_6\right)\right] - \left({l_{13t}}' \right. \\ &\quad \left. + {l_{11x}}' a_3 + {l_{13x}}' a_1 + l_{12} b_3 + l_{13} c_3\right), \\ p_{04} &= \alpha' {l_{12}}^2 \left(\bar{c}_6 - l_2 \bar{b}_6\right) - \left({l_{12x}}' a_2 + l_{12} b_4 + l_{13} c_4\right), \\ p_{05} &= 2\alpha' l_{12} l_{13}\left(\bar{c}_6 - l_2 \bar{b}_6\right) - \left({l_{12x}}' a_3 + {l_{13x}}' a_2 + l_{12} b_5 \right. \\ &\quad \left. + l_{13} c_5\right), \\ p_{06} &= \alpha' {l_{13}}^2 \left(\bar{c}_6 - l_2 \bar{b}_6\right) - \left({l_{13x}}' a_3 + l_{12} b_6 + l_{13} c_6\right), \\ P_1 &= p_{11} + p_{12} y + p_{13} z, \\ p_{11} &= \alpha' \left[\bar{c}_2 - l_2 \bar{b}_2 + l_2\left(\bar{c}_3 - l_3 \bar{b}_3\right) + l_{11}\left(\bar{c}_5 - l_2 \bar{b}_5\right) \right. \\ &\quad \left. + 2 l_2 l_{11}\left(\bar{c}_6 - l_2 \bar{b}_6\right)\right] - \left({l_{2t}}' + {l_{2x}}' a_1\right), \\ p_{12} &= \alpha' \left[l_{12}\left(\bar{c}_5 - l_2 \bar{b}_5\right) + 2 l_2 l_{12}\left(\bar{c}_6 - l_2 \bar{b}_6\right)\right] - {l_{2x}}' a_2, \\ p_{13} &= \alpha' \left[l_{13}\left(\bar{c}_5 - l_2 \bar{b}_5\right) + 2 l_2 l_{13}\left(\bar{c}_6 - l_2 \bar{b}_6\right)\right] - {l_{2x}}' a_3, \\ P_2 &= \alpha' \left[\bar{c}_4 - l_2 \bar{b}_4 + l_2\left(\bar{c}_5 - l_2 \bar{b}_5\right) + {l_2}^2 \left(\bar{c}_6 - l_2 \bar{b}_6\right)\right]. \end{aligned} \tag{12}$$

Lemma 2. *For system (3), suppose $P_2 \neq 0$, $F_1 = x$ and the limit $\lim_{t\longrightarrow 0}(p_{0i}/P_2)$ $(i = 1, 2, ..., 6)$ exist; then*

$$\begin{aligned} &\operatorname*{Lim}_{t\longrightarrow 0}\frac{p_{0j}}{P_2} = 0 \quad (j = 1, 3, 6), \\ &\lim_{t\longrightarrow 0}\frac{p_{02}+p_{11}}{P_2} = 0, \\ &\lim_{t\longrightarrow 0}\frac{p_{05}+p_{13}}{P_2} = 0, \\ &\lim_{t\longrightarrow 0}\frac{p_{04}+p_{12}}{P_2} = -1. \end{aligned} \tag{13}$$

Proof. Using relation (11), we have $\lim_{t\longrightarrow 0}F_2{}^2 + \lim_{t\longrightarrow 0}(P_1/P_2)F_2 + \lim_{t\longrightarrow 0}(P_0/P_2) = 0$, i.e.

$$\begin{aligned} &\lim_{t\longrightarrow 0}F_2{}^2 + \lim_{t\longrightarrow 0}\frac{p_{11}+p_{12}y+p_{13}z}{P_2}F_2 \\ &\quad + \lim_{t\longrightarrow 0}\frac{p_{01}+p_{02}y+p_{03}z+p_{04}y^2+p_{05}yz+p_{06}z^2}{P_2} \\ &= 0, \end{aligned} \tag{14}$$

by $F_2(0, x, y, z) = y$, which implies that the results of Lemma 2 are true. □

Theorem 3. *For system (3), suppose that $\lim_{t\longrightarrow 0}(p_{12}/P_2)+2 > 0$ and all the conditions of Lemmas 1 and 2 are satisfied; then*

$$F_i(t, x, y, z) = f_{i0}(t, x) + f_{i1}(t, x)\, y + f_{i2}(t, x)\, z \quad (i = 2, 3). \tag{15}$$

Proof. As $P_2 \neq 0$, by relation (11), we have

$$\begin{aligned} &F_2{}^2 = -\frac{P_1}{P_2}F_2 - \frac{P_0}{P_2}. \\ &F_2{}^3 = -\frac{P_1}{P_2}F_2{}^2 - \frac{P_0}{P_2}F_2 = \left(\frac{P_1{}^2}{P_2{}^2} - \frac{P_0}{P_2}\right)F_2 + \frac{P_0P_1}{P_2{}^2}. \end{aligned} \tag{16}$$

Differentiating relation (11) with respect to t, we can obtain

$$\begin{aligned} &DP_0 + (DP_1)F_2 + \alpha' P_1 B(\alpha, x, F_2, F_3) + (DP_2)F_2{}^2 \\ &\quad + 2\alpha' P_2 B(\alpha, x, F_2, F_3)F_2 = 0; \end{aligned} \tag{17}$$

Substituting (9) into the above, we get

$$\sum_{i=0}^{3} Q_i F_2{}^i = 0, \tag{18}$$

where

$$\begin{aligned} Q_0 &= DP_0 + \alpha'\mu_1 P_1, \\ Q_1 &= DP_1 + \alpha'\mu_2 P_1 + 2\alpha'\mu_1 P_2, \\ Q_2 &= DP_2 + \alpha'\mu_3 P_1 + 2\alpha'\mu_2 P_2, \\ Q_3 &= 2\alpha'\mu_3 P_2; \\ \mu_1 &= \bar{b}_1 + l_1\bar{b}_3 + l_1{}^2\bar{b}_6, \\ \mu_2 &= \bar{b}_2 + l_2\bar{b}_3 + l_1\bar{b}_5 + 2l_1l_2\bar{b}_6, \\ \mu_3 &= \bar{b}_4 + l_2\bar{b}_5 + l_2{}^2\bar{b}_6; \\ DP_i &= \frac{\partial P_i}{\partial t} + \frac{\partial P_i}{\partial x}A(t, x, y, z) + \frac{\partial P_i}{\partial y}B(t, x, y, z) \\ &\quad + \frac{\partial P_i}{\partial z}C(t, x, y, z) \quad (i = 0, 1, 2). \end{aligned} \tag{19}$$

Substituting (16) into (18), we have

$$R_0 + R_1F_2 = 0, \tag{20}$$

in which

$$\begin{aligned} R_0 &= Q_0 - Q_2\frac{P_0}{P_2} + Q_3\frac{P_0P_1}{P_2{}^2}, \\ R_1 &= Q_1 - Q_2\frac{P_1}{P_2} + Q_3\left(\frac{P_1{}^2}{P_2{}^2} - \frac{P_0}{P_2}\right). \end{aligned} \tag{21}$$

(1) If $R_1 \neq 0$, we get $F_2 = -R_0/R_1$ by relation (20). Through the expression of P_i, Q_j $(i = 0, 1, 2;\ j = 0, 1, 2, 3)$, we know that R_1 is a quadratic polynomial with respect to y, z and R_0 is a cubic polynomial with respect to y, z. Substituting $F_2 = -R_0/R_1$ into (11), we obtain $P_2R_0{}^2 = -R_1(P_0R_1 - P_1R_0)$ which implies that $R_1 \mid R_0$ or $R_1 \mid P_2$ and $F_2 = \sum_{i+j=0}^{3} f_{2ij}(t, x)y^iz^j$. Substituting the relations into (11) and equating the coefficients of the same powers of y and z, we have $f_{2ij}(t, x) = 0$, $i + j > 1$. Finally, we arrive at

$$F_i(t, x, y, z) = f_{i0}(t, x) + f_{i1}(t, x)\, y + f_{i2}(t, x)\, z \quad (i = 2, 3) \tag{22}$$

(2) If $R_1 = 0$, it follows $R_0 = 0$ from (20). By simple computation, we obtain

$$D\frac{P_0}{P_2} = -\alpha'\mu_1\frac{P_1}{P_2} + 2\alpha'\mu_2\frac{P_0}{P_2} - \alpha'\mu_3\frac{P_0P_1}{P_2{}^2}, \tag{23}$$

$$D\frac{P_1}{P_2} = -2\alpha'\mu_1 + \alpha'\mu_2\frac{P_1}{P_2} - \alpha'\mu_3\left(\frac{P_1{}^2}{P_2{}^2} - 2\frac{P_0}{P_2}\right) \tag{24}$$

Let $\Delta = P_1{}^2/P_2{}^2 - 4P_0/P_2$; we have

$$\begin{aligned} D\Delta &= D\left(\frac{P_1{}^2}{P_2{}^2} - \frac{4P_0}{P_2}\right) = 2\frac{P_1}{P_2}D\frac{P_1}{P_2} - 4D\frac{P_0}{P_2} \\ &= 2\alpha'\left(\mu_2 - \mu_3\frac{P_1}{P_2}\right)\Delta \end{aligned} \tag{25}$$

by (23) and (24).

Since

$$\begin{aligned}\Delta &= \frac{{P_1}^2}{{P_2}^2} - \frac{4P_0}{P_2} = \frac{1}{{P_2}^2}\left[\left(p_{11} + p_{12}y + p_{13}z\right)^2\right.\\ &\quad - 4P_2\left(p_{01} + p_{02}y + p_{03}z + p_{04}y^2 + p_{05}yz\right.\\ &\quad \left.\left. + p_{06}z^2\right)\right] = s_1 + s_2y + s_3z + s_4y^2 + s_5yz + s_6z^2\\ &= s_4\left(y + \frac{s_5}{2s_4}z + \frac{s_2}{2s_4}\right)^2 + w(t,x,z),\end{aligned} \tag{26}$$

where

$$\begin{aligned}s_1 &= \frac{1}{{P_2}^2}\left({p_{11}}^2 - 4P_2p_{01}\right),\\ s_2 &= \frac{1}{{P_2}^2}\left(2p_{11}p_{12} - 4P_2p_{02}\right),\\ s_3 &= \frac{1}{{P_2}^2}\left(2p_{11}p_{13} - 4P_2p_{03}\right),\\ s_4 &= \frac{1}{{P_2}^2}\left({p_{12}}^2 - 4P_2p_{04}\right),\\ s_5 &= \frac{1}{{P_2}^2}\left(2p_{12}p_{13} - 4P_2p_{05}\right),\\ s_6 &= \frac{1}{{P_2}^2}\left({p_{13}}^2 - 4P_2p_{06}\right);\end{aligned} \tag{27}$$

$$\begin{aligned}w(t,x,z) &= w_0 + w_1z + w_2z^2,\\ w_0 &= \frac{1}{4s_4}\left(4s_1s_4 - {s_2}^2\right),\\ w_1 &= \frac{1}{2s_4}\left(2s_3s_4 - s_2s_5\right),\\ w_2 &= \frac{1}{4s_4}\left(4s_4s_6 - {s_5}^2\right),\end{aligned}$$

we can obtain $\lim_{t\longrightarrow 0}(4s_4s_6 - {s_5}^2) = 0$, $\lim_{t\longrightarrow 0}(2s_3s_4 - s_2s_5) = 0$, and $\lim_{t\longrightarrow 0}(4s_1s_4 - {s_2}^2) = 0$ by taking advantage of Lemma 2. Thus $\lim_{t\longrightarrow 0}w_1(t,x,z) = 0$. It is easy to verify that

$$\begin{aligned}&D\Delta(t,x,y,z)\big|_{y=-(s_5/2s_4)z-s_2/2s_4}\\ &\quad = \left(\Delta_t + \Delta_xA + \Delta_yB + \Delta_zC\right)\big|_{y=-(s_5/2s_4)z-s_2/2s_4}\\ &\quad = w_t + w_xA\left(t,x,-\frac{s_5}{2s_4}z - \frac{s_2}{2s_4},z\right)\\ &\qquad + w_zC\left(t,x,-\frac{s_5}{2s_4}z - \frac{s_2}{2s_4},z\right).\end{aligned} \tag{28}$$

Using identity (25), we get

$$\begin{aligned}&w_t + w_xA\left(t,x,-\frac{s_5}{2s_4}z - \frac{s_2}{2s_4},z\right)\\ &\quad + w_zC\left(t,x,-\frac{s_5}{2s_4}z - \frac{s_2}{2s_4},z\right)\\ &\quad = 2\alpha' w\left(\mu_2 - \mu_3\frac{P_1}{P_2}\right)\Big|_{y=-(s_5/2s_4)z-s_2/2s_4}.\end{aligned} \tag{29}$$

By the uniqueness of solutions of the initial problem of linear partial differential equations, we have $w = 0$. Therefore

$$\Delta = s_4\left(y + \frac{s_5}{2s_4}z + \frac{s_2}{2s_4}\right)^2. \tag{30}$$

So, using relation (11), we get

$$\begin{aligned}F_2 &= \left(\frac{s_2}{4s_4}\sqrt{s_4} - \frac{P_1}{2P_2}\right) + \frac{1}{2}\sqrt{s_4}y + \frac{s_5}{4s_4}\sqrt{s_4}z\\ &= f_{20}(t,x) + f_{21}(t,x)y + f_{22}(t,x)z.\end{aligned} \tag{31}$$

Finally, we get $F_3 = l_1 + l_2F_2 = f_{30}(t,x) + f_{31}(t,x)y + f_{32}(t,x)z$ by relation (9).

Summarizing the above, the proof is finished. □

Theorem 4. *For system (3), let $F_1(t,x,y,z) = x$, $P_1 \neq 0$, $P_2 = 0$, and $a_i(0,x) = 0$ $(i = 1,2,3)$; then*

$$\begin{aligned}F_2(t,x,y,z) &= -\frac{P_0}{P_1}\\ &= -\frac{p_{01} + p_{02}y + p_{03}z + p_{04}y^2 + p_{05}yz + p_{06}z^2}{p_{11} + p_{12}y + p_{13}z},\\ F_3 &= l_1 + l_2F_2.\end{aligned} \tag{32}$$

Case II ($a_3 \equiv 0$, $a_2 \neq 0$). From relation (8), we have

$$F_2 = \delta_1 + \delta_2y, \tag{33}$$

where

$$\begin{aligned}\delta_1 &= \frac{a_1 - \alpha'\overline{a}_1}{\alpha'\overline{a}_2},\\ \delta_2 &= \frac{a_2}{\alpha'\overline{a}_2}.\end{aligned} \tag{34}$$

Differentiating this identity with respect to t gives

$$\sum_{i=0}^{2}V_i{F_3}^i = 0, \tag{35}$$

where

$$\begin{aligned}
V_0 &= v_{01} + v_{02}y + v_{03}z + v_{04}y^2 + v_{05}yz + v_{06}z^2,\\
v_{01} &= \alpha'\left(\bar{b}_1 + \delta_1\bar{b}_2 + \delta_1{}^2\bar{b}_4\right) - \left(\delta_{1t}{}' + \delta_{1x}{}'a_1 + \delta_2 b_1\right)\\
v_{02} &= \alpha'\left(\delta_2\bar{b}_2 + 2\delta_1\delta_2\bar{b}_4\right)\\
&\quad - \left(\delta_{2t}{}' + \delta_{2x}{}'a_1 + \delta_{1x}{}'a_2 + \delta_2 b_2\right),\\
v_{03} &= -\delta_{1x}{}'a_3 - \delta_2 b_3,\\
v_{04} &= \alpha'\delta_2{}^2\bar{b}_4 - \delta_{2x}{}'a_2 - \delta_2 b_4,\\
v_{05} &= -\delta_{2x}{}'a_3 - \delta_2 b_5,\\
v_{06} &= -\delta_2 b_6.\\
V_1 &= v_{11} + v_{12}y,\\
v_{11} &= \alpha'\left(\bar{b}_3 + \delta_1\bar{b}_5\right),\\
v_{12} &= \alpha'\delta_2\bar{b}_5,\\
V_2 &= \alpha'\bar{b}_6.
\end{aligned} \tag{36}$$

Likewise, we have the following conclusions.

Lemma 5. *Suppose $F_1 = x$, $a_3 \equiv 0$, $a_2 \neq 0$, $b_6 \neq 0$, and the limits $\lim_{t\longrightarrow 0}(v_{0i}/b_6)$ $(i = 1, 2, ..., 5)$ and $\lim_{t\longrightarrow 0}(a_1/\alpha'\bar{a}_2)$ exist; then*

$$\begin{aligned}
&\lim_{t\longrightarrow 0}\frac{v_{0j}}{\alpha'\bar{b}_6} = 0 \quad (j = 1, 2, 4),\\
&\lim_{t\longrightarrow 0}\frac{v_{03} + v_{11}}{\alpha'\bar{b}_6} = 0,\\
&\operatorname{Lim}_{t\longrightarrow 0}\frac{v_{05} + v_{12}}{\alpha'\bar{b}_6} = 0,\\
&\lim_{t\longrightarrow 0}\frac{a_1 - \alpha'\bar{a}_1}{\alpha'\bar{a}_2} = 0,\\
&\lim_{t\longrightarrow 0}\frac{a_2}{\alpha'\bar{a}_2} = \lim_{t\longrightarrow 0}\frac{b_6}{\alpha'\bar{b}_6} = 1.
\end{aligned} \tag{37}$$

Theorem 6. *Suppose that all the conditions of Lemmas 1 and 5 are satisfied; then*

$$\begin{aligned}
F_2 &= \delta_1(t, x) + \delta_2(t, x)\, y,\\
F_3 &= f_{30}(t, x) + f_{31}(t, x)\, y + f_{32}(t, x)\, z.
\end{aligned} \tag{38}$$

Theorem 7. *Suppose that $F_1 = x$, $v_{11}{}^2 + v_{12}{}^2 \neq 0$, $b_3 = b_6 = 0$. $a_i(0, x) = 0$ $(i = 1, 2)$; then*

$$\begin{aligned}
F_2 &= \delta_1(t, x) + \delta_2(t, x)\, y,\\
F_3(t, x, y, z) &= -\frac{V_0}{V_1}\\
&= -\frac{v_{01} + v_{02}y + v_{03}z + v_{04}y^2 + v_{05}yz + v_{06}z^2}{v_{11} + v_{12}y}.
\end{aligned} \tag{39}$$

Theorem 8. *Function $F(t, x, y, z) = (x, f_{21}y + f_{22}z, g_{21}y + g_{22}z)^T$ is the GRF of system (3), if the following conditions are satisfied:*

$$\begin{aligned}
&a_1 - \alpha'\bar{a}_1 = 0,\\
&a_2 - \alpha'\bar{a}_2 f_{21} - \alpha'\bar{a}_3 g_{21} = 0,\\
&a_3 - \alpha'\bar{a}_3 f_{22} - \alpha'\bar{a}_3 g_{22} = 0\\
&\begin{pmatrix} f_{21}\\ f_{22}\end{pmatrix} b_1 + \begin{pmatrix} f_{22}\\ g_{22}\end{pmatrix} c_1 - \alpha'\begin{pmatrix}\bar{b}_1\\ \bar{c}_1\end{pmatrix} = 0,\\
&\begin{pmatrix} f_{21t}{}' & f_{22t}{}'\\ g_{21t}{}' & g_{22t}{}'\end{pmatrix} + \begin{pmatrix} f_{21x}{}' & f_{22x}{}'\\ g_{21x}{}' & g_{22x}{}'\end{pmatrix} a_1\\
&\quad + \begin{pmatrix} f_{21} & f_{22}\\ g_{21} & g_{22}\end{pmatrix}\begin{pmatrix} b_2 & b_3\\ c_2 & c_3\end{pmatrix} - \alpha'\begin{pmatrix}\bar{b}_2 & \bar{b}_3\\ \bar{c}_2 & \bar{c}_3\end{pmatrix}\begin{pmatrix} f_{21} & f_{22}\\ g_{21} & g_{22}\end{pmatrix}\\
&\quad = 0,\\
&\begin{pmatrix} f_{21x}{}'a_2 & f_{22x}{}'a_3\\ g_{21x}{}'a_2 & g_{22x}{}'a_3\end{pmatrix} + \begin{pmatrix} f_{21} & f_{22}\\ g_{21} & g_{22}\end{pmatrix}\begin{pmatrix} b_4 & b_6\\ c_4 & c_6\end{pmatrix}\\
&\quad - \alpha'\begin{pmatrix}\bar{b}_4 & \bar{b}_6\\ \bar{c}_4 & \bar{c}_6\end{pmatrix}\begin{pmatrix} f_{21}{}^2 & f_{22}{}^2\\ g_{21}{}^2 & g_{22}{}^2\end{pmatrix}\\
&\quad - \alpha'\begin{pmatrix} f_{21}g_{21}\bar{b}_5 & f_{22}g_{22}\bar{b}_5\\ f_{21}g_{21}\bar{c}_5 & f_{22}g_{22}\bar{c}_5\end{pmatrix} = 0,\\
&\begin{pmatrix} f_{21x}{}' & f_{22x}{}'\\ g_{21x}{}' & g_{22x}{}'\end{pmatrix}\begin{pmatrix} a_3\\ a_2\end{pmatrix} + \begin{pmatrix} f_{21} & f_{22}\\ g_{21} & g_{22}\end{pmatrix}\begin{pmatrix} b_5\\ c_5\end{pmatrix}\\
&\quad - \alpha'\begin{pmatrix} 2\bar{b}_4 f_{21}f_{22} + \bar{b}_5(f_{21}g_{22} + f_{22}g_{21}) + 2\bar{b}_6 g_{21}g_{22}\\ 2\bar{c}_4 f_{21}f_{22} \quad \bar{c}_5(f_{21}g_{22} + f_{22}g_{21}) + 2\bar{c}_6 g_{21}g_{22}\end{pmatrix}\\
&\quad = 0,\\
&\begin{pmatrix} f_{21} & f_{22}\\ g_{21} & g_{22}\end{pmatrix}\begin{pmatrix} \bar{f}_{21} & \bar{f}_{22}\\ \bar{g}_{21} & \bar{g}_{22}\end{pmatrix} = \begin{pmatrix} f_{21}(0, x) & f_{22}(0, x)\\ g_{21}(0, x) & g_{22}(0, x)\end{pmatrix}\\
&\quad = \begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}.
\end{aligned} \tag{40}$$

Proof. By checkout of the *BR*, it can be proved that the function

$$F = \left(x, f_{21}y + f_{22}z, g_{21}y + g_{22}z\right)^T \tag{41}$$

is the *GRF* of system (3). □

Theorem 9. *Let the hypotheses of Theorem 8 be satisfied, system (3) is 2ω–periodic with respect to t, and there exists a number η in R such that $\alpha(\eta) = 2\omega + \eta$; then all the solutions of system (3) defined on the interval $[\eta, \eta + 2\omega]$ are 2ω–periodic if and only if $f_{21}(2\omega+\eta, x) = 1$, $f_{22}(2\omega+\eta, x) = 0$, $g_{21}(2\omega + \eta, x) = 0$, and $g_{22}(2\omega + \eta, x) = 1$.*

Proof. By Theorem 8, the Poincaré mapping of system (3) is $T(x,y,z) = F(\alpha(\eta),x,y,z) = (x, f_{21}(2\omega+\eta,x)y + f_{22}(2\omega+\eta,x)z, g_{21}(2\omega+\eta,x)y + g_{22}(2\omega+\eta,x)z)^T$.

According to the previous introduction, the solutions of system (3) are 2ω–periodic if and only if $T(x,y,z) \equiv (x,y,z)^T$, which implies that the assertions of the present theorem hold. The proof is finished. □

Example 10. The differential system

$$\begin{aligned}
x' &= \left(1 - e^{\sin t}\left(1 - x^3 \sin t\right)\right) y - x^2 z \sin t \\
y' &= \frac{1}{2} y \cos t \left(x^6 \sin^2 t + x^3 - 1\right) - \frac{1}{2} \\
&\quad \cdot x^2 z \cos t e^{-\sin t}\left(1 + \sin t + x^3 \sin^2 t\right) \\
&\quad + y^2\left(3x^2 \sin t - x^5 \sin^2 t\right) \\
&\quad + yze^{-\sin t}\left(x^4 \sin^2 t - 2x \sin t\right) \\
z' &= \frac{1}{2} x^4 y \cos t e^{\sin t}\left(1 - \sin t + x^3 \sin^2 t\right) \\
&\quad \cdot \left(x^6 \sin^2 t + x^3 - 1\right) + \frac{1}{2} z \cos t \left(t - x^3 - x^6 \sin^2 t\right) \\
&\quad - y^2 e^{\sin t}\left(x^6 \sin^2 t - 4x^3 \sin t\right) \\
&\quad - yze^{-\sin t}\left(3x^2 \sin t - x^5 \sin^2 t\right)
\end{aligned} \tag{42}$$

has *GRF* $F(t,x,y,z) = (x, F_2, F_3)^T$, in which

$$\begin{aligned}
F_2(t,x,y,z) &= \left(1 + \alpha(t) x^3 \sin t\right) y e^{\sin t} \\
&\quad - \alpha(t) x^2 z \sin t, \\
F_3 &= \left(1 - \alpha(t) x^3 \sin t\right) z e^{-\sin t} \\
&\quad + \alpha(t) x^4 y \sin t, \quad \alpha(t) = -t.
\end{aligned} \tag{43}$$

Since this system is a 2π–periodic system, and there exists a number $\eta = -\pi$ in R such that $\alpha(\eta) = 2\pi + \eta$, then $T(x) = F(-\pi, x) = x$; by Theorem 9, all the considered solutions defined on $[-\pi, \pi]$ are 2π–periodic.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

References

[1] V. I. Arnold, *Ordinary Differential Equation*, Science Press, Moscow, Russia, 1971.

[2] V. I. Mironenko, "A method that allows one to determine the initial data of periodic solutions of differential systems and to compare the mappings for a period," *Differential Equations*, vol. 14, no. 11, pp. 1985–1994, 1980.

[3] V. I. Mironenko, *Reflective Function and Periodic Solution of the Differential System*, vol. 20, University Press, Minsk, Russia, 1984.

[4] P. P. Veresovich, "Nonautonomous two-dimensional quadratic systems that are equivalent to linear systems," *Differential Equations*, vol. 34, no. 12, pp. 2257–2259, 1998.

[5] E. V. Musafirov, "Differential systems, the mapping over period for which is represented by a product of three exponential matrixes," *Journal of Mathematical Analysis and Applications*, vol. 329, no. 1, pp. 647–654, 2007.

[6] Z. Zhengxin, "On the qualitative behavior of periodic solutions of differential systems," *Nonlinear Analysis*, vol. 232, no. 2, pp. 600–611, 2009.

[7] Z. Jian, "Differential systems with the same reflecting functions," *Applied Mathematics and Computation*, vol. 218, pp. 3144–3148, 2011.

[8] Z. Zhengxin, "On the first integral and equivalence of nonlinear differential equations," *Applied Mathematics and Computation*, vol. 268, pp. 295–302, 2015.

[9] Z. Zhengxin, "On the structure of the equivalent differential systems and their reflecting integrals," *Bulletin of the Brazilian Mathematical Society, New Series*, vol. 48, no. 3, pp. 439–447, 2017.