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Research Article

Differential Harnack Estimates for a Semilinear Parabolic System

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In this paper, we prove differential Harnack inequalities for positive solutions of a semilinear parabolic system on hyperbolic space. We use the inequalities to construct classical Harnack estimates by integrating along space-time.

1. Introduction

In this paper, we study the following problem:

$$\begin{aligned} f_t &= \Delta f + e^{\mu t} g^p, \quad \mathbb{R}^n \times (0, +\infty), \\ g_t &= \Delta g + e^{\nu t} f^q, \quad \mathbb{R}^n \times (0, +\infty), \\ f(x, 0) &= f_0, \\ g(x, 0) &= g_0, \\ &\mathbb{R}^n, \end{aligned} \tag{1}$$

where p, q, μ, ν are positive constants.

P. Li and S.-T.Yau in [1] were first pioneers to the study of differential Harnack inequalities which were brought to general parabolic geometric flows by R. Hamilton (see [2]). Using these inequalities can derive ancient solutions, bounds on gradient Ricci solitons, Holder continuity. Differential Harnack inequalities are important aspects of properties of partial differential equations. Paper [3] described differential Harnack inequalities to the initial value problem of a semilinear parabolic equation when the semilinear term is $e^{\mu t} f^p, \mu > 0$. There have been numerous interesting results on the properties of solutions of partial differential equations, such as existence of solutions [4–17], nonexistence and blow-up of solutions [18–22], and asymptotic behaviors of solutions [23–28].

Let $(f(x,t), g(x,t))$ be positive smooth solutions to (1) and $(u, v) := (\log f, \log g)$. The main object of our study is the following Harnack quantities:

$$\begin{aligned} H_1 &\equiv \alpha \Delta u + \beta_1 |\nabla u|^2 + c e^{\mu t + p v - u} + \psi_1(t) + \phi_1(x), \\ H_2 &\equiv \alpha \Delta v + \beta_2 |\nabla v|^2 + c e^{\nu t + q u - v} + \psi_2(t) + \phi_2(x), \end{aligned} \tag{2}$$

where $\alpha, \beta_i, c \in \mathbb{R}$, $\alpha > \max\{\beta_1, \beta_2\}$ and ψ_i, ϕ_i will be chosen suitably $i = 1, 2$.

We will derive our differential Harnack estimate.

Theorem 1. *Let $(f(x,t), g(x,t))$ be positive classical solutions to (1), and $(u, v) := (\log f, \log g)$. If α, β_i, k_i, and c satisfy*

$$\begin{aligned} &\alpha > \max\{\beta_1, \beta_2\} \geq 0, \\ &k_1 - p k_2 \geq 0, \\ &k_2 - q k_1 \geq 0, \\ &k_i \geq \frac{n\alpha^2}{2(\alpha - \beta_i)} > 0, \\ &-\beta_2 + \alpha p + c(1 - p) \geq 0, \\ &-\beta_1 + \alpha q + c(1 - q) \geq 0, \\ &\frac{4}{n\alpha^2}\beta_1 c + 1 - \frac{p^2(\alpha - \beta_1)}{-p\beta_2 + \alpha p^2 + cp(1 - p)} \geq 0, \\ &\frac{4}{n\alpha^2}\beta_2 c + 1 - \frac{q^2(\alpha - \beta_2)}{-q\beta_1 + \alpha q^2 + cq(1 - q)} \geq 0, \end{aligned} \tag{3}$$

then we have

$$H_1 \equiv \alpha\Delta u + \beta_1 |\nabla u|^2 + ce^{\mu t+pv-u} + \frac{k_1}{t} \geq 0,$$
$$H_2 \equiv \alpha\Delta v + \beta_2 |\nabla v|^2 + ce^{\mu t+qu-v} + \frac{k_2}{t} \geq 0, \tag{4}$$

for all t.

The paper is organized as follows. In Section 2 we prove Theorem 1 which describes differential Harnack estimate. There are applications of Theorem 1 in Section 3.

2. Harnack Estimate

In this section, we shall first obtain our differential Harnack inequalities, relying on the parabolic maximum principle.

Lemma 2. *Suppose $(f(x,t), g(x,t))$ are positive solutions to (1) and $(u,v) := (\log f, \log g)$ and (H_1, H_2) are defined as in (2). Assume that α, β_1, β_2, and c satisfy*

$$\begin{aligned}
&\alpha > \max\{\beta_1, \beta_2\} \geq 0,\\
&-\beta_2 + \alpha p + c(1-p) \geq 0,\\
&-\beta_1 + \alpha q + c(1-q) \geq 0,\\
&\frac{4}{n\alpha^2}\beta_1 c + 1 - \frac{p^2(\alpha-\beta_1)}{-p\beta_2 + \alpha p^2 + cp(1-p)} \geq 0,\\
&\frac{4}{n\alpha^2}\beta_2 c + 1 - \frac{q^2(\alpha-\beta_2)}{-q\beta_1 + \alpha q^2 + cq(1-q)} \geq 0.
\end{aligned} \tag{5}$$

Then we have

$$\begin{aligned}
\partial_t H_1 &\geq \Delta H_1 + 2\nabla H_1 \cdot \nabla u + h_{11} H_1\\
&\quad + e^{\mu t+pv-u}(pH_2 - H_1) + h_{12},\\
\partial_t H_2 &\geq \Delta H_2 + 2\nabla H_2 \cdot \nabla v + h_{21} H_2\\
&\quad + e^{vt+qu-v}(qH_1 - H_2) + h_{22},
\end{aligned} \tag{6}$$

where

$$\begin{aligned}
h_{11} &= 2(\alpha-\beta_1)\\
&\quad \cdot \frac{1}{n\alpha^2}\left\{H_1 - 2\left(\beta_1|\nabla u|^2 + ce^{\mu t+pv-u} + \psi_1 + \phi_1\right)\right\},\\
h_{12} &= \partial_t\psi_1 - \Delta\phi_1 + \frac{2}{n\alpha^2}(\alpha-\beta_1)(\psi_1+\phi_1)^2\\
&\quad - \frac{n\alpha^2|\nabla\phi_1|^2}{4\beta_1(\alpha-\beta_1)(\psi_1+\phi_1)}\\
&\quad + e^{\mu t+pv-u}(c\mu + \psi_1 + \phi_1 - p(\psi_2+\phi_2)),\\
h_{21} &= 2(\alpha-\beta_2)\\
&\quad \cdot \frac{1}{n\alpha^2}\left\{H_2 - 2\left(\beta_2|\nabla v|^2 + ce^{vt+qu-v} + \psi_2 + \phi_2\right)\right\},\\
h_{22} &= \partial_t\psi_2 - \Delta\phi_2 + \frac{2}{n\alpha^2}(\alpha-\beta_2)(\psi_2+\phi_2)^2\\
&\quad - \frac{n\alpha^2|\nabla\phi_2|^2}{4\beta_2(\alpha-\beta_2)(\psi_2+\phi_2)}\\
&\quad + e^{vt+qu-v}(cv + \psi_2 + \phi_2 - q(\psi_1+\phi_1)).
\end{aligned} \tag{7}$$

Proof. Substituting $(f,g) = (e^u, e^v)$ into (1), we have

$$\begin{aligned}
u_t &= \Delta u + |\nabla u|^2 + e^{\mu t+pv-u},\\
v_t &= \Delta v + |\nabla v|^2 + e^{vt+qu-v}.
\end{aligned} \tag{8}$$

Using the above equations, we have

$$\begin{aligned}
&(\partial_t - \Delta)\Delta u\\
&\quad = 2|\nabla\nabla u|^2 + 2\nabla u \cdot \nabla\Delta u\\
&\qquad + e^{\mu t+pv-u}\left(p\Delta v - \Delta u + |p\nabla v - \nabla u|^2\right),\\
&(\partial_t - \Delta)\left(|\nabla u|^2\right)\\
&\quad = 2\nabla u \cdot \nabla|\nabla u|^2 - 2|\nabla\nabla u|^2 + 2e^{\mu t+pv-u}\nabla u\\
&\qquad \cdot (p\nabla v - \nabla u).
\end{aligned} \tag{9}$$

Furthermore, applying (2) and Cauchy-Schwarz inequality $|\nabla\nabla u|^2 \geq (1/n)(\Delta u)^2$ yields

$$\begin{aligned}
&\partial_t H_1 - \Delta H_1 - 2\nabla H_1 \cdot \nabla u = 2(\alpha-\beta_1)|\nabla\nabla u|^2\\
&\quad + e^{\mu t+pv-u}(pH_2 - H_1)\\
&\quad + e^{\mu t+pv-u}\left\{\left(\alpha p^2 - p\beta_2 + cp(1-p)\right)|\nabla v|^2\right.\\
&\quad \left. - 2p(\alpha-\beta_1)\nabla u \cdot \nabla v\right\} + e^{\mu t+pv-u}(\alpha-\beta_1)|\nabla u|^2\\
&\quad + e^{\mu t+pv-u}(c\mu + \psi_1 + \phi_1 - p(\psi_2+\phi_2)) + \partial_t\psi_1\\
&\quad - \Delta\phi_1 - 2\nabla u \cdot \nabla\phi_1 \geq 2(\alpha-\beta_1)\frac{1}{n\alpha^2}H_1\left\{H_1\right.\\
&\quad \left. - 2\left(\beta_1|\nabla u|^2 + ce^{\mu t+pv-u} + \psi_1 + \phi_1\right)\right\} + 2(\alpha-\beta_1)\\
&\quad \cdot \frac{1}{n\alpha^2}\beta_1\left\{\beta_1|\nabla u|^4 + 2|\nabla u|^2\left(ce^{\mu t+pv-u} + \psi_1 + \phi_1\right)\right\}\\
&\quad + 2(\alpha-\beta_1)\frac{1}{n\alpha^2}\left(ce^{\mu t+pv-u} + \psi_1 + \phi_1\right)^2\\
&\quad + e^{\mu t+pv-u}(pH_2 - H_1)\\
&\quad + e^{\mu t+pv-u}\left\{p(\alpha p - \beta_2 + c(1-p))|\nabla v|^2\right.\\
&\quad \left. - 2p(\alpha-\beta_1)\nabla u \cdot \nabla v\right\} + e^{\mu t+pv-u}(\alpha-\beta_1)|\nabla u|^2\\
&\quad + e^{\mu t+pv-u}(c\mu + \psi_1 + \phi_1 - p(\psi_2+\phi_2)) + \partial_t\psi_1\\
&\quad - \Delta\phi_1 - 2\nabla u \cdot \nabla\phi_1
\end{aligned} \tag{10}$$

If $\alpha p - \beta_2 + c(1-p) \geq 0$, the above inequality is

$$\begin{aligned}
&\geq 2(\alpha-\beta_1)\frac{1}{n\alpha^2} \\
&\quad \cdot H_1\left\{H_1 - 2\left(\beta_1|\nabla u|^2 + ce^{\mu t+pv-u} + \psi_1 + \phi_1\right)\right\} \\
&\quad + e^{\mu t+pv-u}(pH_2 - H_1) + \frac{4\beta_1}{n\alpha^2}(\alpha-\beta_1)(\psi_1+\phi_1) \\
&\quad \cdot |\nabla u|^2 + 2(\alpha-\beta_1) \\
&\quad \cdot \frac{1}{n\alpha^2}\left\{\beta_1^2|\nabla u|^4 + \left(ce^{\mu t+pv-u} + \psi_1 + \phi_1\right)^2\right\} \\
&\quad + \left(\frac{4}{n\alpha^2}\beta_1 c + 1 - \frac{p(\alpha-\beta_1)}{\alpha p - \beta_2 + c(1-p)}\right)(\alpha-\beta_1) \\
&\quad \cdot e^{\mu t+pv-u}|\nabla u|^2 \\
&\quad + e^{\mu t+pv-u}(c\mu + \psi_1 + \phi_1 - p(\psi_2+\phi_2)) + \partial_t\psi_1 \\
&\quad - \Delta\phi_1 - 2\nabla u\cdot\nabla\phi_1
\end{aligned} \tag{11}$$

First note the following inequality:

$$\begin{aligned}
&\frac{4\beta_1}{n\alpha^2}(\alpha-\beta_1)(\psi_1+\phi_1)|\nabla u|^2 - 2\nabla u\cdot\nabla\phi_1 \\
&\quad \geq -\frac{n\alpha^2|\nabla\phi_1|^2}{4\beta_1(\alpha-\beta_1)(\psi_1+\phi_1)}.
\end{aligned} \tag{12}$$

If $\alpha > \beta_1$ and $(4/n\alpha^2)\beta_1 c + 1 - p^2(\alpha-\beta_1)/(-p\beta_2 + \alpha p^2 + cp(1-p)) \geq 0$, the above inequality is

$$\begin{aligned}
&\geq 2(\alpha-\beta_1)\frac{1}{n\alpha^2} \\
&\quad \cdot H_1\left\{H_1 - 2\left(\beta_1|\nabla u|^2 + ce^{\mu t+pv-u} + \psi_1 + \phi_1\right)\right\} \\
&\quad + e^{\mu t+pv-u}(pH_2 - H_1) - \frac{n\alpha^2|\nabla\phi_1|^2}{4\beta_1(\alpha-\beta_1)(\psi_1+\phi_1)} \\
&\quad + e^{\mu t+pv-u}(c\mu + \psi_1 + \phi_1 - p(\psi_2+\phi_2)) + \partial_t\psi_1 \\
&\quad - \Delta\phi_1 + \frac{2}{n\alpha^2}(\alpha-\beta_1)(\psi_1+\phi_1)^2.
\end{aligned} \tag{13}$$

For H_2, we have similar results.

$$\begin{aligned}
&\partial_t H_2 - \Delta H_2 - 2\nabla H_2\cdot\nabla v \geq 2(\alpha-\beta_2)\frac{1}{n\alpha^2}H_2\left\{H_2\right. \\
&\quad \left. - 2\left(\beta_2|\nabla v|^2 + ce^{vt+qu-v} + \psi_2 + \phi_2\right)\right\} + 2(\alpha-\beta_2) \\
&\quad \cdot \frac{1}{n\alpha^2}\beta_2\left\{\beta_2|\nabla v|^4 + 2|\nabla v|^2\left(ce^{vt+qu-v} + \psi_2 + \phi_2\right)\right\} \\
&\quad + 2(\alpha-\beta_2)\frac{1}{n\alpha^2}\left(ce^{vt+qu-v} + \psi_2 + \phi_2\right)^2 \\
&\quad + e^{vt+qu-v}(qH_1 - H_2) \\
&\quad + e^{vt+qu-v}\left\{q(-\beta_1 + \alpha q + c(1-q))|\nabla u|^2\right. \\
&\quad \left. - 2q(\alpha-\beta_2)\nabla u\cdot\nabla v\right\} + e^{vt+qu-v}(\alpha-\beta_2)|\nabla v|^2 \\
&\quad + e^{vt+qu-v}(cv + \psi_2 + \phi_2 - q(\psi_1+\phi_1)) + \partial_t\psi_2 \\
&\quad - \Delta\phi_2 - 2\nabla v\cdot\nabla\phi_2
\end{aligned} \tag{14}$$

If $\alpha q - \beta_1 + c(1-q) \geq 0$, the above inequality is

$$\begin{aligned}
&\geq 2(\alpha-\beta_2)\frac{1}{n\alpha^2} \\
&\quad \cdot H_2\left\{H_2 - 2\left(\beta_2|\nabla v|^2 + ce^{vt+qu-v} + \psi_2 + \phi_2\right)\right\} \\
&\quad + e^{vt+qu-v}(qH_1 - H_2) + \frac{4\beta_2}{n\alpha^2}(\alpha-\beta_2)(\psi_2+\phi_2) \\
&\quad \cdot |\nabla v|^2 + 2(\alpha-\beta_2) \\
&\quad \cdot \frac{1}{n\alpha^2}\left\{\beta_2^2|\nabla v|^4 + \left(ce^{vt+qu-v} + \psi_2 + \phi_2\right)^2\right\} \\
&\quad + \left(\frac{4}{n\alpha^2}\beta_2 c + 1 - \frac{q^2(\alpha-\beta_2)}{-q\beta_1 + \alpha q^2 + cq(1-q)}\right) \\
&\quad \cdot (\alpha-\beta_2)e^{vt+qu-v}|\nabla v|^2 \\
&\quad + e^{vt+qu-v}(cv + \psi_2 + \phi_2 - q(\psi_1+\phi_1)) + \partial_t\psi_2 \\
&\quad - \Delta\phi_2 - 2\nabla v\cdot\nabla\phi_2.
\end{aligned} \tag{15}$$

If $(4/n\alpha^2)\beta_2 c + 1 - q^2(\alpha-\beta_2)/(-q\beta_1 + \alpha q^2 + cq(1-q)) \geq 0$, the above inequality is

$$\begin{aligned}
&\geq 2(\alpha-\beta_2)\frac{1}{n\alpha^2} \\
&\quad \cdot H_2\left\{H_2 - 2\left(\beta_2|\nabla v|^2 + ce^{vt+qu-v} + \psi_2 + \phi_2\right)\right\} \\
&\quad + e^{vt+qu-v}(qH_1 - H_2) - \frac{n\alpha^2|\nabla\phi_2|^2}{4\beta_2(\alpha-\beta_2)(\psi_2+\phi_2)} \\
&\quad + e^{vt+qu-v}(cv + \psi_2 + \phi_2 - q(\psi_1+\phi_1)) + \partial_t\psi_2 \\
&\quad - \Delta\phi_2 + \frac{2}{n\alpha^2}(\alpha-\beta_2)(\psi_2+\phi_2)^2.
\end{aligned} \tag{16}$$

This completes the proof of Lemma 2. □

Next, we need to compute specific $\psi_i(t)$ and $\phi_i(x)$ that guarantee $h_{i2} > 0$ for the maximum principle to be applicable where $i = 1, 2$.

Lemma 3. *Assume $k_1 - pk_2 \geq 0$, $k_2 - qk_1 \geq 0$, $l_1 - pl_2 \geq 0$, $l_2 - ql_1 \geq 0$ and*

$$\begin{aligned}
&k_i \geq \frac{n\alpha^2}{2(\alpha-\beta_i)} > 0, \\
&l_i \geq \frac{n\alpha^2}{2(\alpha-\beta_i)}\left(6 + \frac{n\alpha^2}{\beta_i(\alpha-\beta_i)}\right), \quad i = 1, 2.
\end{aligned} \tag{17}$$

If $\psi_i(t) = k_i/t$, $\phi_i(x) = \sum_{m=1}^{n}(l_i/(x_m - a_m)^2 + l_i/(b_m - x_m)^2)$, $i = 1,2$ *for* $x \in \Omega = [a_1, b_1] \times [a_2, b_2] \times \cdots [a_n, b_n]$, *then for some* $x_0 \in \Omega$, $h_{12}|_{x=x_0} > 0$ *and* $h_{22}|_{x=x_0} > 0$.

Proof. From $k_1 - pk_2 \geq 0$, $k_2 - qk_1 \geq 0$ and $l_1 - pl_2 \geq 0$, $l_2 - ql_1 \geq 0$, we get $c\mu + \psi_1 + \phi_1 - p(\psi_2 + \phi_2) \geq 0$ and $c\nu + \psi_2 + \phi_2 - q(\psi_1 + \phi_1) \geq 0$.

Now applying Lemma 2 yields

$$\begin{aligned} h_{12} &\geq \partial_t \psi_1 - \Delta\phi_1 + \frac{2}{n\alpha^2}(\alpha - \beta_1)(\psi_1 + \phi_1)^2 \\ &\quad - \frac{n\alpha^2 |\nabla\phi_1|^2}{4\beta_1(\alpha - \beta_1)(\psi_1 + \phi_1)}, \\ h_{22} &\geq \partial_t \psi_2 - \Delta\phi_2 + \frac{2}{n\alpha^2}(\alpha - \beta_2)(\psi_2 + \phi_2)^2 \\ &\quad - \frac{n\alpha^2 |\nabla\phi_2|^2}{4\beta_2(\alpha - \beta_2)(\psi_2 + \phi_2)}. \end{aligned} \tag{18}$$

By the definitions of ψ_i and ϕ_i, we obtain

$$\begin{aligned} \Delta\phi_i &= \sum_{m=1}^{n}\left(\frac{6l_i}{(x_m - a_m)^4} + \frac{6l_i}{(b_m - x_m)^4}\right), \\ |\nabla\phi_i|^2 &= \sum_{m=1}^{n}\left(\frac{-2l_i}{(x_m - a_m)^3} + \frac{2l_i}{(b_m - x_m)^3}\right)^2, \end{aligned} \tag{19}$$

and

$$\frac{|\nabla\phi_i|^2}{\psi_i + \phi_i} \leq \sum_{m=1}^{n}\left(\frac{2\sqrt{l_i}}{(x_m - a_m)^2} + \frac{2\sqrt{l_i}}{(b_m - x_m)^2}\right)^2. \tag{20}$$

If

$$\begin{aligned} k_i &\geq \frac{n\alpha^2}{2(\alpha - \beta_i)} > 0, \\ l_i &\geq \frac{n\alpha^2}{2(\alpha - \beta_i)}\left(6 + \frac{n\alpha^2}{\beta_i(\alpha - \beta_i)}\right), \quad i = 1,2, \end{aligned} \tag{21}$$

then $h_{12} > 0$ and $h_{22} > 0$.

This proves Lemma 3. □

Proof of Theorem 1. Choose

$$\begin{aligned} \psi_i(t) &= \frac{k_i}{t}, \\ \phi_i(x) &= \sum_{m=1}^{n}\left(\frac{l_i}{(x_m - a_m)^2} + \frac{l_i}{(b_m - x_m)^2}\right), \end{aligned} \tag{22}$$

$i = 1,2$ for $x \in \Omega = [a_1, b_1] \times [a_2, b_2] \times \cdots [a_n, b_n]$. Note that

$$\begin{aligned} \lim_{t \to 0} H_i = \infty = \lim_{x_m \to a_m} H_i = \lim_{x_m \to b_m} H_i, \\ m = 1, 2, \cdots, n, \ i = 1, 2. \end{aligned} \tag{23}$$

Assume that there exists a first time t_0 and point $x_0 \in \Omega$ where $H_1(x_0, t_0) = 0$ and $H_2(x_0, t_0) > 0$. At (x_0, t_0); we have

$$\begin{aligned} \nabla H_1 &= 0, \\ \Delta H_1 &\geq 0, \\ \partial_t H_1 &\leq 0. \end{aligned} \tag{24}$$

Lemma 3 implies that

$$\begin{aligned} &\partial_t H_1 - \Delta H_1 - 2\nabla H_1 \cdot \nabla u - h_{11} H_1 \\ &\quad - e^{\mu t + p\nu - u}(pH_2 - H_1) \geq h_{12} > 0 \end{aligned} \tag{25}$$

This is a contradiction. Assume that there exist a first time t_0 and point $x_0 \in \Omega$ where $H_2(x_0, t_0) = 0$ and $H_1(x_0, t_0) > 0$. At (x_0, t_0), we have

$$\begin{aligned} \nabla H_2 &= 0, \\ \Delta H_2 &\geq 0, \\ \partial_t H_2 &\leq 0. \end{aligned} \tag{26}$$

Lemma 3 implies that

$$\begin{aligned} &\partial_t H_2 - \Delta H_2 - 2\nabla H_2 \cdot \nabla v - h_{21} H_2 \\ &\quad - e^{\nu t + qu - v}(qH_1 - H_2) \geq h_{22}, \end{aligned} \tag{27}$$

This is a contradiction. Furthermore, $H_1(x_0, t_0) = 0$ and $H_2(x_0, t_0) = 0$ cause the same contradiction as $H_1(x_0, t_0) = 0$ and $H_2(x_0, t_0) > 0$. Thus $H_1(x, t) > 0$ and $H_2(x, t) > 0$ for all $x, t > 0$.

Taking $\Omega \longrightarrow \mathbb{R}^n$ which obtains $\phi_i \longrightarrow 0$ then gives the desired result. □

3. Applications

In this section, we shall give an application of Theorem 1. We integrate along space-time to derive a classical Harnack inequality.

3.1. Classical Harnack Inequality. In this subsection, we integrate our differential Harnack inequality of Theorem 1 along space-time to derive a classical Harnack inequality.

Proposition 4. *Let* $(f(x,t), g(x,t))$ *be positive classical solutions to (1) and* $(u(x,t), v(x,t)) := (\log f, \log g)$. *Suppose that* $x_1, x_2 \in \mathbb{R}^n$ *and* $t_2 > t_1 > 0$. *Assume further that* $\alpha \geq 2\max\{\beta_1, \beta_2\}$, $\alpha \geq c$ *and* $k_i = n\alpha^2/2(\alpha - \beta_i)$, $i = 1,2$. *Then we have*

$$\begin{aligned} f(x_1, t_1) &\leq f(x_2, t_2)\left(\frac{t_2}{t_1}\right)^n \exp\left(\frac{|x_2 - x_1|^2}{2(t_2 - t_1)}\right), \\ g(x_1, t_1) &\leq g(x_2, t_2)\left(\frac{t_2}{t_1}\right)^n \exp\left(\frac{|x_2 - x_1|^2}{2(t_2 - t_1)}\right). \end{aligned} \tag{28}$$

Proof. Define the one-variable functions $w_i : [t_1, t_2] \longrightarrow \mathbb{R}$ as

$$w_1(t) := u(\gamma(t), t),$$

$$w_2(t) := v(\gamma(t), t) \tag{29}$$

for any C^1 path $\gamma : [t_1, t_2] \longrightarrow \mathbb{R}^n$ such that $\gamma(t_1) = x_1, \gamma(t_2) = x_2$.

Applying Theorem 1, we have

$$\Delta u \geq -\alpha^{-1}\left(\beta_1 |\nabla u|^2 + ce^{\mu t + pv - u} + \frac{k_1}{t}\right). \tag{30}$$

It yields that

$$\begin{aligned}
\frac{d}{dt}w_1(t) &= u_t + \nabla u \cdot \frac{d\gamma}{dt} \\
&= \Delta u + |\nabla u|^2 + e^{\mu t + pv - u} + \nabla u \cdot \frac{d\gamma}{dt} \\
&\geq -\alpha^{-1}\left(\beta_1 |\nabla u|^2 + ce^{\mu t + pv - u} + \frac{k_1}{t}\right) + |\nabla u|^2 \\
&\quad + e^{\mu t + pv - u} + \nabla u \cdot \frac{d\gamma}{dt} \\
&\geq \left(\frac{1}{2} - \frac{\beta_1}{\alpha}\right)|\nabla u|^2 + \left(1 - \frac{c}{\alpha}\right)e^{\mu t + pv - u} - \frac{k_1}{\alpha t} \\
&\quad - \frac{1}{2}\left|\frac{d\gamma}{dt}\right|^2 \geq -\frac{1}{2}\left|\frac{d\gamma}{dt}\right|^2 - \frac{n}{t},
\end{aligned} \tag{31}$$

where $\alpha \geq 0, \alpha \geq 2\beta_1, \alpha \geq c$ and $k_1 = n\alpha^2/2(\alpha - \beta_1)$. Similarly,

$$\frac{d}{dt}w_2(t) \geq -\frac{1}{2}\left|\frac{d\gamma}{dt}\right|^2 - \frac{n}{t}, \tag{32}$$

for $\alpha \geq 0, \alpha \geq 2\beta_2, \alpha \geq c$ and $k_2 = n\alpha^2/2(\alpha - \beta_2)$.

By

$$\begin{aligned}
&\int_{t_1}^{t_2}\left|\frac{d\gamma}{dt}\right|^2 dt \geq \frac{|x_2 - x_1|^2}{t_2 - t_1}, \\
&\int_{t_1}^{t_2} -\frac{d}{dt}w_i(t) \leq \inf_{\gamma(t)}\int_{t_1}^{t_2}\left(\frac{1}{2}\left|\frac{d\gamma}{dt}\right|^2 + \frac{n}{t}\right)dt, \quad i = 1, 2,
\end{aligned} \tag{33}$$

applying $(u, v) = (\log f, \log g)$ gives Proposition 4. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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