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Research Article

On the Uniform Convergence of Sine Series with Square Root

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Chaundy and Jolliffe proved that if $\{c_k\}_{k=1}^{\infty}$ is a nonincreasing real sequence with $\lim_{k\longrightarrow\infty} c_k = 0$, then the series $\sum_{k=1}^{\infty} c_k \sin kx$ converges uniformly if and only if $kc_k \longrightarrow 0$. The purpose of this paper is to show that $kc_k \longrightarrow 0$ is a necessary and sufficient condition for the uniform convergence of series $\sum_{k=1}^{\infty} c_k \sin \sqrt{k}\theta$ in $\theta \in [0,\pi]$. However for $\sum_{k=1}^{\infty} c_k \sin k^2\theta$ it is not true in $\theta \in [0,\pi]$.

1. Introduction

Chaundy and Jolliffe [1] proved the following.

Theorem 1. *If $\{c_k\}_{k=1}^{\infty} \subset R_+$ is decreasing to zero, then $\sum_{k=1}^{\infty} c_k \sin kx$ converges uniformly in x if and only if $kc_k \longrightarrow 0$ as $k \longrightarrow \infty$.*

Theorem 1 has had numerous generalizations.

Leindler [2] verified that in Theorem 1 the monotonicity assumption $c_n \geq c_{n+1}$ can be replaced by $c \in RBVS$, i.e., if the conditions $c_n \longrightarrow 0$ and $\sum_{k=n}^{\infty} |c_k - c_{k+1}| \leq Kc_n$ hold for all n with constant $K = K(c)$ which depends only upon c.

The next theorem was indicated in [3].

Theorem 2. *If $\{c_k\}$ belongs to the class MVBVS, i.e., if there exist constants C and $\lambda \geq 2$, depending only on the sequence $\{c_k\}$ such that*

$\sum_{k=n}^{2n} |c_k - c_{k+1}| \leq (C/n) \sum_{k=\lfloor\lambda^{-1}n\rfloor}^{\lfloor\lambda n\rfloor} c_k$ for all $n \geq \lambda$, then series $\sum_{k=1}^{\infty} c_k \sin kx$ converges uniformly in x if and only if $\lim_{k\longrightarrow\infty} kc_k = 0$.

Móricz [4] proves the following theorem.

Theorem 3. *Assume $f : R_+ \longrightarrow [0,\infty)$ with property $xf(x) \in L^1_{loc}(R_+)$. If $f(x)$ is nonincreasing on R_+, then integral $\int_0^{\infty} f(x) \sin tx\,dx$, $t \in R_+$, converges uniformly in t if and only if $xf(x) \longrightarrow 0$ as $x \longrightarrow \infty$.*

A result due to Žak and Šneider [5] holds for double sine series. $\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} c_{jk} \sin jx \sin ky$ is regularly convergent in case of a fixed (x,y) if the rectangular sums $\sum_{j=1}^{m}\sum_{k=1}^{n} c_{jk} \sin jx \sin ky$ converge to a finite number as m and n independently tend to infinity; moreover the row and column series $\sum_{j=1}^{\infty} c_{jn} \sin jx \sin ny$, $n = 1,2,\ldots$, and $\sum_{k=1}^{\infty} c_{mk} \sin mx \sin ky$, $m = 1,2,\ldots$, are convergent.

Theorem 4. *If $\{c_{jk}\}_{j,k=1}^{\infty} \subset R_+$ is a monotonically decreasing double sequence, i.e., sequence of real numbers such that, for $j,k = 1,2,\ldots$,*

$c_{jk} - c_{j+1,k} \geq 0$, $c_{jk} - c_{j,k+1} \geq 0$, and $c_{jk} - c_{j,k+1} - c_{j+1,k} + c_{j+1,k+1} \geq 0$, then $\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} c_{jk} \sin jx \sin ky$ is uniformly regularly convergent in (x,y) if and only if $jkc_{jk} \longrightarrow 0$ as $j + k \longrightarrow \infty$.

Theorem 4 was generalized by Kórus [6]. He has defined new classes of double sequences ($SBVDS_1$) to obtain those generalizations.

Duzinkiewicz and Szal [7] introduce a new class of double sequence called $DGM(\alpha,\beta,\gamma,r)$, which is a generalization of the class considered by Kórus, and they obtain sufficient and necessary conditions for uniform convergence of double sine series.

A series

$$\sum c_k e^{i\lambda_k\theta} \tag{1}$$

was motivation for the generalization of the Theorem 1. Such series were studied by Paley and Wiener who called them nonharmonic Fourier series. They proved the following [8].

Theorem 5. *If $|\lambda_k - k| \le D < 1/\pi^2$ for $-\infty < k < \infty$, then the sequence $\{e^{i\lambda_k\theta}\}$ is closed in $L^2(-\pi,\pi)$ and possesses a unique biorthogonal set $\{h_k(\theta)\}$, such that the series $\sum_{k=-\infty}^{\infty}\{(e^{i\lambda_k\theta}/2\pi)\int_{-\pi}^{\pi} f(t)e^{-ikt}dt - e^{i\lambda_k\theta}\int_{-\pi}^{\pi} f(t)h_k(t)dt\}$ converges uniformly to zero over interval $(-\pi+\delta,\pi-\delta)$ for any positive δ, and over any such interval the summability properties of $\sum_{k=-\infty}^{\infty} e^{i\lambda_k\theta}\int_{-\pi}^{\pi} f(t)h_k(t)dt$ are uniformly the same as those of the Fourier series of $f(\theta)$.*

We will consider a special case of the series (1) for $\lambda_k = \sqrt{k}$ and $\lambda_k = k^2$, $k \ge 1$, which does not meet the assumptions of the above theorem.

2. Main Results

Theorem 6. *If $\{c_k\}_{k=1}^{\infty} \subset R_+$ is nonincreasing, then the series $\sum_{k=1}^{\infty} c_k \sin\sqrt{k}\theta$ converges uniformly in $\theta \in [0,\pi]$ if and only if $\lim_{k\longrightarrow\infty} kc_k = 0$.*

Proof (necessary condition). Suppose that a series $\sum_{k=1}^{\infty} c_k \sin\sqrt{k}\theta$ converges uniformly on $[0,\pi]$. Let $\theta = \pi/\alpha$. We consider $\alpha = \sqrt{n}$ for $n = 4r$ and $r \in N = \{1,2,3,\ldots\}$: $\sum_{k=5n/4}^{9n/4} c_k \sin(\sqrt{k}\pi/\alpha) = \sum_{k=5n/4}^{9n/4} c_k \sin(\sqrt{k}\pi/\sqrt{n}) = \sum_{k=5n/4}^{9n/4} c_k \sin(\pi - \sqrt{k/n}\pi) \le \sum_{k=5n/4}^{9n/4} 2c_k(1-\sqrt{k}/\sqrt{n}) = 2(c_{5n/4}(1-\sqrt{5}/2)+\ldots+c_{9n/4}(1-3/2)) \le 2(1-\sqrt{5}/2)(c_{5n/4}+\ldots+c_{9n/4})) \le (1-\sqrt{5}/2)nc_{9n/4} = 4/9(1-\sqrt{5}/2)9rc_{9r}$.

Hence,

$$\forall r \in N \quad \left|\sum_{k=5r}^{9r} c_k \sin\sqrt{\frac{\pi^2 k}{4r}}\right| \ge \frac{4}{9}\left(\sqrt{5}/2 - 1\right) 9rc_{9r}. \tag{2}$$

After considering that $\sum_{k=1}^{\infty} c_k \sin\sqrt{k}\theta$ converges uniformly we obtain $|\sum_{k=5r}^{k=9r} c_k \sin\sqrt{\pi^2 k/4r}| < \epsilon$ for sufficiently large r. Thus, in view of inequality (2), we obtain $4/9(\sqrt{5}/2-1)9rc_{9r} < \epsilon$ for sufficiently large r, so

$$\lim_{r\longrightarrow\infty} 9rc_{9r} = 0. \tag{3}$$

After considering that the sequence $\{c_k\}$ is nonincreasing we have

$$\begin{aligned} 9rc_{9r} &\le 9rc_{9r-1} \le \ldots \le 9rc_{9r-8} \le 9rc_{9(r-1)} \\ &= 9(r-1)c_{9(r-1)} + 9c_{9(r-1)}. \end{aligned} \tag{4}$$

Thus

$$\forall s = 0,1,\ldots,8 \quad \lim_{r\longrightarrow\infty} 9rc_{9r-s} = 0 \tag{5}$$

by (3). In view of (5), we obtain $\forall s = 0,1,\ldots,8$ $\lim_{r\longrightarrow\infty}(9r-s)c_{9r-s} = \lim_{r\longrightarrow\infty} 9rc_{9r-s} - \lim_{r\longrightarrow\infty} sc_{9r-s} = 0$ and $\lim_{m\longrightarrow\infty} mc_m = 0$. □

Proof (sufficient condition). Let $\theta = \pi/\alpha$.

Case 1:

$$\alpha \ge \sqrt{p} \ge \sqrt{n}. \tag{6}$$

After considering $\sqrt{k}/\alpha \le \sqrt{p}/\sqrt{p} = 1$ and $\sin(\pi\sqrt{k}/\alpha) \le \pi(\sqrt{k}/\alpha)$ we obtain

$$\begin{aligned} 0 &\le \sum_{k=n}^{p} c_k \sin\frac{\pi\sqrt{k}}{\alpha} \le \frac{\pi}{\alpha}\sum_{k=n}^{p} c_k\sqrt{k} \\ &= \frac{\pi}{\alpha}\left(\frac{nc_n}{\sqrt{n}} + \ldots + \frac{pc_p}{\sqrt{p}}\right) \\ &\le \frac{\pi}{\alpha}\sup_{k\ge n}\{kc_k\}\left(\frac{1}{\sqrt{n}} + \ldots + \frac{1}{\sqrt{p}}\right) \\ &\le \pi \sup_{k\ge n}\{kc_k\}\frac{1}{\sqrt{p}}\sum_{k=1}^{p}\frac{1}{\sqrt{k}}. \end{aligned} \tag{7}$$

This follows from (6). Note that the following condition is fulfilled:

$$1 + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{p}} < \frac{1}{\sqrt{p+1}}\frac{\sqrt{p}}{\sqrt{p+1}-\sqrt{p}} \quad \text{for } p \ge 1. \tag{8}$$

In view of (8) the following inequality is satisfied for $b_p = (1/\sqrt{p})\sum_{k=1}^{p}(1/\sqrt{k})$:

$$\begin{aligned} \frac{b_{p+1}}{b_p} &= \frac{1}{\sqrt{p+1}}\left(\sqrt{p} + \frac{\sqrt{p}}{\sqrt{p+1}}\frac{1}{1+1/\sqrt{2}+\ldots+1/\sqrt{p}}\right) \\ &> \frac{1}{\sqrt{p+1}}\left(\sqrt{p} + \frac{\sqrt{p}}{\sqrt{p+1}}\frac{\sqrt{p+1}-\sqrt{p}}{\sqrt{p}/\sqrt{p+1}}\right) = 1. \end{aligned} \tag{9}$$

Thus the sequence b_p is increasing with respect to p and $\lim_{p\longrightarrow\infty} b_p = 2$.

$$\forall n \le p \quad \frac{1}{\sqrt{p}}\sum_{k=n}^{p}\frac{1}{\sqrt{k}} \le \lim_{p\longrightarrow\infty}\frac{1}{\sqrt{p}}\sum_{k=1}^{p}\frac{1}{\sqrt{k}} = 2. \tag{10}$$

This follows from (9). Finally for $n \le p$ and $\alpha \ge \sqrt{p}$,

$$0 \le \sum_{k=n}^{p} c_k \sin\frac{\pi\sqrt{k}}{\alpha} \le 2\pi\sup_{k\ge n}\{kc_k\}. \tag{11}$$

This follows from (7), (10).

To prove the case $\alpha \leq \sqrt{p}$ we first observe the following.

Lemma 2.2. Let $\alpha \geq 1$ and $m \in N$. Let $\lfloor\ \rfloor$ denote the floor function, i.e., $\lfloor x \rfloor = z \Longleftrightarrow z \in Z$ and $x - 1 < z \leq x$. Then

$$\forall s = 1,2,3,\ldots,\left\lfloor \alpha^2\left(m+\frac{3}{4}\right)\right\rfloor - 1$$

$$\alpha\left(2m+\frac{3}{2}\right) \leq \sqrt{\left\lfloor \alpha^2 (m+1)^2 \right\rfloor - s} + \sqrt{\left\lfloor \alpha^2 (m+1)^2 \right\rfloor + s} \leq 2\alpha (m+1) \quad (12)$$

$$\forall s = 1,2,3,\ldots,\left\lfloor \alpha^2\left(m+\frac{3}{4}\right)\right\rfloor - 1$$

$$-\alpha \leq \sqrt{\left\lfloor \alpha^2 (m+1)^2 \right\rfloor - s} - \sqrt{\left\lfloor \alpha^2 (m+1)^2 \right\rfloor + s} \leq 0 \quad (13)$$

$$\forall s = \left\lfloor \alpha^2\left(m+\frac{3}{4}\right)\right\rfloor + 2,\ldots,\left\lfloor \alpha^2 (2m+1)\right\rfloor - 1$$

$$\alpha(2m+3) \geq \sqrt{\left\lfloor \alpha^2 (m+1)^2 \right\rfloor + 1 - s} + \sqrt{\left\lfloor \alpha^2 (m+1)^2 + 2\alpha^2 \right\rfloor + 1 + s} \geq 2\alpha(m+1) \quad (14)$$

$$\forall s = \left\lfloor \alpha^2\left(m+\frac{3}{4}\right)\right\rfloor + 2,\ldots,\left\lfloor \alpha^2 (2m+1)\right\rfloor - 1$$

$$-2\alpha \leq \sqrt{\left\lfloor \alpha^2 (m+1)^2 \right\rfloor + 1 - s} - \sqrt{\left\lfloor \alpha^2 (m+1)^2 + 2\alpha^2 \right\rfloor + 1 + s} \leq -\alpha. \quad (15)$$

The proof of (12).

Note that for $s = 0,1,2,\ldots,\alpha^2(m+3/4)$ the following condition is fulfilled:

$$\sqrt{\alpha^2 (m+1)^2 - s} + \sqrt{\alpha^2 (m+1)^2 + s} \leq 2\alpha (m+1), \quad (16)$$

which follows from the relationship:

$$\alpha^2 (m+1)^2 - s + \alpha^2 (m+1)^2 + s + 2\sqrt{\alpha^4 (m+1)^4 - s^2} \leq 4\alpha^2 (m+1)^2 \quad \text{and } \alpha^4 (m+1)^4 - s^2 \leq \alpha^4 (m+1)^4. \quad (17)$$

For $s = 1,2,\ldots,\lfloor \alpha^2(m+3/4)\rfloor - 1$,

$$\sqrt{\left\lfloor \alpha^2 (m+1)^2 \right\rfloor - s} + \sqrt{\left\lfloor \alpha^2 (m+1)^2 \right\rfloor + s} \leq \sqrt{\alpha^2 (m+1)^2 - s} + \sqrt{\alpha^2 (m+1)^2 + s} \leq 2\alpha (m+1). \quad (18)$$

This follows from (16).

The proof of (14).

Note that for $s = \alpha^2(m+3/4),\ldots,\alpha^2(2m+1)$ the following condition is fulfilled:

$$\sqrt{\alpha^2 (m+1)^2 - s} + \sqrt{\alpha^2 (m+1)^2 + 2\alpha^2 + s} \geq 2\alpha (m+1), \quad (19)$$

which follows from the relationship:

$$2\alpha^2 + 2\alpha^2 (m+1)^2 + 2\sqrt{\alpha^4 (m+1)^4 - s^2 - 2\alpha^2 s + 2\alpha^4 (m+1)^2} \geq 4\alpha^2 (m+1)^2 \quad (20)$$

$$\text{and } \alpha^4 (m+1)^4 - s^2 - s\alpha^2 s + 2\alpha^4 (m+1)^2 \geq \alpha^4 (m+1)^4 - 2\alpha^4 (m+1)^2 + \alpha^4 \quad (21)$$

for $s \in [0, \alpha^2(2m+1)]$.

For $s = \lfloor \alpha^2(m+3/4)\rfloor + 2,\ldots,\lfloor \alpha^2(2m+1)\rfloor - 1$,

$$\sqrt{\left\lfloor \alpha^2 (m+1)^2 \right\rfloor + 1 - s} + \sqrt{\left\lfloor \alpha^2 (m+1)^2 + 2\alpha^2 \right\rfloor + 1 + s} \geq \sqrt{\alpha^2 (m+1)^2 - s} + \sqrt{\alpha^2 (m+1)^2 + 2\alpha^2 + s} \geq 2\alpha (m+1). \quad (22)$$

This follows from (19). The proof of the rest of Lemma 2.2 is obvious.

Case 2:

$$1 \leq \alpha \leq \sqrt{n} \leq \sqrt{p}. \quad (23)$$

Case 2':

$$\sqrt{\frac{p}{n}} < 7. \quad (24)$$

Therefore,

$$\sum_{k=n}^{p} c_k \left|\sin \frac{\pi\sqrt{k}}{\alpha}\right| \leq (p-n+1)\, c_n \leq 49 n c_n \leq 49 \sup_{k\geq n} \{k c_k\}. \quad (25)$$

Case 2":

$$\sqrt{\frac{p}{n}} \geq 7. \tag{26}$$

In view of (23), for all α there are

an odd number $m_n'(\alpha) \geq 1$ that:

$$m_n'(\alpha) - 2 < \frac{\sqrt{n}}{\alpha} \leq m_n'(\alpha), \tag{27}$$

an even number $m_n''(\alpha) \geq 2$ that:

$$m_n''(\alpha) - 2 < \frac{\sqrt{n}}{\alpha} \leq m_n''(\alpha). \tag{28}$$

Note that

$$\begin{aligned}
\sum_{k=n}^{\lfloor \alpha^2 m_n'^2(\alpha)\rfloor} c_k \left|\sin\frac{\pi\sqrt{k}}{\alpha}\right| &\leq \sum_{k=n}^{\lfloor \alpha^2 m_n'^2(\alpha)\rfloor} c_k \\
&\leq \left(\alpha^2 m_n'^2(\alpha) - n + 1\right) c_n \\
&= \left(\alpha^2 \left(m_n'(\alpha) - 2\right)^2 + 4\alpha^2 m_n'(\alpha) - 4\alpha^2 - n + 1\right) c_n \\
&\leq \left(n + 4\alpha^2 \left(m_n'(\alpha) - 1\right) - n + 1\right) c_n \\
&= \left(4\alpha^2 \left(m_n'(\alpha) - 2\right) + 4\alpha^2 + 1\right) c_n \\
&\leq \left(4\alpha\sqrt{n} + 4\alpha^2 + 1\right) c_n \leq 9nc_n \leq 9\sup_{k\geq n}\{kc_k\},
\end{aligned} \tag{29}$$

which follows from (23) and (27). The proof of

$$\sum_{k=n}^{\lfloor \alpha^2 m_n''^2(\alpha)\rfloor} c_k \left|\sin\frac{\pi\sqrt{k}}{\alpha}\right| \leq 9\sup_{k\geq n}\{kc_k\} \tag{30}$$

is similar. This follows from (23) and (28). Moreover for all α there are

an odd number $m_p'(\alpha) \geq 1$ that:

$$m_p'(\alpha) \leq \frac{\sqrt{p}}{\alpha} < m_p'(\alpha) + 2, \tag{31}$$

an even number $m_p''(\alpha) \geq 2$ that:

$$m_p''(\alpha) \leq \frac{\sqrt{p}}{\alpha} < m_p''(\alpha) + 2. \tag{32}$$

Note that

$$\begin{aligned}
&\sum_{k=\lfloor \alpha^2 m_p'^2(\alpha)\rfloor+1}^{p} c_k \left|\sin\frac{\pi\sqrt{k}}{\alpha}\right| \\
&\leq \left(p - \left\lfloor \alpha^2 m_p'(\alpha)\right\rfloor\right) c_{\lfloor \alpha^2 m_p'^2(\alpha)\rfloor+1} \\
&\leq \left(p - \alpha^2 m_p'(\alpha) + 1\right) c_{\lfloor \alpha^2 m_p'^2(\alpha)\rfloor+1} \\
&\leq \frac{p - \alpha^2 m_p'^2(\alpha) + 1}{\left\lfloor \alpha^2 m_p'^2(\alpha)\right\rfloor + 1}\left(\left\lfloor \alpha^2 m_p'^2(\alpha)\right\rfloor + 1\right) c_{\lfloor \alpha^2 m_p'^2(\alpha)\rfloor+1}.
\end{aligned} \tag{33}$$

Let us observe that $\sqrt{p}/\alpha - \sqrt{n}/\alpha = (\sqrt{n}/\alpha)(\sqrt{p/n} - 1) \geq 6$. This follows from (23) and (26). Hence,

$$\frac{\sqrt{p}}{\alpha} - 4 \geq \frac{\sqrt{n}}{\alpha} + 2. \tag{34}$$

Therefore,

$$m_p'(\alpha) - 2 > \frac{\sqrt{p}}{\alpha} - 4 \geq \frac{\sqrt{n}}{\alpha} + 2 > m_n'(\alpha). \tag{35}$$

This follows from (27), (31), and (34). The proof of

$$m_p''(\alpha) - 2 > m_n''(\alpha) \tag{36}$$

is similar. This follows from (28), (32), and (34).

Furthermore $\lfloor \alpha^2 m_p'^2(\alpha)\rfloor + 1 \geq n$. This follows from (27), (35). Thus,

$$\left(\left\lfloor \alpha^2 m_p'^2(\alpha)\right\rfloor + 1\right) c_{\lfloor \alpha^2 m_p'^2(\alpha)\rfloor+1} \leq \sup_{k\geq n}\{kc_k\}. \tag{37}$$

Note that

$$\begin{aligned}
\frac{p - \alpha^2 m_p'^2(\alpha) + 1}{\left\lfloor \alpha^2 m_p'^2(\alpha)\right\rfloor + 1} &\leq \frac{p}{\alpha^2 m_p'^2(\alpha)} - 1 + \frac{2}{\alpha^2 m_p'^2(\alpha)} \\
&\leq \frac{p}{\left(\alpha\left(m_p'(\alpha) + 2\right) - 2\alpha\right)^2} + 1 \\
&\leq \frac{p}{\left(\sqrt{p} - 2\alpha\right)^2} + 1 \\
&= \frac{1}{\left(1 - 2\alpha/\sqrt{p}\right)^2} + 1 \\
&\leq \frac{1}{\left(1 - 2\sqrt{n}/\sqrt{p}\right)^2} + 1 \\
&\leq \frac{1}{\left(1 - 2/7\right)^2} + 1 < 3,
\end{aligned} \tag{38}$$

which follows from the relationship $\alpha(m_p'(\alpha) + 2) > \sqrt{p} \geq 7\sqrt{n} \geq 7\alpha, \alpha \geq 1$ and $2\alpha \leq 2\sqrt{n} \leq (2/7)\sqrt{p}$ (this follows from (23), (26), and (31)).

Therefore, for $\sqrt{p/n} \geq 7$,

$$\sum_{k=\lfloor \alpha^2 m_p'^2(\alpha)\rfloor+1}^{p} c_k \left|\sin\frac{\pi\sqrt{k}}{\alpha}\right| < 3\sup_{k\geq n}\{kc_k\}. \tag{39}$$

This follows from (33), (37), and (38). In analogy with (33), (37), and (38) we have

$$\begin{aligned}
&\sum_{k=\lfloor \alpha^2 m_p''^2(\alpha)\rfloor+1}^{p} c_k \left|\sin\frac{\pi\sqrt{k}}{\alpha}\right| \\
&\quad\leq \frac{p - \alpha^2 m_p''^2(\alpha) + 1}{\left\lfloor \alpha^2 m_p''^2(\alpha)\right\rfloor + 1}\left(\left\lfloor \alpha^2 m_p''^2(\alpha)\right\rfloor + 1\right) \\
&\quad\cdot c_{\lfloor \alpha^2 m_p''^2(\alpha)\rfloor+1};
\end{aligned} \tag{40}$$

$$\left(\left\lfloor \alpha^2 m_p''^2(\alpha)\right\rfloor + 1\right) c_{\lfloor \alpha^2 m_p''^2(\alpha)\rfloor+1} \leq \sup_{k\geq n}\{kc_k\}, \tag{41}$$

this follows from (28) and (36);

$$\frac{p - \alpha^2 m_p''^2(\alpha) + 1}{\lfloor \alpha^2 m_p''^2(\alpha) \rfloor + 1} < 3, \tag{42}$$

which follows from the relationship

$\alpha(m_p''(\alpha) + 2) > \sqrt{p} \geq 7\sqrt{n} \geq 7\alpha, \alpha \geq 1$ and $2\alpha \leq 2\sqrt{n} \leq (2/7)\sqrt{p}$ (this follows from (23), (26), and (32)). Therefore, for $\sqrt{p/n} \geq 7$,

$$\sum_{k=\lfloor \alpha^2 m_p''^2(\alpha) \rfloor + 1}^{p} c_k \left| \sin \frac{\pi\sqrt{k}}{\alpha} \right| < 3 \sup_{k \geq n} \{k c_k\}. \tag{43}$$

This follows from (40), (41), and (42). Denote by S the sum $\sum_{k=n}^{p} c_k \sin(\pi\sqrt{k}/\alpha)$. Let us observe that

$$S = \sum_{k=n}^{\lfloor \alpha^2 m_n'^2(\alpha) \rfloor} c_k \sin \frac{\pi\sqrt{k}}{\alpha} + \sum_{k=\lfloor \alpha^2 m_p'^2(\alpha) \rfloor}^{p} c_k \sin \frac{\pi\sqrt{k}}{\alpha} + S',$$
$$\text{where } S' = \sum_{\substack{m=m_n'(\alpha) \\ m \in \text{Odd numbers}}}^{m_p'(\alpha)-2} \left\{ \sum_{z=\lfloor \alpha^2 m^2 \rfloor + 1}^{\lfloor \alpha^2 (m+1/2)^2 \rfloor} + \sum_{z=\lfloor \alpha^2 (m+1/2)^2 \rfloor + 1}^{\lfloor \alpha^2 (m+1)^2 \rfloor} + \sum_{z=\lfloor \alpha^2 (m+1)^2 \rfloor + 1}^{\lfloor \alpha^2 (m+3/2)^2 \rfloor} + \sum_{z=\lfloor \alpha^2 (m+3/2)^2 \rfloor + 1}^{\lfloor \alpha^2 (m+2)^2 \rfloor} c_z \sin \frac{\pi\sqrt{z}}{\alpha} \right\}. \tag{44}$$

Then,

$$S \leq \sum_{k=n}^{\lfloor \alpha^2 m_n'^2(\alpha) \rfloor} c_k \left| \sin \frac{\pi\sqrt{k}}{\alpha} \right| + \sum_{k=\lfloor \alpha^2 m_p'^2(\alpha) \rfloor}^{p} c_k \left| \sin \frac{\pi\sqrt{k}}{\alpha} \right| + S' \leq 12 \sup_{k \geq n} \{k c_k\} + S'. \tag{45}$$

This follows from (29) and (39). Note that for any odd number m and $\forall \alpha \geq 1$ we have

$$c_{\lfloor \alpha^2 (m+1)^2 \rfloor} \left(\sum_{z=\lfloor \alpha^2 m^2 \rfloor + 1}^{\lfloor \alpha^2 (m+1/2)^2 \rfloor} \sin \frac{\pi\sqrt{z}}{\alpha} + \sum_{z=\lfloor \alpha^2 (m+1/2)^2 \rfloor + 1}^{\lfloor \alpha^2 (m+1)^2 \rfloor} \sin \frac{\pi\sqrt{z}}{\alpha} \right) \geq \sum_{z=\lfloor \alpha^2 m^2 \rfloor + 1}^{\lfloor \alpha^2 (m+1/2)^2 \rfloor} c_z \sin \frac{\pi\sqrt{z}}{\alpha} + \sum_{z=\lfloor \alpha^2 (m+1/2)^2 \rfloor + 1}^{\lfloor \alpha^2 (m+1)^2 \rfloor} c_z \sin \frac{\pi\sqrt{z}}{\alpha}, \tag{46}$$

which follows from the relationship

$$\sin \frac{\pi\sqrt{\lfloor \alpha^2 m^2 \rfloor + 1}}{\alpha} \leq 0, \ldots, \sin \frac{\pi\sqrt{\lfloor \alpha^2 (m+1)^2 \rfloor}}{\alpha} \leq 0. \tag{47}$$

On the other hand, for any odd number m and $\forall \alpha \geq 1$, we have

$$c_{\lfloor \alpha^2 (m+1)^2 \rfloor} \left(\sum_{z=\lfloor \alpha^2 (m+1)^2 \rfloor + 1}^{\lfloor \alpha^2 (m+3/2)^2 \rfloor} \sin \frac{\pi\sqrt{z}}{\alpha} + \sum_{z=\lfloor \alpha^2 (m+3/2)^2 \rfloor + 1}^{\lfloor \alpha^2 (m+2)^2 \rfloor} \sin \frac{\pi\sqrt{z}}{\alpha} \right) \geq \sum_{z=\lfloor \alpha^2 (m+1)^2 \rfloor + 1}^{\lfloor \alpha^2 (m+3/2)^2 \rfloor} c_z \sin \frac{\pi\sqrt{z}}{\alpha} + \sum_{z=\lfloor \alpha^2 (m+3/2)^2 \rfloor + 1}^{\lfloor \alpha^2 (m+2)^2 \rfloor} c_z \sin \frac{\pi\sqrt{z}}{\alpha}, \tag{48}$$

which follows from the relationship

$$\sin \frac{\pi\sqrt{\lfloor \alpha^2 (m+1)^2 \rfloor + 1}}{\alpha} \geq 0, \ldots, \sin \frac{\pi\sqrt{\lfloor \alpha^2 (m+2)^2 \rfloor}}{\alpha} \geq 0. \tag{49}$$

Thus

$$S' \leq \sum_{\substack{m=m_n'(\alpha) \\ m \in \text{Odd numbers}}}^{m_p'(\alpha)-2} c_{\lfloor \alpha^2 (m+1)^2 \rfloor} \sum_{z=\lfloor \alpha^2 m^2 \rfloor + 1}^{\lfloor \alpha^2 (m+2)^2 \rfloor} \sin \frac{\pi\sqrt{z}}{\alpha}. \tag{50}$$

This follows from (46) and (48). Note that $\forall m \in N$ we have

$$\sum_{s=1}^{\lfloor \alpha^2 (m+3/4) \rfloor - 1} \left\{ \sin \frac{\pi\sqrt{\lfloor \alpha^2 (m+1)^2 \rfloor - s}}{\alpha} + \sin \frac{\pi\sqrt{\lfloor \alpha^2 (m+1)^2 \rfloor + s}}{\alpha} \right\} = 2 \sum_{s=1}^{\lfloor \alpha^2 (m+3/4) \rfloor - 1} \left\{ \sin \frac{\pi\left(\sqrt{\lfloor \alpha^2 (m+1)^2 \rfloor - s} + \sqrt{\lfloor \alpha^2 (m+1)^2 \rfloor + s}\right)}{2\alpha} \cos \frac{\pi\left(\sqrt{\lfloor \alpha^2 (m+1)^2 \rfloor - s} - \sqrt{\lfloor \alpha^2 (m+1)^2 \rfloor + s}\right)}{2\alpha} \right\}. \tag{51}$$

We see that $\forall s = 1, 2, \ldots, \lfloor \alpha^2 (m+3/4) \rfloor - 1$, $m \in N$ and $\alpha \ge 1$,

$$\left(m + \frac{3}{4}\right)\pi \le \frac{\pi}{2\alpha}\left(\sqrt{\lfloor \alpha^2 (m+1)^2 \rfloor - s} + \sqrt{\lfloor \alpha^2 (m+1)^2 \rfloor + s}\right) \le (m+1)\pi. \quad (52)$$

This follows from (12). Thus $\forall s = 1, 2, \ldots, \lfloor \alpha^2 (m+3/4) \rfloor - 1$, $\alpha \ge 1$, and for any odd number m we have

$$-1 \le \sin \frac{\pi\left(\sqrt{\lfloor \alpha^2 (m+1)^2 \rfloor - s} + \sqrt{\lfloor \alpha^2 (m+1)^2 \rfloor + s}\right)}{2\alpha} \le 0. \quad (53)$$

Moreover $\forall s = 1, 2, \ldots, \lfloor \alpha^2 (m+3/4) \rfloor - 1$, $m \in N$ and $\alpha \ge 1$, we get

$$-\frac{\pi}{2} \le \frac{\pi}{2\alpha}\left(\sqrt{\lfloor \alpha^2 (m+1)^2 \rfloor - s} - \sqrt{\lfloor \alpha^2 (m+1)^2 \rfloor + s}\right) \le 0. \quad (54)$$

This follows from (13). Thus $\forall s = 1, 2, \ldots, \lfloor \alpha^2 (m+3/4) \rfloor - 1$, $\alpha \ge 1$, and for any odd number m we have

$$0 \le \cos \frac{\pi\left(\sqrt{\lfloor \alpha^2 (m+1)^2 \rfloor - s} - \sqrt{\lfloor \alpha^2 (m+1)^2 \rfloor + s}\right)}{2\alpha} \le 1. \quad (55)$$

In view of (51), (53), and (55), for any odd number m, the following inequality is satisfied:

$$\sum_{s=1}^{\lfloor \alpha^2 (m+3/4) \rfloor - 1} \left\{ \sin \frac{\pi \sqrt{\lfloor \alpha^2 (m+1)^2 \rfloor - s}}{\alpha} + \sin \frac{\pi \sqrt{\lfloor \alpha^2 (m+1)^2 \rfloor + s}}{\alpha} \right\} \le 0. \quad (56)$$

Note that $\forall m \in N$

$$\begin{aligned} &\sum_{s=\lfloor \alpha^2 (m+3/4) \rfloor + 2}^{\lfloor \alpha^2 (2m+1) \rfloor - 1} \left\{ \sin \frac{\pi \sqrt{\lfloor \alpha^2 (m+1)^2 \rfloor + 1 - s}}{\alpha} + \sin \frac{\pi \sqrt{\lfloor \alpha^2 (m+1)^2 + 2\alpha^2 \rfloor + 1 + s}}{\alpha} \right\} \\ &= 2 \sum_{s=\lfloor \alpha^2 (m+3/4) \rfloor + 2}^{\lfloor \alpha^2 (2m+1) \rfloor - 1} \sin \frac{\pi\left(\sqrt{\lfloor \alpha^2 (m+1)^2 \rfloor + 1 - s} + \sqrt{\lfloor \alpha^2 (m+1)^2 + 2\alpha^2 \rfloor + 1 + s}\right)}{2\alpha} \\ &\quad \cdot \cos \frac{\pi\left(\sqrt{\lfloor \alpha^2 (m+1)^2 \rfloor + 1 - s} - \sqrt{\lfloor \alpha^2 (m+1)^2 + 2\alpha^2 \rfloor + 1 + s}\right)}{2\alpha}. \end{aligned} \quad (57)$$

We see that $\forall s = \lfloor \alpha^2 (m+3/4) \rfloor + 2, \ldots, \lfloor \alpha^2 (2m+1) \rfloor - 1$, $m \in N$ and $\alpha \ge 1$,

$$(m+1)\pi \le \frac{\pi}{2\alpha}\left(\sqrt{\lfloor \alpha^2 (m+1)^2 \rfloor + 1 - s} + \sqrt{\lfloor \alpha^2 (m+1)^2 + 2\alpha^2 \rfloor + 1 + s}\right) \le \left(m + \frac{3}{2}\right)\pi. \quad (58)$$

This follows from (14). Thus $\forall s = \lfloor \alpha^2 (m+3/4) \rfloor + 2, \ldots, \lfloor \alpha^2 (2m+1) \rfloor - 1$, $\alpha \ge 1$, and for any odd number m we have

$$0 \le \sin\left(\frac{\pi \sqrt{\lfloor \alpha^2 (m+1)^2 \rfloor + 1 - s}}{2\alpha} + \frac{\pi \sqrt{\lfloor \alpha^2 (m+1)^2 + 2\alpha^2 \rfloor + 1 + s}}{2\alpha}\right) \le 1. \quad (59)$$

We see that $\forall s = \lfloor \alpha^2 (m+3/4) \rfloor + 2, \ldots, \lfloor \alpha^2 (2m+1) \rfloor - 1$, $m \in N$ and $\alpha \ge 1$,

$$-\pi \le \frac{\pi}{2\alpha}\left(\sqrt{\lfloor \alpha^2 (m+1)^2 \rfloor + 1 - s} - \sqrt{\lfloor \alpha^2 (m+1)^2 + 2\alpha^2 \rfloor + 1 + s}\right) \le -\frac{\pi}{2}. \quad (60)$$

This follows from (15). Thus $\forall s = \lfloor \alpha^2 (m+3/4) \rfloor + 2, \ldots, \lfloor \alpha^2 (2m+1) \rfloor - 1$, $\alpha \ge 1$, and for any odd number m we have

$$-1 \le \cos\left(\frac{\pi \sqrt{\lfloor \alpha^2 (m+1)^2 \rfloor + 1 - s}}{2\alpha} - \frac{\pi \sqrt{\lfloor \alpha^2 (m+1)^2 + 2\alpha^2 \rfloor + 1 + s}}{2\alpha}\right) \le 0. \quad (61)$$

In view of (57), (59), and (61), for any odd number m, the following inequality is satisfied:

$$\sum_{s=\lfloor\alpha^2(m+3/4)\rfloor+2}^{\lfloor\alpha^2(2m+1)\rfloor-1}\left\{\sin\frac{\pi\sqrt{\lfloor\alpha^2(m+1)^2\rfloor+1-s}}{\alpha}+\sin\frac{\pi\sqrt{\lfloor\alpha^2(m+1)^2+2\alpha^2\rfloor+1+s}}{\alpha}\right\}\leq 0. \tag{62}$$

Now we see that $\forall m\in N$

$$\begin{aligned}
&\sin\frac{\pi\sqrt{\lfloor\alpha^2m^2\rfloor+1}}{\alpha}+\ldots+\sin\frac{\pi\sqrt{\lfloor\alpha^2(m+2)^2\rfloor}}{\alpha}\\
&=\sum_{s=1}^{\lfloor\alpha^2(m+3/4)\rfloor-1}\left\{\sin\frac{\pi\sqrt{\lfloor\alpha^2(m+1)^2\rfloor-s}}{\alpha}+\sin\frac{\pi\sqrt{\lfloor\alpha^2(m+1)^2\rfloor+s}}{\alpha}\right\}\\
&+\sum_{s=\lfloor\alpha^2(m+3/4)\rfloor+2}^{\lfloor\alpha^2(2m+1)\rfloor-1}\left\{\sin\frac{\pi\sqrt{\lfloor\alpha^2(m+1)^2\rfloor+1-s}}{\alpha}+\sin\frac{\pi\sqrt{\lfloor\alpha^2(m+1)^2+2\alpha^2\rfloor+1+s}}{\alpha}\right\}\\
&+\sin\frac{\pi\sqrt{\lfloor\alpha^2m^2\rfloor+1}}{\alpha}+\sin\frac{\pi\sqrt{\lfloor\alpha^2m^2\rfloor+2}}{\alpha}+\sin\frac{\pi\sqrt{\lfloor\alpha^2(m+1)^2\rfloor-\lfloor\alpha^2(m+3/4)\rfloor}}{\alpha}\\
&+\sin\frac{\pi\sqrt{\lfloor\alpha^2(m+1)^2\rfloor}}{\alpha}+\sin\frac{\pi\sqrt{\lfloor\alpha^2(m+1)^2\rfloor+\lfloor\alpha^2(m+3/4)\rfloor}}{\alpha}+\ldots\\
&+\sin\frac{\pi\sqrt{\lfloor\alpha^2(m+1)^2+2\alpha^2\rfloor+\lfloor\alpha^2(m+3/4)\rfloor+2}}{\alpha}+\sin\frac{\pi\sqrt{\lfloor\alpha^2(m+2)^2\rfloor}}{\alpha},
\end{aligned}\tag{63}$$

which follows from the relationship

$$\begin{aligned}
&\lfloor\alpha^2(m+1)^2\rfloor-\lfloor\alpha^2(2m+1)\rfloor+2\\
&\quad<\alpha^2\left(m^2+2m+1-2m-1\right)+3=\alpha^2m^2+3\\
&\quad<\lfloor\alpha^2m^2\rfloor+4,\\
&\lfloor\alpha^2(m+1)^2\rfloor-\lfloor\alpha^2(2m+1)\rfloor+2>\alpha^2m^2+1\\
&\quad\geq\lfloor\alpha^2m^2\rfloor+1,\\
&\text{and }\lfloor\alpha^2(m+2)^2\rfloor-\lfloor\alpha^2(m+1)^2+2\alpha^2\rfloor\\
&\quad-\lfloor\alpha^2(2m+1)\rfloor-1+1\leq 1.
\end{aligned}\tag{64}$$

Note that, for some numbers m or α, some components will cease to exist in formula (63). As an example, let $\alpha=m=1$. Then there are not $\sum_{s=1}^{\lfloor\alpha^2(m+3/4)\rfloor-1}\{\ldots\},\sum_{s=\lfloor\alpha^2(m+3/4)\rfloor+2}^{\lfloor\alpha^2(2m+1)\rfloor-1}\{\ldots\}$, $\sin(\pi\sqrt{\lfloor\alpha^2m^2\rfloor+2}/\alpha)$, $\sin(\pi\sqrt{\lfloor\alpha^2(m+2)^2\rfloor}/\alpha)$ in formula (63). However, an estimation of the number of components of (63) shall be sufficient for further consideration. Denote by X the set

$$\begin{aligned}
&\left\{\lfloor\alpha^2m^2\rfloor+1,\lfloor\alpha^2m^2\rfloor+2,\lfloor\alpha^2(m+1)^2\rfloor\right.\\
&\quad-\left\lfloor\alpha^2\left(m+\frac{3}{4}\right)\right\rfloor,\lfloor\alpha^2(m+1)^2\rfloor,\lfloor\alpha^2(m+1)^2\rfloor\\
&\quad+\left\lfloor\alpha^2\left(m+\frac{3}{4}\right)\right\rfloor,\ldots,\lfloor\alpha^2(m+1)^2+2\alpha^2\rfloor\\
&\quad\left.+\left\lfloor\alpha^2\left(m+\frac{3}{4}\right)\right\rfloor+2,\lfloor\alpha^2(m+2)^2\rfloor\right\}.
\end{aligned}\tag{65}$$

We calculate

$$\begin{aligned}
&\forall m\in N\\
&2\alpha^2+1<\lfloor\alpha^2(m+1)^2+2\alpha^2\rfloor+\left\lfloor\alpha^2\left(m+\frac{3}{4}\right)\right\rfloor+2\\
&\qquad-\lfloor\alpha^2(m+1)^2\rfloor-\left\lfloor\alpha^2\left(m+\frac{3}{4}\right)\right\rfloor+1\\
&\quad<2\alpha^2+5
\end{aligned}\tag{66}$$

and thus

$$|X|<2\alpha^2+10. \tag{67}$$

Note that $\forall\alpha\geq 1$ and $\forall m\in$ Odd numbers

$$\begin{aligned}
&\sin\frac{\pi\sqrt{\lfloor\alpha^2m^2\rfloor+1}}{\alpha}+\ldots+\sin\frac{\pi\sqrt{\lfloor\alpha^2(m+2)^2\rfloor}}{\alpha}\\
&\quad\leq 2\alpha^2+10\leq 12\alpha^2.
\end{aligned}\tag{68}$$

This follows from (56), (62), (63), and (67).

Note that

$$S' \leq 12\alpha^2 \sum_{\substack{m=m_n'(\alpha) \\ m\in\{\text{Odd numbers}\}}}^{m_p'(\alpha)-2} c_{\lfloor \alpha^2(m+1)^2 \rfloor}$$
$$\leq 12\alpha^2 \left\{ c_{\lfloor \alpha^2(m_n'(\alpha)+1)^2 \rfloor} + c_{\lfloor \alpha^2(m_n'(\alpha)+3)^2 \rfloor} + c_{\lfloor \alpha^2(m_n'(\alpha)+5)^2 \rfloor} + \dots + c_{\lfloor \alpha^2(m_p'(\alpha)-1)^2 \rfloor} \right\}$$
$$\leq 12\alpha^2 \sum_{t=1}^{\infty} c_{\lfloor (\alpha m_n'(\alpha)+(2t-1)\alpha)^2 \rfloor}$$
$$\leq 12\alpha^2 \sum_{t=1}^{\infty} c_{\lfloor (\sqrt{n}+(2t-1)\alpha)^2 \rfloor} = 12\alpha^2 \sum_{t=1}^{\infty} c_{\lfloor (\sqrt{n}+(2t-1)\alpha)^2 \rfloor} \cdot \frac{\lfloor (\sqrt{n}+(2t-1)\alpha)^2 \rfloor}{\lfloor (\sqrt{n}+(2t-1)\alpha)^2 \rfloor}$$
$$\leq 12\alpha^2 \sup_{t\geq 1} \left\{ c_{\lfloor (\sqrt{n}+(2t-1)\alpha)^2 \rfloor} \left\lfloor (\sqrt{n}+(2t-1)\alpha)^2 \right\rfloor \right\} \cdot \sum_{t=1}^{\infty} \frac{1}{\lfloor (\sqrt{n}+(2t-1)\alpha)^2 \rfloor}. \tag{69}$$

This follows from (27), (35), (50), and (68). After considering that $\forall t \geq 1 \lfloor (\sqrt{n}+(2t-1)\alpha)^2 \rfloor \geq n$ we obtain

$$\sup_{t\geq 1} \left\{ c_{\lfloor (\sqrt{n}+(2t-1)\alpha)^2 \rfloor} \left\lfloor (\sqrt{n}+(2t-1)\alpha)^2 \right\rfloor \right\} \leq \sup_{k\geq n} \{k c_k\}. \tag{70}$$

Thus

$$S' \leq 12 \sup_{k\geq n} \{k c_k\} \alpha^2 \sum_{t=1}^{\infty} \frac{1}{(\sqrt{n}+(2t-1)\alpha)^2 - 1} \leq 12 \sup_{k\geq n} \{k c_k\} \sum_{t=1}^{\infty} \frac{1}{4t^2-1} \leq 2\pi^2 \sup_{k\geq n} \{k c_k\}. \tag{71}$$

This follows from (23). In view of (45) and (71) the following inequality is satisfied:

$$S \leq 32 \sup_{k\geq n} \{k c_k\}. \tag{72}$$

On the other hand

$$S = \sum_{k=n}^{\lfloor \alpha^2 m_n''^2(\alpha) \rfloor} c_k \sin\frac{\pi\sqrt{k}}{\alpha} + \sum_{k=\lfloor \alpha^2 m_p''^2(\alpha) \rfloor}^{p} c_k \sin\frac{\pi\sqrt{k}}{\alpha} + S'',$$
$$\text{where } S'' = \sum_{\substack{m=m_n''(\alpha) \\ m\in\text{Even numbers}}}^{m_p''(\alpha)-2} \left\{ \sum_{z=\lfloor \alpha^2 m^2 \rfloor+1}^{\lfloor \alpha^2(m+1/2)^2 \rfloor} + \sum_{z=\lfloor \alpha^2 (m+1/2)^2 \rfloor+1}^{\lfloor \alpha^2(m+1)^2 \rfloor} + \sum_{z=\lfloor \alpha^2 (m+1)^2 \rfloor+1}^{\lfloor \alpha^2(m+3/2)^2 \rfloor} + \sum_{z=\lfloor \alpha^2 (m+3/2)^2 \rfloor+1}^{\lfloor \alpha^2(m+2)^2 \rfloor} c_z \sin\frac{\pi\sqrt{z}}{\alpha} \right\}.. \tag{73}$$

We see that

$$S \geq -\sum_{k=n}^{\lfloor \alpha^2 m_n''^2(\alpha) \rfloor} c_k \left|\sin\frac{\pi\sqrt{k}}{\alpha}\right| - \sum_{k=\lfloor \alpha^2 m_p''^2(\alpha) \rfloor}^{p} c_k \left|\sin\frac{\pi\sqrt{k}}{\alpha}\right| + S'' \geq -12 \sup_{k\geq n} \{k c_k\} + S''. \tag{74}$$

This follows from (30) and (43). Note that for any even number m and $\forall \alpha \geq 1$

$$c_{\lfloor \alpha^2(m+1)^2 \rfloor} \sum_{z=\lfloor \alpha^2 m^2 \rfloor+1}^{\lfloor \alpha^2(m+1)^2 \rfloor} \sin\frac{\pi\sqrt{z}}{\alpha} \leq \sum_{z=\lfloor \alpha^2 m^2 \rfloor+1}^{\lfloor \alpha^2(m+1)^2 \rfloor} c_z \sin\frac{\pi\sqrt{z}}{\alpha} \tag{75}$$

which follows from the relationship

$$\sin\frac{\pi\sqrt{\lfloor \alpha^2 m^2 \rfloor + 1}}{\alpha} \geq 0, \dots, \sin\frac{\pi\sqrt{\lfloor \alpha^2 (m+1)^2 \rfloor}}{\alpha} \geq 0. \tag{76}$$

On the other hand, $\forall \alpha \geq 1$ and for any even number m and, we have

$$c_{\lfloor \alpha^2(m+1)^2 \rfloor} \sum_{z=\lfloor \alpha^2 (m+1)^2 \rfloor+1}^{\lfloor \alpha^2(m+2)^2 \rfloor} \sin\frac{\pi\sqrt{z}}{\alpha} \leq \sum_{z=\lfloor \alpha^2 (m+1)^2 \rfloor+1}^{\lfloor \alpha^2(m+2)^2 \rfloor} c_z \sin\frac{\pi\sqrt{z}}{\alpha}, \tag{77}$$

which follows from the relationship

$$\sin\frac{\pi\sqrt{\lfloor \alpha^2 (m+1)^2 \rfloor + 1}}{\alpha} \leq 0, \dots, \sin\frac{\pi\sqrt{\lfloor \alpha^2 (m+2)^2 \rfloor}}{\alpha} \leq 0. \tag{78}$$

Thus

$$S'' \geq c_{\lfloor \alpha^2(m+1)^2 \rfloor} \sum_{z=\lfloor \alpha^2 m^2 \rfloor+1}^{\lfloor \alpha^2(m+2)^2 \rfloor} \sin \frac{\pi \sqrt{z}}{\alpha}. \quad (79)$$

This follows from (75), (77). Note that for $s = 1, 2, \ldots, \lfloor \alpha^2(m+3/4) \rfloor - 1$ and for any even number m

$$0 \leq \sin \frac{\pi \left(\sqrt{\lfloor \alpha^2 (m+1)^2 \rfloor - s} + \sqrt{\lfloor \alpha^2 (m+1)^2 \rfloor + s} \right)}{2\alpha} \leq 1 \quad (80)$$

$$0 \leq \cos \frac{\pi \left(\sqrt{\lfloor \alpha^2 (m+1)^2 \rfloor - s} - \sqrt{\lfloor \alpha^2 (m+1)^2 \rfloor + s} \right)}{2\alpha} \leq 1. \quad (81)$$

This follows from (52) and (54). Hence if m is an even number, then

$$\sum_{s=1}^{\lfloor \alpha^2(m+3/4) \rfloor - 1} \left\{ \sin \frac{\pi \sqrt{\lfloor \alpha^2 (m+1)^2 \rfloor - s}}{\alpha} + \sin \frac{\pi \sqrt{\lfloor \alpha^2 (m+1)^2 \rfloor + s}}{\alpha} \right\} \geq 0. \quad (82)$$

This follows from (51), (80), and (81). Note that for $s = \lfloor \alpha^2(m+3/4) \rfloor + 2, \ldots, \lfloor \alpha^2(2m+1) \rfloor - 1$ and for any even number m

$$\sin \frac{\pi \left(\sqrt{\lfloor \alpha^2 (m+1)^2 \rfloor + 1 - s} + \sqrt{\lfloor \alpha^2 (m+1)^2 + 2\alpha^2 \rfloor + 1 + s} \right)}{2\alpha} \quad (83)$$

and

$$\cos \frac{\pi \left(\sqrt{\lfloor \alpha^2 (m+1)^2 \rfloor + 1 - s} - \sqrt{\lfloor \alpha^2 (m+1)^2 + 2\alpha^2 \rfloor + 1 + s} \right)}{2\alpha} \quad (84)$$

belong to $[-1, 0]$. This follows from (58) and (60). In view of (57), (83), and (84) and for any even number m, the following inequality is satisfied:

$$\sum_{s=\lfloor \alpha^2(m+3/4) \rfloor + 2}^{\lfloor \alpha^2(2m+1) \rfloor - 1} \left\{ \sin \frac{\pi \sqrt{\lfloor \alpha^2 (m+1)^2 \rfloor + 1 - s}}{\alpha} + \sin \frac{\pi \sqrt{\lfloor \alpha^2 (m+1)^2 + 2\alpha^2 \rfloor + 1 + s}}{\alpha} \right\} \geq 0. \quad (85)$$

Hence, if m is an even number, then

$$\sum_{\lfloor \alpha^2 m^2 \rfloor + 1}^{\lfloor \alpha^2(m+2)^2 \rfloor} \sin \frac{\pi \sqrt{z}}{\alpha} \geq -2\alpha^2 - 10 \geq -12\alpha^2. \quad (86)$$

This follows from (63), (67), (82), and (85). Thus

$$S'' \geq -12\alpha^2 \sum_{\substack{m=m_n''(\alpha) \\ m \in \{\text{Even numbers}\}}}^{m_p''(\alpha)-2} c_{\lfloor \alpha^2(m+1)^2 \rfloor}. \quad (87)$$

This follows from (79) and (86). In analogy with (69) and (71) we have

$$12\alpha^2 \sum_{\substack{m=m_n''(\alpha) \\ m \in \{\text{Even numbers}\}}}^{m_p''(\alpha)-2} c_{\lfloor \alpha^2(m+1)^2 \rfloor} \leq 2\pi^2 \sup_{k \geq n} \{k c_k\}. \quad (88)$$

This follows from (23), (28), and (36). In view of (87) and (88), the following inequality is satisfied:

$$S'' \geq -2\pi^2 \sup_{k \geq n} \{k c_k\}. \quad (89)$$

Hence

$$S \geq -32 \sup_{k \geq n} \{k c_k\}. \quad (90)$$

This follows from (74) and (89). Thus

$$\text{For } \sqrt{\frac{p}{n}} \geq 7 \quad \text{we have}: \left| \sum_{k=n}^{p} c_k \sin \frac{\pi \sqrt{k}}{\alpha} \right| \leq 32 \sup_{k \geq n} \{k c_k\}. \quad (91)$$

This follows from (72) and (90). In view of (25) and (91) the following inequality is satisfied:

$$\forall p \geq n \quad \forall \alpha \in [1, \sqrt{n}] \quad \left| \sum_{k=n}^{p} c_k \sin \frac{\pi \sqrt{k}}{\alpha} \right| \leq 49 \sup_{k \geq n} \{k c_k\}. \quad (92)$$

Case 3:

$$\sqrt{n} \leq \alpha \leq \sqrt{p}. \quad (93)$$

In view of (24) and (25) there is one case to consider, namely, $\sqrt{p/n} \geq 7$. Then

$$\exists r \in N \text{ that } r > n \text{ and } \sqrt{n} \leq r - 1 \leq \alpha^2 \leq r \leq p. \quad (94)$$

Note that

$$\sum_{k=n}^{p} c_k \sin \frac{\pi \sqrt{k}}{\alpha} = \sum_{k=n}^{r-1} c_k \sin \frac{\pi \sqrt{k}}{\alpha} + \sum_{k=r}^{p} c_k \sin \frac{\pi \sqrt{k}}{\alpha}. \quad (95)$$

We now give an estimate for $\sum_{k=n}^{r-1} c_k \sin(\pi\sqrt{k}/\alpha)$. Note that $0 < \pi(\sqrt{k}/\alpha) < \pi$ and $0 \le \sin(\pi\sqrt{k}/\alpha) \le \pi(\sqrt{k}/\alpha)$ for $k = n, \ldots, r-1$. This follows from (94). Hence

$$\begin{aligned} 0 &\le \sum_{k=n}^{r-1} c_k \sin\frac{\pi\sqrt{k}}{\alpha} \le \sum_{k=n}^{r-1} c_k \pi \frac{\sqrt{k}}{\alpha} \\ &\le \pi \sup_{k\ge n}\{kc_k\} \sum_{k=n}^{r-1} \frac{1}{\sqrt{k}\alpha} \le \pi \sup_{k\ge n}\{kc_k\} \sum_{k=n}^{r-1} \frac{1}{\sqrt{k}\sqrt{r-1}} \\ &\le 2\pi \sup_{k\ge n}\{kc_k\}. \end{aligned} \tag{96}$$

This follows from (9) and (94).

We now give an estimate for $\sum_{k=r}^{p} c_k \sin(\pi\sqrt{k}/\alpha)$. There are two cases to consider:

Case 3’: $\sqrt{p/r} < 7$.

Then, by (94) and (96), we have

$$\begin{aligned} \left|\sum_{k=n}^{p} c_k \sin\frac{\pi\sqrt{k}}{\alpha}\right| &\le \sum_{k=n}^{r-1} c_k \sin\frac{\pi\sqrt{k}}{\alpha} + \sum_{k=r}^{p} c_k \sin\frac{\pi\sqrt{k}}{\alpha} \\ &\le (2\pi + 49) \sup_{k\ge n}\{kc_k\}. \end{aligned} \tag{97}$$

Case 3”: $\sqrt{p/r} \ge 7$.

Note that the replacement of n with r in (23) and (26) gives us

$$\left|\sum_{k=r}^{p} c_k \sin\frac{\pi\sqrt{k}}{\alpha}\right| \le 32 \sup_{k\ge n}\{kc_k\} \tag{98}$$

and thus in case 3 we proved that $\forall n, p$, where $n \le p$, and $\forall \alpha \in [\sqrt{n}, \sqrt{p}]$

$$\left|\sum_{k=n}^{p} c_k \sin\frac{\pi\sqrt{k}}{\alpha}\right| \le 56 \sup_{k\ge n}\{kc_k\}. \tag{99}$$

This follows from (96), (97), and (98). In view of cases 1, 2, and 3 we obtain

$$\begin{aligned} &\forall \alpha \ge 1\ \forall n, p \\ &\text{where } n \le p : \left|\sum_{k=n}^{p} c_k \sin\frac{\pi\sqrt{k}}{\alpha}\right| \\ &\quad \le \max\{2\pi, 49, 56\} \sup_{k\ge n}\{kc_k\}. \end{aligned} \tag{100}$$

This follows from (11), (92), and (99). If $\lim_{n\to\infty} nc_n = 0$, then $\forall \epsilon > 0\ \exists M\ \forall n > M\ \sup_{k\ge n}\{kc_k\} < \epsilon$, so

$$\begin{aligned} &\forall n, p \\ &\text{where } n \le p : \left|\sum_{k=n}^{p} c_k \sin\frac{\pi\sqrt{k}}{\alpha}\right| \le 56\epsilon \\ &\qquad \text{for all } \alpha \ge 1 \end{aligned} \tag{101}$$

provided that $n > M$. This finishes the proof of the sufficient condition. This completes the proof of Theorem 6. □

Remark 7. If $\{c_k\}_{k=1}^{\infty} \subset R_+$ is nonincreasing, then $\sum_{k=1}^{\infty} c_k \sin k^2\theta$ converges uniformly in $\theta \in [0, \pi]$ if and only if $\sum_{k=1}^{\infty} c_k$ is convergent.

Proof. Proof of the sufficient condition is obvious.

Proof of the necessary condition. Let

$$\sum_{k=1}^{\infty} c_k \sin\frac{\pi k^2}{8} \tag{102}$$

be a convergent series. Note that for any even number m we have

$$\begin{aligned} &\sum_{k=4m}^{4(m+2)-1} c_k \sin\frac{\pi k^2}{8} \\ &= c_{4m+1}\sin\pi\left(2m^2 + m + \frac{1}{8}\right) \\ &\quad + c_{4m+2}\sin\pi\left(2m^2 + 2m + \frac{4}{8}\right) \\ &\quad + c_{4m+3}\sin\pi\left(2m^2 + 3m + \frac{9}{8}\right) \\ &\quad + c_{4m+5}\sin\pi\left(2m^2 + 5m + \frac{25}{8}\right) \\ &\quad + c_{4m+6}\sin\pi\left(2m^2 + 6m + \frac{36}{8}\right) \\ &\quad + c_{4m+7}\sin\pi\left(2m^2 + 7m + \frac{49}{8}\right) \\ &\ge c_{4m+3}\left(\sin\frac{\pi}{8} + \sin\frac{9\pi}{8}\right) \\ &\quad + c_{4m+5}\left(\sin\frac{4\pi}{8} + \sin\frac{25\pi}{8}\right) > \frac{1}{2}c_{4m+5}. \end{aligned} \tag{103}$$

Thus

$$\begin{aligned} \sum_{k=8}^{\infty} c_k \sin\frac{k^2\pi}{8} &= \sum_{\substack{m\in\text{Even numbers}\\ m\ge 2}} \sum_{k=4m}^{4m+7} c_k \sin\frac{k^2\pi}{8} \\ &> \sum_{\substack{m\in\text{Even numbers}\\ m\ge 2}} \frac{c_{4m+5}}{2}. \end{aligned} \tag{104}$$

This follows from (103). After considering that $\sum_{k=8}^{\infty} c_k \sin(k^2\pi/8)$ is convergent, we obtain the convergent series $c_{13} + c_{21} + c_{29} + \ldots + c_{8r+5} + \ldots$. This follows from (104). After considering that $\{c_k\}_{k=1}^{\infty}$ is nonincreasing, we have that $\forall s = 0, 1, \ldots, 7\ \sum_{r=1}^{\infty} c_{8r+5+s}$ is convergent and $\sum_{r=1}^{\infty} c_r$ is convergent. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares no conflicts of interest.

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