Hindawi
Journal of Function Spaces
Volume 2019, Article ID 1389787, 8 pages
https://doi.org/10.1155/2019/1389787

Research Article

Distributed Control for Time-Fractional Differential System Involving Schrödinger Operator

Abd-Allah Hyder [1,2] and M. EL-Badawy [3]

[1] *King Khalid University, College of Science, Department of Mathematics, P.O. Box 9004, 61413 Abha, Saudi Arabia*
[2] *Department of Engineering Mathematics and Physics, Faculty of Engineering, Al-Azhar University, Cairo, Egypt*
[3] *Mathematics Department, Al-Azhar University, Nasr City, Cairo, Egypt*

Correspondence should be addressed to Abd-Allah Hyder; abahahmed@kku.edu.sa

Received 6 March 2019; Accepted 18 June 2019; Published 22 July 2019

Academic Editor: Alberto Fiorenza

In this paper, we investigate the distributed optimal control problem for time-fractional differential system involving Schrödinger operator defined on $\mathbb{R}^n$. The time-fractional derivative is considered in the Riemann-Liouville sense. By using the Lax-Milgram lemma, we prove the existence and uniqueness of the solution of this system. For the fractional Dirichlet problem with linear quadratic cost functional, we give some equations and inequalities which provide the necessary and sufficient optimality conditions. Moreover, we provide specific application examples to demonstrate the effectiveness of our results.

1. Introduction

This paper is devoted to the study of the distributed control problem for the time-fractional differential system involving Schrödinger operator. This problem leads us to the minimization of the quadratic cost functional:

$$J(v) = \|y(v) - z_d\|_{L^2(\mathcal{Q})} + (Nv, v)_{L^2(\mathcal{Q})}, \quad \text{for all } v \in \mathcal{U}_{ad} \tag{1}$$

subject to the state equation

$$\begin{aligned} &D_t^\alpha y(x,t;u) + (-\Delta + q(x))\, y(x,t;u) \\ &\quad = a\, y(x,t;u) + f(x,t) + u \quad \text{in } \mathcal{Q}, \\ &y(x,t;u) \longrightarrow 0 \quad \text{as } |x| \longrightarrow \infty, \text{ for all } t \in (0,T) \\ &y(x,t;u) = 0 \quad \text{on } \Sigma, \\ &I_+^{1-\alpha} y(x,0;u) = 0 \quad \text{in } \mathbb{R}^n. \end{aligned} \tag{2}$$

For a fixed time $T > 0$, we set $\mathcal{Q} = \mathbb{R}^n \times (0,T)$ with boundary $\Sigma = \Gamma \times (0,T)$ and $\Gamma = \partial\mathbb{R}^n$. Here, $f \in L^2(\mathcal{Q})$, a is a given real number, $q : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a positive function which tends to ∞ as $|x|$ tends to ∞, u is a given control belonging to the set of admissible controls $\mathcal{U}_{ad}$, y is the state of the system which is to be controlled, $z_d : \mathcal{Q} \longrightarrow \mathbb{R}$ is a given desired state, N is a Hermitian positive definite operator, and finally the operators I_+^α, D_t^α are the Riemann-Liouville fractional integral and derivative, respectively.

In recent years, fractional order partial differential equations (FPDEs) models have been proposed and investigated in many research fields, such as biology, plasma physics, finance, chemistry, quantum theory, and fluids [1–6]. In [7], the numerical solution for the time-fractional advection-diffusion problem in one-dimension with the initial boundary condition is studied. A new approach to solving a type of fractional order differential equations without singularity is introduced in [8]. In [9], a type of Fokker-Planck equation (FPE) with Caputo-Fabrizio fractional derivative is considered; they present a numerical approach which is based on the Ritz method with known basis functions to transform this equation into an optimization problem. In [10], the time-fractional Klein-Gordon type equation is solved by applying two decomposition methods, the Adomian decomposition method and a well-established

iterative method. The developments in time-fractional quantum mechanics can be considered a newly emerging and attractive application of fractional calculus to quantum theory. Time-fractional quantum mechanics helps to understand the significance and importance of the fundamentals of quantum mechanics such as Hamilton operator, unitarity of evolution operator, existence of stationary energy levels of quantum mechanical system, quantum superposition law, and conservation of quantum probability. In addition, time-fractional quantum mechanics invokes new mathematical tools, which have never been utilized in quantum theory; for more details see [11–13]. The necessary and sufficient conditions to optimal control for integer order PDEs of elliptic, parabolic, and hyperbolic type have been studied by Lions in [14, 15]. This discussion was extended to time-fractional systems in the sense of Riemann-Liouville in [16–18] and in the sense of Caputo in [19]. Recently, the distributed optimal control problem for systems involving Schrödinger operator has been studied for elliptic systems in [20], parabolic systems in [21], and hyperbolic systems in [22].

The main target of this paper is to study the distributed control problem for the cooperative time-fractional differential system involving Schrödinger operator, which is formulated in system (2). This theme is based on the abstract variational formulation method, the Lax-Milgram lemma, and the adjoint problem technique. With the help of Lax-Milgram lemma, we prove existence and uniqueness of solution for the time-fractional differential system involving Schrödinger operator. By employing the adjoint problem, we characterize the optimal control. With regard to the fractional Dirichlet problem with linear quadratic cost functional, we produce some equations and inequalities which provide the necessary and sufficient optimality conditions. Further, we provide specific application examples to demonstrate the effectiveness of our results.

This paper is organised as follows. In Section 2, we formulate the time-fractional differential system and recall some related results and function spaces. In Section 3, the existence and uniqueness of solution of our problem are proved. In Section 4, we formulate the control problem, show that our time-fractional control problem holds, and drive the optimality conditions for the system. In Section 5, we give some examples for our theorem. Section 6 is devoted to summary and discussion.

2. Time-Fractional Differential System and Related Results

This section is divided into two subsections. In Subsection 2.1, we introduce the time-fractional differential system involving Schrödinger operator, where the time-fractional derivative is introduced in the Riemann-Liouville sense. In Subsection 2.2, we exhibit some results related to the eigenvalue problem and some function spaces with their embedding properties. These function spaces include the solution of our problem.

2.1. Formulation of the Problem. Before giving the precise formulation of our problem, we recall some notations from the fractional calculus theory.

Definition 1 (see [6, 16, 17]). Let $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ be a continuous function on $\mathbb{R}^+$. The expression $\mathrm{I}_+^\alpha f$ is called the left Riemann-Liouville fractional integral of f of order $\alpha > 0$, which is given by

$$I_+^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \quad t > 0. \tag{3}$$

Definition 2 (see [6, 16, 17]). Let $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ be a continuous function on $\mathbb{R}^+$. The expression $D_t^\alpha f$ is the left Riemann-Liouville fractional derivative of f of order $\alpha \in (0,1)$, which is given by

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(s)}{(t-s)^\alpha} ds, \quad t > 0. \tag{4}$$

Definition 3 (see [6, 16, 17]). Let $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ be a continuous function on $\mathbb{R}^+$. The left Caputo fractional derivative of f of order $\alpha \in (0,1)$ is defined by

$${}_0^cD_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^\alpha} ds, \quad t > 0, \tag{5}$$

and the right Caputo fractional derivative of order $\alpha \in (0,1)$ of f is defined by

$${}_t^cD_T^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_t^T \frac{f'(s)}{(s-t)^\alpha} ds, \quad t > 0. \tag{6}$$

Lemma 4 (see [16, 17, 19]). *Let $T > 0$, $f \in \mathscr{C}^m([0,T])$, $p \in (m-1, m)$, $m \in \mathscr{N}$, and $g \in \mathscr{C}^1([0,T])$. Then for $t \in [0,T]$, $0 \le \beta \le 1$, the following properties hold*

$$\begin{aligned} D_t^p g(t) &= \frac{d}{dt} I_+^{1-p} g(t), \quad m = 1, \\ D_t^p I_+^p g(t) &= g(t); \\ I_+^p\, {}_0^cD_t^p f(t) &= f(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} f^{(k)}(0); \\ \lim_{t \longrightarrow 0^+} {}_0^cD_t^p f(t) &= \lim_{t \longrightarrow 0^+} I_+^p f(t). \end{aligned} \tag{7}$$

From now on we denote ${}_t^cD_T^\alpha f(t) = D_c^\alpha f(t)$.

Now, we can formulate our problem. Assume that the time-fractional derivative D_t^α and the fractional integral I_+^α are considered in the Riemann-Liouville sense; then the time-fractional differential system with Schrödinger operator can be given in the form

$$\begin{aligned} &D_t^\alpha y(x,t) + (-\Delta + q(x))\, y(x,t) = a\, y(x,t) + f(x,t) \quad \text{in } \mathcal{Q}, \\ &y(x,t) \longrightarrow 0 \quad \text{as } |x| \longrightarrow \infty, \\ &y(x,t) = 0 \quad \text{on } \Sigma, \\ &I_+^{1-\alpha} y(x,0) = 0 \quad \text{in } \mathbb{R}^n. \end{aligned} \tag{8}$$

2.2. Eigenvalue Problem. Let $V_q(\mathbb{R}^n)$ be the associated variational space, which is the completion of the test function space $\mathcal{D}(\mathbb{R}^n)$, with respect to the norm

$$\|y\|_q^2 = \int_{\mathbb{R}^n} \left(|\nabla y|^2 + q|y|^2\right) dx. \tag{9}$$

We have that the embedding $V_q(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$ is compact [23].

Consider the eigenvalue problem:

$$\begin{aligned} (-\Delta + q(x))\, y &= \lambda(q)\, y \quad \text{in } \mathbb{R}^n, \\ y(x) &\longrightarrow 0 \quad \text{as } |x| \longrightarrow \infty, \ y > 0. \end{aligned} \tag{10}$$

Since the embedding $V_q(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$ is compact, then the operator $(-\Delta + q)$ considered as an operator in $L^2(\mathbb{R}^n)$ is positive self-adjoint with compact inverse. Hence its spectrum consists of an infinite sequence of positive eigenvalues, tending to infinity; moreover the smallest one which is called the principal eigenvalue is denoted by $\lambda(q)$ which is simple and its corresponding eigenfunction does not change sign in $\mathbb{R}^n$. Also it is characterized by the following inequality:

$$\lambda(q) \int_{\mathbb{R}^n} |y|^2 dx \le \int_{\mathbb{R}^n} \left(|\nabla y|^2 + q|y|^2\right) dx, \quad \forall y \in V_q(\mathbb{R}^n). \tag{11}$$

For more details about the eigenvalue problem (10) and its principal eigenvalue, we refer the reader to [23].

To investigate our problem, we need to the following function space:

$$\begin{aligned} \mathcal{W}(0,T) = \{ y : y \in L^2(0,T; V_q(\mathbb{R}^n)), D_t^\alpha y(t) \\ \in L^2(0,T; V_q^*(\mathbb{R}^n)) \}, \end{aligned} \tag{12}$$

with the embedding given by

$$\begin{aligned} L^2(0,T; V_q(\mathbb{R}^n)) &\hookrightarrow L^2(0,T;\mathbb{R}^n) \\ &\hookrightarrow L^2(0,T; V_q^*(\mathbb{R}^n)), \end{aligned} \tag{13}$$

which is continuous and compact, where $V_q^*(\mathbb{R}^n)$ is the dual of $V_q(\mathbb{R}^n)$.

3. Existence and Uniqueness of Solutions

In this section, we prove the existence and uniqueness of solutions of problem (8) by using the Lax-Milgram lemma.

For each $t \in (0,T)$ and $y, \phi \in L^2(\mathbb{R}^n)$, we define the bilinear form $\pi(t; y, \phi)$ by

$$\pi(t; y, \phi) = (\mathcal{A} y, \phi)_{L^2(\mathbb{R}^n)}, \tag{14}$$

where the differential operator $\mathcal{A}$ maps $V_q(\mathbb{R}^n)$ onto $V_q^*(\mathbb{R}^n)$ and takes the following form.

$$\mathcal{A} y = (-\Delta + q - a)\, y \tag{15}$$

Then, for all $y, \phi \in V_q(\mathbb{R}^n)$, we have

$$\pi(t; y, \phi) = \int_{\mathbb{R}^n} (\nabla y \nabla \phi + q y \phi)\, dx - \int_{\mathbb{R}^n} a y \phi dx. \tag{16}$$

Theorem 5. *The bilinear form $\pi(t; y, \phi) = (\mathcal{A} y, \phi)$ is coercive on $V_q(\mathbb{R}^n)$.*

Proof. Replacing ϕ with y in (16), we have

$$\pi(t; y, y) = \int_{\mathbb{R}^n} \left(|\nabla y|^2 + q|y|^2\right) dx - \int_{\mathbb{R}^n} a|y|^2 dx. \tag{17}$$

By using (9), we obtain

$$\pi(t; y, y) = \|y\|_q^2 - \int_{\mathbb{R}^n} a|y|^2 dx. \tag{18}$$

Then, if (9) and $a \leqslant \lambda(q)$ are satisfied, then there exists $C > 0$ such that

$$\pi(t; y, y) \geqslant C \|y\|_q^2, \tag{19}$$

and hence the bilinear form $\pi(t; y, \phi)$ is coercive on $V_q(\mathbb{R}^n)$.

According to (16), we have that the function $t \longmapsto \pi(t; y, \phi)$ is continuously differentiable on $(0,T)$ and symmetric; that is,

$$\pi(t; y, \phi) = \pi(t; \phi, y) \quad \text{for all } y, \phi \in V_q(\mathbb{R}^n). \tag{20}$$

And to prove the existence of a unique solution of our problem by Lax-Milgram lemma, we need the following two lemmas, which provide the fractional Green's formula. □

Lemma 6 (see [16, 17]). *Let $\alpha \in (0,1)$. Then, for any $\phi \in \mathcal{C}^\infty(\overline{\mathcal{Q}})$, we have*

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^n} \left(D_t^\alpha y(x,t) + (-\Delta + q)\, y(x,t) - a y(x,t)\right) \\ &\quad \cdot \phi(x,t)\, dx\, dt = \int_{\mathbb{R}^n} \phi(x,T)\, I_+^{1-\alpha} y(x,T)\, dx \\ &\quad - \int_{\mathbb{R}^n} \phi(x,0)\, I_+^{1-\alpha} y(x,0)\, dx - \int_0^T \int_{\mathbb{R}^n} y(x,t) \\ &\quad \cdot D_c^\alpha \phi(x,t)\, dx\, dt - \int_0^T \int_\Gamma \frac{\partial y}{\partial \nu} \phi\, d\Gamma\, dt \\ &\quad + \int_0^T \int_\Gamma y \frac{\partial \phi}{\partial \nu} d\Gamma\, dt - \int_0^T \int_{\mathbb{R}^n} y(x,t) \\ &\quad \cdot \left((-\Delta + q)\, \phi(x,t)\right) dx\, dt - a \int_0^T \int_{\mathbb{R}^n} y(x,t) \\ &\quad \cdot \phi(x,t)\, dx\, dt. \end{aligned} \tag{21}$$

Lemma 7 (see [16, 17]). *Let $\alpha \in (0,1)$. Then, for any $\phi \in \mathscr{C}^{\infty}(\overline{\mathcal{Q}})$ such that $\phi(x,T) = 0$ in $\mathbb{R}^n$ and $\phi(x,t) = 0$ on Σ, we have*

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^n} &\left(D_t^{\alpha} y(x,t) + (-\Delta + q)\, y(x,t) - a y(x,t)\right) \\ &\cdot \phi(x,t)\, dx\, dt = -\int_{\mathbb{R}^n} \phi(x,0)\, I_+^{1-\alpha} y(x,0)\, dx \\ &- \int_0^T \int_{\mathbb{R}^n} y(x,t)\, D_c^{\alpha} \phi(x,t)\, dx\, dt \\ &- \int_0^T \int_{\Gamma} y \frac{\partial \phi}{\partial \nu} d\Gamma\, dt + \int_0^T \int_{\mathbb{R}^n} y(x,t) \\ &\cdot \left[(-\Delta + q)\, \phi(x,t)\right] dx\, dt - a \int_0^T \int_{\mathbb{R}^n} y(x,t) \\ &\cdot \phi(x,t)\, dx\, dt.\end{aligned} \tag{22}$$

Now, with the help of Lemmas 6 and 7, and Lax-Milgram lemma, we state and prove the existence and uniqueness theorems for the solution of problem (8).

Theorem 8. *If (19) and (20) are satisfied, then the problem (8) admits a unique solution $y \in \mathscr{W}(0,T)$.*

Proof. Define the mapping $L : L^2(\mathcal{Q}) \longrightarrow \mathbb{R}$ by

$$\begin{aligned} L(\phi) = \int_{\mathcal{Q}} f(x,t)\, \phi(x)\, dx\, dt, \\ \forall \phi(x) \in L^2\left(0,T; V_q(\mathbb{R}^n)\right). \end{aligned} \tag{23}$$

It is easy to see that L forms a continuous linear form defined on $L^2(\mathcal{Q})$. Then, by Lax-Milgram lemma, there exists a unique solution $y \in L^2(0,T; V_q(\mathbb{R}^n))$ such that

$$\begin{aligned} \left(D_t^{\alpha} y, \phi\right)_{L^2(\mathrm{Q})} + \pi(t; y, \phi)_{L^2(\mathrm{Q})} = L(\phi) \\ \forall \phi \in L^2\left(0,T; V_q(\mathbb{R}^n)\right), \end{aligned} \tag{24}$$

which is equivalent to the existence of a unique solution $y \in L^2(0,T; V_q(\mathbb{R}^n))$ for the equation

$$\begin{aligned} \left(D_t^{\alpha} y, \phi\right)_{L^2(\mathrm{Q})} + \left(\mathscr{A} y, \phi\right)_{L^2(\mathrm{Q})} = L(\phi) \\ \forall \phi \in L^2\left(0,T; V_q(\mathbb{R}^n)\right). \end{aligned} \tag{25}$$

Equation (25) can be rewritten as

$$\begin{aligned} \int_{\mathrm{Q}} \left(D_t^{\alpha} y + \mathscr{A} y\right) \phi(x)\, dx\, dt = \int_{\mathcal{Q}} f \phi\, dx\, dt \\ \forall \phi \in L^2\left(0,T; V_q(\mathbb{R}^n)\right), \end{aligned} \tag{26}$$

which produces the fractional partial differential equation

$$D_t^{\alpha} y + \mathscr{A} y = f. \tag{27}$$

Now, multiplying both sides of (27) by $\phi(x)$ and integrating over $\mathcal{Q}$, we have

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^n} &\left(D_t^{\alpha} y(x,t) + (-\Delta + q(x))\, y(x,t)\right) \\ &\cdot \phi(x,t)\, dx\, dt - a \int_0^T \int_{\mathbb{R}^n} y(x,t) \\ &\cdot \phi(x,t)\, dx\, dt = \int_0^T \int_{\mathbb{R}^n} f(x,t)\, \phi(x)\, dx\, dt. \end{aligned} \tag{28}$$

According to Lemmas 6 and 7, we have

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^n} &f(x,t)\, \phi(x)\, dx\, dt \\ &= -\int_{\mathbb{R}^n} \phi(x,0)\, I_+^{1-\alpha} y(x,0)\, dx \\ &- \int_0^T \int_{\mathbb{R}^n} y(x,t)\, D_c^{\alpha} \phi(x,t)\, dx\, dt \\ &+ \int_0^T \int_{\Gamma} y \frac{\partial \phi}{\partial \nu} d\Gamma\, dt \\ &- \int_0^T \int_{\mathbb{R}^n} y(x,t) \left((-\Delta + q(x))\, \phi(x,t)\right) dx\, dt \\ &- a \int_0^T \int_{\mathbb{R}^n} y(x,t)\, \phi(x,t)\, dx\, dt. \end{aligned} \tag{29}$$

By the comparison between (23), (26), and (29) we get

$$-\int_{\mathbb{R}^n} \phi(x,0)\, I_+^{1-\alpha} y(x,0)\, dx + \int_0^T \int_{\Gamma} y \frac{\partial \phi}{\partial \nu} d\Gamma\, dt = 0, \tag{30}$$

and, then, we deduce that

$$\begin{aligned} y(x,t) = 0 \quad \text{on } \Sigma, \\ \text{and } I_+^{1-\alpha} y(x,0) = 0 \quad \text{in } \mathbb{R}^n. \end{aligned} \tag{31}$$

This completes the proof. □

4. Formulation of the Control Problem

This section is a main object of this paper; we drive the adjoint state of our problem and the first order necessary and sufficient optimality conditions. Let $\mathcal{U} = L^2(\mathcal{Q})$ be the space of controls. For a control $u \in L^2(\mathcal{Q})$, the state of system (2) is given by $y(u) \in L^2(0,T; V_q(\mathbb{R}^n))$, and the observation equation is given by $z(u) = y(u)$. For a given $z_d \in L^2(\mathcal{Q})$, the cost functional is given by (1), where $N \in \mathscr{L}(L^2(\mathcal{Q}), L^2(\mathcal{Q}))$ is a positive definite Hermitian operator that satisfies the condition

$$(Nu, u) \geq \beta \|u\|^2_{L^2(\mathcal{Q})}, \quad \beta > 0. \tag{32}$$

Let $\mathcal{U}_{ad}$ be a closed convex subset of $L^2(\mathcal{Q})$, and then the control problem is to find

$$\begin{aligned} u \in \mathcal{U}_{ad}, \\ \text{such that } J(u) \leq J(v), \quad \forall v \in \mathcal{U}_{ad}. \end{aligned} \tag{33}$$

Using the general theory of Lions [14], the following result gives the existence and uniqueness of optimal control. Moreover it gives the first order necessary and sufficient optimality conditions for the distributed optimal control problem (1) and (2), formulated to the time-fractional differential system (8).

Theorem 9. *Assume that (19) and (32) are satisfied. If the cost functional is given by (1), then there exists an optimal control $u \in \mathcal{U}_{ad}$ and this control is characterized by the following equations and inequalities:*

$$\begin{aligned} -D_c^\alpha p(u) + (-\Delta + q)\, p(u) - ap(u) &= y(u) - z_d \quad \text{in } \mathcal{Q}, \\ p &\longrightarrow 0 \quad \text{as } |x| \longrightarrow \infty, \\ p(u) &= 0 \quad \text{on } \Sigma, \\ p(x, T; u) &= 0 \quad \text{in } \mathbb{R}^n. \end{aligned} \tag{34}$$

In addition,

$$\int_0^T \int_{\mathbb{R}^n} (p(u) + Nu)(v - u)\, dxdt \geqslant 0 \quad \forall u, v \in \mathcal{U}_{ad}, \tag{35}$$

where $p(u) \in L^2(0, T; V_q(\mathbb{R}^n))$ is the adjoint state.

Proof. If $u \in L^2(\mathcal{Q})$ is the optimal control, then it is characterized by [14]

$$J'(u)(v - u) \geq 0 \quad \forall v \in \mathcal{U}_{ad}, \tag{36}$$

which is equivalent to

$$(y(u) - z_d, y(v) - y(u))_{L^2(\mathcal{Q})} + (Nu, v - u)_{L^2(\mathcal{Q})} \geq 0. \tag{37}$$

Since $(\mathcal{B}^* p, y) = (p, \mathcal{B} y)$, where

$$\mathcal{B} y = D_t^\alpha y + (-\Delta + q)\, y - ay \tag{38}$$

for $y \in L^2(0, T; V_q(\mathbb{R}^n))$, then

$$(p, \mathcal{B} y) = (p, D_t^\alpha y + (-\Delta + q)\, y - ay). \tag{39}$$

By using the fractional Green's formula, we get

$$(\mathcal{B}^* p, y) = (-D_c^\alpha p + (-\Delta + q)\, p - ap, y), \tag{40}$$

hence,

$$\mathcal{B}^* p = -D_c^\alpha p + (-\Delta + q)\, p - ap \tag{41}$$

and, then, the adjoint equation takes the form:

$$-D_c^\alpha p + (-\Delta + q)\, p - ap = y(u) - z_d. \tag{42}$$

Now, multiplying (42) by $(y(v) - y(u))$ and applying fractional Green's formula, we obtain

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^n} (y(u) - z_d)(y(v) - y(u))\, dx\, dt \\ &= \int_0^T \int_{\mathbb{R}^n} (-D_c^\alpha p(u) + (-\Delta + q)\, p(u) - ap(u)) \\ &\cdot (y(v) - y(u))\, dx\, dt = -\int_{\mathbb{R}^n} p(x, 0) \\ &\cdot \mathrm{I}_+^{1-\alpha} (y(v; x, 0) - y(u; x, 0))\, dx \\ &+ \int_0^T \int_{\mathbb{R}^n} p(u) (D_t^\alpha + (-\Delta + q) - a) \\ &\cdot (y(v) - y(u))\, dx\, dt + \int_0^T \int_\Gamma p(u) \\ &\cdot \left(\frac{\partial y(v)}{\partial \nu} - \frac{\partial y(u)}{\partial \nu} \right) d\Gamma\, dt \\ &- \int_0^T \int_\Gamma \frac{\partial p(u)}{\partial \nu} (y(v) - y(u))\, d\Gamma\, dt. \end{aligned} \tag{43}$$

Since by using (2) we obtain

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^n} (y(u) - z_d)(y(v) - y(u))\, dx\, dt \\ &= \int_0^T \int_{\mathbb{R}^n} p(u)(v - u)\, dx\, dt, \end{aligned} \tag{44}$$

(37) is equivalent to

$$\int_0^T \int_{\mathbb{R}^n} p(u)(v - u)\, dx\, dt + (Nu, v - u) \geqslant 0, \tag{45}$$

which is reduced to

$$\int_0^T \int_{\mathbb{R}^n} (p(u) + Nu)(v - u)\, dx\, dt \geqslant 0. \tag{46}$$

Thus the proof is complete. □

5. Applications

In this section, we provide specific application examples to demonstrate the effectiveness of our results and to justify the real contribution of these results.

Example 1. In the case of no constraints on the control, i.e., $\mathcal{U}_{ad} = \mathcal{U}$, (35) reduces to

$$P + Nu = 0, \tag{47}$$

and we may then put $u = -N^{-1}p$ from (2) and (34). Then the optimal control is given by the following system of fractional partial differential equations:

$$\begin{aligned} &D_t^\alpha y + (-\Delta + q)\, y - a\, y + N^{-1} p = f \quad \text{in } \mathcal{Q}, \\ &-D_c^\alpha p + (-\Delta + q)\, p - ap - y = -z_d \quad \text{in } \mathcal{Q}, \\ &p, y \longrightarrow 0 \quad \text{as } |x| \longrightarrow \infty, \\ &y = 0, \\ &p = 0 \quad \text{on } \Sigma, \\ &I_+^{1-\alpha} y(x,0) = 0, \\ &p(x,T) = 0 \quad \text{in } \mathbb{R}^n. \end{aligned} \tag{48}$$

Example 2. If we take

$$\begin{aligned} &\mathcal{U}_{ad} \\ &= \left\{ v \mid v \in L^2(Q), v \geq 0 \text{ almost everywhere in } \mathcal{Q} \right\}, \end{aligned} \tag{49}$$

then (35) is equivalent to

$$\begin{aligned} u &\geq 0, \quad \text{almost every where n } \mathcal{Q}, \\ p(u) + Nu &\geq 0, \quad \text{almost every where n } \mathcal{Q}, \\ u(p(u) + Nu) &= 0, \quad \text{almost every where n } \mathcal{Q}. \end{aligned} \tag{50}$$

And hence u may be eliminated from (2) and (34) in the following way:

$$\begin{aligned} &D_t^\alpha y + (-\Delta + q)\, y - ay - f \geq 0 \text{ in } \mathcal{Q}, \\ &-D_c^\alpha p + (-\Delta + q)\, p - ap - y = -z_d \quad \text{in } \mathcal{Q}, \\ &p + N\left(D_t^\alpha y + (-\Delta + q)\, y - a\, y - f\right) \geq 0 \quad \text{in } \mathcal{Q}, \\ &\left(D_t^\alpha y + (-\Delta + q)\, y - a\, y - f\right) \\ &\quad \cdot \left[p + N\left(D_t^\alpha y + (-\Delta + q)\, y - a\, y - f\right)\right] = 0 \\ &\qquad \text{in } \mathcal{Q}, \\ &y, p \longrightarrow 0 \quad \text{as } |x| \longrightarrow \infty, \\ &y = 0, \\ &p = 0 \quad \text{on } \Sigma, \\ &I_+^{1-\alpha} y(x,0) = 0 \\ &p(x,T) = 0 \quad \text{in } \mathbb{R}^n. \end{aligned} \tag{51}$$

Clearly, from the last condition of (50), either $u = 0$ and $p + Nu = p > 0$, or $p+Nu = 0$ and $u > 0$, or $u = 0$ and $p+Nu = 0$. In the last case, we have $y = z_d$ and

$$D_t^\alpha y + (-\Delta + q)\, y - ay = f. \tag{52}$$

Hence, assuming that

$$D_t^\alpha z_d + (-\Delta + q)\, z_d - az_d \neq f, \tag{53}$$

only the first two cases are possible. If

$$N = v\mathcal{I}, \quad v > 0, \tag{54}$$

where $\mathcal{I}$ is the identity operator, then we deduce

$$u = -\frac{1}{v} \inf\{o, p\} \quad \text{almost everywhere in } \mathcal{Q}. \tag{55}$$

Then the optimal control u is given by (55), where p is furnished by the solution of the following nonlinear boundary value problem:

$$\begin{aligned} &D_t^\alpha y + (-\Delta + q)\, y - a\, y + \frac{1}{v} \inf(o, p) = f \quad \text{in } \mathcal{Q}, \\ &-D_c^\alpha p + (-\Delta + q)\, p - ap - y = -z_d \quad \text{in } \mathcal{Q}, \\ &p, y \longrightarrow 0 \quad \text{as } |x| \longrightarrow \infty, \\ &y = 0, \\ &p = 0 \quad \text{on } \Sigma, \\ &I_+^{1-\alpha} y(x,0) = 0, \\ &p(x,T) = 0 \quad \text{in } \mathbb{R}^n. \end{aligned} \tag{56}$$

Example 3. Consider the case where

$$\begin{aligned} \mathcal{U}_{ad} = \{ &v \mid \xi_0(x,t) \leq v(x,t) \leq \xi_1(x,t), \\ &\text{almost everywhere in } \mathcal{Q}, \xi_0, \\ &\xi_1 \text{ given functions in } L^\infty(Q) \}. \end{aligned} \tag{57}$$

Then (35) is equivalent to the local condition

$$\begin{aligned} &\left(p(x,t;u) + Nu(x,t)\right)(\xi - u(x,t)) \geq 0, \\ &\qquad \forall \xi \in \left[\xi_0(x,t), \xi_1(x,t)\right]. \end{aligned} \tag{58}$$

Further assume that

$$N = v\mathcal{I}, \quad v > 0. \tag{59}$$

Hence the last condition (58) is simplified to the following conditions:

$$\begin{aligned} &p(x,t;u) + vu(x,t) > 0, \quad u(x,t) = \xi_0(x,t), \\ &p(x,t;u) + vu(x,t) < 0, \quad u(x,t) = \xi_1(x,t), \\ &p(x,t;u) + vu(x,t) = 0, \quad u(x,t) = -\frac{1}{v} p(x,t;u). \end{aligned} \tag{60}$$

6. Summary and Conclusion

The motivation of this paper is to present a new planner to study the distributed control problem for the fractional differential systems. We pay our attention to the time-fraction differential cooperative system involving Schrödinger operator, which is formulated in system (2). This planner is based on the abstract variational formulation method, the Lax-Milgram lemma, and the adjoint problem technique. With the help of Lax-Milgram lemma, we prove existence and uniqueness of solution for the time-fractional differential system involving Schrödinger operator. By employing the adjoint problem, we characterize the optimal control. With regard to the fractional Dirichlet problem with linear quadratic cost functional, we produce some equations and inequalities which provide the necessary and sufficient optimality conditions. Further, we provide specific application examples to demonstrate the effectiveness of our results. Moreover, our technique proposed in this paper can be analogously applied to many fractional systems such as boundary control for time-fractional differential system involving Schrödinger operator with nature of control Dirichlet, Neumann, or Robin. Also, If $\alpha \longrightarrow 1$, our results reduce to classical optimal control theory obtained by Lions in [14, 15] and Bahaa in [21] for the differential system involving Schrödinger operator.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The authors contributed equally and significantly to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgments

The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through research groups program under grant number (R. G. P. 1/ 160/40 /2019).

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