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Research Article

New Types of $F-$Contraction for Multivalued Mappings and Related Fixed Point Results in Abstract Spaces

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In this article, we establish fixed point results for a pair of multivalued mappings satisfying generalized contraction on a sequence in dislocated b-quasi metric spaces and $F\rho_s^*$ Khan type contraction on a sequence in b-quasi metric spaces. An example and an application have been discussed. Our results modify and generalize many existing results in literature.

1. Introduction and Preliminaries

A point v is said to be a fixed point of a multivalued/self-mapping E, if $v \in Ev/v = Ev$. Fixed point theory has a large number of applications, for example, [1–4]. Czewick [5] initiated the study of fixed point in b-metric spaces. Many authors used the concept of b-metric spaces to prove the existence and the uniqueness of a fixed point for several contraction mappings [6–9]. Furthermore, dislocated quasi-metric spaces [10–13] generalized abstract spaces such as dislocated metric spaces [14] and quasi-metric spaces [15–17]. Recently, Klin-eam and Suanoom [18] introduced the concept of dislocated b-quasi metric spaces. Fixed point results in complete dislocated b-quasi metric spaces can be seen in [19, 20].

Wardowski [21] generalized many fixed point results in a beautiful way by introducing $F-$contraction (see also [6, 22–30]). Nadler [31] extended Banach's contraction mapping principle to a fundamental fixed point theorem for multivalued mappings. Since then, an interesting and rich fixed point theory for such mappings was developed in many directions; see [32–36]. The results of single valued mappings can be generalized by using multivalued mappings. Results for multivalued mappings have applications in engineering, Nash equilibria, and game theory [37–40]. Rasham et al. [41] obtained fixed point results for a pair of multivalued $F-$contractive mappings, which are extensions of some multivalued fixed point results.

This paper introduces new types of $F-$contractions on a sequence and generalizes many recent results. An example has been given to show how our results are valid when the others fail. An application has been given to obtain a solution of a system of integral equations.

Definition 1 (see [18]). Let Y be a nonempty set and $s \geq 1$ a real number. A mapping $d_{qb} : Y \times Y \longrightarrow [0, \infty)$ is called a dislocated quasi b-metric (or simply d_{qb}-metric), if the following conditions hold for any $x, y, z \in Y$:

(a) If $d_{qb}(x, y) = d_{qb}(y, x) = 0$, then $x = y$;

(b) $d_{qb}(x, y) \leq s[d_{qb}(x, z) + d_{qb}(z, y)]$.

The pair (Y, d_{qb}) is called a dislocated quasi b-metric space (in short dislocated b-quasi-metric space).

The following remarks can be observed:

(a) If $s = 1$, then a dislocated b-quasi-metric space becomes a dislocated quasi-metric space [12];

(b) if $s = 1$ and $x = y$ implies $d_{qb}(x, y) = d_{qb}(y, x) = 0$, then (Y, d_{qb}) becomes a quasi-metric space [17];

(c) if $d_{qb}(x, y) = d_{qb}(y, x)$ and $x = y$ implies $d_{qb}(x, y) = 0$, then (Y, d_{qb}) becomes a b-metric space [9].

Example 2 (see [20], let $Y = R^{+}$ and $p > 1$). Define $d_{qb} : Y \times Y \longrightarrow R^{+}$ by $d_{qb}(x, y) = |x - y| + |x|$ for $x, y \in X$. Then (Y, d_{qb}) is a d_{qb}-metric space with $s = 2^{p} > 1$. But it is not a quasi b-metric space. Also it is not a dislocated b-metric space. It is obvious that (Y, d_{qb}) is neither b-metric space nor dislocated quasi-metric space.

Definition 3 (see [11]). Let (Y, d_{qb}) be a dislocated b-quasi-metric space. Let $\{y_n\}$ be a sequence in (Y, d_{qb}), and then

(a) $\{y_n\}$ is called Cauchy if $\forall\, \varepsilon > 0$, $\exists\, n_0 \in N$ such that $\forall\, n > m \geq n_0$ (respectively $\forall\, m > n \geq n_0$), $d_{qb}(y_m, y_n) < \varepsilon$.

(b) $\{y_n\}$ dislocated quasi b-converges (for short d_{qb} -converges) to $y \in Y$, if $\lim_{n\longrightarrow\infty} d_{qb}(y_n, y) = \lim_{n\longrightarrow\infty} d_{qb}(y, y_n) = 0$ or for any $\varepsilon > 0$, there exists $n_0 \in N$, such that for all $n > n_0$, $d_{qb}(y, y_n) < \varepsilon$ and $d_{qb}(y_n, y) < \varepsilon$. In this case y is called a d_{qb}-limit of $\{y_n\}$.

(c) (Y, d_{qb}) is called complete if every Cauchy sequence in Y converges to a point $y \in Y$.

Definition 4 (see [12]). Let (Y, d_{qb}) be a dislocated b-quasi metric space. Let K be a nonempty subset of Y and let $x \in Y$. An element $y_0 \in K$ is called a best approximation in K if

$$\begin{aligned} &d_{qb}(x, K) = d_{qb}(x, y_0), \\ &\qquad\qquad \text{where } d_{qb}(x, K) = \inf_{y\in K} d_{qb}(x, y) \\ &\text{and } d_{qb}(K, x) = d_{qb}(y_0, x), \\ &\qquad\qquad \text{where } d_{qb}(K, x) = \inf_{y\in K} d_{qb}(y, x). \end{aligned} \tag{1}$$

If each $x \in Y$ has at least one best approximation in K, then K is called a proximinal set.

It is clear that if $d_{qb}(x, K) = d_{qb}(K, x) = 0$, then $x \in K$. But if $x \in K$, then $d_{qb}(x, K)$ or $d_{qb}(K, x)$ may not equal zero. We denote $P(Y)$ by the set of all proximinal subsets of Y.

Definition 5 (see [12]). The function $H_{d_q} : P(Y) \times P(Y) \longrightarrow \mathbb{R}_{+}$, defined by

$$H_{d_{qb}}(A, B) = \max\left\{\sup_{a\in A} d_q(a, B), \sup_{b\in B} d_{qb}(A, b)\right\} \tag{2}$$

is called dislocated quasi Hausdorff b metric on $P(Y)$. Also $(P(Y), H_{d_{qb}})$ is known as dislocated quasi Hausdorff b-metric space, where $P(Y)$ is the proximinal subset of Y.

Ali et al. [6] extended the family of mapping $\mathscr{F}$ defined by [21] to the family $\mathscr{F}_S$ of all functions $F : \mathbb{R}_{+} \longrightarrow \mathbb{R}$ such that

(F1) F is strictly increasing, that is, for all $x, y \in \mathbb{R}_{+}$ such that $x < y$ implies $F(x) < F(y)$;

(F2) for each sequence $\{\vartheta_n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n\longrightarrow\infty} \vartheta_n = 0$ if and only if $\lim_{n\longrightarrow\infty} F(\vartheta_n) = -\infty$;

(F3) there exists $k \in (0, 1)$ such that $\lim_{\vartheta\longrightarrow 0^{+}} \vartheta^{k} F(\vartheta) = 0$.

(F4) For each sequence $\{\vartheta_n\}$ of positive real numbers and such that $\tau + F(s\vartheta_n) \leq F(\vartheta_{n-1})$ for each $n \in \mathbb{N}$, and some $\tau > 0$, we have $\tau + F(s^{n}\vartheta_n) \leq F(s^{n-1}\vartheta_{n-1})$, for each $n \in \mathbb{N}$.

Lemma 6. *Let (Y, d_{qb}, s) be a dislocated b-quasi-metric space. Let $(P(Y), H_{d_{qb}})$ be the dislocated quasi Hausdorff b-metric space on $P(Y)$. Then, for all $A, B \in P(Y)$ and for each $a \in A$, there exists $b_a \in B$, such that $H_{d_{qb}}(A, B) \geq d_{qb}(a, b_a)$ and $H_{d_q}(B, A) \geq d_{qb}(b_a, a)$, where $d_{qb}(a, B) = d_{qb}(a, b_a)$ and $d_{qb}(B, a) = d_{qb}(b_a, a)$.*

Lemma 7 (see [6]). *Let (Y, d_b, s) be a b-metric space and let $\{y_n\}$ be any sequence in Y for which there exist $\tau > 0$ and $F \in \mathscr{F}_S$ such that $\tau + F(sd_{qb}(y_n, y_{n+1})) \leq F(d_{qb}(y_{n-1}, y_n))$, $n \in \mathbb{N}$. Then $\{y_n\}$ is a Cauchy sequence in Y.*

Lemma 8. *Let (X, d_{qb}, s) be a dislocated b-quasi metric space, and let $\{x_n\}$ be any sequence in X for which there exist $\tau > 0$ and $F \in \mathscr{F}_S$ such that*

$$\begin{aligned} \tau &+ F\left(s \max\left\{d_{qb}(y_n, y_{n+1}), d_{qb}(y_{n+1}, y_n)\right\}\right) \\ &\leq F\left(\max\left\{d_{qb}(y_{n-1}, y_n), d_{qb}(y_n, y_{n-1})\right\}\right) \end{aligned} \tag{3}$$

for each $n \in \mathbb{N}$. Then $\{y_n\}$ is a Cauchy sequence in X.

Proof. Let $\vartheta_n = \max\{d_{qb}(y_n, y_{n+1}), d_{qb}(y_{n+1}, y_n)\}$, for each $n \in \mathbb{N}$. Thus, by (3) and property (F4), we get

$$\tau + F(s^{n}\vartheta_n) \leq F\left(s^{n-1}\vartheta_{n-1}\right), \quad n \in \mathbb{N}. \tag{4}$$

Following similar arguments as given in [6], we obtain $\{y_n\}$ is a Cauchy sequence in X. □

2. Main Result

Let (Y, d_{qb}) be a dislocated b-quasi metric space, $y_0 \in Y$ and $S, T : Y \longrightarrow P(Y)$ be multifunctions on Y. Let $y_1 \in Sy_0$ be an element such that $d_{qb}(y_0, Sy_0) = d_{qb}(y_0, y_1)$, $d_{qb}(Sy_0, y_0) = d_{qb}(y_1, y_0)$. Let $y_2 \in Ty_1$ be such that $d_{qb}(y_1, Ty_1) = d_{qb}(y_1, y_2)$, $d_{qb}(Ty_1, y_1) = d_{qb}(y_2, y_1)$. Let $y_3 \in Sy_2$ be such that $d_{qb}(y_2, Sy_2) = d_{qb}(y_2, y_3)$ and so on. Thus, we construct a sequence y_n of points in Y such that $y_{2n+1} \in Sy_{2n}$ and $y_{2n+2} \in Ty_{2n+1}$, with $d_{qb}(y_{2n}, Sy_{2n}) = d_{qb}(y_{2n}, y_{2n+1})$, $d_{qb}(Sy_{2n}, y_{2n}) = d_{qb}(y_{2n+1}, y_{2n})$, and $d_{qb}(y_{2n+1}, Ty_{2n+1}) = d_{qb}(y_{2n+1}, y_{2n+2})$, $d_{qb}(Ty_{2n+1}, y_{2n+1}) = d_{qb}(y_{2n+2}, y_{2n+1})$, where $n = 0, 1, 2, \cdots$. We denote this iterative sequence by $\{TS(y_n)\}$. We say that $\{TS(y_n)\}$ is a sequence in Y generated by y_0. If $T = S$, then we say that $\{XT(y_n)\}$ is a sequence in Y generated by y_0.

Let us introduce the following definition:

Definition 9. Let (Y, d_{qb}, s) be a dislocated b-quasi-metric space and $S, T : Y \longrightarrow P(Y)$ be two multivalued mappings. The pair (S, T) is called a DQF−contraction, if there exists $F \in \mathscr{F}_S$ and $\tau, a > 0$ such that for every two consecutive points x, y belonging to the range of an iterative sequence $\{TS(y_n)\}$ with $\max\{H_{d_{qb}}(Sx, Ty), H_{d_{qb}}(Ty, Sx), D_{qb}(x, y), D_{qb}(y, x)\} > 0$, we have

$$\begin{aligned} \tau &+ \max\left\{F\left(sH_{d_{qb}}(Sx, Ty)\right), F\left(sH_{d_{qb}}(Ty, Sx)\right)\right\} \\ &\leq \min\left\{F\left(D_{qb}(x, y)\right), F\left(D_{qb}(y, x)\right)\right\} \end{aligned} \tag{5}$$

where

$$D_{qb}(x,y) = \max\Big\{d_{qb}(x,y), \frac{d_{qb}(x,Sx).d_{qb}(y,Ty)}{a+\max\{d_{qb}(x,y),d_{qb}(y,x)\}}, d_{qb}(x,Sx), d_{qb}(y,Ty)\Big\}, \quad (6)$$

And we now prove the following main result.

Theorem 10. *Let (Y, d_{qb}, s) be a complete dislocated b-quasi-metric with $s \geq 1$ and (S,T) be a DQF−contraction. Then $\{TS(y_n)\} \longrightarrow u \in Y$. Also, if (5) holds for each $x, y \in \{u\}$, then S and T have a common fixed point u in Y and $d_{qb}(u,u) = 0$.*

Proof. Let $\{TS(y_p)\}$ be the iterative sequence in Y generated by a point $y_0 \in Y$. If

$$\max\big\{H_{d_{qb}}(Sy_{2p'}, Ty_{2p'+1}), H_{d_{qb}}(Ty_{2p'+1}, Sy_{2p'}), D_{qb}(y_{2p'}, y_{2p'+1}), D_{qb}(y_{2p'+1}, y_{2p'})\big\} \ngtr 0 \quad (7)$$

for some $p' \in \mathbb{N} \cup \{0\}$, then

$$\begin{aligned} H_{d_{qb}}(Sy_{2p'}, Ty_{2p'+1}) &= H_{d_{qb}}(Ty_{2p'+1}, Sy_{2p'}) \\ &= D_{qb}(y_{2p'}, y_{2p'+1}) \\ &= D_{qb}(y_{2p'+1}, y_{2p'}) = 0 \end{aligned} \quad (8)$$

Clearly, if $D_{qb}(y_{2p'}, y_{2p'+1}) = 0$, then $d_{qb}(y_{2p'}, y_{2p'+1}) = 0$. Also $D_{qb}(y_{2p'+1}, y_{2p'}) = 0$ implies $d_{qb}(y_{2p'+1}, y_{2p'}) = 0$. So, $y_{2p'} = y_{2p'+1}$ and $y_{2p'} \in Sy_{2p'}$. Now, $H_{d_{qb}}(Sy_{2p'}, Ty_{2p'+1}) = 0$ implies $d_{qb}(y_{2p'+1}, Ty_{2p'+1}) = 0$ and $H_{d_{qb}}(Ty_{2p'+1}, Sy_{2p'}) = 0$ implies $d_{qb}(Ty_{2p'+1}, y_{2p'+1}) = 0$. So, $y_{2p'+1} \in Ty_{2p'+1}$ and $y_{2p'}$ is a common fixed point of S and T. So the proof is completed in this case. Now, let

$$\max\big\{H_{d_{qb}}(Sy_{2p}, Ty_{2p+1}), H_{d_{qb}}(Ty_{2p+1}, Sy_{2p}), D_{qb}(y_{2p}, y_{2p+1}), D_{qb}(y_{2p+1}, y_{2p})\big\} > 0, \quad (9)$$

for all $p \in \mathbb{N} \cup \{0\}$. By Lemma 6, we have

$$\begin{aligned} d_{qb}(y_{2p}, y_{2p+1}) &\leq H_{d_{qb}}(Ty_{2p-1}, Sy_{2p}), \\ d_{qb}(y_{2p+1}, y_{2p}) &\leq H_{d_{qb}}(Sy_{2p}, Ty_{2p-1}), \end{aligned} \quad (10)$$

and

$$\begin{aligned} d_{qb}(y_{2p+1}, y_{2p+2}) &\leq H_{d_{qb}}(Sy_{2p}, Ty_{2p+1}), \\ d_{qb}(y_{2p+2}, y_{2p+1}) &\leq H_{d_{qb}}(Ty_{2p+1}, Sy_{2p}). \end{aligned} \quad (11)$$

From (11), (F1) and using condition (5), we get

$$\begin{aligned} F(sd_{qb}(y_{2p+1}, y_{2p+2})) &\leq F(sH_{d_{qb}}(Sy_{2p}, Ty_{2p+1})) \\ &\leq \max\{F(sH_{d_{qb}}(Sy_{2p}, Ty_{2p+1})), F(sH_{d_{qb}}(Ty_{2p+1}, Sy_{2p}))\} \\ &\leq \min\{F(D_{qb}(y_{2p}, y_{2p+1})), F(D_{qb}(y_{2p+1}, y_{2p}))\} \\ &\quad - \tau \leq F(D_{qb}(y_{2p}, y_{2p+1})) - \tau, \end{aligned} \quad (12)$$

From (6), we have

$$\begin{aligned} D_{qb}(y_{2p}, y_{2p+1}) &= \max\Big\{d_{qb}(y_{2p}, y_{2p+1}), \frac{d_{qb}(y_{2p}, Sy_{2p}).d_{qb}(y_{2p+1}, Ty_{2p+1})}{a+\max\{d_{qb}(y_{2p}, y_{2p+1}), d_{qb}(y_{2p+1}, y_{2p})\}}, d_{qb}(y_{2p}, Sy_{2p}), d_{qb}(y_{2p+1}, Ty_{2p+1})\Big\} \\ &= \max\Big\{d_{qb}(y_{2p}, y_{2p+1}), \frac{d_{qb}(y_{2p}, y_{2p+1}).d_{qb}(y_{2p+1}, y_{2p+2})}{a+\max\{d_{qb}(y_{2p}, y_{2p+1}), d_{qb}(y_{2p+1}, y_{2p})\}}, d_{qb}(y_{2p}, y_{2p+1}), d_{qb}(y_{2p+1}, y_{2p+2})\Big\} \\ &\leq \max\{d_{qb}(y_{2p}, y_{2p+1}), d_{qb}(y_{2p+1}, y_{2p+2})\}. \end{aligned} \quad (13)$$

If $\max\{d_{qb}(y_{2p}, y_{2p+1}), d_{qb}(y_{2p+1}, y_{2p+2})\} = d_{qb}(y_{2p+1}, y_{2p+2})$, then

$$F(sd_{qb}(y_{2p+1}, y_{2p+2})) \leq F(d_{qb}(y_{2p+1}, y_{2p+2})) - \tau, \quad (14)$$

which is a contradiction due to (F1) and $s \geq 1$. Therefore,

$$F(sd_{qb}(y_{2p+1}, y_{2p+2})) \leq F(d_{qb}(y_{2p}, y_{2p+1})) - \tau, \quad (15)$$

$$\begin{aligned} &F(sd_{qb}(y_{2p+1}, y_{2p+2})) \\ &\quad \leq F(\max\{d_{qb}(y_{2p}, y_{2p+1}), d_{qb}(y_{2p+1}, y_{2p})\}) \\ &\qquad - \tau. \end{aligned} \quad (16)$$

From (11), (F1) and using condition (5), we get

$$\begin{aligned} F\left(sd_{qb}\left(y_{2p+2}, y_{2p+1}\right)\right) &\leq F\left(sH_{d_{qb}}\left(Ty_{2p+1}, Sy_{2p}\right)\right) \\ &\leq \max\left\{F\left(sH_{d_{qb}}\left(Sy_{2p}, Ty_{2p+1}\right)\right),\right. \\ &\left. F\left(sH_{d_{qb}}\left(Ty_{2p+1}, Sy_{2p}\right)\right)\right\} \\ &\leq \min\left\{F\left(D_{qb}\left(y_{2p}, y_{2p+1}\right)\right), F\left(D_{qb}\left(y_{2p+1}, y_{2p}\right)\right)\right\} \\ &- \tau \leq F\left(D_{qb}\left(y_{2p}, y_{2p+1}\right)\right) - \tau \\ &= F\left(\max\left\{d_{qb}\left(y_{2p}, y_{2p+1}\right), d_{qb}\left(y_{2p+1}, y_{2p+2}\right)\right\}\right) \\ &- \tau \end{aligned} \tag{17}$$

By using (15) and (F1), we get

$$\begin{aligned} &F\left(sd_{qb}\left(y_{2p+2}, y_{2p+1}\right)\right) \\ &\quad\leq F\left(\max\left\{d_{qb}\left(y_{2p}, y_{2p+1}\right), \frac{1}{s}d_{qb}\left(y_{2p}, y_{2p+1}\right)\right\}\right) \\ &\quad\quad- \tau = F\left(d_{qb}\left(y_{2p}, y_{2p+1}\right)\right) - \tau \\ &\quad\leq F\left(\max\left\{d_{qb}\left(y_{2p}, y_{2p+1}\right), d_{qb}\left(y_{2p+1}, y_{2p}\right)\right\}\right) \\ &\quad\quad- \tau. \end{aligned} \tag{18}$$

$$\begin{aligned} &F\left(sd_{qb}\left(y_{2p+2}, y_{2p+1}\right)\right) \\ &\quad\leq F\left(\max\left\{d_{qb}\left(y_{2p}, y_{2p+1}\right), d_{qb}\left(y_{2p+1}, y_{2p}\right)\right\}\right) \\ &\quad\quad- \tau. \end{aligned} \tag{19}$$

Combining (16) and (19), we get

$$\begin{aligned} &\max\left\{F\left(sd_{qb}\left(y_{2p+2}, y_{2p+1}\right)\right), F\left(sd_{qb}\left(y_{2p+1}, y_{2p+2}\right)\right)\right\} \\ &\quad\leq F\left(\max\left\{d_{qb}\left(y_{2p}, y_{2p+1}\right), d_{qb}\left(y_{2p+1}, y_{2p}\right)\right\}\right) \\ &\quad- \tau. \end{aligned} \tag{20}$$

By using (10) and (5), we have

$$\begin{aligned} F\left(sd_{qb}\left(y_{2p}, y_{2p+1}\right)\right) &\leq F\left(sH_{d_{qb}}\left(Ty_{2p-1}, Sy_{2p}\right)\right) \\ &\leq \max\left\{F\left(sH_{d_{qb}}\left(Sy_{2p}, Ty_{2p-1}\right)\right),\right. \\ &\left. F\left(sH_{d_{qb}}\left(Ty_{2p-1}, Sy_{2p}\right)\right)\right\} \\ &\leq \min\left\{F\left(D_{qb}\left(y_{2p-1}, y_{2p}\right)\right), F\left(D_{qb}\left(y_{2p}, y_{2p-1}\right)\right)\right\} \\ &- \tau \leq F\left(D_{qb}\left(y_{2p}, y_{2p-1}\right)\right) - \tau. \end{aligned} \tag{21}$$

From (6), we have

$$\begin{aligned} D_{qb}\left(y_{2p}, y_{2p-1}\right) &= \max\left\{d_{qb}\left(y_{2p}, y_{2p-1}\right),\right. \\ &\frac{d_{qb}\left(y_{2p}, y_{2p+1}\right) . d_{qb}\left(y_{2p-1}, y_{2p}\right)}{a + \max\left\{d_{qb}\left(y_{2p}, y_{2p-1}\right), d_{qb}\left(y_{2p-1}, y_{2p}\right)\right\}}, \\ &\left. d_{qb}\left(y_{2p}, y_{2p+1}\right), d_{qb}\left(y_{2p-1}, y_{2p}\right)\right\} \\ &\leq \max\left\{d_{qb}\left(y_{2p}, y_{2p-1}\right), d_{qb}\left(y_{2p-1}, y_{2p}\right),\right. \\ &\left. d_{qb}\left(y_{2p}, y_{2p+1}\right)\right\}. \end{aligned} \tag{22}$$

If $\max\{d_{qb}(y_{2p}, y_{2p-1}), d_{qb}(y_{2p-1}, y_{2p}), d_{qb}(y_{2p}, y_{2p+1})\} = d_{qb}(y_{2p}, y_{2p+1})$, then we obtain

$$F\left(sd_{qb}\left(y_{2p}, y_{2p+1}\right)\right) \leq F\left(d_{qb}\left(y_{2p}, y_{2p+1}\right)\right) - \tau, \tag{23}$$

which is a contradiction due to (F1). Therefore,

$$\begin{aligned} &F\left(sd_{qb}\left(y_{2p}, y_{2p+1}\right)\right) \\ &\quad\leq F\left(\max\left\{d_{qb}\left(y_{2p-1}, y_{2p}\right), d_{qb}\left(y_{2p}, y_{2p-1}\right)\right\}\right) \\ &\quad\quad- \tau. \end{aligned} \tag{24}$$

By using (10) and (5), we have

$$\begin{aligned} F\left(sd_{qb}\left(y_{2p+1}, y_{2p}\right)\right) &\leq F\left(sH_{d_{qb}}\left(Sy_{2p}, Ty_{2p-1}\right)\right) \\ &\leq F\left(D_{qb}\left(y_{2p}, y_{2p-1}\right)\right) - \tau \\ &\leq F\left(\max\left\{d_{qb}\left(y_{2p}, y_{2p-1}\right), d_{qb}\left(y_{2p-1}, y_{2p}\right),\right.\right. \\ &\left.\left. d_{qb}\left(y_{2p}, y_{2p+1}\right)\right\}\right) - \tau. \end{aligned} \tag{25}$$

From (24), $d_{qb}(y_{2p}, y_{2p+1}) < \max\{d_{qb}(y_{2p-1}, y_{2p}), d_{qb}(y_{2p}, y_{2p-1})\}$, so

$$\begin{aligned} &F\left(sd_{qb}\left(y_{2p+1}, y_{2p}\right)\right) \\ &\quad\leq F\left(\max\left\{d_{qb}\left(y_{2p}, y_{2p-1}\right), d_{qb}\left(y_{2p-1}, y_{2p}\right)\right\}\right) \\ &\quad\quad- \tau. \end{aligned} \tag{26}$$

Combining (24) and (26), we get

$$\begin{aligned} &\max\left\{F\left(sd_{qb}\left(y_{2p}, y_{2p+1}\right)\right), F\left(sd_{qb}\left(y_{2p+1}, y_{2p}\right)\right)\right\} \\ &\quad\leq \max\left\{d_{qb}\left(y_{2p}, y_{2p-1}\right), d_{qb}\left(y_{2p-1}, y_{2p}\right)\right\} - \tau. \end{aligned} \tag{27}$$

Combining (20) and (27), we get

$$\begin{aligned} &\tau + F\left(s\max\left\{d_{qb}\left(y_{n}, y_{n+1}\right), d_{qb}\left(y_{n+1}, y_{n}\right)\right\}\right) \\ &\quad\leq F\left(\max\left\{d_{qb}\left(y_{n-1}, y_{n}\right), d_{qb}\left(y_{n}, y_{n-1}\right)\right\}\right) \end{aligned} \tag{28}$$

By Lemma 8, $\{TS(y_n)\}$ is a Cauchy sequence in (Y, d_{qb}). Since (Y, d_{qb}) is a complete dislocated b-quasi-metric space, so there exists $u \in Y$ such that $\{TS(y_n)\} \longrightarrow u$; that is,

$$\lim_{n\longrightarrow\infty} d_{qb}(y_n, u) = \lim_{n\longrightarrow\infty} d_{qb}(u, y_n) = 0. \tag{29}$$

Now, suppose $d_{qb}(u, Tu) > 0$, and then $D_{qb}(y_{2n}, u) > 0$, so

$$\begin{aligned} &\max\left\{H_{d_{qb}}(Sy_{2n}, Tu), H_{d_{qb}}(Tu, Sy_{2n}), D_{qb}(y_{2n}, u),\right.\\ &\left.D_{qb}(u, y_{2n})\right\} > 0. \end{aligned} \tag{30}$$

By using Lemma 6 and (5), we have

$$\begin{aligned} &\tau + F\left(sd_{qb}(y_{2n+1}, Tu)\right) \leq \tau \\ &\quad + \max\left\{F\left(sH_{d_{qb}}(Sy_{2n}, Tu)\right),\right.\\ &\quad \left. F\left(sH_{d_{qb}}(Tu, Sy_{2n})\right)\right\} \leq \min\left\{F\left(D_{qb}(y_{2n}, u)\right),\right.\\ &\quad \left. F\left(D_{qb}(u, y_{2n})\right)\right\} \leq F\left(D_{qb}(y_{2n}, u)\right). \end{aligned} \tag{31}$$

Since F is strictly increasing, we have

$$sd_{qb}(y_{2n+1}, Tu) < D_{qb}(y_{2n}, u). \tag{32}$$

Taking $\lim_{n\longrightarrow\infty}$ on both sides, we get

$$\lim_{n\longrightarrow\infty} sd_{qb}(y_{2n+1}, Tu) < \lim_{n\longrightarrow\infty} D_{qb}(y_{2n}, u) \tag{33}$$

From (6)

$$\begin{aligned} D_{qb}(y_{2n}, u) = \max&\left\{d_{qb}(y_{2n}, u),\right.\\ &\frac{d_{qb}(y_{2n}, y_{2n+1}) . d_{qb}(u, Tu)}{a + \max\left\{d_{qb}(y_{2n}, u), d_{qb}(u, y_{2n})\right\}},\\ &\left. d_{qb}(y_{2n}, y_{2n+1}), d_{qb}(u, Tu)\right\}. \end{aligned} \tag{34}$$

Taking limit as $n \longrightarrow \infty$, and by using (29), we get

$$\lim_{n\longrightarrow\infty} D_{qb}(y_{2n}, u) = d_{qb}(u, Tu). \tag{35}$$

Using inequality (35) in (33), we get

$$\lim_{n\longrightarrow\infty} sd_{qb}(y_{2n+1}, Tu) < d_{qb}(u, Tu). \tag{36}$$

Now,

$$d_{qb}(u, Tu) \leq sd_{qb}(u, y_{2n+1}) + sd_{qb}(y_{2n+1}, Tu). \tag{37}$$

Taking limit as $n \longrightarrow \infty$,

$$\begin{aligned} d_{qb}(u, Tu) \leq s &\lim_{n\longrightarrow\infty} d_{qb}(u, y_{2n+1}) \\ &+ \lim_{n\longrightarrow\infty} sd_{qb}(y_{2n+1}, Tu). \end{aligned} \tag{38}$$

Using inequalities (29) and (36) in (38), we get

$$d_{qb}(u, Tu) < d_{qb}(u, Tu). \tag{39}$$

This is a contradiction, so $d_{qb}(u, Tu) = 0$. Now, suppose $d_{qb}(Tu, u) > 0$, and then there exists $n_0 \in \mathbb{N}$ such that $d_{qb}(Tu, y_{2n+1}) > 0$ for all $n \geq n_0$. By Lemma 6 $d_{qb}(Tu, y_{2n+1}) \leq H_{d_{qb}}(Tu, Sy_{2n})$, so

$$\begin{aligned} &\max\left\{H_{d_{qb}}(Sy_{2n}, Tu), H_{d_{qb}}(Tu, Sy_{2n}), D_{qb}(y_{2n}, u),\right.\\ &\left.D_{qb}(u, y_{2n})\right\} > 0. \end{aligned} \tag{40}$$

for all $n \geq n_0$. Following similar arguments as above, we get

$$\lim_{n\longrightarrow\infty} sd_{qb}(Tu, y_{2n+1}) < d_{qb}(u, Tu) = 0. \tag{41}$$

Now,

$$d_{qb}(Tu, u) \leq sd_{qb}(Tu, y_{2n+1}) + sd_{qb}(y_{2n+1}, u). \tag{42}$$

Taking limit as $n \longrightarrow \infty$, and using inequalities (29) and (41), we get

$$d_{qb}(Tu, u) \leq 0 \tag{43}$$

which is a contradiction, so $d_{qb}(Tu, u) = 0$. Hence $u \in Tu$. Similarly by using (29), Lemma 6, and the inequality

$$\tau + d_{qb}(Su, y_{2n+2}) \leq \tau + F\left(H_{d_{qb}}(Su, Ty_{2n+1})\right), \tag{44}$$

we can show that $d_{qb}(Su, u) = 0$. Similarly, $d_{qb}(u, Su) = 0$. Hence, the pair (S, T) has a common fixed point u in (Y, d_{qb}). Now,

$$d_{qb}(u, u) \leq d_{qb}(u, Tu) + d_{qb}(Tu, u) \leq 0. \tag{45}$$

This implies that $d_{qb}(u, u) = 0$. Hence the proof is completed. □

Now, let us introduce the following example.

Example 11. Let $Y = \{0\} \cup \mathbb{Q}^+$ and $d_{qb}(x, y) = (x + 2y)^2$ if $x \neq y$, and $d_{qb}(x, y) = 0$, if $x = y$. Then (Y, d_{qb}) is a dislocated b-quasi-metric space with $s = 2$. Define the mappings $S, T : Y \longrightarrow P(Y)$ as follows:

$$\begin{aligned} &S(y) \\ &= \begin{cases} \left[\frac{1}{4}y, \frac{2}{5}y\right] \cap \mathbb{Q}^+, & \text{for all } y \in \left\{0, 7, \frac{7}{4}, \frac{7}{12}, \frac{7}{48}, \dots\right\}, \\ \left[y + 1, y + 4\right] \cap \mathbb{Q}^+, & \text{otherwise.} \end{cases} \\ &T(y) \\ &= \begin{cases} \left[\frac{1}{3}y, \frac{3}{8}y\right] \cap \mathbb{Q}^+, & \text{for all } y \in \left\{0, 7, \frac{7}{4}, \frac{7}{12}, \frac{7}{48}, \dots\right\}, \\ \left[y + 3, y + 6\right] \cap \mathbb{Q}^+, & \text{otherwise.} \end{cases} \end{aligned} \tag{46}$$

Case 1. If $\tau + \max\{F(sH_{d_{qb}}(Sx, Ty)), F(sH_{d_{qb}}(Tx, Sy))\} = \tau + F(sH_{d_{qb}}(Sx, Ty)) \leq \min\{F(D_{qb}(x, y)), F(D_{qb}(y, x))\}$ holds. Define the function $F : R^+ \longrightarrow R$ by $F(x) = \ln(x)$ for all

$x \in R^{+}$ and $\tau > 0$. As $x, y \in Y$, $\tau = \ln(1.2)$ and by taking $y_0 = 7$, we define the sequence $\{TS(y_n)\} = \{7, 7/4, 7/12, 7/48, \cdots\}$ in Y generated by $y_0 = 7$. Also, $\{TS(y_n)\} \longrightarrow 0$. Now, if $x, y \in \{TS(y_n)\} \cup \{0\}$, we have

$$\begin{aligned} sH_{d_{qb}}(Sx, Ty) &= 2H_{d_{qb}}\left(\left[\frac{1}{4}x, \frac{2}{5}x\right], \left[\frac{1}{3}y, \frac{3}{8}y\right]\right) = 2 \\ &\cdot \max\left[\left\{\sup_{a \in Sx} d_{qb}\left(a, \left[\frac{1}{3}y, \frac{3}{8}y\right]\right),\right.\right. \\ &\left.\left.\sup_{b \in Ty} d_{qb}\left(\left[\frac{1}{4}x, \frac{2}{5}x\right], b\right)\right\}\right] = 2\max\left\{d_{qb}\left(\frac{2x}{5}, \frac{y}{3}\right),\right. \\ &\left. d_{qb}\left(\frac{x}{4}, \frac{3}{8}y\right)\right\} = 2\max\left\{\left(\frac{2x}{5} + \frac{2y}{3}\right)^2, \left(\frac{x}{4} + \frac{3}{4}\right.\right. \\ &\left.\left.\cdot y\right)^2\right\}. \end{aligned} \tag{47}$$

Also

$$\begin{aligned} D_{q_b}(x, y) &= \max\left\{d_{qb}(x, y),\right. \\ &\frac{d_{qb}(x, [x/4, 2x/5]) . d_{qb}(y, [y/3, 3y/8])}{1 + \max\left\{d_{qb}(x, y), d_{qb}(y, x)\right\}}, \\ &\left. d_{qb}\left(x, \left[\frac{x}{4}, \frac{2x}{5}\right]\right), d_{qb}\left(y, \left[\frac{y}{3}, \frac{3y}{8}\right]\right)\right\} \\ &= \max\left\{d_{qb}(x, y), \frac{d_{qb}(x, x/4) . d_{qb}(y, y/3)}{1 + \max\left\{d_{qb}(x, y), d_{qb}(y, x)\right\}},\right. \\ &\left. d_{qb}\left(x, \frac{x}{4}\right), d_{qb}\left(y, \frac{y}{3}\right)\right\} = \max\left\{(x + 2y)^2,\right. \\ &\left.\frac{(5xy)^2}{4\left(1 + (x + 2y)^2\right)}, \left(\frac{3x}{2}\right)^2, \left(\frac{5y}{3}\right)^2\right\} = (x + 2y)^2. \end{aligned} \tag{48}$$

Case (i). If $\max\{(2x/5 + 2y/3)^2, (x/4 + (3/4)y)^2\} = (x/4 + (3/4)y)^2$, and $\tau = \ln(1.2)$, then we have

$$\begin{aligned} 3(x + 3y)^2 &\leq 20(x + 2y)^2 \\ \frac{6}{5}\left(\frac{x}{4} + \frac{3}{4}y\right)^2 &\leq (x + 2y)^2 \\ \ln(1.2) + \ln\left(\frac{x}{4} + \frac{3}{4}y\right)^2 &\leq \ln(x + 2y)^2. \end{aligned} \tag{49}$$

This implies that

$$\tau + F(sH_{d_{qb}}(Sx, Ty) \leq F\left(D_{q_b}(x, y)\right). \tag{50}$$

Case (ii). Similarly, if $\max\{(2x/5 + 2y/3)^2, (x/4 + (3/4)y)^2\} = (2x/5 + 2y/3)^2$, and $\tau = \ln(1.2)$, then we have

$$\begin{aligned} 48(3x + 5y)^2 &\leq 1125(x + 2y)^2 \\ \frac{6}{5}\left(\frac{2x}{5} + \frac{2y}{3}\right)^2 &\leq (x + 2y)^2 \\ \ln(1.2) + \ln\left(\frac{2x}{5} + \frac{2y}{3}\right)^2 &\leq \ln(x + 2y)^2. \end{aligned} \tag{51}$$

Hence,

$$\tau + F(sH_{d_{qb}}(Sx, Ty) \leq F\left(D_{q_b}(x, y)\right). \tag{52}$$

Case 2. If $\max\{\tau + F(sH_{d_{qb}}(Sx, Ty)), \tau + F(sH_{d_{qb}}(Tx, Sy))\} = \tau + F(sH_{d_{qb}}(Tx, Sy))$ holds.

$$\begin{aligned} sH_{d_{qb}}(Tx, Sy) &= 2\max\left[\left\{\sup_{b \in Tx} d_{qb}(b, Sy),\right.\right. \\ &\left.\left.\sup_{a \in Sy} d_{qb}(Tx, a)\right\}\right] = 2 \\ &\cdot \max\left[\left\{\sup_{b \in Tx} d_{qb}\left(b, \left[\frac{1}{4}y, \frac{2}{5}y\right]\right),\right.\right. \\ &\left.\left.\sup_{a \in Sy} d_{qb}\left(\left[\frac{1}{3}x, \frac{3}{8}x\right], a\right)\right\}\right] = 2\max\left\{d_{qb}\left(\frac{3x}{8}, \frac{y}{4}\right),\right. \\ &\left. d_{qb}\left(\frac{x}{3}, \frac{2y}{5}\right)\right\} = 2\max\left\{\left(\frac{3x}{8} + \frac{2y}{4}\right)^2, \left(\frac{x}{3}\right.\right. \\ &\left.\left.+ \frac{4y}{5}\right)^2\right\}, \end{aligned} \tag{53}$$

where

$$\begin{aligned} D_{q_b}(y, x) &= \max\left\{d_{qb}(y, x),\right. \\ &\frac{d_{qb}(x, [x/4, 2x/5]) . d_{qb}(y, [y/3, 3y/8])}{1 + \max\left\{d_{qb}(x, y), d_{qb}(y, x)\right\}}, \\ &\left. d_{qb}\left(x, \left[\frac{x}{4}, \frac{2x}{5}\right]\right), d_{qb}\left(y, \left[\frac{y}{3}, \frac{3y}{8}\right]\right)\right\} \\ &= \max\left\{d_{qb}(y, x),\right. \\ &\frac{d_{qb}(x, x/4) . d_{qb}(y, y/3)}{1 + \max\left\{d_{qb}(x, y), d_{qb}(y, x)\right\}}, d_{qb}\left(x, \frac{x}{4}\right), \\ &\left. d_{qb}\left(y, \frac{y}{3}\right)\right\} \end{aligned}$$

$$D_{q_b}(y,x) = \max\left\{(y+2x)^2, \frac{(5xy)^2}{4\left(1+(y+2x)^2\right)}, \left(\frac{3x}{2}\right)^2, \left(\frac{5y}{3}\right)^2\right\} = (y+2x)^2. \quad (54)$$

Case (i). If $\max\{(3x/8+2y/4)^2, (x/3+4y/5)^2\} = (x/3+4y/5)^2$, and $\tau = \ln(1.2)$, then we have

$$\begin{aligned} 12(5x+12y)^2 &\leq 1125(y+2x)^2 \\ \frac{6}{5}\left(\frac{x}{3}+\frac{4y}{5}\right)^2 &\leq (y+2x)^2 \\ \ln(1.2) + \ln\left(\left(\frac{x}{3}+\frac{4y}{5}\right)^2\right. &\leq \ln(y+2x)^2, \end{aligned} \quad (55)$$

so

$$\tau + F\left(sH_{d_{qb}}(Tx,Sy)\right. \leq F\left(D_{q_b}(y,x)\right). \quad (56)$$

Case (ii). Similarly, if $\max\{(3x/8+2y/4)^2, (x/3+4y/5)^2\} = (3x/8+2y/4)^2$, and $\tau = \ln(1.2)$, then we have

$$\begin{aligned} 12(3x+4y)^2 &\leq 320(y+2x)^2 \\ \frac{6}{5}\left(\frac{3x}{8}+\frac{2y}{4}\right)^2 &\leq (y+2x)^2 \\ \ln(1.2) + \ln\left(\frac{3x}{8}+\frac{2y}{4}\right)^2 &\leq \ln(y+2x)^2. \end{aligned} \quad (57)$$

Hence,

$$\tau + F\left(sH_{d_{qb}}(Tx,Sy)\right. \leq F\left(D_{q_b}(y,x)\right). \quad (58)$$

Now, if $x, y \notin \{TS(y_n)\}$, then the contraction does not hold. Hence all the hypotheses of Theorem 10 are satisfied so S and T have a common fixed point.

If we take $S = T$ in Theorem 10, then we obtain the following theorem.

Theorem 12. *Let (Y, d_{qb}) be a complete dislocated b-quasi-metric space with $s \geq 1$ and $S : Y \longrightarrow P(Y)$ be a multivalued mapping such that for every two consecutive points x, y belonging to the range of an iterative sequence $\{S(y_n)\}$ with $D_{q_b}(x,y) > 0$, $F \in \mathcal{F}_S$, $\tau, a > 0$*

$$\tau + F\left(sH_{q_b}(Sx,Sy)\right) \leq F\left(D_{q_b}(x,y)\right), \quad (59)$$

where

$$D_{q_b}(x,y) = \max\left\{d_{qb}(x,y), \frac{d_{qb}(x,Sx) . d_{qb}(y,Sy)}{a + d_{qb}(x,y)}, d_{qb}(x,Sx), d_{qb}(y,Sy)\right\}. \quad (60)$$

Then $\{S(y_n)\} \longrightarrow u \in Y$. Moreover, if (59) also holds for u, then S has a fixed point u in Y and $d_{qb}(u,u) = 0$.

Remark 13. By setting the different values of $D_{qb}(x,y)$ in (6), we can obtain different results on multivalued $F-$contractions as corollaries of Theorem 10.

3. $F\rho_s^*$ -Khan Type Contraction in Quasi b-Metric Spaces

Piri et al. [42] extended the results of Khan [43] and Fisher [44] by introducing a new general contractive condition with rational expressions. Recently, Piri et al. [30] improved some fixed point results of F_k-Khan type self-mapping on complete metric spaces. In this section, we introduce a new type of contraction satisfying an inequality of rational expressions and prove a new fixed point theorem concerning this type of contraction. Our result is real generalization of Khan fixed point theorem; we introduced $F\rho_s^*$-Khan type multivalued for two mappings in b-quasi-metric space. We start this section with the following definitions.

Definition 14. Let Y be a nonempty set, $s \geq 1$, and $\rho_s : X \times X \longrightarrow [0, +\infty)$ be a mapping such that $\rho_s(x,y) \geq s$ and $\rho_s(y,x) \geq s$, implying $x = y$. Let $M \subseteq Y$ define $\rho_s^*(x,M)=\inf\{\rho_s(x,a), a \in M\}$ and $\rho_s^*(M,y) = \inf\{\rho_s(b,y), b \in M\}$. Let $S,T : Y \longrightarrow P(Y)$ be the multivalued mappings; then the pair (S,T) is said to be ρ_s^*-Alt multivalued mapping; if $x \in Y$, then

$$\begin{aligned} &\text{(a)}\ \rho_s^*(x,Sx) \geq s, \\ &\quad q_b(x,Sx) = q_b(x,y) \\ &\text{and } q_b(Sx,x) = q_b(y,x) \quad \text{implies } \rho_s^*(Sy,y) \geq s, \\ &\text{(b)}\ \rho_s^*(Sx,x) \geq s, \\ &\quad q_b(x,Tx) = q_b(x,y) \\ &\text{and } q_b(Tx,x) = q_b(y,x) \quad \text{implies } \rho_s^*(y,Sy) \geq s. \end{aligned} \quad (61)$$

Definition 15 (see [30]). Let (X,d) be a metric space. A mapping $T : X \longrightarrow X$ is said to be F-Khan type contraction if there exists $\tau \in (0,\infty)$ and $F \in \mathcal{F}_k$ such that

$$\tau + F(d(Tx,Ty)) \leq F\left(\frac{d(x,Tx) . d(x,Ty) + d(y,Ty) . d(y,Tx)}{\max\{d(x,Ty), d(y,Tx)\}}\right), \quad (62)$$

for all $x,y \in X$, and if $\max\{d(x,Ty), d(y,Tx)\} \neq 0$, then $Tx \neq Ty$ and if $\max\{d(x,Ty), d(y,Tx)\} = 0$, then $Tx = Ty$.

Definition 16. Let (Y, q_b, s) be a b-quasi-metric space and (S,T) be a pair of ρ_s^* multivalued mappings. Then (S,T) is called $F\rho_s^*$ Khan type contraction, if there exists $F \in \mathcal{F}_S$ and $\tau > 0$ such that for every two consecutive points x,y belonging to the range of an iterative sequence $\{TS(y_n)\}$ with $\rho_s^*(Sy,y) \geq s$, $\rho_s^*(x,Sx) \geq s$, and

$\max\{H_{q_b}(Sx,Ty), H_{q_b}(Ty,Sx), q_b(x,y), q_b(y,x)\} > 0$, we have

$$\tau + \max\left\{F\left(sH_{q_b}(Sx,Ty)\right), F\left(sH_{q_b}(Ty,Sx)\right.\right. \\ \leq \min\left\{F\left(Q_b(x,y)\right), F\left(Q_b(y,x)\right)\right\}, \tag{63}$$

where

$$Q_b(x,y) \\ = \frac{q_b(x,Sx)\,q_b(x,Ty) + q_b(y,Ty)\,q_b(y,Sx)}{\max\left\{q_b(x,Ty), q_b(y,Sx)\right\}}. \tag{64}$$

Theorem 17. *Let (Y,q_b,s) be a complete b-quasi-metric space with $s \geq 1$. Let $\rho_s : Y \times Y \longrightarrow [0,+\infty)$ and (S,T) be a pair of $F\rho_s^*$ Khan type contractions and the set $G(S) = \{x : \rho_s^*(x,Sx) \geq s\}$ is closed and contained y_0. Then $\{TS(y_n)\} \longrightarrow u \in Y$. Also, if (63) holds for each $x, y \in \{u\}$, then S and T have a common fixed point u in Y and $q_b(u,u) = 0$.*

Proof. As y_0 is an arbitrary element of $G(S)$, from condition of the theorem $\rho_s^*(y_0,Sy_0) \geq s$. Let $\{TS(y_n)\}$ be the iterative sequence in Y generated by a point $y_0 \in Y$. Let $y_{2p'}, y_{2p'+1}$ be elements of this sequence. Clearly, if

$$\max\left\{H_{q_b}\left(Sy_{2p'}, Ty_{2p'+1}\right), H_{q_b}\left(Ty_{2p'+1}, Sy_{2p'}\right),\right. \\ \left. q_b\left(y_{2p'}, y_{2p'+1}\right), q_b\left(y_{2p'+1}, y_{2p'}\right)\right\} \ngtr 0, \tag{65}$$

for some $p' \in \mathbb{N} \cup \{0\}$, then

$$H_{q_b}\left(Sy_{2p'}, Ty_{2p'+1}\right) = H_{q_b}\left(Ty_{2p'+1}, Sy_{2p'}\right) \\ = q_b\left(y_{2p'}, y_{2p'+1}\right) \\ = q_b\left(y_{2p'+1}, y_{2p'}\right) = 0. \tag{66}$$

As $q_b(y_{2p'}, y_{2p'+1}) = q_b(y_{2p'+1}, y_{2p'}) = 0$, so $y_{2p'} = y_{2p'+1}$ and $y_{2p'} \in Sy_{2p'}$. Now, $H_{q_b}(Sy_{2p'}, Ty_{2p'+1}) = 0$ implies $q_b(y_{2p'+1}, Ty_{2p'+1}) = 0$ and $H_{q_b}(Ty_{2p'+1}, Sy_{2p'}) = 0$ implies $q_b(Ty_{2p'+1}, y_{2p'+1}) = 0$. So, $y_{2p'+1} \in Ty_{2p'+1}$ and $y_{2p'}$ is a common fixed point of S and T. So the proof is done. In order to find common fixed point of both S and T, when

$$\max\left\{H_{q_b}\left(Sy_{2p}, Ty_{2p+1}\right), H_{q_b}\left(Ty_{2p+1}, Sy_{2p}\right),\right. \\ \left. q_b\left(y_{2p}, y_{2p+1}\right), q_b\left(y_{2p+1}, y_{2p}\right)\right\} > 0, \tag{67}$$

for all $p \in \{0\} \cup \mathbb{N}$. Since $\rho_s^*(y_0,Sy_0) \geq s$, $q_b(y_0,Sy_0) = q_b(y_0,y_1)$ and $q_b(Sy_0,y_0) = q_b(y_1,y_0)$. As (S,T) is ρ_s^* multivalued mapping, $\rho_s^*(Sy_1,y_1) \geq s$. Now, $\rho_s^*(Sy_1,y_1) \geq s$, $q_b(y_1,Ty_1) = q_b(y_1,y_2)$ and $q_b(Ty_1,y_1) = q_b(y_2,y_1)$ implies that $\rho_s^*(y_2,Sy_2) \geq s$. By induction we deduce that $\rho_s^*(y_{2p},Sy_{2p}) \geq s$ and $\rho_s^*(Sy_{2p+1},y_{2p+1}) \geq s$, for all $p = 0,1,2,\cdots$. Now, by Lemma 6, we have

$$q_b\left(y_{2p}, y_{2p+1}\right) \leq H_{q_b}\left(Ty_{2p-1}, Sy_{2p}\right), \\ q_b\left(y_{2p+1}, y_{2p}\right) \leq H_{q_b}\left(Sy_{2p}, Ty_{2p-1}\right) \tag{68}$$

and

$$q_b\left(y_{2p+1}, y_{2p+2}\right) \leq H_{q_b}\left(Sy_{2p}, Ty_{2p+1}\right), \\ q_b\left(y_{2p+2}, y_{2p+1}\right) \leq H_{q_b}\left(Ty_{2p+1}, Sy_{2p}\right). \tag{69}$$

As $s \geq 1$, then (69) implies

$$F\left(sq_b\left(y_{2p+1}, y_{2p+2}\right)\right) \leq F\left(sH_{q_b}\left(Sy_{2p}, Ty_{2p+1}\right)\right) \\ \leq \max\left\{F\left(sH_{q_b}\left(Sy_{2p}, Ty_{2p+1}\right)\right),\right. \\ \left. F\left(sH_{q_b}\left(Ty_{2p+1}, Sy_{2p}\right)\right)\right\}. \tag{70}$$

As $y_{2p}, y_{2p+1} \in \{TS(y_n)\}$, $\rho_s^*(y_{2p},Sy_{2p}) \geq s$ and $\rho_s^*(Sy_{2p+1},y_{2p+1}) \geq s$, then by using the condition (63), we get

$$F\left(sq_b\left(y_{2p+1}, y_{2p+2}\right)\right) \\ \leq \min\left\{F\left(Q_b\left(y_{2p}, y_{2p+1}\right)\right), F\left(Q_b\left(y_{2p+1}, y_{2p}\right)\right)\right\} \\ - \tau \leq F\left(Q_b\left(y_{2p}, y_{2p+1}\right)\right) - \tau. \tag{71}$$

From (64), we get

$$Q_b\left(y_{2p}, y_{2p+1}\right) = \frac{q_b\left(y_{2p}, Sy_{2p}\right) q_b\left(y_{2p}, Ty_{2p+1}\right) + q_b\left(y_{2p+1}, Ty_{2p+1}\right) q_b\left(y_{2p+1}, Sy_{2p}\right)}{\max\left\{q_b\left(y_{2p}, Ty_{2p+1}\right), q_b\left(Sy_{2p}, y_{2p+1}\right)\right\}} \\ = \frac{q_b\left(y_{2p}, y_{2p+1}\right) . q_b\left(y_{2p}, Ty_{2p+1}\right) + q_b\left(y_{2p+1}, y_{2p+2}\right) \times 0}{\max\left\{q_b\left(y_{2p}, Ty_{2p+1}\right), 0\right\}} = q_b\left(y_{2p}, y_{2p+1}\right). \tag{72}$$

Therefore,

$$F\left(sq_b\left(y_{2p+1}, y_{2p+2}\right) \leq F\left(q_b\left(y_{2p}, y_{2p+1}\right)\right) - \tau \tag{73}$$

and this implies

$$F\left(sq_b\left(y_{2p+1}, y_{2p+2}\right)\right) \\ \leq F\left(\max\left\{q_b\left(y_{2p}, y_{2p+1}\right), q_b\left(y_{2p+1}, y_{2p}\right)\right\}\right) - \tau. \tag{74}$$

As $s \geq 1$, then (69) implies

$$\begin{aligned} F\left(sq_b\left(y_{2p+2}, y_{2p+1}\right)\right) &\leq F\left(sH_{q_b}\left(Ty_{2p+1}, Sy_{2p}\right)\right) \\ &\leq \max\left\{F\left(sH_{q_b}\left(Ty_{2p+1}, Sy_{2p}\right)\right),\right. \\ &\left.F\left(sH_{q_b}\left(Sy_{2p}, Ty_{2p+1}\right)\right)\right\} \end{aligned} \tag{75}$$

As $y_{2p+1}, y_{2p} \in \{TS(y_n)\}$, $\rho_s^*(Sy_{2p+1,}, y_{2p+1}) \geq s$ and $\rho_s^*(y_{2p}, Sy_{2p}) \geq s$, then using condition (63), we get

$$\begin{aligned} &F\left(sq_b\left(y_{2p+2}, y_{2p+1}\right)\right) \\ &\quad\leq \min\left\{F\left(Q_b\left(y_{2p}, y_{2p+1}\right)\right), F\left(Q_b\left(y_{2p+1}, y_{2p}\right)\right)\right\} \\ &\quad\quad- \tau \leq F\left(Q_b\left(y_{2p}, y_{2p+1}\right)\right) - \tau \\ &\quad= F\left(q_b\left(y_{2p}, y_{2p+1}\right)\right) - \tau. \end{aligned} \tag{76}$$

Therefore,

$$\begin{aligned} &F\left(sq_b\left(y_{2p+2}, y_{2p+1}\right)\right) \\ &\quad\leq F\left(\max\left\{q_b\left(y_{2p}, y_{2p+1}\right), q_b\left(y_{2p+1}, y_{2p}\right)\right\}\right) - \tau. \end{aligned} \tag{77}$$

Combining (74) and (77), we get

$$\begin{aligned} &\max\left\{F\left(sq_b\left(y_{2p+1}, y_{2p+2}\right)\right), F\left(sq_b\left(y_{2p+2}, y_{2p+1}\right)\right)\right\} \\ &\quad\leq F\left(\max\left\{q_b\left(y_{2p}, y_{2p+1}\right), q_b\left(y_{2p+1}, y_{2p}\right)\right\}\right) - \tau. \end{aligned} \tag{78}$$

As $s \geq 1$, then (68) implies

$$\begin{aligned} F\left(sq_b\left(y_{2p}, y_{2p+1}\right)\right) &\leq \left(sH_{q_b}\left(Ty_{2p-1}, Sy_{2p}\right)\right) \\ &\leq \max\left\{F\left(sH_{q_b}\left(Sy_{2p}, Ty_{2p-1}\right)\right),\right. \\ &\left.F\left(sH_{q_b}\left(Ty_{2p-1}, Sy_{2p}\right)\right)\right\} \end{aligned} \tag{79}$$

As $y_{2p}, y_{2p-1} \in \{TS(y_n)\}$, $\rho_s^*(y_{2p,}, Sy_{2p}) \geq s$ and $\rho_s^*(Sy_{2p-1}, y_{2p-1}) \geq s$, then by using condition (63), we get

$$\begin{aligned} &F\left(sq_b\left(y_{2p}, y_{2p+1}\right)\right) \leq \min\left\{F\left(Q_b\left(y_{2p}, y_{2p-1}\right)\right), F\left(Q_b\left(y_{2p-1}, y_{2p}\right)\right)\right\} - \tau \leq F\left(Q_b\left(y_{2p}, y_{2p-1}\right)\right) - \tau. \\ &F\left(sq_b\left(y_{2p}, y_{2p+1}\right)\right) \leq F\left(\frac{q_b\left(y_{2p}, Sy_{2p}\right).q_b\left(y_{2p}, Ty_{2p-1}\right) + q_b\left(y_{2p-1}, Ty_{2p-1}\right).q_b\left(y_{2p-1}, Sy_{2p}\right)}{\max\left\{q_b\left(y_{2p}, Ty_{2p-1}\right), q_b\left(y_{2p-1}, Sy_{2p}\right)\right\}}\right) - \tau \\ &\qquad\leq F\left(\frac{q_b\left(y_{2p}, y_{2p+1}\right).q_b\left(y_{2p}, y_{2p}\right) + q_b\left(y_{2p-1}, y_{2p}\right).q_b\left(y_{2p-1}, Sy_{2p}\right)}{\max\left\{0, q_b\left(y_{2p-1}, Sy_{2p}\right)\right\}}\right) - \tau \\ &\qquad\leq F\left(q_b\left(y_{2p-1}, y_{2p}\right)\right) - \tau. \end{aligned} \tag{80}$$

Therefore,

$$\begin{aligned} &F\left(sq_b\left(y_{2p}, y_{2p+1}\right)\right) \\ &\quad\leq F\left(\max\left\{q_b\left(y_{2p-1}, y_{2p}\right), q_b\left(y_{2p}, y_{2p-1}\right)\right\}\right) - \tau. \end{aligned} \tag{81}$$

Similarly, by using (63), (64), and (68), we get

$$\begin{aligned} &F\left(sq_b\left(y_{2p+1}, y_{2p}\right)\right) \\ &\quad\leq F\left(\max\left\{q_b\left(y_{2p-1}, y_{2p}\right), q_b\left(y_{2p}, y_{2p-1}\right)\right\}\right) - \tau. \end{aligned} \tag{82}$$

Combining (81) and (82), we get

$$\begin{aligned} &\tau + F\left(s\max\left\{q_b\left(y_{2p}, y_{2p+1}\right)\right) F\left(q_b\left(y_{2p+1}, y_{2p}\right)\right)\right. \\ &\quad\leq F\left(\max\left\{q_b\left(y_{2p-1}, y_{2p}\right), q_b\left(y_{2p}, y_{2p-1}\right)\right\}\right). \end{aligned} \tag{83}$$

Combining (78) and (83), we get

$$\begin{aligned} &\tau + F\left(s\max\left\{q_b\left(y_n, y_{n+1}\right), q_b\left(y_{n+1}, y_n\right)\right\}\right) \\ &\quad\leq F\left(\max\left\{q_b\left(y_{n-1}, y_n\right), q_b\left(y_n, y_{n-1}\right)\right\}\right). \end{aligned} \tag{84}$$

By Lemma 8, $\{TS(y_n)\}$ is a Cauchy sequence in $(Y, q_b).\rho_s^*(y_{2p,}, Sy_{2p}) \geq s$ for all $p \in \mathbb{N}$. So $\{y_{2p}\}$ is a subsequence of $\{TS(y_n)\}$ contained in $G(S)$. As $G(S)$ is closed, there exists $u \in G(S)$ such that $\{y_{2p}\} \longrightarrow u$, that is,

$$\lim_{n\longrightarrow\infty} q_b\left(y_n, u\right) = \lim_{n\longrightarrow\infty} q_b\left(u, y_n\right) = 0. \tag{85}$$

Also

$$\rho_s^*\left(u, Su\right) \geq s. \tag{86}$$

Now, we show that u is a fixed point for S. We claim that $q_b(Su, u) = q_b(u, Su) = 0$. On the contrary, we assume that $q_b(u, Su) > 0$. Now

$$q_b\left(u, Su\right) \leq s\left(q_b\left(u, y_{2n}\right) + q_b\left(y_{2n}, Su\right)\right). \tag{87}$$

So, there exists $n_0 \in \mathbb{N}$ such that $q_b(y_{2n}, Su) > 0$ for all $n \geq n_0$. By Lemma 6, we have $0 < q_b(y_{2n}, Su) \leq H_{q_b}(Ty_{2n-1}, Su)$ for all $n \geq n_0$, so

$$\begin{aligned} &\max\left\{H_{q_b}\left(Ty_{2n-1}, Su\right), H_{q_b}\left(Su, Ty_{2n-1}\right), q_b\left(u, y_{2n-1}\right),\right. \\ &\quad\left.q_b\left(y_{2n-1}, u\right)\right\} > 0, \end{aligned} \tag{88}$$

for all $n \geq n_0$. By Lemma 6, and $s \geq 1$, we get

$$\begin{aligned}\tau + F\left(sq_b\left(y_{2n}, Su\right)\right) \\ \leq \tau \\ + F\left(s\max\left\{H_{q_b}\left(Ty_{2n-1}, Su\right), H_{q_b}\left(Su, Ty_{2n-1}\right)\right\}\right).\end{aligned} \tag{89}$$

Now, $\rho_s^*(u, Su) \geq s$ and $\rho_s^*(Sy_{2n-1}, y_{2n-1}) \geq s$, and then by (64), we get

$$\tau + F\left(sq_b\left(y_{2n}, Su\right)\right) \leq F\left(Q_b\left(y_{2n-1}, u\right)\right). \tag{90}$$

Since F is strictly increasing, we have

$$sq_b\left(y_{2n}, Su\right) < Q_b\left(y_{2n-1}, u\right). \tag{91}$$

Taking limit as $n \longrightarrow \infty$, on both sides of inequality (91), we get

$$\lim_{n\longrightarrow\infty} sq_b\left(y_{2n}, Su\right) < \lim_{n\longrightarrow\infty} Q_b\left(y_{2n-1}, u\right) \tag{92}$$

Since $q_b(u, Ty_{2n-1}) \leq q_b(u, y_{2n})$, taking limit as $n \longrightarrow \infty$, on both sides, we get

$$\lim_{n\longrightarrow\infty} q_b\left(u, Ty_{2n-1}\right) = 0 \tag{93}$$

By (64), we have

$$\begin{aligned}Q_b\left(y_{2n-1}, u\right) \\ = \frac{q_b\left(y_{2n-1}, Ty_{2n-1}\right) q_b\left(y_{2n-1}, Su\right) + q_b\left(u, Su\right) q_b\left(u, Ty_{2n-1}\right)}{\max\left\{q_b\left(y_{2n-1}, Su\right), q_b\left(u, Ty_{2n-1}\right)\right\}}\end{aligned} \tag{94}$$

Taking limit as $n \longrightarrow \infty$ and using inequality (93), we have

$$\lim_{n\longrightarrow\infty} Q_b\left(y_{2n-1}, u\right) = \lim_{n\longrightarrow\infty} q_b\left(y_{2n-1}, y_{2n}\right) = 0. \tag{95}$$

Now, inequality (92) implies

$$\lim_{n\longrightarrow\infty} sq_b\left(y_{2n}, Su\right) < 0. \tag{96}$$

Taking limit as $n \longrightarrow \infty$ on both sides of inequality (87) and using the above inequality, we have

$$q_b\left(u, Su\right) < 0. \tag{97}$$

So our assumption is wrong and $q_b(u, Su) = 0$. Now assume that $q_b(Su, u) > 0$, and then there exists $n_1 \in \mathbb{N}$ such that $q_b(Su, y_{2n}) > 0$ for all $n \geq n_1$. By Lemma 6 $q_b(Su, y_{2n}) \leq H_{q_b}(Su, Ty_{2n-1})$, so

$$\begin{aligned}\max\left\{H_{q_b}\left(Ty_{2n-1}, Su\right), H_{q_b}\left(Su, Ty_{2n-1}\right), q_b\left(u, y_{2n-1}\right),\right. \\ \left. q_b\left(y_{2n-1}, u\right)\right\} > 0,\end{aligned} \tag{98}$$

for all $n \geq n_1$. Following similar arguments as above, we get

$$\lim_{n\longrightarrow\infty} sq_b\left(Su, y_{2n}\right) < 0. \tag{99}$$

Now,

$$q_b\left(Su, u\right) \leq sq_b\left(Su, y_{2n}\right) + sq_b\left(y_{2n}, u\right). \tag{100}$$

Taking limit as $n \longrightarrow \infty$, on both sides of inequality (100) and using (85) and (99), we get

$$q_b\left(Su, u\right) < 0 \tag{101}$$

which is a contradiction, so $q_b(Su, u) = 0$. Hence $u \in Su$. As $\rho_s^*(u, Su) \geq s$ and $q_b(u, Su) = q_b(Su, u) = q_b(0, 0)$, then Definition 14 implies

$$\rho_s^*\left(Su, u\right) \geq s. \tag{102}$$

Now, we show that u is a fixed point for T. We claim that $q_b(u, Tu) = 0$. On the contrary, we assume that $q_b(u, Tu) > 0$, and then there exists $n_2 \in \mathbb{N}$ such that $q_b(y_{2n+1}, Tu) > 0$ for all $n \geq n_2$. By Lemma 6, $0 < q_b(y_{2n+1}, Tu) \leq H_{q_b}(Sy_{2n}, Tu)$, so

$$\begin{aligned}\max\left\{H_{q_b}\left(Sy_{2n}, Tu\right), H_{q_b}\left(Tu, Sy_{2n}\right), q_b\left(y_{2n}, u\right),\right. \\ \left. q_b\left(u, y_{2n}\right)\right\} > 0,\end{aligned} \tag{103}$$

for all $n \geq n_2$. By Lemma 6, and $s \geq 1$, we get

$$\begin{aligned}\tau + F\left(sq_b\left(y_{2n+1}, Tu\right)\right) \leq \tau \\ + \max\left\{F\left(sH_{q_b}\left(Sy_{2n}, Tu\right)\right), F\left(sH_{q_b}\left(Tu, Sy_{2n}\right)\right)\right\}.\end{aligned} \tag{104}$$

Now, $\rho_s^*(y_{2n}, Sy_{2n}) \geq s$ and $\rho_s^*(Su, u) \geq s$, and then by (64), we get

$$\tau + F\left(sq_b\left(y_{2n+1}, Tu\right)\right) \leq F\left(Q_b\left(y_{2n}, u\right)\right). \tag{105}$$

Since F is strictly increasing, we have

$$sq_b\left(y_{2n+1}, Tu\right) < Q_b\left(y_{2n}, u\right). \tag{106}$$

Taking limit $n \longrightarrow \infty$, on both sides of inequality (106), we get

$$\lim_{n\longrightarrow\infty} sq_b\left(y_{2n+1}, Tu\right) < \lim_{n\longrightarrow\infty} Q_b\left(y_{2n}, u\right). \tag{107}$$

Since $q_b(u, Sy_{2n}) \leq q_b(u, y_{2n+1})$, taking limit $n \longrightarrow \infty$, on both sides, we get

$$\lim_{n\longrightarrow\infty} q_b\left(u, Sy_{2n}\right) = 0 \tag{108}$$

By using (64), we get

$$\begin{aligned}Q_b\left(y_{2n}, u\right) \\ = \frac{q_b\left(y_{2n}, y_{2n+1}\right) q_b\left(y_{2n}, Tu\right) + q_b\left(u, Tu\right) q_b\left(u, Sy_{2n}\right)}{\max\left\{q_b\left(y_{2n}, Tu\right), q_b\left(u, Sy_{2n}\right)\right\}}.\end{aligned} \tag{109}$$

Taking limit as $n \longrightarrow \infty$ and using inequality (108), we have

$$\lim_{n\longrightarrow\infty} Q_b\left(y_{2n}, u\right) = \lim_{n\longrightarrow\infty} q_b\left(y_{2n}, y_{2n+1}\right) = 0. \tag{110}$$

Now, inequality (107) implies

$$\lim_{n\longrightarrow\infty} sq_b\left(y_{2n+1}, Tu\right) < 0 \tag{111}$$

Now

$$q_b(u,Tu) \le sq_b(u,y_{2n+1}) + sq_b(y_{2n+1},Tu). \quad (112)$$

Taking limit as $n \longrightarrow \infty$,

$$\begin{aligned} q_b(u,Tu) \le{}& s\lim_{n\longrightarrow\infty} q_b(u,_{2n+1}) \\ &+ \lim_{n\longrightarrow\infty} sq_b(y_{2n+1},Tu). \end{aligned} \quad (113)$$

Using inequalities (85) and (111) in (113), we get

$$q_b(u,Tu) < 0. \quad (114)$$

This is a contradiction, so $q_b(u,Tu) = 0$. Now assume that $q_b(Tu,u) > 0$, and then there exists $n_3 \in \mathbb{N}$ such that $q_b(Tu,y_{2n+1}) > 0$ for all $n \ge n_3$. By Lemma 6 $q_b(Tu,y_{2n+1}) \le H_{q_b}(Tu,Sy_{2n})$, so

$$\begin{aligned} &\max\left\{H_{q_b}(Sy_{2n},Tu), H_{q_b}(Tu,Sy_{2n}), q(y_{2n},u),\right. \\ &\left. q_b(u,y_{2n})\right\} > 0. \end{aligned} \quad (115)$$

for all $n \ge n_3$. Following similar arguments as above, we get

$$q_b(Tu,u) < 0. \quad (116)$$

So $q_b(Tu,u) = 0$. Hence $u \in Tu$. As $\rho_s^*(Su,u) \ge s$ and $q_b(u,Tu) = q_b(Tu,u) = q_b(0,0)$, then Definition 14 implies

$$\rho_s^*(u,Su) \ge s. \quad (117)$$

Hence, the pair (S,T) has a common fixed point u in (Y,q_b). Hence the proof is completed. □

Corollary 18 (see [30]). *Let (X,d) be a complete metric space and $T: X \longrightarrow X$ be an F-Khan contraction. Then, T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}$ converges to x^*.*

4. Single Valued Result with Application to System of Integral Equations

Let $S,T: Y \longrightarrow Y$ be two self-mappings and $x_0 \in Y$. Let $x_1 = Sx_0$, $x_2 = Tx_1$, $x_3 = Sx_2$ and so on. In this way, we construct a sequence x_n in X such that

$$\begin{aligned} &x_{2p+1} = Sx_{2p} \\ &\text{and } x_{2p+2} = Tx_{2p+1}, \\ &\qquad (\text{where } p = 0,1,2,\ldots). \end{aligned} \quad (118)$$

We say that $\{TS(x_n)\}$ is a sequence in Y generated by x_0.

The following result is obtained by replacing the multi-valued mappings with the single valued mappings in Theorem 10. Our result generalizes Theorem 24 in [41]. Also, we prove uniqueness of common fixed point in our result.

Theorem 19. *Let (Y,d_{q_b}) be a complete dislocated b-quasi-metric space with constant $s \ge 1$ and $S,T: Y \longrightarrow Y$ be two self-mappings. If there exists $F \in \mathcal{F}_S$ and $\tau, a > 0$ such that for every two consecutive points x,y belonging to the range of an iterative sequence $\{TS(y_n)\}$ with $\max\{d_{qb}(Sx,Ty), d_{qb}(Ty,Sx), D_{qb}(x,y), D_{qb}(y,x)\} > 0$, we have*

$$\begin{aligned} &\tau + \max\left\{F\left(sd_{qb}(Sx,Ty)\right), F\left(sd_{qb}(Ty,Sx)\right)\right\} \\ &\quad \le \min\left\{F\left(D_{q_b}(x,y)\right), F\left(D_{q_b}(y,x)\right)\right\}, \end{aligned} \quad (119)$$

where

$$\begin{aligned} D_{q_b}(x,y) = \max\Bigg\{&d_{qb}(x,y), \\ &\frac{d_{qb}(x,Sx).d_{qb}(y,Ty)}{a + \max\left\{d_{qb}(x,y), d_{qb}(y,x)\right\}}, d_{qb}(x,Sx), \\ &d_{qb}(y,Ty)\Bigg\}, \end{aligned} \quad (120)$$

then $\{TS(y_n) \longrightarrow u \in X$. Also, if u satisfies (119), then S and T have a unique common fixed point u in X and $d_{q_b}(u,u) = 0$.

Proof. Now, we have to prove uniqueness only. Let x^* be another common fixed point of S,T. Suppose $d_{qb}(Su,Tx^*) > 0$. Then, we have

$$\begin{aligned} \tau + F\left(sd_{qb}(Su,Tx^*)\right) \le F\Bigg(\max\Bigg\{&d_{qb}(u,x^*), \\ &\frac{d_{qb}(u,Su).d_{qb}(x^*,Tx^*)}{1 + \max\left\{d_{qb}(u,x^*), d_{qb}(x^*,u)\right\}}, d_{qb}(u,Su), \\ &d_{qb}(x^*,Tx^*)\Bigg\}\Bigg), \end{aligned} \quad (121)$$

which implies that

$$sd_{qb}(u,x^*) < d_{qb}(u,x^*) \quad (122)$$

which is contradiction. Then $d_{qb}(Su,Tx^*) = 0$. Also

$$\begin{aligned} \tau + F\left(sd_{qb}(Sx^*,Tu)\right) \le F\Bigg(\max\Bigg\{&d_{qb}(x^*,u), \\ &\frac{d_{qb}(x^*,Sx^*).d_{qb}(u,Tu)}{1 + \max\left\{d_{qb}(x^*,u), d_{qb}(u,x^*)\right\}}, d_{qb}(x^*,Sx^*), \\ &d_{qb}(u,Tu)\Bigg\}\Bigg), \end{aligned} \quad (123)$$

And then, we get $d_{qb}(Sx^*,Tu) = 0$. So, $x^* = u$. Now, we deduce the following main result. □

Corollary 20. *Let (Y,d_{q_b}) be a complete dislocated b metric space with constant $s \ge 1$ and $S,T: Y \longrightarrow Y$ be* *two self-mappings. If there exists* $F \in \mathcal{F}_S$ *and* $\tau, a > 0$ *such that for every two consecutive points* x, y *belonging to the range of an iterative sequence* $\{TS(y_n)\}$ *with* $\max\{d_{qb}(Sx, Ty), D_{qb}(x, y)\} > 0$*, we have*

$$\tau + F\left(sd_{qb}(Sx, Ty)\right) \leq F\left(D_{q_b}(x, y)\right), \quad (124)$$

where

$$D_{q_b}(x, y) = \max\left\{d_{qb}(x, y), \frac{d_{qb}(x, Sx) . d_{qb}(y, Ty)}{a + d_{qb}(x, y)}, d_{qb}(x, Sx), d_{qb}(y, Ty)\right\}, \quad (125)$$

then $\{TS(y_n) \longrightarrow u \in X$*. Also, if* u *satisfies (124), then* S *and* T *have a unique common fixed point* u *in* X *and* $d_{q_b}(u, u) = 0$.

Let $\mathcal{F}$ be the set of all functions $F : \mathbb{R}_+ \longrightarrow \mathbb{R}$ defined by [21]. Then, we have the following new result.

Corollary 21. *Let* (Y, d_{q_b}) *be a complete dislocated quasi-metric space and* $S, T : Y \longrightarrow Y$ *be two self-mappings. If there exists* $F \in \mathcal{F}$ *and* $\tau, a > 0$ *such that for every two consecutive points* x, y *belonging to the range of an iterative sequence* $\{TS(y_n)\}$ *with* $\max\{d_{qb}(Sx, Ty), d_{qb}(Ty, Sx), D_{qb}(x, y), D_{qb}(y, x)\} > 0$*, we have*

$$\tau + \max\left\{F\left(d_{qb}(Sx, Ty)\right), F\left(d_{qb}(Ty, Sx)\right)\right\} \leq \min\left\{F\left(D_{q_b}(x, y)\right), F\left(D_{q_b}(y, x)\right)\right\}, \quad (126)$$

where

$$D_{q_b}(x, y) = \max\left\{d_{qb}(x, y), \frac{d_{qb}(x, Sx) . d_{qb}(y, Ty)}{a + \max\left\{d_{qb}(x, y), d_{qb}(y, x)\right\}}, d_{qb}(x, Sx), d_{qb}(y, Ty)\right\}, \quad (127)$$

then $\{TS(y_n) \longrightarrow u \in X$*. Also, if* u *satisfies (126), then* S *and* T *have a unique common fixed point* u *in* X *and* $d_{q_b}(u, u) = 0$.

Now, as an application, we discuss the application of Theorem 19 to find solution of the system of Volterra type integral equations. Consider the following integral equations:

$$u(t) = \int_0^t K_1(t, s, u(s))\, ds, \quad (128)$$

$$v(t) = \int_0^t K_2(t, s, v(s))\, ds \quad (129)$$

for all $t \in [0, 1]$. We find the solution of (128) and (129). Let $X = C([0, 1], \mathbb{R}_+)$ be the set of all continuous functions on $[0, 1]$, endowed with the complete dislocated b-quasi-metric. For $u \in C([0, 1], \mathbb{R}_+)$, define supremum norm as $\|u\|_\tau = \sup_{t\in[0,1]}\{u(t)e^{-\tau t}\}$, where $\tau > 0$ is taken arbitrarily. Then define

$$d_\tau(u, v) = \left[\sup_{t\in[0,1]}\left\{(u(t) + 2v(t))\, e^{-\tau t}\right\}\right]^2 = \|u + 2v\|_\tau^2 \quad (130)$$

for all $u, v \in C([0, 1], \mathbb{R}_+)$, and with these settings, $(C([0, 1], \mathbb{R}_+), d_\tau)$ becomes a dislocated b-quasi-metric space.

Now we prove the following theorem to ensure the existence of solution of integral equations.

Theorem 22. *Assume the following conditions are satisfied:*

(i) $K_1, K_2 : [0, 1] \times [0, 1] \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *and* $f, g : [0, 1] \longrightarrow \mathbb{R}_+$ *are continuous.*

(ii) Define

$$Su(t) = \int_0^t K_1(t, s, u(s))\, ds, \qquad Tv(t) = \int_0^t K_2(t, s, v(s))\, ds. \quad (131)$$

Suppose there exist $\tau > 1$*, such that*

$$\max\{K_1(t, s, u) + 2K_2(t, s, v), K_2(t, s, v) + 2K_1(t, s, u)\} \leq \sqrt{\tau e^{2\tau s - \tau} \min\{M(u, v), M(v, u)\}}, \quad (132)$$

for all $t, s \in [0, 1]$ *and* $u, v \in C([0, 1], \mathbb{R})$*, where*

$$M(u, v) = \max\left\{\|u + 2v\|^2, \frac{\|u + 2Su\|^2 \|v + 2Tv\|^2}{a + \max\left\{\|u + 2v\|^2, \|v + 2u\|^2\right\}}, \|u + 2Su\|^2, \|v + 2Tv\|^2\right\}. \quad (133)$$

Then integral equations (128) and (129) have a unique solution.

Proof. By assumption (ii) and (132), we have

$$\max\{Su + 2Tv, Tv + 2Su\} = \max\left\{\int_0^t (K_1(t, s, u) + 2K_2(t, s, v))\, ds, \int_0^t (K_2(t, s, v) + 2K_1(t, s, u))\, ds\right\} \leq \int_0^t \sqrt{\tau e^{2\tau s - \tau} \min\{M(u, v), M(v, u)\}} ds \quad (134)$$

$$(\max\{Su + 2Tv, Tv + 2Su\})^2 \leq \tau e^{-\tau} \min\{M(u, v), M(v, u)\} \int_0^t e^{2\tau s} ds \leq \frac{1}{2} e^{-\tau} \min\{M(u, v), M(v, u)\} \cdot e^{2\tau t}.$$

This implies

$$\left(\max\{Su+2Tv,Tv+2Su\}e^{-\tau t}\right)^2 \leq \frac{1}{2}e^{-\tau}\min\{M(u,v),M(v,u)\}. \tag{135}$$

That is,

$$2\|\max\{Su+2Tv,Tv+2Su\}\|_\tau^2 \leq e^{-\tau}\min\{M(u,v),M(v,u)\}, \tag{136}$$

which further implies

$$\begin{aligned}&\tau+2\ln\|\max\{Su+2Tv,Tv+2Su\}\|_\tau^2\\&\quad\leq \ln\min\{M(u,v),M(v,u)\},\\&\tau+\max\left\{s\ln\|Su+2Tv\|_\tau^2, s\ln\|Tv+2Su\|_\tau^2\right\}\\&\quad\leq \ln\min\{M(u,v),M(v,u)\}.\end{aligned} \tag{137}$$

So, all the conditions of Theorem 19 are satisfied for $(a)=\ln a$, $d_\tau(u,v)=\|u+2v\|_\tau^2$, $s=2$. Hence integral equations given in (128)and (129) have a common unique solution. □

Remark 23. By setting different values of $M(u,v)$ in (132), we can obtain different weak contractive inequalities and results as corollaries of Theorem 22.

5. Conclusion

In this work, we have discussed the notion of dislocated b-quasi-metric space and given an application to find the solutions of the nonlinear integral equations in such spaces. New results in b-quasi-metric, quasi-metric, quasi dislocated metric, dislocated metric, and metric can be obtained as corollaries of our theorems, which are still not present in the literature. The notions of ρ_s^*−Alt multivalued mapping and $F\rho_s^*$ Khan type contraction on a sequence have been introduced. Our observation is that the fixed points of mappings which are contractive only on a sequence can be ensured by the fixed point results. Our results extend the results given in [41, 45].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

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References

[1] A. B. Amar, A. Jeribi, and M. Mnif, “Some fixed point theorems and application to biological model,” *Numerical Functional Analysis and Optimization*, vol. 29, no. 1-2, pp. 1–23, 2008.

[2] A. Jeribi, B. Krichen, and B. Mefteh, “Existence of solutions of a two-dimensional boundary value problem for a system of nonlinear equations arising in growing cell populations,” *Journal of Biological Dynamics*, vol. 7, no. 1, pp. 218–232, 2013.

[3] K. Leibovic, “The principle of contration mapping in nonlinear and adoptive controle systems,” *IEEE Transactions on Automatic Control*, vol. 9, pp. 393–398, 1964.

[4] G. Medrano-cerda, “A fixed point formulation to parameter estimation problems,” in *Proceedings of the 26th IEEE Conference on Decision and Control*, pp. 1914-1915, The University of California, Los Angeles, Calif, USA, December 1987.

[5] S. Czerwik, “Nonlinear set-valued contraction mappings in b-metric spaces,” *Atti del Seminario Matematico e Fisico dell'Università di Modena*, vol. 46, no. 2, pp. 263–276, 1998.

[6] M. U. Ali, T. Kamran, and M. Postolache, “Solution of Volterra integral inclusion in b-metric spaces via new fixed point theorem,” *Nonlinear Analysis: Modelling and Control*, vol. 2017, no. 1, pp. 17–30, 2017.

[7] Q. Mahmood, A. Shoaib, T. Rasham, and M. Arshad, “Fixed point results for the family of multivalued f-contractive mappings on closed ball in complete dislocated b-metric spaces,” *Mathematics*, vol. 7, no. 1, p. 56, 2019.

[8] T. Rasham, A. Shoaib, N. Hussain, B. A. S. Alamri, and M. Arshad, “Multivalued fixed point results in dislocated b- metric spaces with application to the system of nonlinear integral equations,” *Symmetry*, vol. 11, no. 1, p. 40, 2019.

[9] X. Zhao, F. He, and Y. Sun, “Common fixed point results for quasi-contractions of Ciric type in b-metric spaces with Qt-functions,” *Journal of Nonlinear Functional Analysis*, vol. 2017, article 43, pp. 1–18, 2017.

[10] M. Arshad, Z. Kadelburg, S. Radenovic, A. Shoaib, and S. Shukla, “Fixed points of α-dominated mappings on dislocated quasi metric spaces,” *Filomat*, vol. 31, no. 11, pp. 3041–3056, 2017.

[11] I. Beg, M. Arshad, and A. Shoaib, “Fixed point on a closed ball in ordered dislocated quasi metric spaces,” *Fixed Point Theory*, vol. 16, no. 2, pp. 195–206, 2015.

[12] A. Shoaib, “Fixed point results for α^*-ψ -multivalued mappings,” *Bulletin of Mathematical Analysis and Applications*, vol. 8, no. 4, pp. 43–55, 2016.

[13] A. Shoaib, M. Arshad, T. Rasham, and M. Abbas, “Unique fixed points results on closed ball for dislocated quasi G-metric spaces,” *Transactions of A. Razmadze Mathematical Institute*, vol. 171, no. 2, pp. 221–230, 2017.

[14] M. A. Alghamdi, N. Hussain, and P. Salimi, “Fixed point and coupled fixed point theorems on b-metric-like spaces,” *Journal of Inequalities and Applications*, vol. 2013, article 402, 2013.

[15] I. A. Bakhtin, “The contraction mapping principle in almost metric space,” *Functional Analysis and its Applications*, vol. 30, pp. 26–37, 1989.

[16] A. Felhi, S. Sahmim, and H. Aydi, “Ulam-hyers stability and well-posedness of fixed point problems for α-λ-metric spaces,” *Fixed Point Theory and Applications*, vol. 2016, no. 1, 20 pages, 2016.

[17] H.-C. Wu, “Coincidence point and common fixed point theorems in the product spaces of quasi-ordered metric spaces,” *Journal of Nonlinear and Variational Analysis*, vol. 1, no. 2, pp. 175–199, 2017.

[18] C. Klin-eam and C. Suanoom, "Dislocated quasi-b-metric spaces and fixed point theorems for cyclic contractions," *Fixed Point Theory and Applications*, vol. 2015, article 74, 2015.

[19] M. Abbas, V. Rakocevic, and B. Leyew, "Common fixed points of (α-ψ)- generalized rational multivalued contractions in dislocated quasi b-metric spaces and applications," *Filomat*, vol. 31, no. 11, pp. 3263–3284, 2017.

[20] M. U. Rahman and M. Sarwar, "Dislocated quasi b-metric space and fixed point theorems," *Electronic Journal of Mathematical Analysis and Applications*, vol. 4, no. 2, pp. 16–24, 2016.

[21] D. Wardowski, "Fixed points of a new type of contractive mappings in complete metric spaces," *Fixed Point Theory and Applications*, vol. 2012, article 94, 2012.

[22] Ö. Acar and I. Altun, "A fixed point theorem for multivalued mappings with δ-distance," *Abstract and Applied Analysis*, vol. 2014, Article ID 497092, 5 pages, 2014.

[23] Ö. Acar, G. Durmaz, and G. Minak, "Generalized multivalued F-contractions on complete metric spaces," *Bull. Iranian Math. Society*, vol. 40, no. 6, pp. 1469–1478, 2014.

[24] J. Ahmad, A. Al-Rawashdeh, and A. Azam, "New fixed point theorems for generalized F-contractions in complete metric spaces," *Fixed Point Theory and Applications*, vol. 2015, no. 1, article no. 80, 2015.

[25] M. Arshad, S. U. Khan, and J. Ahmad, "Fixed point results for F-contractions involving some new rational expressions," *Journal of Fixed Point Theory and Applications*, vol. 11, no. 1, pp. 79–97, 2016.

[26] N. Hussain, J. Ahmad, and A. Azam, "On Suzuki-Wardowski type fixed point theorems," *Journal of Nonlinear Sciences and Applications*, vol. 8, no. 6, pp. 1095–1111, 2015.

[27] A. Hussain, M. Arshad, and S. U. Khan, "τ-generalization of fixed point results for F-contraction," *Bangmod International Journal of Mathematical & Computational Science*, vol. 1, no. 1, pp. 127–137, 2015.

[28] S. U. Khan, M. Arshad, A. Hussain, and M. Nazam, "Two new types of fixed point theorems for F-contraction," *Journal of Advanced Studies in Topology*, vol. 7, no. 4, pp. 251–260, 2016.

[29] E. Karap nar, H. Piri, and H. H. Alsulami, "Fixed points of modified F-contractive mappings in complete metric-like spaces," *Journal of Function Spaces*, vol. 2015, Article ID 270971, 9 pages, 2015.

[30] H. Piri, S. Rahrovi, H. Marasi, and P. Kumam, "Fixed point theorem for F-Khan-contractions on complete metric spaces and application to the integral equations," *Journal of Nonlinear Sciences and Applications*, vol. 10, no. 9, pp. 4564–4573, 2017.

[31] J. Nadler, "Multi-valued contraction mappings," *Pacific Journal of Mathematics*, vol. 30, pp. 475–488, 1969.

[32] H. Aydi, M.-F. Bota, E. Karapinar, and S. Mitrović, "A fixed point theorem for set-valued quasi-contractions in b-metric spaces," *Fixed Point Theory and Applications*, vol. 2012, article 88, 2012.

[33] I. Beg, A. R. Butt, and S. Radenović, "The contraction principle for set valued mappings on a metric space with a graph," *Computers & Mathematics with Applications*, vol. 60, no. 5, pp. 1214–1219, 2010.

[34] A. Shoaib, A. Hussain, M. Arshad, and A. Azam, "Fixed point results for α^*-ψ-Ciric type multivalued mappings on an intersection of a closed ball and a sequence with graph," *Journal of Mathematical Analysis*, vol. 7, no. 3, pp. 41–50, 2016.

[35] A. Shoaib, A. Azam, M. Arshad, and A. Shahzad, "Fixed point results for the multivalued mapping on closed ball in dislocated fuzzy metric space," *Journal of Mathematical Analysis*, vol. 8, no. 2, pp. 98–106, 2017.

[36] A. Shoaib, A. Azam, and A. Shahzad, "Common fixed point results for the family of multivalued mappings satisfying contractions on a sequence in Hausdorff fuzzy metric space," *Journal of Computational Analysis and Applications*, vol. 24, no. 4, pp. 692–699, 2018.

[37] J. Aubin, *Mathematical Methods of Game and Economic Theory*, North-Holland Publishing, Amsterdam, The Netherlands, 1979.

[38] S. Bohnenblust and S. Karlin, *Contributions to the Theory of Games*, Princeton University Press, Princeton, NJ, USA, 1950.

[39] S. Carl and S. Heikkila, *Fixed Point Theory in Ordered Sets and Applications: From Differential and Integral Equations to Game Theory*, Springer, New York, NY, USA, 2010.

[40] J. Li, "Several extensions of the Abian-Brown fixed point theorem and their applications to extended and generalized Nash equilibria on chain-complete posets," *Journal of Mathematical Analysis and Applications*, vol. 409, no. 2, pp. 1084–1092, 2014.

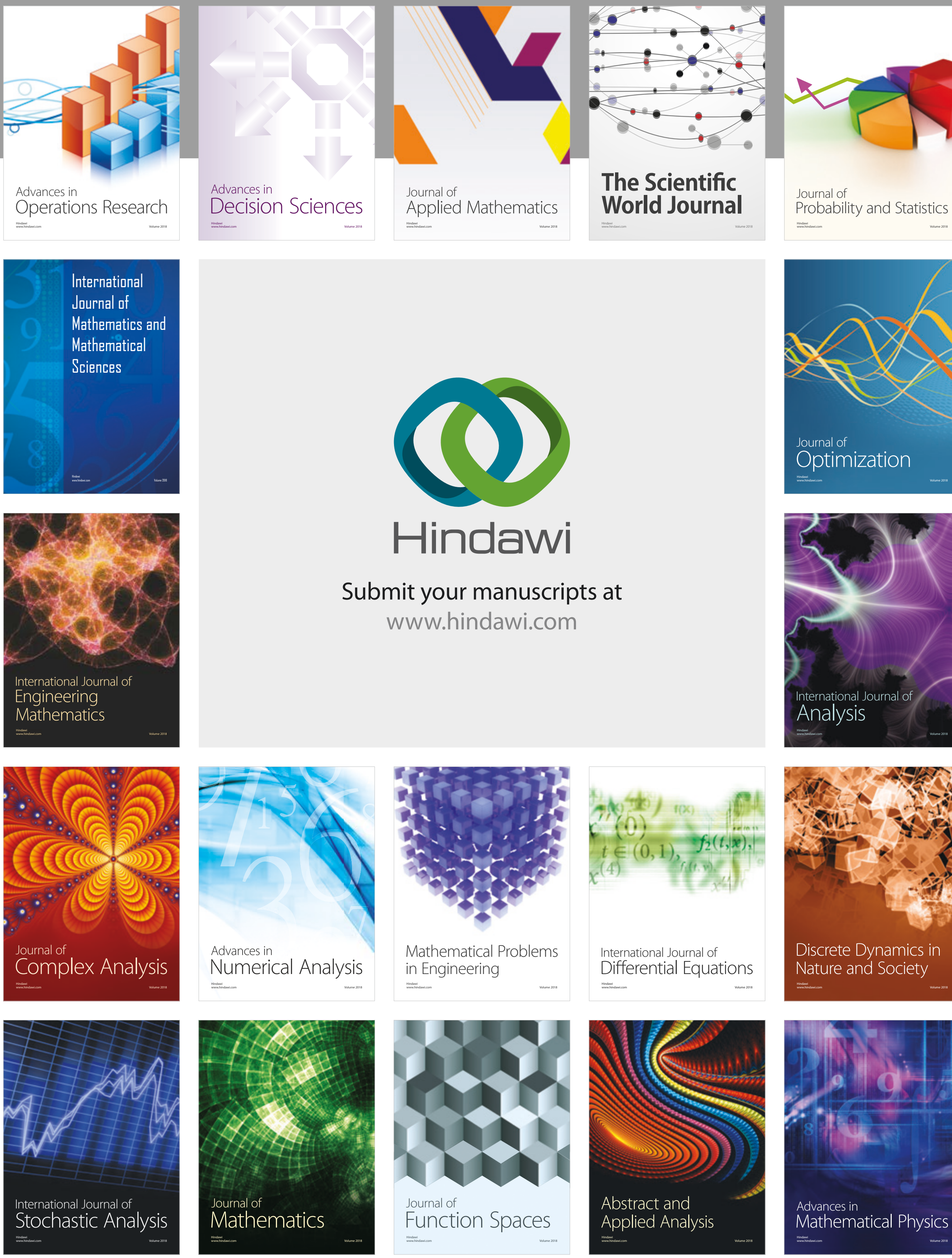
[41] T. Rasham, A. Shoaib, N. Hussain, M. Arshad, and S. U. Khan, "Common fixed point results for new Ciric-type rational multivalued F-contraction with an application," *Journal of Fixed Point Theory and Applications*, vol. 20, no. 1, article 45, 16 pages, 2018.

[42] H. Piri, S. Rahrovi, and P. Kumam, "Generalization of Khan fixed point theorem," *Journal of Mathematics and Computer Science*, vol. 17, no. 1, pp. 76–83, 2017.

[43] M. S. Khan, "A fixed point theorem for metric spaces," *Rendiconti dell'Istituto di Matematica dell'Università di Trieste*, vol. 8, no. 1, pp. 69–72, 1976.

[44] B. Fisher, "On a theorem of Khan," *Rivista di Matematica della Università di Parma*, vol. 4, pp. 135–137, 2016.

[45] I. Altun, N. Al Arifi, M. Jleli, A. Lashin, and B. Samet, "A new approach for the approximations of solutions to a common fixed point problem in metric fixed point theory," *Journal of Function Spaces*, vol. 2016, Article ID 6759320, 5 pages, 2016.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