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Research Article

C^*-Basic Construction from the Conditional Expectation on the Drinfeld Double

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Let $D(G)$ be the Drinfeld double of a finite group G and $D(G;H)$ be the crossed product of $C(G)$ and $\mathbb{C}H$, where H is a subgroup of G. Then the sets $D(G)$ and $D(G;H)$ can be made C^*-algebras naturally. Considering the C^*-basic construction $C^*\langle D(G),e\rangle$ from the conditional expectation E of $D(G)$ onto $D(G;H)$, one can construct a crossed product C^*-algebra $C(G/H\times G)\rtimes\mathbb{C}G$, such that the C^*-basic construction $C^*\langle D(G),e\rangle$ is C^*-algebra isomorphic to $C(G/H\times G)\rtimes\mathbb{C}G$.

1. Introduction

Index theory for subfactors was initiated by Jones ([1]) and has experienced rapid progress beyond the framework of operator algebras. For example, Jones' index theory has found important applications in knots theory, conformal field theory, algebraic quantum field theory, and so on ([2–8]). For a nontechnical but broad overview of the subject including a lot of important connections with other areas, the readers can refer to [9].

Let $N\subseteq M$ be factors of type II_1 with finite Jones index and tr the faithful normal tracial state on M. Denoted by $L^2(M,\mathrm{tr})$ the Hilbert space closure of M is with respect to the norm $\langle x,y\rangle = \mathrm{tr}(y^*x)$. Then M acts on $L^2(\mathrm{M},\mathrm{tr})$ by the left multiplication. The involution $x\longrightarrow x^*$ extends to an isometric conjugate linear operator on $L^2(M,\mathrm{tr})$ denoted by J. The remarkable discovery of Jones is that the possible values for the index are $[M:N]=4\cos^2(\pi/n)$, $n=3,4,\ldots$, or $[M:N]\geq 4$. It is rather easy to construct a reducible inclusion of factors with any index value larger or equal to 4. All the values $4\cos^2(\pi/n)$ in the discrete series are realized by means of the basic construction. To be precise, let E_N be a conditional expectation from M onto N associated with the trace, such that $\mathrm{tr}(E_N(x)y)=\mathrm{tr}(xy)$ for $x\in M$ and $y\in N$. The extension of E_N to $L^2(M,\mathrm{tr})$, denoted by e_N, is the orthogonal projection of $L^2(M,\mathrm{tr})$ onto the closure of N regarded as a subspace of $L^2(M,\mathrm{tr})$. Then $M_1\triangleq\langle M,e_N\rangle$, the von Neumann algebra generated by M and e_N on $L^2(M,\mathrm{tr})$, is called the basic construction, and $\langle M,e_N\rangle = JN'J$, which is a perfect result. Subsequently Jones used the basic construction to obtain an increasing sequence of type II_1 factors, $N\subseteq M\subseteq M_1\subseteq M_2\subseteq M_3\subseteq\cdots$ which is called the Jones tower, iteratively by adding the Jones projections $\{e_n : n\geq 1, e_1 = e_N\}$ which satisfy the Temperley-Lieb relations. Finally, Jones used this structure to construct an example for any possible index value below 4.

The Jones index theory for subfactors of type II_1 has been extended to unital C^*-algebras by Watatani ([10]), and many interesting results of C^*-index theory can be found in [11, 12]. Note that in [10] Watatani showed that if α is an outer action of a finite group G on a simple C^*-algebra A and E is the condition expectation from A onto the fixed point subalgebra A^α given by the average over G, then the basic construction is identified with the crossed product $A\rtimes G$. It was also shown in [12] that if A is a pure Hopf algebra acting outerly on a factor M, then $M\rtimes A$ is a factor and therefore the Jones basic construction $M_1 = M\rtimes A$. However, different from the basic construction for type II_1 factors, the C^*-basic construction $C^*\langle B,\gamma_A\rangle$ does not have the concrete form in general, where Γ is a conditional expectation of a C^*-algebra

B onto a C^*-subalgebra A. The reason is that any factor of type II_1 possesses the faithful trace which is a state of this kind for which the Gelfand-Naimark-Segal construction may be performed, while for general C^*-algebras, the existence of this functional is uncertain.

Letting G be a finite group and H its subgroup, denoted by $H \leq G$, then $D(G;H)$ is defined as the crossed product of $C(G)$, the algebra of complex functions on G, and group algebra $\mathbb{C}H$ with respect to the adjoint action of the latter on the former. In particular, $H = G$; then $D(G;H) \triangleq D(G)$ is the Drinfeld double of G. It was shown in [13] that there is the conditional expectation $E : D(G) \longrightarrow D(G;H)$ of index-finite type. In this paper, we prove that the C^*-basic construction from the conditional expectation E can be described as a crossed product $C(G/H \times G) \rtimes \mathbb{C}G$.

The paper is organized as follows. In Section 2, we give a brief description of the C^*-basic construction for C^*-algebras, and we also collect the necessary definitions and facts about the set $D(G;H)$. In Section 3, we define an action of $\mathbb{C}G$ on $C(G/H \times G)$, and obtain the resulting crossed product $C(G/H \times G) \rtimes \mathbb{C}G$. Theorem 10 is our main result which means that the C^*-basic construction $C^*\langle D(G), e\rangle$ is C^*-algebra isomorphic to the C^*-algebras $C(G/H\times G)\rtimes\mathbb{C}G$.

2. The C^*-Basic Construction of the Drinfeld Double

In this section, suppose that B is a C^*-algebra over $\mathbb{C}$ and A a C^*-subalgebra with a common unit 1. Let us recall these definitions and facts which can be found in the works of Watatani ([10]) and Jiang ([13]).

Definition 1. A linear map $\Gamma : B \longrightarrow A$ is called a conditional expectation if it satisfies the following conditions:

(1) $\Gamma(1) = 1$;

(2) (bimodular property) for $a_1, a_2 \in A$, $b \in B$,

$$\Gamma(a_1 b a_2) = a_1 \Gamma(b) a_2; \tag{1}$$

(3) Γ is positive; i.e., $\Gamma(b^* b)$ is a positive element in A for any $b \in B$.

Actually, if Γ is a conditional expectation from B onto A, then Γ is a projection of norm one ([14]).

Definition 2. Let $\Gamma : B \longrightarrow A$ be a conditional expectation. A finite family $\{(u_1, u_1^*), (u_2, u_2^*), \ldots, (u_n, u_n^*)\} \subseteq B \times B$ is called a quasi-basis for Γ if for all $x \in B$,

$$\sum_{i=1}^{n} u_i \Gamma(u_i^* x) = x = \sum_{i=1}^{n} \Gamma(x u_i) u_i^*. \tag{2}$$

Furthermore, if there exists a quasi-basis for Γ, we call Γ of index-finite type. In this case we define the index of Γ by

$$\text{Index } \Gamma = \sum_{i=1}^{n} u_i u_i^*. \tag{3}$$

If Γ is of index-finite type, then Index Γ is in the center of B and does not depend on the choice of quasi-basis ([10]).

In the following we recall the C^*-basic construction from the conditional expectation $\Gamma : B \longrightarrow A$.

Notice that B is an A-bimodule algebra; one can consider the module tensor product algebra $B \otimes_A B$, the product of which depends on Γ as follows:

$$(b_1 \otimes b_2)(b_3 \otimes b_4) = b_1 \Gamma(b_2 b_3) \otimes b_4 \tag{4}$$

for $b_i \in B$, $i = 1, 2, 3, 4$. Then $B \otimes_A B$ turns out to be an algebra. And define an involution $*$ by

$$(x \otimes y)^* = y^* \otimes x^* \tag{5}$$

for $x, y \in B$. The involution $*$ is well defined by considering conjugate operation in C^*-algebras. Thus $B \otimes_A B$ becomes an $*$-algebra.

Recall that $L(\mathcal{X})$ denotes the algebra of all bounded linear operators on a Hilbert space $\mathcal{X}$.

Lemma 3. *Let $\Gamma : B \longrightarrow A$ be a conditional expectation. Let $\phi : B \otimes_A B \longrightarrow L(\mathcal{X})$ be a $*$-representation of $B \otimes_A B$ on a Hilbert space $\mathcal{X}$. Consider a conditional expectation $\Gamma \otimes id : B \otimes M_n \longrightarrow A \otimes M_n$. Then there exists a $*$-representation ϕ' : $(B \otimes M_n) \otimes_{A \otimes M_n} (B \otimes M_n) \longrightarrow L(\mathcal{X}) \otimes M_n$ such that*

$$\phi'\left(\left(x_{ij}\right)_{ij} \otimes \left(y_{jk}\right)_{jk}\right) = \left(\sum_j \phi\left(x_{ij} \otimes y_{jk}\right)\right)_{ik} \tag{6}$$

for $(x_{ij})_{ij}, (y_{jk})_{jk} \in B \otimes M_n$.

For $c = \sum_i x_i \otimes y_i \in B \otimes_A B$, set

$$\begin{aligned} &\|c\|_{max} \\ &= \sup\left\{\left\|\phi(c)\right\| : \phi \text{ is a } *-\text{representation of } B \otimes_A B\right\}, \end{aligned} \tag{7}$$

then

$$\begin{aligned} \|c\|_{max} = &\left\|\left(\left(\Gamma\left(x_i^* x_j\right)\right)_{ij}\right)^{1/2}\left(\left(\Gamma\left(y_i y_j^*\right)\right)_{ij}\right)^{1/2}\right\| \\ &< +\infty. \end{aligned} \tag{8}$$

Thus $\|\cdot\|_{max}$ is a C^*-seminorm on $B \otimes_A B$ ([10]).

Definition 4. The completion of $B \otimes_A B$ by the norm $\|\cdot\|_{\max}$ (after taking quotient by the ideal $\{c \in B \otimes_A B : \|c\|_{\max} = 0\}$ if necessary) is called the C^*-basic construction from the conditional expectation Γ. We denote this C^*-algebra by $C^*\langle B, \gamma_A\rangle$.

Now assume that G is a finite group with a subgroup H and $t_1 = u, t_2, \ldots, t_k$ is a left coset representation of H in G, namely, $G = \bigcup_{i=1}^{k} t_i H$ and $i \neq j$ induces that $t_i H \cap t_j H = \emptyset$, where u is the unit of G. Let us begin with the following definitions.

Definition 5. $D(G;H)$ is the crossed product of $C(G)$ and group algebra $\mathbb{C}H$, where $C(G)$ denotes the set of complex functions on G, with respect to the adjoint action of the latter on the former.

Using the basis elements (g,h) of $D(G;H)$, the multiplication rule is given as follows:

$$(g_1,h_1)(g_2,h_2)=\delta_{g_1h_1,h_1g_2}(g_1,h_1h_2). \tag{9}$$

The unit of $D(G;H)$ is $\sum_{g\in G}(g,u)\triangleq I$. $D(G;H)$ becomes a unital $*$-algebra by defining the $*$-operation $(g,h)^*=(h^{-1}gh,h^{-1})$ on the basis elements and extending antilinearly to $D(G;H)$.

The coproduct $\triangle$, the counit ε, and the antipode S are defined by

$$\begin{aligned}\triangle(g,h)&=\sum_{t\in H}(t,h)\otimes\left(t^{-1}g,h\right),\\ \varepsilon(g,h)&=\delta_{g,u},\\ S(g,h)&=\left(h^{-1}g^{-1}h,h^{-1}\right),\end{aligned} \tag{10}$$

where $g\in G,h\in H$, and $\delta_{g,h}=\begin{cases}1, & \text{if } h=g\\ 0, & \text{if } h\neq g.\end{cases}$ One can prove $D(G;H)$ becomes a Hopf $*$-algebra ([15]). There exists a unique element $z_H=(1/|H|)\sum_{h\in H}(u,h)$, called a cointegral, satisfying $\forall a\in D(G;H)$, $az_H=z_Ha=\varepsilon(a)z_H$, and $\varepsilon(z_H)=1$. As a result, $D(G;H)$ is semisimple with a natural $*$-structure. Consequently it can be a C^*-algebra of finite dimension.

In particular, if $H=G$, then $D(G;H)\triangleq D(G)$ is the Drinfeld double of G. For more information about $D(G)$ one can refer to [16–18]. The main difference between $D(G)$ and $D(G;H)$ is that the former is a quasitriangular Hopf algebra while the latter is not ([19]).

Considering a linear map

$$\begin{aligned}E:\quad & D(G)\longrightarrow D(G;H)\\ & \sum_{g,h\in G}k_{g,h}(g,h)\longmapsto\sum_{g\in G,h\in H}k_{g,h}(g,h)\end{aligned} \tag{11}$$

where $k_{g,h}\in\mathbb{C}$, one can show that it is the conditional expectation from $D(G)$ onto $D(G;H)$. Moreover, setting $u_i=\sum_{g\in G}(g,t_i)$, then $u_i^*=\sum_{g\in G}(g,t_i^{-1})$ and $\{(u_i,u_i^*)\}$ is a quasibasis for E. Thus Index $E=[G:H]I$. In this case, E is of index-finite type.

Definition 6. The completion of $D(G)\otimes_{D(G;H)}D(G)$ with respect to the norm $\|\cdot\|_{\max}$ (after taking quotient by the ideal $\{c\in D(G)\otimes_{D(G;H)}D(G):\|c\|_{\max}=0\}$ if necessary) is called the C^*-basic construction from the conditional expectation E of $D(G)$ onto $D(G;H)$. We denote this C^*-algebra by $C^*\langle D(G),e\rangle$.

Note that $\sum_{g\in G}\sum_{i=1}^k(g,t_i)\otimes(g,t_i)^*$ is the unit of $C^*\langle D(G),e\rangle$.

3. The Construction of a Crossed Product C^*-Algebra

Let us continue to suppose that G is a finite group and $H\le G$. Denoted by G/H the set of all left cosets of H, i.e., $G/H=\{[t_1],[t_2],\dots,[t_k]\}$. Let $C(G/H\times G)$ and $\mathbb{C}G$ be the algebra of complex functions on $G/H\times G$ and the group algebra over $\mathbb{C}$, respectively.

The set $\{\chi_{[t_i]}:\ i=1,2,\dots,k\}$ is a linear basis of $C(G/H)$ where $\chi_{[t_i]}[t_j]=\begin{cases}1, & \text{if } j=i,\\ 0, & \text{if } j\neq i,\end{cases}$ while the set $\{\delta_g:\ g\in G\}$ is a linear basis of $C(G)$ where $\delta_g(h)=\begin{cases}1, & \text{if } h=g,\\ 0, & \text{if } h\neq g.\end{cases}$ Since G is a finite group and $C(G/H\times G)\cong C(G/H)\otimes C(G)$, then $\{(t_i,g):\ i=1,2,\dots,k;\ g\in G\}$ can be regarded as a linear basis of $C(G/H\times G)$, where we write (t_i,g) instead of $(\chi_{[t_i]},\delta_g)$ for notational convenience.

For any $h\in G$, we can define the map $\sigma_h: C(G/H\times G)\longrightarrow C(G/H\times G)$ given on the basis elements (t_i,g) of $C(G/H\times G)$ as

$$\sigma_h(t_i,g)=\left(ht_i,hgh^{-1}\right), \tag{12}$$

which can be linearly extended in $C(G/H\times G)$. One can verify that σ_h is an automorphic map, and then σ defines an automorphic action of $\mathbb{C}(G)$ on $C(G/H\times G)$.

Assume that $C(G/H\times G)$ acts on a Hilbert space $l^2(G/H\times G)$, and an action σ of $C(G)$ on $C(G/H\times G)$ is given as above. Let $\mathscr{H}=l^2(G/H\times G)\otimes l^2(G)$. We view $\mathscr{H}$ as the Hilbert space of all square summable $l^2(G/H\times G)$-valued functions on G, and define:

$$\begin{aligned}(\pi(t_i,g)\xi)(h)&=\sigma_{h^{-1}}(t_i,g)\xi(h),\\ (\lambda(g)\xi)(h)&=\xi\left(g^{-1}h\right),\end{aligned} \tag{13}$$

for $g,h\in G,\xi\in\mathscr{H}, i=1,2,\dots,k$.

Lemma 7. *π is a faithful $*$-representation of $C(G/H\times G)$ on $\mathscr{H}$, while λ is a unitary representation of G on $\mathscr{H}$. And*

$$\begin{aligned}\lambda(h)\pi(t_i,g)\lambda(h)^*=\pi(\sigma_h(t_i,g)),\\ g,h\in G,\ i=1,2,\dots,k.\end{aligned} \tag{14}$$

Definition 8. The associative C^*-algebra on $\mathscr{H}$ generated by $\{\pi(t_i,g),\lambda(h):\ g,h\in G;\ i=1,2,\dots,k\}$ is called the crossed product of $C(G/H\times G)$ by $\mathbb{C}G$ with respect to σ, and we denote it by $C(G/H\times G)\rtimes_\sigma\mathbb{C}G$ (or simply by $C(G/H\times G)\rtimes\mathbb{C}G$).

Here and from now on, by $h(t_i,g)$, (t_i,g) and h we always denote $\sigma_h(t_i,g)$, $\pi(t_i,g)$ and $\lambda(h)$, respectively.

Lemma 9.

(1) *The element $\sum_{g\in G}(t_1,g)$ in $C(G/H\times G)$ is a projection on $\mathscr{H}$, i.e.,*

$$\sum_{g\in G}(t_1,g)=\left(\sum_{g\in G}(t_1,g)\right)^2=\left(\sum_{g\in G}(t_1,g)\right)^*. \tag{15}$$

(2) *For $g,h\in G$, we have*

$$\begin{aligned}&\left(\sum_{\alpha\in G}(t_1,\alpha)\right)\left(\sum_{i=1}^k(t_i,g)h\right)\left(\sum_{\beta\in G}(t_1,\beta)\right)\\ &\quad=\delta_{[t_1],[h]}\left(\sum_{i=1}^k(t_i,g)h\right)\left(\sum_{\gamma\in G}(t_1,\gamma)\right).\end{aligned} \tag{16}$$

(3) *Let $h \in G$, then $h \in H$ if and only if*

$$\left(\sum_{i=1}^{k}(t_i, g)h\right)\sum_{\gamma\in G}(t_1,\gamma) = \sum_{\gamma\in G}(t_1,\gamma)\left(\sum_{i=1}^{k}(t_i, g)h\right). \quad (17)$$

Proof. (1) For any $\xi \in \mathscr{H}$ and $h \in G$, observe that

$$\begin{aligned}\left(\sum_{g\in G}(t_1, g)\xi\right)(h) &= h^{-1}\left(\sum_{g\in G}(t_1,g)\right)\xi(h) \\ &= \left(\sum_{g\in G}\left(h^{-1}, h^{-1}gh\right)\right)\xi(h),\end{aligned} \quad (18)$$

and

$$\begin{aligned}&\left(\left(\sum_{g\in G}(t_1, g)\right)^2\xi\right)(h) \\ &= h^{-1}\left(\sum_{f\in G}(t_1,f)\right)\left(\left(\sum_{g\in G}(t_1,g)\right)\xi\right)(h) \\ &= h^{-1}\left(\sum_{f\in G}(t_1,f)\right)\left(h^{-1}\left(\sum_{g\in G}(t_1,g)\right)\xi(h)\right) \\ &= h^{-1}\left(\sum_{f\in G}(t_1,f)\right)\left(h^{-1}\left(\sum_{g\in G}(t_1,g)\right)\right)\xi(h) \\ &= \left(\sum_{f\in G}\left(h^{-1},h^{-1}fh\right)\right)\left(\sum_{g\in G}\left(h^{-1},h^{-1}gh\right)\right)\xi(h) \\ &= \left(\sum_{g\in G}\left(h^{-1},h^{-1}gh\right)\right)\xi(h),\end{aligned} \quad (19)$$

From the above equalities, we conclude that $\sum_{g\in G}(t_1,g) = (\sum_{g\in G}(t_1,g))^2$. And

$$\begin{aligned}&\left\langle\left(\sum_{g\in G}(t_1,g)\right)\xi,\eta\right\rangle \\ &= \sum_{x\in G}\sum_{g\in G}\langle((t_1,g)\xi)(x), \\ &\eta(x)\rangle \\ &= \sum_{x\in G}\sum_{g\in G}\sum_{m\in G/H}\sum_{n\in G}((t_1,g)\xi)(x)(m,n)\overline{\eta(x)(m,n)} \\ &= \sum_{x\in G}\sum_{g\in G}\sum_{m\in G/H}\sum_{n\in G}\left(x^{-1}(t_1,g)\right)\xi(x)(m,n)\overline{\eta(x)(m,n)} \\ &= \sum_{x\in G}\sum_{g\in G}\sum_{m\in G/H}\sum_{n\in G}\left(\left(x^{-1},x^{-1}gx\right)(m,n)\right)\xi(x)(m,n)\overline{\eta(x)(m,n)} \\ &= \sum_{x\in G}\sum_{g\in G}\sum_{m\in G/H}\sum_{n\in G}\xi(x)(m,n)\overline{\left(\left(x^{-1},x^{-1}gx\right)(m,n)\right)\eta(x)(m,n)} \\ &= \sum_{x\in G}\sum_{g\in G}\sum_{m\in G/H}\sum_{n\in G}\xi(x)(m,n)\overline{\left(x^{-1}(t_1,g)\right)\eta(x)(m,n)} \\ &= \sum_{x\in G}\sum_{g\in G}\langle\xi(x), \\ &((t_1,g)\eta)(x)\rangle = \left\langle\xi,\left(\sum_{g\in G}(t_1,g)\right)\eta\right\rangle,\end{aligned} \quad (20)$$

which shows that $\sum_{g\in G}(t_1,g) = (\sum_{g\in G}(t_1,g))^*$.

(2) Let $g, h \in G$, we can compute that

$$\begin{aligned}&\left(\sum_{\alpha\in G}(t_1,\alpha)\right)\left(\sum_{i=1}^{k}(t_i,g)h\right)\left(\sum_{\beta\in G}(t_1,\beta)\right) \\ &= \sum_{\alpha\in G}\sum_{\beta\in G}\sum_{i=1}^{k}(t_1,\alpha)((t_i,g)h)(t_1,\beta) \\ &= \sum_{\alpha\in G}\sum_{\beta\in G}\sum_{i=1}^{k}(t_1,\alpha)(t_i,g)(h(t_1,\beta)) \\ &= \sum_{\alpha\in G}\sum_{\beta\in G}\sum_{i=1}^{k}(t_1,\alpha)(t_i,g)\left(h,h\beta h^{-1}\right)h \\ &= \sum_{\beta\in G}(t_1,g)\left(h,h\beta h^{-1}\right)h = \delta_{[t_1],[h]}(t_1,g)h,\end{aligned} \quad (21)$$

and

$$\begin{aligned}&\left(\sum_{i=1}^{k}(t_i,g)h\right)\left(\sum_{\gamma\in G}(t_1,\gamma)\right) \\ &= \sum_{\gamma\in G}\sum_{i=1}^{k}((t_i,g)h)(t_1,\gamma) \\ &= \sum_{\gamma\in G}\sum_{i=1}^{k}(t_i,g)\left(h,h\gamma h^{-1}\right)h \\ &= \sum_{\gamma\in G}\sum_{i=1}^{k}\delta_{[t_i],[h]}\delta_{gh,h\gamma}(t_i,g)h = (t_1,g)h.\end{aligned} \quad (22)$$

(3) If $h \in H$, then $(\sum_{\alpha\in G}(t_1,\alpha))(\sum_{i=1}^{k}(t_i,g)h) = \sum_{\alpha\in G}\sum_{i=1}^{k}(t_1,\alpha)(t_i,g)h = (t_1,g)h$ and

$$\left(\sum_{i=1}^{k}(t_i,g)h\right)\left(\sum_{\alpha\in G}(t_1,\alpha)\right)$$

$$= \sum_{i=1}^{k} \sum_{\alpha \in G} ((t_i, g) h) (t_1, \alpha)$$
$$= \sum_{i=1}^{k} \sum_{\alpha \in G} (t_i, g) \left(h, h\alpha h^{-1}\right) h = (t_1, g) h. \tag{23}$$

Thus $\sum_{i=1}^{k} (t_i, g)h$ commutes with $\sum_{\alpha \in G} (t_1, \alpha)$.

Conversely, if $h \in G$ and $(\sum_{i=1}^{k} (t_i, g)h) \sum_{\gamma \in G} (t_1, \gamma) = \sum_{\gamma \in G} (t_1, \gamma)(\sum_{i=1}^{k} (t_i, g)h)$, then $(h, g)h = (t_1, g)h$ which implies that $(h, g) = (t_1, g)$. This shows that $h \in H$. □

From the proof of Lemma 9, we can obtain $(t_i, g)^* = (t_i, g)$ for any $g \in G$, $i = 1, 2, \ldots, k$.

Now we give the main theorem of this paper.

Theorem 10. *There exists an isometric $*$-isomorphism of $C^* \langle D(G), e\rangle$ onto $C(G/H \times G) \rtimes \mathbb{C}G$.*

Proof. By the definition of $\|\cdot\|_{\max}$, it suffices to show that there exists an $*$-isomorphism Φ of $D(G) \otimes_{D(G;H)} D(G)$ on $\mathscr{H}$.

The map $\Phi : D(G) \otimes_{D(G;H)} D(G) \longrightarrow C(G/H \times G) \rtimes \mathbb{C}G$ is defined on the generators of $D(G) \otimes_{D(G;H)} D(G)$ by

$$\begin{aligned} &\Phi ((g, \alpha) \otimes (h, \beta)) \\ &\quad = \left(\sum_{i=1}^{k} (t_i, g) \alpha \right) \left(\sum_{f \in G} (t_1, f) \right) \left(\sum_{j=1}^{k} (t_j, h) \beta \right) \\ &\quad = \delta_{g\alpha, \alpha h} (\alpha, g) \alpha\beta, \end{aligned} \tag{24}$$

for (g, α), (h, β) in $D(G)$. Since $\sum_{i=1}^{k} (t_i, g)h$ and $\sum_{\gamma \in G} (t_1, \gamma)$ commute, Φ is a well defined map. Then it can be linearly extended in $D(G) \otimes_{D(G;H)} D(G)$. Again since

$$\begin{aligned} &\Phi \left(((g, \alpha) \otimes (h, \beta))^* \right) = \Phi \left((h, \beta)^* \otimes (g, \alpha)^* \right) \\ &\quad = \Phi \left(\left(\beta^{-1} h\beta, \beta^{-1}\right) \otimes \left(\alpha^{-1} g\alpha, \alpha^{-1}\right) \right) \\ &\quad = \delta_{g\alpha, \alpha h} \left(\beta^{-1}, \beta^{-1} h\beta\right) \beta^{-1} \alpha^{-1}, \end{aligned} \tag{25}$$

and

$$\begin{aligned} &(\Phi ((g, \alpha) \otimes (h, \beta)))^* = \delta_{g\alpha, \alpha h} ((\alpha, g) \alpha\beta)^* \\ &\quad = \delta_{g\alpha, \alpha h} \beta^{-1} \alpha^{-1} (\alpha, g)^* \\ &\quad = \delta_{g\alpha, \alpha h} \left(\beta^{-1} \alpha^{-1} \alpha, \beta^{-1} \alpha^{-1} g\alpha\beta\right) \\ &\quad = \delta_{g\alpha, \alpha h} \left(\beta^{-1}, \beta^{-1} h\beta\right) \beta^{-1} \alpha^{-1}. \end{aligned} \tag{26}$$

Then

$$\Phi \left(((g, \alpha) \otimes (h, \beta))^* \right) = (\Phi ((g, \alpha) \otimes (h, \beta)))^* . \tag{27}$$

Hence, combining with Lemma 9, we have that the map Φ is a $*$-homomorphism.

For $(t_i, g)h \in C(G/H \times G) \rtimes \mathbb{C}G$, choose $(g, t_i) \otimes (t_i^{-1} g t_i, t_i^{-1} h) \in D(G) \otimes_{D(G;H)} D(G)$ such that $\Phi((g, t_i) \otimes (t_i^{-1} g t_i, t_i^{-1} h)) = (t_i, g)h$. This shows that Φ is surjective. Finally, we will verify that Φ is isometric.

Let $\psi : D(G; H) \longrightarrow C(G/H \times G) \rtimes \mathbb{C}G \subseteq L(\mathscr{H})$ be a map given as $\psi(g, \alpha) = (t_1, g)\alpha$. It is easy to see that ψ is a injective $*$-homomorphism. Define $\psi' : D(G; H) \otimes M_n \longrightarrow L(\mathscr{H}) \otimes M_n$ by $\psi' = \psi \otimes \mathrm{id}$. Then ψ' is also injective. Again, by Lemma 3, there exists

$$\begin{aligned} \Phi' : &(D (G) \otimes M_n) \otimes_{D(G;H) \otimes M_n} (D (G) \otimes M_n) \\ &\longrightarrow L (\mathscr{H}) \otimes M_n \end{aligned} \tag{28}$$

such that $\Phi'((x_{ij})_{ij} \otimes (y_{jk})_{jk}) = (\sum_j \Phi(x_{ij} \otimes y_{jk}))_{ik}$. Since $\psi'((x_{ij})_{ij}) = \Phi'((x_{ij})_{ij} \otimes 1)$,

$$\begin{aligned} \|\Phi (c)\|^2 &= \left\| \sum_i \Phi (x_i \otimes y_i) \right\|^2 \\ &= \left\| \begin{pmatrix} \sum_i \Phi (x_i \otimes y_i) & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \right\|^2 \\ &= \left\| \Phi' \left(\begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ 0 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \right.\right. \\ &\quad \left.\left. \otimes \begin{pmatrix} y_1 & 0 & 0 & \cdots & 0 \\ y_2 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ y_n & 0 & 0 & \cdots & 0 \end{pmatrix} \right) \right\|^2 = \left\| \Phi' (X \otimes Y) \right\|^2 \\ &= \left\| \Phi' (X \otimes Y)^* \Phi' (X \otimes Y) \right\| = \left\| \Phi' (Y^* \otimes X^*) \right. \\ &\quad \left. \cdot \Phi' (X \otimes Y) \right\| = \left\| \Phi' (Y^* F (X^* X) \otimes Y) \right\| \\ &= \left\| \Phi' \left(Y^* F (X^* X)^{1/2} \otimes 1 \right) \Phi' \left(1 \right.\right. \\ &\quad \left.\left. \otimes F (X^* X)^{1/2} Y \right) \right\| = \left\| \Phi' \left(1 \otimes F (X^* X)^{1/2} Y \right)^* \right. \\ &\quad \left. \cdot \Phi' \left(1 \otimes F (X^* X)^{1/2} Y \right) \right\| = \left\| \Phi' \left(1 \right.\right. \\ &\quad \left.\left. \otimes F (X^* X)^{1/2} Y \right) \Phi' \left(1 \otimes F (X^* X)^{1/2} Y \right)^* \right\| \end{aligned}$$

$$
\begin{aligned}
&= \left\| \Phi' \left(F\left(X^{*}X\right)^{1/2} F\left(YY^{*}\right) F\left(X^{*}X\right)^{1/2} \otimes 1 \right) \right\| \\
&= \left\| F\left(X^{*}X\right)^{1/2} F\left(YY^{*}\right) F\left(X^{*}X\right)^{1/2} \otimes 1 \right\| \\
&= \left\| F\left(X^{*}X\right)^{1/2} F\left(YY^{*}\right)^{1/2} \right\|^{2} \\
&= \left\| \left(\left(E\left(x_i^{*} x_j\right) \right)_{ij} \right)^{1/2} \left(\left(E\left(y_i y_j^{*}\right) \right)_{ij} \right)^{1/2} \right\|,
\end{aligned}
\tag{29}
$$

for $c = \sum_i x_i \otimes y_i \in D(G) \otimes_{D(G;H)} D(G)$, where $F = E \otimes \mathrm{id} : D(G) \otimes M_n \longrightarrow D(G;H) \otimes M_n$ is a conditional expectation and $X = \begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ 0 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$, $Y = \begin{pmatrix} y_1 & 0 & 0 & \cdots & 0 \\ y_2 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ y_n & 0 & 0 & \cdots & 0 \end{pmatrix}$. Thus, we have that $\|\Phi(c)\| = \|c\|$. Now we have shown Φ is an isometric $*$-isomorphism. □

Example 11. (1) Let $C^*(G)$ (or $C^*(H)$) be the group C^*-algebra of G (or H), namely, the C^*-subalgebra of $L(l^2(G))$ generated by the left regular representation of G (or H). Consider the basic construction from the conditional expectation $E : C^*(G) \longrightarrow C^*(H)$ defined by

$$E\left(\sum_{g \in G} k_g \lambda_g \right) = \sum_{h \in H} k_h \lambda_h \tag{30}$$

where $k_g \in \mathbb{C}$. Let $\tau : \mathbb{C}G \longrightarrow \mathrm{Aut}\, C(G/H)$ be the action induced by translation from left, where $\mathrm{Aut}\, C(G/H)$ stands for the group of all automorphism of $C(G/H)$. In [10], Watatani showed that the C^*-basic construction $C^*\langle C^*(G), e\rangle$ is C^*-algebra isomorphic to $C(G/H) \rtimes_\tau \mathbb{C}G$. This result is a special case of Theorem 10. In fact, there is an inclusion $i : C^*(G) \longrightarrow D(G)$ given by $i(\lambda_g) = \sum_{f \in G}(f, g)$.

(2) If G is a finite abelian group, then $D(G)$ reduces to a symmetry group $\widehat{G} \times G$, where $\widehat{G}$ denotes the Pontryagin dual of G. Let $E : \widehat{G} \times G \longrightarrow \widehat{G} \times H$ be a conditional expectation defined by

$$E\left(\sum_{g,h \in G} k_{g,h} (g, h) \right) = \sum_{g \in G, h \in H} k_{g,h} (g, h) . \tag{31}$$

We can define the map $\varsigma : \mathbb{C}G \times C(G/H \times G) \longrightarrow C(G/H \times G)$ by

$$\varsigma \left(h \times (t_i, g) \right) = (ht_i, g) \tag{32}$$

for all $(t_i, g) \in C(G/H \times G)$. Then ς is an automorphic action of $\mathbb{C}G$ on $C(G/H \times G)$. Then the C^*-basic construction $C^*\langle \widehat{G} \times G, e\rangle$ is C^*-algebra isomorphic to $C(G/H \times G) \rtimes_\varsigma \mathbb{C}G$. Furthermore, the C^*-basic construction $C^*\langle \widehat{G} \times G, e\rangle$ can be described as $\widehat{G} \times (C(G/H) \rtimes_\tau \mathbb{C}G)$, where τ is the action induced by translation from left.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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