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Research Article

Convergence Analysis of an Accelerated Iteration for Monotone Generalized α-Nonexpansive Mappings with a Partial Order

Yi-An Chen and Dao-Jun Wen

College of Mathematics and Statistics, Chongqing Technology and Business University, Chongqing 400067, China

Correspondence should be addressed to Dao-Jun Wen; daojunwen@163.com

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In this paper, we introduce a new accelerated iteration for finding a fixed point of monotone generalized α-nonexpansive mapping in an ordered Banach space. We establish some weak and strong convergence theorems of fixed point for monotone generalized α-nonexpansive mapping in a uniformly convex Banach space with a partial order. Further, we provide a numerical example to illustrate the convergence behavior and effectiveness of the proposed iteration process.

1. Introduction

Let $(E, \leq)$ be an ordered Banach space endowed with the partial order $\leq$ and K be a nonempty closed convex subset of E. A mapping $T : K \longrightarrow K$ is called monotone if $Tx \leq Ty$ whenever $x \leq y$ for all $x, y \in K$. Moreover, T is said to be as follows:

(1) Monotone nonexpansive if T is monotone and such that

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x \leq y. \tag{1}$$

(2) Monotone quasinonexpansive if T is monotone with $F(T) \neq \emptyset$ such that

$$\|Tx - p\| \leq \|x - p\|, \quad \forall p \leq x \text{ or } x \leq p, \tag{2}$$

where $p \in F(T)$, the set of fixed points of T, i.e., $F(T) = \{x \in K,\ Tx = x\}$.

(3) Monotone α-nonexpansive if T is monotone and there exists a constant $\alpha < 1$ such that

$$\begin{aligned} \|Tx - Ty\|^2 &\leq \alpha \|Tx - y\|^2 + \alpha \|Ty - x\|^2 \\ &\quad + (1 - 2\alpha) \|x - y\|^2, \quad \forall x \leq y. \end{aligned} \tag{3}$$

(4) Suzuki's generalized nonexpansive if T satisfy condition (C), i.e., $(1/2)\|x - Tx\| \leq \|x - y\|$ implies

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in K, \tag{4}$$

which is an interesting generalization of nonexpansive mapping because it is weaker than nonexpansiveness and stronger than quasinonexpansiveness [1].

(5) Monotone generalized α-nonexpansive if T is monotone and exists a constant $\alpha \in [0, 1)$ such that $(1/2)\|x - Tx\| \leq \|x - y\|$ implies

$$\begin{aligned} \|Tx - Ty\| &\leq \alpha \|Tx - y\| + \alpha \|Ty - x\| \\ &\quad + (1 - 2\alpha) \|x - y\|, \quad \forall x \leq y. \end{aligned} \tag{5}$$

Obviously, a monotone α-nonexpansive mapping includes monotone nonexpansive (0-nonexpansive) mapping as special case. Every monotone mapping satisfying condition (C) is a monotone generalized α-nonexpansive mapping, but the converse is not true. Moreover, a monotone generalized α-nonexpansive mapping includes nonexpansive, firmly nonexpansive, Suzuki's generalized nonexpansive mapping as special cases and partially extends monotone α-nonexpansive mapping [2].

In 1965, Browder [3] proved that every nonexpansive self-mapping of a closed convex and bounded subset has a fixed point in a uniformly convex Banach space. Since then, a number of iteration methods have been developed to approximate fixed point of nonexpansive mappings and some other relevant problems; see [4–16] and the references therein. In these algorithms, Mann iteration is a fundamental method approximating fixed points of nonexpansive mappings, which is defined by

$$x_{n+1} = (1-\alpha_n) x_n + \alpha_n T x_n, \tag{6}$$

where $\alpha_n \in (0,1)$ and T is a nonexpansive mapping. The other important iteration widely used to approximate fixed point of nonexpansive mapping is Ishikawa iteration, which is defined by

$$\begin{aligned} y_n &= (1-\beta_n) x_n + \beta_n T x_n, \\ x_{n+1} &= (1-\alpha_n) x_n + \alpha_n T y_n, \end{aligned} \tag{7}$$

where $\alpha_n, \beta_n \in (0,1)$. Note that Ishikawa iteration (7) improves the rate of convergence of Mann iteration process for an increasing function due to Ishikawa [17] and Rhoades [18].

In 2007, Agrawal et al. [19] modified (7) and considered the following two-step iteration process: for an arbitrary $x_1 \in K$, the sequence of $\{x_n\}$ is defined in the following manner:

$$\begin{aligned} y_n &= (1-\beta_n) x_n + \beta_n T x_n, \\ x_{n+1} &= (1-\alpha_n) T x_n + \alpha_n T y_n, \end{aligned} \tag{8}$$

where $\alpha_n, \beta_n \in (0,1)$ and T is a nearly asymptotically nonexpansive mapping. They claimed that this iteration process converges faster than the Mann iteration for some contractions.

Recently, Noor [20] modified (7) and further studied a three-step iteration process to solve the general variational inequalities: for an arbitrary $x_1 \in K$ defined a sequence $\{x_n\}$ by

$$\begin{aligned} z_n &= (1-\gamma_n) x_n + \gamma_n T x_n, \\ y_n &= (1-\beta_n) x_n + \beta_n T z_n, \\ x_{n+1} &= (1-\alpha_n) x_n + \alpha_n T y_n, \end{aligned} \tag{9}$$

where $\alpha_n, \beta_n, \gamma_n \in (0,1)$ and T is a strong monotone mapping involved variational inequalities. Very recently, Abbas-Nazir [21] and Thakur et al. [22] modified Noor iteration (9) and introduced a new faster iteration process for solving the constrained minimization and feasibility problems and for finding the fixed point of Suzuki's generalized nonexpansive mappings, respectively.

On the other hand, in 2004, Ran-Reurings [23] firstly introduced a fixed point theorem in a partially ordered metric space and some applications to matrix equations. They developed a new field only for comparable elements instead of the nonexpansive (or Lipschitz) condition in a partially ordered metric space, which has been successfully applied to solve not only the existence of fixed points but also a positive or negative solution of ordinary differential equations [24].

In 2015, Bin Dehaish-Khamsi [25] applied the Mann iteration (6) to the case of a monotone nonexpansive mapping in a Banach space endowed with a partial order. Moreover, they proved that $\{x_n\}$ generated by (6) weakly converges to $x^* \in F(T)$ and x^* and x_1 are comparable.

In 2016, Song et al. [26] further extended the Mann iteration (6) to monotone α-nonexpansive mappings and obtained some weak and strong convergence theorems in an ordered Banach space, which complemented the fixed point results of α-nonexpansive mappings in Aoyama-Kohsaka [27]. However, in general, the monotone condition on comparable elements is a weaker assumption. In particular, the continuity property probably is not valid, which not only reduces the efficiency of numerical approach but also increases the difficulty of convergence analysis. This is also the main reason why Mann iteration has become popular in approximating the fixed point of monotone-type mappings [2, 25, 26]. Therefore, it is important and interesting to construct an iterative accelerator method for finding the fixed points problem of such class of monotone-type mappings.

Inspired and motivated by research going on in this area, we modify the iteration process (6), (8), and (9) to the case of monotone generalized α-nonexpansive mappings and introduce a new accelerated iteration: for an arbitrary $x_1 \in K$, sequence $\{x_n\}$ is defined by

$$\begin{aligned} z_n &= (1-\gamma_n) x_n + \gamma_n T x_n, \\ y_n &= (1-\beta_n) T x_n + \beta_n T z_n, \\ x_{n+1} &= T\left[(1-\alpha_n) y_n + \alpha_n z_n\right], \end{aligned} \tag{10}$$

Our purpose is not only to extend Mann iteration of Bin Dehaish-Khamsi [25] and Song et al. [26] to an accelerated iteration for monotone generalized α-nonexpansive mappings, but also to establish some weak and strong convergence theorems of fixed point for monotone generalized α-nonexpansive mapping in a uniformly convex Banach space with a partial order. Furthermore, we provide a numerical example to illustrate the convergence behavior and effectiveness of the proposed iteration. The method and results presented in this paper extend and improve the corresponding results of [2, 17, 19, 20, 25, 26] and some others previously.

2. Preliminaries

Recall that a Banach space E with the norm $\|.\|$ is called uniformly convex if, for all $\varepsilon \in (0,2]$, there exists a constant $\delta > 0$ for which $\|x\| \le 1$, $\|y\| \le 1$ and $\|x-y\| \ge \varepsilon$ implies

$$\frac{1}{2}\|x+y\| \le 1-\delta. \tag{11}$$

A Banach space E is said to satisfy the Opial property [5] if for each weakly convergent sequence $\{x_n\}$ in E with weak limit x,

$$\limsup_{n\to\infty}\|x_n - x\| < \limsup_{n\to\infty}\|x_n - y\| \tag{12}$$

holds, for all $y \in E$ with $y \ne x$.

Let K be a nonempty subset of a Banach space E and $\{x_n\}$ be a bounded sequence in E. For each $x \in E$, we define the following:

(i) Asymptotic radius of $\{x_n\}$ at x by $r(x,\{x_n\}) := \limsup_{n\to\infty}\|x_n - x\|$.

(ii) Asymptotic radius of $\{x_n\}$ relative to K by $r(K, \{x_n\}) := \inf\{r(x, x_n) : x \in K\}$.

(iii) Asymptotic center of $\{x_n\}$ relative to K by $A(K, \{x_n\}) := \{x \in K : r(x, \{x_n\}) = r(K, \{x_n\})\}$.

Note that $A(K, \{x_n\})$ is nonempty. Further, if E is uniformly convex, then $A(K, \{x_n\})$ has exactly one point [28].

Recall also that an order interval $[a, b]$ is defined by

$$[a, b] = \{x \in E : a \le x \le b\} = [a, \longrightarrow) \cap (\longleftarrow, b], \quad (13)$$

where $[a, \longrightarrow) = \{x \in E : a \le x\}$ and $(\longleftarrow, b] = \{x \in E : x \le b\}$. Throughout, we assume that the order intervals are closed and convex in an ordered Banach space $(E, \le)$.

Lemma 1 (see [1]). *Let K be a nonempty closed convex subset of an ordered Banach space $(E, \le)$ and $T : K \longrightarrow K$ be a monotone generalized α-nonexpansive mapping. Then, for all $x, y \in K$ with $x \le y$, the following inequalities hold.*

(i) $\|Tx - T^2x\| \le \|x - Tx\|$,

(ii) $\|Tx - Ty\| \le \alpha\|Tx - y\| + \alpha\|Ty - x\| + (1 - 2\alpha)\|x - y\|$ *or*
$\|T^2x - Ty\| \le \alpha\|T^2x - y\| + \alpha\|Tx - Ty\| + (1 - 2\alpha)\|Tx - y\|$.

Lemma 2 (see [2]). *Let K be a nonempty subset of an ordered Banach space $(E, \le)$ and $T : K \longrightarrow K$ be a generalized α-nonexpansive mapping. Then $F(T)$ is closed.*

Lemma 3 (see [2]). *Let K be a nonempty closed convex subset of a uniformly convex ordered Banach spaces $(E, \le)$. Let $T : K \longrightarrow K$ be a monotone generalized α-nonexpansive mapping. Then $F(T) \ne \emptyset$ if and only if $\{T^n(x)\}$ is a bounded sequence for some $x \in K$ with $x \le Tx$.*

Lemma 4 (see [29]). *A Banach space E is uniformly convex if and only if there exists a continuous strictly increasing convex function $f : [0, +\infty) \longrightarrow [0, +\infty)$ with $f(0) = 0$ such that*

$$\begin{aligned} \|\lambda x + (1 - \lambda) y\|^2 \le \lambda \|x\|^2 + (1 - \lambda) \|y\|^2 \\ - \lambda (1 - \lambda) f(\|x - y\|), \end{aligned} \quad (14)$$

where $\lambda \in [0, 1]$ and $x, y \in B_r(0) = \{x \in E : \|x\| \le r,\ r > 0\}$.

Lemma 5 (see [30]). *Let E be a uniformly convex Banach space and $\{\lambda_n\}$ be a sequence with $0 < \liminf_{n\longrightarrow\infty}\lambda_n \le \limsup_{n\longrightarrow\infty}\lambda_n < 1$. Suppose $\{x_n\}$ and $\{y_n\}$ are two sequences of E such that $\limsup_{n\longrightarrow\infty}\|x_n\| \le r$, $\limsup_{n\longrightarrow\infty}\|y_n\| \le r$ and $\lim_{n\longrightarrow\infty}\|\lambda_n x_n + (1 - \lambda_n)y_n\| = r$. Then*

$$\lim_{n\longrightarrow\infty} \|x_n - y_n\| = 0. \quad (15)$$

Lemma 6. *Let K be a nonempty closed convex subset of an ordered Banach space $(E, \le)$ and $T : K \longrightarrow K$ be a monotone generalized α-nonexpansive mapping. Then*

(i) $\|Tx - Ty\| \le (2/(1 - \alpha))\|x - Tx\| + \|x - y\|$, $\forall x, y \in K$ *with $x \le y$.*

(ii) *T is monotone quasinonexpansive if $F(T) \ne \emptyset$ and $p \in F(T)$ with $x \le p$ or $p \le x$.*

Proof. (i) From Lemma 1 (ii), in the first case, we have

$$\begin{aligned} \|Tx - Ty\| &\le \alpha \|Tx - y\| + \alpha \|x - Ty\| \\ &\quad + (1 - 2\alpha) \|x - y\| \\ &\le \alpha [\|Tx - x\| + \|x - y\|] \\ &\quad + \alpha [\|x - Tx\| + \|Tx - Ty\|] \\ &\quad + (1 - 2\alpha) \|x - y\| \\ &= 2\alpha \|x - Tx\| + \alpha \|Tx - Ty\| \\ &\quad + (1 - 2\alpha) \|x - y\|, \end{aligned} \quad (16)$$

which implies that

$$\|Tx - Ty\| \le \frac{2\alpha}{1 - \alpha} \|x - Tx\| + \|x - y\|. \quad (17)$$

In the other case of Lemma 1 (ii), we further have

$$\begin{aligned} \|Tx - Ty\| &\le \|Tx - T^2x\| + \|T^2x - Ty\| \\ &\le \|x - Tx\| + \alpha \|Tx - Ty\| + \alpha \|T^2x - y\| \\ &\quad + (1 - 2\alpha) \|Tx - y\| \\ &\le \|x - Tx\| + \alpha \|Tx - Ty\| \\ &\quad + \alpha [\|T^2x - Tx\| + \|Tx - y\|] \\ &\quad + (1 - 2\alpha) \|Tx - y\| \\ &\le (1 + \alpha) \|x - Tx\| + \alpha \|Tx - Ty\| \\ &\quad + (1 - \alpha) [\|Tx - x\| + \|x - y\|] \\ &= 2 \|x - Tx\| + \alpha \|Tx - Ty\| \\ &\quad + (1 - \alpha) \|x - y\|, \end{aligned} \quad (18)$$

which implies that

$$\|Tx - Ty\| \le \frac{2}{1 - \alpha} \|x - Tx\| + \|x - y\|. \quad (19)$$

The desired conclusion follows immediately from (17) and (19) for all $x, y \in K$ and $\alpha \in [0, 1)$.

(ii) By the definition of monotone generalized α-nonexpansive mapping, we have

$$\begin{aligned} \|Tx - p\| &= \|Tx - Tp\| \\ &\le \alpha \|Tx - p\| + \alpha \|Tp - x\| \\ &\quad + (1 - 2\alpha) \|x - p\| \\ &\le \alpha \|Tx - p\| + (1 - \alpha) \|x - p\|, \end{aligned} \quad (20)$$

where $p \in F(T)$, and so $\|Tx - p\| \le \|x - p\|$; that is, T is monotone quasinonexpansive. $\square$

3. Main Results

Lemma 7. *Let K be a nonempty closed convex subset of an ordered Banach space $(E, \leq)$ and $T : K \longrightarrow K$ be a monotone mapping. Assume that the sequence $\{x_n\}$ is defined by the iteration (10) and $x_1 \leq Tx_1$. Then*

(i) $x_n \leq z_n \leq Tx_n \leq y_n \leq Tz_n \leq x_{n+1} \leq Ty_n$;

(ii) $\{x_n\}$ has at most one weak-cluster point $x \in K$. Moreover, $x_n \leq x$ for all $n \geq 1$ provided $\{x_n\}$ weakly converges to a point $x \in K$.

Proof. (i) Note that if $c_1, c_2 \in K$ are such that $c_1 \leq c_2$, then $c_1 \leq \lambda c_1 + (1-\lambda)c_2 \leq c_2$ holds from the convex property defined on order intervals. This allows us to focus only on the proof of $x_n \leq Tx_n$ for any $n \geq 1$. By $x_1 \leq Tx_1$, we suppose that $x_n \leq Tx_n$ for $n \geq 2$. From (10), we have

$$\begin{aligned} x_n &\leq (1-\gamma_n) x_n + \gamma_n T x_n = z_n \\ &\leq (1-\gamma_n) T x_n + \gamma_n T x_n = T x_n. \end{aligned} \tag{21}$$

Since T is monotone, we obtain $x_n \leq z_n \leq Tx_n \leq Tz_n$. Using (10) again, we obtain

$$\begin{aligned} Tx_n &= (1-\beta_n) Tx_n + \beta_n Tx_n \leq (1-\beta_n) Tx_n + \beta_n Tz_n \\ &= y_n \leq (1-\beta_n) Tz_n + \beta_n Tz_n = Tz_n, \end{aligned} \tag{22}$$

which implies that $x_n \leq z_n \leq Tx_n \leq y_n \leq Tz_n$. Similarly, we have

$$z_n \leq (1-\alpha_n) y_n + \alpha_n z_n \leq (1-\alpha_n) y_n + \alpha_n y_n = y_n. \tag{23}$$

It follows from (23) that $Tz_n \leq x_{n+1} \leq Ty_n$. Consequently, $x_n \leq z_n \leq Tx_n \leq y_n \leq Tz_n \leq x_{n+1} \leq Ty_n$, which further implies that $x_{n+1} \leq Tx_{n+1}$.

(ii) The desired conclusion follows from (i) and Lemma 3.1 in Bin Dehaish-Khamsi [25]. □

Theorem 8. *Let K be a nonempty closed convex subset of a uniformly convex ordered Banach space $(E, \leq)$ and $T : K \longrightarrow K$ be a monotone generalized α-nonexpansive mapping. Suppose that the sequence $\{x_n\}$ defined by (10) is bounded and* $\liminf_{n\longrightarrow\infty}\|x_n - Tx_n\| = 0$. *Then $F(T) \neq \emptyset$.*

Proof. Since $\{x_n\}$ is a bounded sequence and $\liminf_{n\longrightarrow\infty}\|x_n - Tx_n\| = 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{k\longrightarrow\infty} \left\| x_{n_k} - Tx_{n_k} \right\| = 0. \tag{24}$$

The asymptotic center of $\{x_{n_k}\}$ with respect to K is denoted by $A(K, \{x_{n_k}\}) = \{x^*\}$ such that $x_{n_k} \leq x^*$ for all $n \in \mathbb{N}$, such x^* is unique. From the definition of asymptotic radius, we have

$$r\left(Tx^*, \left\{x_{n_k}\right\}\right) = \limsup_{k\longrightarrow\infty} \left\| x_{n_k} - Tx^* \right\|. \tag{25}$$

Using Lemma 6 (i) and (24), we further obtain

$$\begin{aligned} & r\left(Tx^*, \left\{x_{n_k}\right\}\right) \\ &\quad \leq \limsup_{k\longrightarrow\infty} \left[\left\| x_{n_k} - Tx_{n_k} \right\| + \left\| Tx_{n_k} - Tx^* \right\| \right] \\ &\quad = \limsup_{k\longrightarrow\infty} \left\| Tx_{n_k} - Tx^* \right\| \\ &\quad \leq \limsup_{k\longrightarrow\infty} \left[\frac{2}{1-\alpha} \left\| x_{n_k} - Tx_{n_k} \right\| + \left\| x_{n_k} - x^* \right\| \right] \\ &\quad = r\left(x^*, \left\{x_{n_k}\right\}\right). \end{aligned} \tag{26}$$

It follows from the uniqueness of x^* that $Tx^* = x^*$, which shows that $F(T) \neq \emptyset$. □

Theorem 9. *Let K be a nonempty closed convex subset of a uniformly convex ordered Banach space $(E, \leq)$ with Opial property. Let $T : K \longrightarrow K$ be a monotone generalized α-nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that there exists a $x_1 \in K$ such that $x_1 \leq Tx_1$, then the sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions*

(i) $\alpha_n \in [a, b] \subset (0, 1)$, $\beta_n \in (0, 1)$;

(ii) $\gamma_n \in (0, 1)$ *and* $0 < \liminf_{n\longrightarrow\infty}\gamma_n \leq \limsup_{n\longrightarrow\infty}\gamma_n < 1$.

Then the sequence $\{x_n\}$ generated by (10) weakly converges to a fixed point $q \in F(T)$.

Proof. Firstly, we show $\{x_n\}$ is bounded. Taking $p \in F(T)$, without loss of generality, we assume $x_1 \leq p$. Associating with the monotone property of T, we find

$$x_1 \leq Tx_1 \leq Tp = p. \tag{27}$$

From (10) and (27), we have

$$\begin{aligned} z_1 &= (1-\gamma_1) x_1 + \gamma_1 T x_1 \leq p, \\ Tz_1 &\leq Tp = p, \\ y_1 &= (1-\beta_1) Tx_1 + \beta_1 Tz_1 \leq p, \\ Ty_1 &\leq Tp = p, \\ x_2 &= T\left[(1-\alpha_1) y_1 + \alpha_1 z_1\right] \leq Tp = p, \\ Tx_2 &\leq Tp = p. \end{aligned} \tag{28}$$

Continuing in this way, we can assume that $x_n \leq p$, we get $Tx_n \leq Tp = p$. Similarly, we have $y_n \leq p, Ty_n \leq Tp = p$ and $z_n \leq p, Tz_n \leq Tp = p$. By Lemma 7 (i), we obtain

$$x_n \leq z_n \leq Tx_n \leq y_n \leq Tz_n \leq x_{n+1} \leq Ty_n \leq p, \tag{29}$$

which implies that $x_{n+1} \leq p$. Therefore, the sequence $\{x_n\}$ is bounded, and so $\{y_n\}$ and $\{z_n\}$ are also bounded.

Secondly, we prove that $\lim_{n\longrightarrow\infty}\|x_n - Tx_n\| = 0$. From (10) and Lemma 6 (ii), we have

$$\begin{aligned}\|z_n - p\| &= \|(1-\gamma_n)x_n + \gamma_n Tx_n - p\| \\ &\le (1-\gamma_n)\|x_n - p\| + \gamma_n\|Tx_n - p\| \\ &\le \|x_n - p\|.\end{aligned} \tag{30}$$

Similarly, from (10) and (30), we have

$$\begin{aligned}\|y_n - p\| &\le (1-\beta_n)\|Tx_n - p\| + \beta_n\|Tz_n - p\| \\ &\le (1-\beta_n)\|x_n - p\| + \beta_n\|z_n - p\| \\ &\le \|x_n - p\|.\end{aligned} \tag{31}$$

Combining (10), (30), and (31), we obtain

$$\begin{aligned}\|x_{n+1} - p\| &= \|T[(1-\alpha_n)y_n + \alpha_n z_n] - p\| \\ &\le \|(1-\alpha_n)y_n + \alpha_n z_n - p\| \\ &\le (1-\alpha_n)\|y_n - p\| + \alpha_n\|z_n - p\| \\ &\le \|x_n - p\|,\end{aligned} \tag{32}$$

which implies the limit of $\{\|x_n - p\|\}$ exists, i.e., $\lim_{n\longrightarrow\infty}\|x_n - p\| = r$. Also, it follows from (30) that

$$\limsup_{n\longrightarrow\infty}\|z_n - p\| \le \limsup_{n\longrightarrow\infty}\|x_n - p\| = r. \tag{33}$$

Together (30), (31) with (32), we have

$$\begin{aligned}\|x_{n+1} - p\| &\le (1-\alpha_n)\|y_n - p\| + \alpha_n\|z_n - p\| \\ &\le (1-\alpha_n)\|x_n - p\| + \alpha_n\|z_n - p\|,\end{aligned} \tag{34}$$

which implies that

$$\begin{aligned}&\|x_{n+1} - p\| - \|x_n - p\| \le \frac{\|x_{n+1} - p\| - \|x_n - p\|}{\alpha_n} \\ &\quad\le \|z_n - p\| - \|x_n - p\|.\end{aligned} \tag{35}$$

Hence, $\|x_{n+1} - p\| \le \|z_n - p\|$. Noting that $\alpha_n \in [a,b] \subset (0,1)$, we obtain

$$r = \liminf_{n\longrightarrow\infty}\|x_{n+1} - p\| \le \liminf_{n\longrightarrow\infty}\|z_n - p\|. \tag{36}$$

Moreover, from (33) and (36), we can get

$$\begin{aligned}&\lim_{n\longrightarrow\infty}\|z_n - p\| \\ &\quad = \lim_{n\longrightarrow\infty}\|(1-\gamma_n)(x_n - p) + \gamma_n(Tx_n - p)\| = r.\end{aligned} \tag{37}$$

On the other hand, by the nonexpansive property defined on T, we have

$$\limsup_{n\longrightarrow\infty}\|Tx_n - p\| \le \limsup_{n\longrightarrow\infty}\|x_n - p\| = r. \tag{38}$$

It follows from (37), (38) and Lemma 5 that

$$\lim_{n\longrightarrow\infty}\|x_n - Tx_n\| = 0. \tag{39}$$

Finally, we show that $\{x_n\}$ weakly converges to $q \in F(T)$. By the boundness of $\{x_n\}$, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ weakly converging $q \in C$ and $x_1 \le x_{n_k} \le q$. From Lemma 6 (i) and (39), we can obtain

$$\begin{aligned}&\limsup_{k\longrightarrow\infty}\|Tx_{n_k} - Tq\| \\ &\quad\le \limsup_{k\longrightarrow\infty}\left[\frac{2}{1-\alpha}\|x_{n_k} - Tx_{n_k}\| + \|x_{n_k} - q\|\right] \\ &\quad= \limsup_{k\longrightarrow\infty}\|x_{n_k} - q\|.\end{aligned} \tag{40}$$

Arguing by contradiction, we suppose that $q \ne Tq$. It follows from the Opial property of E that

$$\begin{aligned}\limsup_{k\longrightarrow\infty}\|x_{n_k} - q\| &< \limsup_{k\longrightarrow\infty}\|x_{n_k} - Tq\| \\ &\le \limsup_{k\longrightarrow\infty}\|x_{n_k} - Tx_{n_k}\| \\ &\quad+ \limsup_{k\longrightarrow\infty}\|Tx_{n_k} - Tq\| \\ &\le \limsup_{k\longrightarrow\infty}\|x_{n_k} - q\|.\end{aligned} \tag{41}$$

This is a contradiction. Therefore, we conclude $q = Tq$; that is, $q \in F(T)$. Moreover, if there exists another subsequence $\{x_{n_j}\} \subset \{x_n\}$ weakly converges $w \ne q$. Similarly, we have $w \in F(T)$. Note that $\lim_{n\longrightarrow\infty}\|x_n - w\|$ exists and

$$\begin{aligned}\lim_{n\longrightarrow\infty}\|x_n - q\| &= \limsup_{k\longrightarrow\infty}\|x_{n_k} - q\| \\ &< \limsup_{k\longrightarrow\infty}\|x_{n_k} - w\| = \lim_{n\longrightarrow\infty}\|x_n - w\| \\ &= \limsup_{j\longrightarrow\infty}\|x_{n_j} - w\| \\ &< \limsup_{j\longrightarrow\infty}\|x_{n_j} - q\| = \lim_{n\longrightarrow\infty}\|x_n - q\|,\end{aligned} \tag{42}$$

This is a contradiction again. Consequently, $w = q$ and $\{x_n\}$ weakly converges to $q \in F(T)$. □

Theorem 10. *Let K be a nonempty closed convex subset of a uniformly convex ordered Banach space $(E, \le)$ and $T : K \longrightarrow K$ be a monotone generalized α-nonexpansive mapping with $F(T) \ne \emptyset$. Suppose that there exists a $x_1 \in K$ such that $x_1 \le Tx_1$, the sequences $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ satisfy the following conditions*

(i) $\alpha_n \in [a,b] \subset (0,1)$, $\beta_n \in (0,1)$;

(ii) $\gamma_n \in (0,1)$ *and* $0 < \liminf_{n\longrightarrow\infty}\gamma_n \le \limsup_{n\longrightarrow\infty}\gamma_n < 1$.

Then the sequence $\{x_n\}$ generated by (10) strong converges to a fixed point $q \in F(T)$ if and only if $\liminf_{n\longrightarrow\infty} d(x_n, F(T)) = 0$, where $d(x, F(T))$ denotes the distance from x to $F(T)$.

Proof. Necessity is obvious. We only prove the sufficiency. Suppose that $\liminf_{n\longrightarrow\infty} d(x_n, F(T)) = 0$. From (31), $\lim_{n\longrightarrow\infty} d(x_n, F(T)$ exists. Thus

$$\lim_{n\longrightarrow\infty} d(x_n, F(T)) = 0. \tag{43}$$

By Theorem 9, we have that $\{x_n\}$ is bounded with $x_n \le p$. Without loss of generality, let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ such that $\|x_{n_j} - p_j\| \le 1/2^j$ for all $j \ge 1$, where $\{p_j\}$ is a sequence in $F(T)$. Combining with (31), we have

$$\left\|x_{n_j+1} - p_j\right\| \le \left\|x_{n_j} - p_j\right\| \le \frac{1}{2^j}. \tag{44}$$

It follows from (44) that

$$\begin{aligned}\left\|p_{j+1} - p_j\right\| &\le \left\|p_{j+1} - x_{n_j+1}\right\| + \left\|x_{n_j} - p_j\right\| \\ &\le \frac{1}{2^{j+1}} + \frac{1}{2^j} \le \frac{1}{2^{j-1}}.\end{aligned} \tag{45}$$

This shows that $\{p_j\}$ is a Cauchy sequence in $F(T)$. By Lemma 2, $F(T)$ is closed, so $\{p_j\}$ converges to some $q \in F(T)$. Moreover, by the triangle inequality, we have

$$\left\|x_{n_j+1} - q\right\| \le \left\|x_{n_j} - p_j\right\| + \left\|p_j - q\right\|. \tag{46}$$

Taking $j \longrightarrow \infty$ implies that $\{x_{n_j}\}$ converges strongly to q. From (31) again, $\lim_{n\longrightarrow\infty}\|x_n - q\|$ exists, and the sequence $\{x_n\}$ converges strongly to $q \in F(T)$. □

4. Numerical Example

Example 1. Define a mapping $T : [0,1] \longrightarrow [0,1]$ by

$$Tx = \begin{cases} x + \dfrac{3}{5}, & x \in \left[0, \dfrac{1}{5}\right) \\ \dfrac{x+4}{5}, & x \in \left[\dfrac{1}{5}, 1\right]. \end{cases} \tag{47}$$

Note that T is not continuous. Setting $x = 18/100$, $y = 1/5$, we obtain

$$\|Tx - Ty\| = \frac{6}{100} > \frac{2}{100} = \|x - y\|, \tag{48}$$

that is, T is not a nonexpansive mapping. However, T is a monotone mapping with $x \le Tx$ and a monotone generalized 3/8-nonexpansive mapping.

Numerical Results 4.2. To illustrate the convergence of the proposed algorithm, we provide some numerical results of Example 1 and comparison with the other iterations previously.

Firstly, we show the convergence behavior of scheme (10) with different initial points. To do this, we take $\alpha_n = n/(n+1), \beta_n = 1/(n+5), \gamma_n = n/\sqrt{(2n+9)^3}$ and set $\|x_n - x^*\| < 10^{-6}$ as stop criterion. From given $x_1 = 0.05, 0.50, 0.75, 0.95$, convergence behaviors of scheme (10) are displayed in Figure 1.

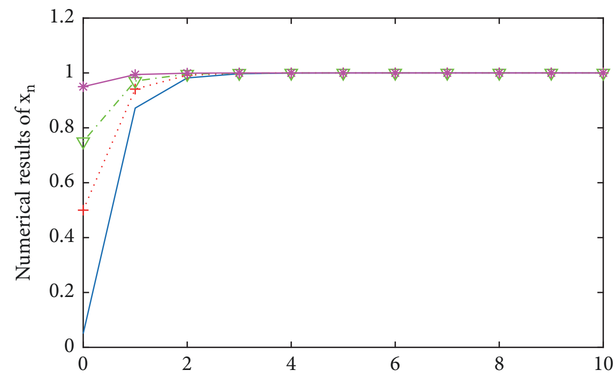

FIGURE 1: No. of iteration for initial $x_1 = 0.05, 0.5, 0.75, 0.95$.

Figure 1 shows that the given point x_1 has a little effect on convergence and scheme (10) is good in strong convergence and operational reliability. Moreover, numerical results show that the increasing of initial point x_1 has a little effect on the speed of convergence; that is, the sequence $\{x_n\}$ generated by (10) will converge faster to a fixed point of Example 1 when x_1 is increased.

Secondly, we further show the stability of scheme (10) based on the different iteration parameters. To complete it, we take $\alpha_n, \beta_n, \gamma_n \in (0,1)$ in the following manner.

(1) $\alpha_n = n/(n+1), \beta_n = 1/(n+5)$ and $\gamma_n = n/\sqrt{(2n+9)^3}$.

(2) $\alpha_n = n/(n+1), \beta_n = n/(n+5)$ and $\gamma_n = 1/\sqrt{2n+7}$.

(3) $\alpha_n = n/(2n+1), \beta_n = n/(3n+5)$ and $\gamma_n = n/(4n+2)$.

(4) $\alpha_n = 1/\sqrt{n+5}, \beta_n = n/(n+1)$ and $\gamma_n = n/\sqrt{2n^2+7}$.

(5) $\alpha_n = 1/(n+1)^2, \beta_n = \sqrt{n}/(n+5)^3$ and $\gamma_n = n/(n+2)$.

(6) $\alpha_n = 1/(n+1)^2, \beta_n = \sqrt{n}/(n+5)^3$ and $\gamma_n = n/(n+2)^4$.

Moreover, we also set $\|x_n - x^*\| < 10^{-6}$ as stop criterion. From a given point $x_1 = 0.20$, computing results of scheme (10) are listed in Table 1.

Table 1 shows that the different parameters $\alpha_n, \beta_n, \gamma_n$ have an effect on iteration and scheme (10) is good in strong convergence and stability. Moreover, for the same initial point $x_1 = 0.2$, numerical results imply that the sequence $\{x_n\}$ generated by (10) will converge faster to a fixed point of Example 1 when parameter α_n is decreased or β_n is increased. In addition, $x_{n(5)}$ and $x_{n(6)}$ imply that parameters γ_n have almost no effect on convergence and iteration.

Finally, we compare the iteration numbers of new proposed method with the others known previously. To make it more obviously, we set $\|x_n - x^*\| < 10^{-10}$ as stop criterion. For given $x_1 = 0.05, 0.20, 0.50, 0.75, 0.95$, iteration numbers of scheme (10) and the known method are listed in Table 2 with some different parameters $\alpha_n, \beta_n, \gamma_n$ in Parameter 1, 3, 5.

Table 2 shows that the different parameters $\alpha_n, \beta_n, \gamma_n$ have a little effect on iteration and scheme (10) is good in strong convergence and effectiveness. Moreover, in Parameter 5, the numerical results imply that computing costs of Mann,

Table 1: Stability of iteration (10) with the different parameters α_n, β_n, γ_n.

Iter.(n)	$x_{n(1)}$	$x_{n(2)}$	$x_{n(3)}$	$x_{n(4)}$	$x_{n(5)}$	$x_{n(6)}$
0	0.200000	0.200000	0.200000	0.200000	0.200000	0.200000
1	0.905813	0.926044	0.932800	0.922392	0.946696	0.936396
2	0.986621	0.991600	0.993918	0.992965	0.997397	0.996334
3	0.997943	0.998943	0.999434	0.999382	0.999886	0.999817
4	0.999670	0.999858	0.999947	0.999947	0.999995	0.999992
5	0.999945	0.999980	0.999995	0.999995	1.000000	1.000000
6	0.999991	0.999997	1.000000	1.000000	1.000000	1.000000
7	0.999998	1.000000	1.000000	1.000000	1.000000	1.000000
8	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000

Table 2: Iteration numbers of scheme (10) and the others known previously.

Param.-point	Mann	Ishikawa	Agarwal	Noor	New method
Parameter 1:					
0.05	20	19	15	19	13
0.20	19	19	14	19	13
0.50	19	19	14	19	13
0.75	19	18	13	18	12
0.95	17	17	12	17	11
Parameter 3:					
0.05	48	45	14	45	10
0.20	48	44	14	44	10
0.50	47	43	14	43	10
0.75	45	42	13	42	10
0.95	42	39	12	39	9
Parameter 5:					
0.05	+	+	15	+	8
0.20	+	+	15	+	8
0.50	+	+	14	+	8
0.75	+	+	14	+	8
0.95	+	+	13	+	7

+ means the number of iterations over 1000.

Ishikawa, and Noor are too heavy. However, our scheme (10) is very advantageous for a wide range of parameters. In addition, scheme (10) requires the less number of iteration for the convergence than Agarwal's when the parameters α_n and β_n are decreased.

The computations are performed by Matlab R2016b running on a PC Desktop Intel(R) Core(TM)i5-5200U CPU @2.20GHz 2.20GHz, 8.00GB RAM.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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