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Research Article

Composition Operators and the Closure of Morrey Space in the Bloch Space

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In this paper, we characterize the closure of the Morrey space in the Bloch space. Furthermore, the boundedness and compactness of composition operators from the Bloch space to the closure of the Morrey space in the Bloch space are investigated.

1. Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the space of analytic functions on $\mathbb{D}$. For a fixed point $a \in \mathbb{D}$, let $\sigma_a(z) = (a - z)/(1 - \overline{a}z)$ denote the Möbius transformation on $\mathbb{D}$. Recall that the Bloch space, denoted by $\mathscr{B} = \mathscr{B}(\mathbb{D})$, is the space of all $f \in H(\mathbb{D})$ for which

$$\|f\|_{\mathscr{B}} = |f(0)| + \sup_{z\in\mathbb{D}} \left(1 - |z|^2\right) \left|f'(z)\right| < \infty. \tag{1}$$

It is a Banach space with the above norm $\|\cdot\|_{\mathscr{B}}$. The little Bloch space $\mathscr{B}_0$ consists of all $f \in H(\mathbb{D})$ such that

$$\lim_{|z|\longrightarrow 1} \left(1 - |z|^2\right) \left|f'(z)\right| = 0. \tag{2}$$

It is easy to see that the little Bloch space $\mathscr{B}_0$ is the subspace of $\mathscr{B}$. It is well known that $\mathscr{B}_0$ is the closure of polynomials in $\mathscr{B}$.

For $0 < p < \infty$, the Hardy space $H^p(\mathbb{D})$ consists of all functions $f \in H(\mathbb{D})$ with

$$\|f\|_{H^p}^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^p d\theta < \infty. \tag{3}$$

Denote by $H^\infty = H^\infty(\mathbb{D})$ the space of bounded analytic functions on $\mathbb{D}$.

For an arc $I \subset \partial\mathbb{D}$, let $|I| = (1/2\pi)\int_I |d\zeta|$ be the normalized length of I and $S(I)$ be the corresponding Carleson box; i.e.,

$$S(I) = \left\{z \in \mathbb{D} : 1 - |I| \le |z| < 1 \text{ and } \frac{z}{|z|} \in I\right\}. \tag{4}$$

Clearly, if $I = \partial\mathbb{D}$, then $S(I) = \mathbb{D}$. Let $s > 0$. A nonnegative measure μ on $\mathbb{D}$ is said to be an s-Carleson measure (see [1]) if

$$\sup_{I\subset\partial\mathbb{D}} \frac{\mu(S(I))}{|I|^s} < \infty. \tag{5}$$

If $s = 1$, a bounded s-Carleson measure is the classical Carleson measure.

For $0 < \lambda \le 1$, the Morrey space $\mathscr{L}^{2,\lambda}(\mathbb{D})$ is the set of all $f \in H^2(\mathbb{D})$ such that

$$\sup_{I\subset\partial\mathbb{D}} \frac{1}{|I|^\lambda} \int_I \left|f(\zeta) - f_I\right|^2 \frac{|d\zeta|}{2\pi} < \infty. \tag{6}$$

Here $f_I = (1/|I|)\int_I f(\zeta)(|d\zeta|/2\pi)$. Clearly, $\mathscr{L}^{2,1}(\mathbb{D}) = BMOA$, the space of analytic functions whose boundary

functions have bounded mean oscillation. From [2] or [3], the norm of functions $f \in \mathscr{L}^{2,\lambda}(\mathbb{D})$ can be defined as follows.

$$\|f\|_{\mathscr{L}^{2,\lambda}} = |f(0)| + \sup_{I\subset\partial\mathbb{D}} \sqrt{\frac{1}{|I|^{\lambda}} \int_{S(I)} |f'(z)|^2 \left(1-|z|^2\right) dA(z)}. \tag{7}$$

We remark that $\mathscr{B} \not\subseteq \mathscr{L}^{2,\lambda}$ and $\mathscr{L}^{2,\lambda} \not\subseteq \mathscr{B}$. It is well known that the function

$$f(z) = \sum_{n=0}^{\infty} z^{2^n} \in \mathscr{B}, \quad \text{but } f \notin \mathscr{L}^{2,\lambda}. \tag{8}$$

After a calculation, we see that $g(z) = (\log(1/(1-z)))^2 \notin \mathscr{B}$, but $g \in \mathscr{L}^{2,\lambda}$. See [2–6] for the study of Morrey space and related operators.

For every self-map φ on $\mathbb{D}$, the composition operator C_φ is defined on $H(\mathbb{D})$ by

$$C_\varphi(f)(z) = f(\varphi(z)), \quad z \in \mathbb{D}. \tag{9}$$

It is a simple consequence of the Schwartz-Pick lemma that any analytic self-mapping φ of $\mathbb{D}$ induces a bounded composition operator C_φ on the Bloch space. Madigan and Matheson in [7] proved that $C_\varphi : \mathscr{B} \longrightarrow \mathscr{B}$ is compact if and only if $\lim_{|\varphi(z)|\longrightarrow 1} |\varphi^{\sharp}(z)| = 0$. Here and henceforth

$$\varphi^{\sharp}(z) = \frac{1-|z|^2}{1-|\varphi(z)|^2} \varphi'(z). \tag{10}$$

See, for example, [7–14] for more characterizations of the boundedness and compactness of composition operators on the Bloch space.

In 1974, Anderson, Clunie, and Pommerenke posed the problem of how to describe the closure of H^∞ in the Bloch norm (see [15]). This is still an open problem. Anderson in [16] mentioned that Jones gave a characterization of $\mathscr{C}_{\mathscr{B}}(BMOA)$, the closure of $BMOA$ in the Bloch norm (an unpublished result). A complete proof was provided by Ghatage and Zheng in [17]. Zhao in [18] studied the closures of some Möbius invariant spaces in the Bloch space. Monreal Galán and Nicolau in [19] characterized the closure in the Bloch norm of the space H^p for $1 < p < \infty$. Later, Galanopoulos et al. in [20] studied the closure in the Bloch norm of H^p on the unit ball in $\mathbb{C}^n$. Moreover, they have extended this result to the whole range $0 < p < \infty$. Bao and Göğüş [21] studied the closure of Dirichlet type spaces $\mathscr{D}_\alpha$ $(-1 < \alpha \le 1)$ in the Bloch space. See [22–26] for some related results.

It is well known that (when $0 < \lambda < 1$)

$$H^\infty \subsetneq BMOA \subsetneq \mathscr{L}^{2,\lambda} \subsetneq H^2 \quad \text{and } H^\infty \subsetneq BMOA \subsetneq \mathscr{B}. \tag{11}$$

Hence,

$$\mathscr{C}_{\mathscr{B}}(BMOA) \subseteq \mathscr{C}_{\mathscr{B}}\left(\mathscr{L}^{2,\lambda} \cap \mathscr{B}\right) \subseteq \mathscr{C}_{\mathscr{B}}\left(H^2 \cap \mathscr{B}\right). \tag{12}$$

From [19], we see that a Bloch function f is in $\mathscr{C}_{\mathscr{B}}(H^2 \cap \mathscr{B})$ if and only if, for every $\epsilon > 0$,

$$\int_{\partial\mathbb{D}} \int_{\Gamma_a(\xi)\cap\Omega_\epsilon(f)} \frac{dA(z)}{\left(1-|z|^2\right)^2} |d\xi| < \infty. \tag{13}$$

Here

$$\Gamma_a(\xi) = \{z \in \mathbb{D} : |z-\xi| < a(1-|z|)\}, \quad a > 1,\ \xi \in \partial\mathbb{D}, \tag{14}$$

and

$$\Omega_\epsilon(f) = \left\{z \in \mathbb{D} : \left(1-|z|^2\right)|f'(z)| \ge \epsilon\right\}. \tag{15}$$

From [16, 17], we see that a Bloch function f is in $\mathscr{C}_{\mathscr{B}}(BMOA)$ if and only if, for every $\epsilon > 0$,

$$\sup_{a\in\mathbb{D}} \int_{\Omega_\epsilon(f)} \frac{1-|\sigma_a(z)|^2}{1-|z|^2} \frac{dA(z)}{1-|z|^2} < \infty. \tag{16}$$

It is natural to ask what is $\mathscr{C}_{\mathscr{B}}(\mathscr{L}^{2,\lambda} \cap \mathscr{B})$, the closure of the Morrey type space $\mathscr{L}^{2,\lambda}$ $(0 < \lambda < 1)$ in the Bloch norm?

The purpose of this paper is to characterize $\mathscr{C}_{\mathscr{B}}(\mathscr{L}^{2,\lambda} \cap \mathscr{B})$. Moreover, we study the boundedness and compactness of composition operators $C_\varphi : \mathscr{B}(\mathscr{B}_0) \longrightarrow \mathscr{C}_{\mathscr{B}}(\mathscr{L}^{2,\lambda} \cap \mathscr{B})$ and $C_\varphi : \mathscr{C}_{\mathscr{B}}(\mathscr{L}^{2,\lambda} \cap \mathscr{B}) \longrightarrow \mathscr{C}_{\mathscr{B}}(\mathscr{L}^{2,\lambda} \cap \mathscr{B})$.

Throughout this paper, we say that $A \lesssim B$ if there exists a constant C such that $A \le CB$. The symbol $A \approx B$ means that $A \lesssim B \lesssim A$.

2. Main Results and Proofs

In this section we give our main results and proofs. For this purpose, we need the following well-known estimate which can be found in [27] or [18].

Lemma 1. *Let $s > -1, r, t > 0$ and $r+t-s > 2$. If $t < s+2 < r$, then*

$$\int_{\mathbb{D}} \frac{\left(1-|\eta|^2\right)^s}{|1-\overline{\eta}z|^r |1-\overline{\eta}\xi|^t} dA(\eta) \le \frac{C}{\left(1-|z|^2\right)^{r-s-2} |1-\overline{\xi}z|^t}. \tag{17}$$

The following lemma is Lemma 3.1.1 in [1].

Lemma 2. *Let $s, t \in (0,\infty)$ and a nonnegative measure μ on $\mathbb{D}$. Then μ is a s-Carleson measure if and only if*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{\left(1-|a|^2\right)^t}{|1-\overline{a}z|^{s+t}} d\mu(z) < \infty \tag{18}$$

Lemma 3. *Let $0 < \lambda < 1$. Then $f \in \mathscr{L}^{2,\lambda}$ if and only if*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 \left(1-|z|^2\right)^{1-\lambda} \left(1-|\sigma_a(z)|^2\right)^{\lambda} dA(z) < \infty. \tag{19}$$

Moreover,

$$\|f\|_{\mathscr{L}^{2,\lambda}} \approx |f(0)| + \left(\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 \left(1-|z|^2\right)^{1-\lambda} \cdot \left(1-|\sigma_a(z)|^2\right)^{\lambda} dA(z)\right)^{1/2} < \infty. \tag{20}$$

Proof. Denote

$$d\mu(z) = |f'(z)|^2 \left(1-|z|^2\right) dA(z). \tag{21}$$

Then, from (7) we have that $f \in \mathscr{L}^{2,\lambda}$ if and only if $d\mu$ is a λ-Carleson measure. Hence, by Lemma 2, we get that $f \in \mathscr{L}^{2,\lambda}$ if and only if

$$\begin{aligned}\sup_{a\in\mathbb{D}} &\int_{\mathbb{D}} \frac{\left(1-|a|^2\right)^t}{|1-\overline{a}z|^{\lambda+t}} d\mu(z) \\ &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{\left(1-|a|^2\right)^t}{|1-\overline{a}z|^{\lambda+t}} |f'(z)|^2 \left(1-|z|^2\right) dA(z) \\ &< \infty.\end{aligned} \tag{22}$$

Let $t = \lambda$. We get the desired result. □

Now we present and prove our main results in this paper.

Theorem 4. *Let $0 < \lambda < 1$ and $f \in \mathscr{B}$. Then $f \in \mathscr{C}_{\mathscr{B}}(\mathscr{L}^{2,\lambda} \cap \mathscr{B})$ if and only if, for any $\epsilon > 0$,*

$$\sup_{a\in\mathbb{D}} \int_{\Omega_\epsilon(f)} \left(\frac{1-|\sigma_a(z)|^2}{1-|z|^2}\right)^{\lambda} \frac{dA(z)}{1-|z|^2} < \infty. \tag{23}$$

Proof. Take $f \in \mathscr{C}_{\mathscr{B}}(\mathscr{L}^{2,\lambda} \cap \mathscr{B})$ and $\epsilon > 0$. Then there exists a $g \in \mathscr{L}^{2,\lambda} \cap \mathscr{B}$ such that $\|f-g\|_{\mathscr{B}} \le \epsilon/2$. Since

$$\begin{aligned}\left(1-|z|^2\right)|f'(z)| &\le \sup_{w\in\mathbb{D}} \left(1-|w|^2\right)|f'(w)-g'(w)| \\ &\quad + \left(1-|z|^2\right)|g'(z)| \\ &\le \frac{\epsilon}{2} + \left(1-|z|^2\right)|g'(z)|, \quad z \in \mathbb{D},\end{aligned} \tag{24}$$

we see that $\Omega_\epsilon(f) \subseteq \Omega_{\epsilon/2}(g)$. Then by Lemma 3 we get

$$\begin{aligned}&\sup_{a\in\mathbb{D}} \int_{\Omega_\epsilon(f)} \left(\frac{1-|\sigma_a(z)|^2}{1-|z|^2}\right)^{\lambda} \frac{dA(z)}{1-|z|^2} \\ &\le \sup_{a\in\mathbb{D}} \int_{\Omega_{\epsilon/2}(g)} \frac{\left(1-|\sigma_a(z)|^2\right)^{\lambda} \left(1-|z|^2\right)^{2-\lambda} |g'(z)|^2}{\left(1-|z|^2\right)^2 |g'(z)|^2} \frac{dA(z)}{1-|z|^2} \\ &\le \frac{4}{\epsilon^2} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |g'(z)|^2 \left(1-|z|^2\right)^{1-\lambda} \left(1-|\sigma_a(z)|^2\right)^{\lambda} dA(z) \\ &< \infty,\end{aligned} \tag{25}$$

as desired.

Conversely, suppose that (23) holds. Fix $\epsilon > 0$ and let f satisfy (23). Without loss of generality, we may assume that $f(0) = 0$. For any $z \in \mathbb{D}$, by Proposition 4.27 in [28],

$$f(z) = \frac{1}{\beta+1} \int_{\mathbb{D}} \frac{f'(w)\left(1-|w|^2\right)^{1+\beta}}{(1-z\overline{w})^{2+\beta}\,\overline{w}} dA(w), \tag{26}$$

where $\beta > 0$. Following [18], we decompose f as $f(z) = f_1(z) + f_2(z)$, where

$$f_1(z) = \frac{1}{\beta+1} \int_{\Omega_\epsilon(f)} \frac{f'(w)\left(1-|w|^2\right)^{1+\beta}}{(1-z\overline{w})^{2+\beta}\,\overline{w}} dA(w) \tag{27}$$

and

$$f_2(z) = \frac{1}{\beta+1} \int_{\mathbb{D}\setminus\Omega_\epsilon(f)} \frac{f'(w)\left(1-|w|^2\right)^{1+\beta}}{(1-z\overline{w})^{2+\beta}\,\overline{w}} dA(w). \tag{28}$$

After a calculation, we get

$$f_1'(z) = \frac{\beta+2}{\beta+1} \int_{\Omega_\epsilon(f)} \frac{f'(w)\left(1-|w|^2\right)^{1+\beta}}{(1-z\overline{w})^{3+\beta}} dA(w) \tag{29}$$

and

$$f_2'(z) = \frac{\beta+2}{\beta+1} \int_{\mathbb{D}\setminus\Omega_\epsilon(f)} \frac{f'(w)\left(1-|w|^2\right)^{1+\beta}}{(1-z\overline{w})^{3+\beta}} dA(w). \tag{30}$$

Let $g = f_1 - f_1(0)$. Then $g(0) = 0$; we obtain

$$\begin{aligned}\|f-g\|_{\mathscr{B}} &= \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)|f_2'(z)| \lesssim \sup_{z\in\mathbb{D}} \left(1-|z|^2\right) \\ &\quad \cdot \int_{\mathbb{D}\setminus\Omega_\epsilon(f)} \frac{|f'(w)|\left(1-|w|^2\right)^{1+\beta}}{|1-z\overline{w}|^{3+\beta}} dA(w) \lesssim \epsilon \\ &\quad \cdot \sup_{z\in\mathbb{D}} \left(1-|z|^2\right) \int_{\mathbb{D}} \frac{\left(1-|w|^2\right)^{\beta}}{|1-z\overline{w}|^{3+\beta}} dA(w).\end{aligned} \tag{31}$$

Then by Lemma 3.10 of [28] we get

$$\|f-g\|_{\mathscr{B}} \lesssim \epsilon. \tag{32}$$

Hence $g \in \mathscr{B}$. Applying Fubini's theorem and Lemma 1, we deduce that

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|g'(z)\right|^2 \left(1-|z|^2\right)^{1-\lambda} \left(1-|\sigma_a(z)|^2\right)^{\lambda} dA(z) \\
&\quad = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|f_1'(z)\right|^2 \\
&\quad \cdot \left(1-|z|^2\right)^{1-\lambda} \left(1-|\sigma_a(z)|^2\right)^{\lambda} dA(z) \\
&\quad \lesssim \|f_1\|_{\mathscr{B}} \sup_{a\in\mathbb{D}} \int_{\Omega_\epsilon(f)} \left|f'(w)\right| \\
&\quad \cdot \left(1-|w|^2\right)^{1+\beta} dA(w) \int_{\mathbb{D}} \frac{\left(1-|z|^2\right)^{-\lambda} \left(1-|\sigma_a(z)|^2\right)^{\lambda}}{|1-z\overline{w}|^{3+\beta}} dA(z) \\
&\quad \lesssim \sup_{a\in\mathbb{D}} \int_{\Omega_\epsilon(f)} \frac{\left(1-|\sigma_a(w)|^2\right)^{\lambda} \left(1-|w|^2\right)^{\beta} |1-\overline{a}w|^{2\lambda}}{\left(1-|w|^2\right)^{\lambda}} dA(w) \\
&\quad \cdot \int_{\mathbb{D}} \frac{dA(z)}{|1-\overline{z}w|^{3+\beta} |1-\overline{z}a|^{2\lambda}} \\
&\quad \lesssim \sup_{a\in\mathbb{D}} \int_{\Omega_\epsilon(f)} \frac{\left(1-|\sigma_a(w)|^2\right)^{\lambda}}{\left(1-|w|^2\right)^{\lambda}} \frac{dA(w)}{1-|w|^2} \\
&\quad < \infty;
\end{aligned} \tag{33}$$

that is, $g \in \mathscr{L}^{2,\lambda}$. Thus, for any $\epsilon > 0$, there exists a function $g \in \mathscr{L}^{2,\lambda} \cap \mathscr{B}$ such that $\|f-g\|_{\mathscr{B}} \lesssim \epsilon$; i.e., $f \in \mathscr{C}_{\mathscr{B}}(\mathscr{L}^{2,\lambda} \cap \mathscr{B})$. The proof is complete. □

Next, we consider the boundedness and compactness of composition operators from $\mathscr{B}$ to $\mathscr{C}_{\mathscr{B}}(\mathscr{L}^{2,\lambda} \cap \mathscr{B})$.

Theorem 5. *Let $0 < \lambda < 1$ and let φ be an analytic self-map of $\mathbb{D}$. Then $C_\varphi : \mathscr{B} \longrightarrow \mathscr{C}_{\mathscr{B}}(\mathscr{L}^{2,\lambda} \cap \mathscr{B})$ is bounded if and only if, for any $\epsilon > 0$,*

$$\sup_{a\in\mathbb{D}} \int_{|\varphi^\sharp(z)|\geq\epsilon} \left(\frac{1-|\sigma_a(z)|^2}{1-|z|^2}\right)^{\lambda} \frac{dA(z)}{1-|z|^2} < \infty. \tag{34}$$

Proof. Assume that $C_\varphi : \mathscr{B} \longrightarrow \mathscr{C}_{\mathscr{B}}(\mathscr{L}^{2,\lambda} \cap \mathscr{B})$ is bounded. From [29], we see that there exists two functions $f_1, f_2 \in \mathscr{B}$ such that

$$\left|f_1'(z)\right| + \left|f_2'(z)\right| \geq \frac{1}{1-|z|^2}. \tag{35}$$

By the boundedness of C_φ, we get $f_1 \circ \varphi, f_2 \circ \varphi \in \mathscr{C}_{\mathscr{B}}(\mathscr{L}^{2,\lambda} \cap \mathscr{B})$. Hence, Theorem 4 implies that, for any $\epsilon > 0$,

$$\sup_{a\in\mathbb{D}} \int_{\Omega_{\epsilon/2}(f_1\circ\varphi)} \left(\frac{1-|\sigma_a(z)|^2}{1-|z|^2}\right)^{\lambda} \frac{dA(z)}{1-|z|^2} < \infty \tag{36}$$

and

$$\sup_{a\in\mathbb{D}} \int_{\Omega_{\epsilon/2}(f_2\circ\varphi)} \left(\frac{1-|\sigma_a(z)|^2}{1-|z|^2}\right)^{\lambda} \frac{dA(z)}{1-|z|^2} < \infty. \tag{37}$$

When $|\varphi^\sharp(z)| \geq \epsilon$, we get

$$\begin{aligned}
&\left(\left|(f_1\circ\varphi)'(z)\right| + \left|(f_2\circ\varphi)'(z)\right|\right)\left(1-|z|^2\right) \\
&\quad = \left(\left|f_1'(\varphi(z))\right| + \left|f_2'(\varphi(z))\right|\right)\left|\varphi'(z)\right|\left(1-|z|^2\right) \\
&\quad = \left(\left|f_1'(\varphi(z))\right| + \left|f_2'(\varphi(z))\right|\right)\left(1-\left|\varphi'(z)\right|^2\right) \\
&\quad \cdot \left|\varphi'(z)\right| \frac{1-|z|^2}{1-\left|\varphi'(z)\right|^2} \geq \left|\varphi^\sharp(z)\right| \geq \epsilon,
\end{aligned} \tag{38}$$

which implies that either

$$\left|(f_1\circ\varphi)'(z)\right|\left(1-|z|^2\right) \geq \frac{\epsilon}{2} \tag{39}$$

or

$$\left|(f_2\circ\varphi)'(z)\right|\left(1-|z|^2\right) \geq \frac{\epsilon}{2}. \tag{40}$$

Hence,

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} \int_{|\varphi^\sharp(z)|\geq\epsilon} \left(\frac{1-|\sigma_a(z)|^2}{1-|z|^2}\right)^{\lambda} \frac{dA(z)}{1-|z|^2} \\
&\quad \leq \sup_{a\in\mathbb{D}} \int_{\Omega_{\epsilon/2}(f_1\circ\varphi)\cup\Omega_{\epsilon/2}(f_2\circ\varphi)} \left(\frac{1-|\sigma_a(z)|^2}{1-|z|^2}\right)^{\lambda} \frac{dA(z)}{1-|z|^2} \\
&\quad \leq \sup_{a\in\mathbb{D}} \int_{\Omega_{\epsilon/2}(f_1\circ\varphi)} \left(\frac{1-|\sigma_a(z)|^2}{1-|z|^2}\right)^{\lambda} \frac{dA(z)}{1-|z|^2} \\
&\quad + \sup_{a\in\mathbb{D}} \int_{\Omega_{\epsilon/2}(f_2\circ\varphi)} \left(\frac{1-|\sigma_a(z)|^2}{1-|z|^2}\right)^{\lambda} \frac{dA(z)}{1-|z|^2} < \infty.
\end{aligned} \tag{41}$$

Conversely, suppose that (34) holds. Let $f \in \mathscr{B}$. Then

$$\begin{aligned}
&\left|(f\circ\varphi)'(z)\right|\left(1-|z|^2\right) \\
&\quad = \left|f'(\varphi(z))\right|\left(1-|\varphi(z)|^2\right) \frac{\left|\varphi'(z)\right|\left(1-|z|^2\right)}{1-|\varphi(z)|^2} \\
&\quad \leq \|f\|_{\mathscr{B}} \left|\varphi^\sharp(z)\right|.
\end{aligned} \tag{42}$$

Therefore, for any $\delta > 0$, we obtain

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} \int_{\Omega_\delta(f\circ\varphi)} \left(\frac{1-|\sigma_a(z)|^2}{1-|z|^2}\right)^{\lambda} \frac{dA(z)}{1-|z|^2} \\
&\quad \leq \sup_{a\in\mathbb{D}} \int_{|\varphi^\sharp(z)|\geq\epsilon} \left(\frac{1-|\sigma_a(z)|^2}{1-|z|^2}\right)^{\lambda} \frac{dA(z)}{1-|z|^2} < \infty.
\end{aligned} \tag{43}$$

From Theorem 4, we have $f \circ \varphi \in \mathscr{C}_{\mathscr{B}}(\mathscr{L}^{2,\lambda} \cap \mathscr{B})$, i.e., $C_\varphi : \mathscr{B} \longrightarrow \mathscr{C}_{\mathscr{B}}(\mathscr{L}^{2,\lambda} \cap \mathscr{B})$ is bounded. The proof is complete. □

Theorem 6. *Let $0 < \lambda < 1$ and let φ be an analytic self-map of $\mathbb{D}$. Then $C_\varphi : \mathscr{B}_0 \longrightarrow \mathscr{C}_{\mathscr{B}}(\mathscr{L}^{2,\lambda} \cap \mathscr{B})$ is automatically bounded.*

Proof. Since $\varphi \in H^\infty \subsetneq BMOA \subsetneq \mathscr{L}^{2,\lambda}$ and $\varphi \in H^\infty \subsetneq \mathscr{B}$, we see that $\varphi \in \mathscr{C}_{\mathscr{B}}(\mathscr{L}^{2,\lambda} \cap \mathscr{B})$. Let $f \in \mathscr{B}_0$. For any $\epsilon > 0$, there is a constant r $(0 < r < 1)$ such that

$$\left|f'(z)\right|\left(1-|z|^2\right) < \frac{\epsilon}{2}; \tag{44}$$

whenever $|z| > r$. Let $z \in \Omega_\epsilon(f \circ \varphi)$. Then, by the assumption and Schwarz-Pick Lemma, we have

$$\begin{aligned}\epsilon &\le \left|f'(\varphi(z))\right|\left|\varphi'(z)\right|\left(1-|z|^2\right)\\ &\le \left|f'(\varphi(z))\right|\left(1-|\varphi(z)|^2\right)\frac{\left|\varphi'(z)\right|\left(1-|z|^2\right)}{1-|\varphi(z)|^2}\\ &\le \left|f'(\varphi(z))\right|\left(1-|\varphi(z)|^2\right),\end{aligned} \tag{45}$$

which implies that $|\varphi(z)| < r$. Thus,

$$\begin{aligned}\epsilon &\le \left|f'(\varphi(z))\right|\left|\varphi'(z)\right|\left(1-|z|^2\right)\\ &\le \|f\|_{\mathscr{B}}\frac{\left|\varphi'(z)\right|\left(1-|z|^2\right)}{1-|\varphi(z)|^2}\\ &\le \frac{\|f\|_{\mathscr{B}}}{1-r^2}\left|\varphi'(z)\right|\left(1-|z|^2\right).\end{aligned} \tag{46}$$

Let $\delta = \epsilon(1-r^2)/\|f\|_{\mathscr{B}}$. Then $|\varphi'(z)|(1-|z|^2) \ge \delta$. Hence, $\Omega_\epsilon(f \circ \varphi) \subseteq \Omega_\delta(\varphi)$. Since $\varphi \in \mathscr{C}_{\mathscr{B}}(\mathscr{L}^{2,\lambda} \cap \mathscr{B})$, by Theorem 4 we get

$$\begin{aligned}&\sup_{a\in\mathbb{D}}\int_{\Omega_\epsilon(f\circ\varphi)}\left(\frac{1-|\sigma_a(z)|^2}{1-|z|^2}\right)^\lambda\frac{dA(z)}{1-|z|^2}\\ &\quad\le \sup_{a\in\mathbb{D}}\int_{\Omega_\delta(\varphi)}\left(\frac{1-|\sigma_a(z)|^2}{1-|z|^2}\right)^\lambda\frac{dA(z)}{1-|z|^2} < \infty.\end{aligned} \tag{47}$$

By Theorem 4 again, we see that $f \circ \varphi \in \mathscr{C}_{\mathscr{B}}(\mathscr{L}^{2,\lambda} \cap \mathscr{B})$. Hence $C_\varphi : \mathscr{B}_0 \longrightarrow \mathscr{C}_{\mathscr{B}}(\mathscr{L}^{2,\lambda} \cap \mathscr{B})$ is bounded. The proof is complete. □

Theorem 7. *Let $0 < \lambda < 1$ and let φ be an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(i) $C_\varphi : \mathscr{B} \longrightarrow \mathscr{C}_{\mathscr{B}}(\mathscr{L}^{2,\lambda} \cap \mathscr{B})$ *is compact;*

(ii) $C_\varphi : \mathscr{B}_0 \longrightarrow \mathscr{C}_{\mathscr{B}}(\mathscr{L}^{2,\lambda} \cap \mathscr{B})$ *is compact;*

(iii)

$$\lim_{|\varphi(z)|\longrightarrow 1}\frac{\left|\varphi'(z)\right|\left(1-|z|^2\right)}{1-|\varphi(z)|^2} = 0. \tag{48}$$

Proof. $(i) \Longrightarrow (ii)$. It is clear.

$(ii) \Longrightarrow (iii)$. Since $\mathscr{C}_{\mathscr{B}}(\mathscr{L}^{2,\lambda} \cap \mathscr{B}) \subseteq \mathscr{B}$, we see that $C_\varphi : \mathscr{B}_0 \longrightarrow \mathscr{B}$ is compact. Using [9, Theorem 1], (48) follows.

$(iii) \Longrightarrow (i)$. By the assumption, we see that there exists r $(0 < r < 1)$, such that

$$\left|\varphi^\sharp(w)\right| < \frac{\epsilon}{2}, \quad \text{when } \left|\varphi(w)\right| \ge r. \tag{49}$$

Let $z \in \mathbb{D}$ such that $|\varphi^\sharp(z)| \ge \epsilon$. Then, $|\varphi(z)| < r$. Therefore,

$$\epsilon \le \left|\varphi^\sharp(z)\right| \le \frac{\left|\varphi'(z)\right|\left(1-|z|^2\right)}{1-r^2}, \tag{50}$$

which implies that

$$\epsilon\left(1-r^2\right) \le \left|\varphi'(z)\right|\left(1-|z|^2\right). \tag{51}$$

Let $\delta = \epsilon(1-r^2)$. Then $z \in \Omega_\delta(\varphi)$. Hence

$$\begin{aligned}&\sup_{a\in\mathbb{D}}\int_{|\varphi^\sharp(z)|\ge\epsilon}\left(\frac{1-|\sigma_a(z)|^2}{1-|z|^2}\right)^\lambda\frac{dA(z)}{1-|z|^2}\\ &\quad\le \sup_{a\in\mathbb{D}}\int_{\Omega_\delta(\varphi)}\left(\frac{1-|\sigma_a(z)|^2}{1-|z|^2}\right)^\lambda\frac{dA(z)}{1-|z|^2}.\end{aligned} \tag{52}$$

Since $\varphi \in \mathscr{C}_{\mathscr{B}}(\mathscr{L}^{2,\lambda} \cap \mathscr{B})$, by Theorem 4 we have

$$\sup_{a\in\mathbb{D}}\int_{\Omega_\delta(\varphi)}\left(\frac{1-|\sigma_a(z)|^2}{1-|z|^2}\right)^\lambda\frac{dA(z)}{1-|z|^2} < \infty. \tag{53}$$

By (52), (53), and Theorem 5, $C_\varphi : \mathscr{B} \longrightarrow \mathscr{C}_{\mathscr{B}}(\mathscr{L}^{2,\lambda} \cap \mathscr{B})$ is bounded.

Since (48) holds, from [7, Theorem 2], we see that $C_\varphi : \mathscr{B} \longrightarrow \mathscr{B}$ is compact. Therefore $C_\varphi : \mathscr{B} \longrightarrow \mathscr{C}_{\mathscr{B}}(\mathscr{L}^{2,\lambda} \cap \mathscr{B})$ is compact. The proof is complete. □

Theorem 8. *Let $0 < \lambda < 1$ and let φ be an analytic self-map of $\mathbb{D}$. Then $C_\varphi : \mathscr{C}_{\mathscr{B}}(\mathscr{L}^{2,\lambda} \cap \mathscr{B}) \longrightarrow \mathscr{C}_{\mathscr{B}}(\mathscr{L}^{2,\lambda} \cap \mathscr{B})$ is compact if and only if*

$$\lim_{|\varphi(z)|\longrightarrow 1}\frac{\left|\varphi'(z)\right|\left(1-|z|^2\right)}{1-|\varphi(z)|^2} = 0. \tag{54}$$

Proof. Suppose that (54) holds. By Theorem 7, $C_\varphi : \mathscr{B} \longrightarrow \mathscr{C}_{\mathscr{B}}(\mathscr{L}^{2,\lambda} \cap \mathscr{B})$ is compact. Since $\mathscr{C}_{\mathscr{B}}(\mathscr{L}^{2,\lambda} \cap \mathscr{B}) \subseteq \mathscr{B}$, we get that $C_\varphi : \mathscr{C}_{\mathscr{B}}(\mathscr{L}^{2,\lambda} \cap \mathscr{B}) \longrightarrow \mathscr{C}_{\mathscr{B}}(\mathscr{L}^{2,\lambda} \cap \mathscr{B})$ is compact, as desired.

Conversely, assume that $C_\varphi : \mathscr{C}_{\mathscr{B}}(\mathscr{L}^{2,\lambda} \cap \mathscr{B}) \longrightarrow \mathscr{C}_{\mathscr{B}}(\mathscr{L}^{2,\lambda} \cap \mathscr{B})$ is compact. It is clear that $\varphi \in \mathscr{C}_{\mathscr{B}}(\mathscr{L}^{2,\lambda} \cap \mathscr{B})$ since $z \in \mathscr{C}_{\mathscr{B}}(\mathscr{L}^{2,\lambda} \cap \mathscr{B})$. Since $\mathscr{B}_0$ is closure of all polynomials in $\mathscr{B}$ and the space $\mathscr{L}^{2,\lambda}$ contains all polynomials, hence, $C_\varphi : \mathscr{B}_0 \longrightarrow \mathscr{C}_{\mathscr{B}}(\mathscr{L}^{2,\lambda} \cap \mathscr{B})$ is compact. By Theorem 7 we see that (54) holds. The proof is complete. □

From Theorems 7 and 8 and [13, Theorem 3], we immediately get the following corollary.

Corollary 9. *Let $0 < \lambda < 1$ and let φ be an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(i) $C_\varphi : \mathscr{B} \longrightarrow \mathscr{C}_{\mathscr{B}}(\mathscr{L}^{2,\lambda} \cap \mathscr{B})$ *is compact;*

(ii) $C_\varphi : \mathscr{B}_0 \longrightarrow \mathscr{C}_{\mathscr{B}}(\mathscr{L}^{2,\lambda} \cap \mathscr{B})$ *is compact;*

(iii) $C_\varphi : \mathscr{B} \longrightarrow \mathscr{B}$ *is compact;*

(iv) $C_\varphi : \mathscr{B}_0 \longrightarrow \mathscr{B}$ *is compact;*

(v) $C_\varphi : \mathscr{C}_{\mathscr{B}}(\mathscr{L}^{2,\lambda} \cap \mathscr{B}) \longrightarrow \mathscr{C}_{\mathscr{B}}(\mathscr{L}^{2,\lambda} \cap \mathscr{B})$ *is compact;*

(vi) $\lim_{n\longrightarrow\infty}\|\varphi^n\|_{\mathscr{B}} = 0$;

(vii) $\lim_{|\varphi(z)|\longrightarrow 1}(|\varphi'(z)|(1-|z|^2)/(1-|\varphi(z)|^2)) = 0$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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