Hindawi
Journal of Function Spaces
Volume 2019, Article ID 4042965, 13 pages
https://doi.org/10.1155/2019/4042965

Research Article

On New Picard-Mann Iterative Approximations with Mixed Errors for Implicit Midpoint Rule and Applications

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Received 9 December 2018; Revised 25 January 2019; Accepted 5 February 2019; Published 3 March 2019

Guest Editor: Vita Leonessa

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In order to solve (partial) differential equations, implicit midpoint rules are often employed as a powerful numerical method. The purpose of this paper is to introduce and study a class of new Picard-Mann iteration processes with mixed errors for the implicit midpoint rules, which is different from existing methods in the literature, and to analyze the convergence and stability of the proposed method. Further, some numerical examples and applications to optimal control problems with elliptic boundary value constraints are considered via the new Picard-Mann iterative approximations, which shows that the new Picard-Mann iteration process with mixed errors for the implicit midpoint rule of nonexpansive mappings is brand new and more effective than other related iterative processes.

1. Introduction

In science and engineering fields, such as Stefan-Boltzmann radiation law, Lotka-Voterra model in population dynamics, etc., more and more problems can be modeled by optimal control problems with the constraints of (partial) differential equations. Moreover, multidimensional dynamical systems can be frequently formulated by partial differential equations, which generally depend on space and time, i.e., parabolic or evolutionary type equations, and are treated with emphasis on various real-world applications in (thermo) mechanics of solids and fluids, electrical devices, engineering, chemistry, biology, etc.

Thus, it is very significative and considerable to study existence of solutions for optimal control problems constrained by (partial) differential equations. Some algorithms were provided for the following optimal control problem in [1]:

$$\min J(u), \tag{1}$$

where control variable $u \in U = L^2(\partial\Omega)$, control space, satisfies some suitable conditions or constraints; $\Omega \subset \mathbb{R}^2$ is a bounded convex region. For example, Liu and Sun [2] considered the optimal boundary control problem (1) of the following elliptic equation constraints:

$$\begin{aligned} -\Delta y &= f(x,u), \quad x \in \Omega, \\ u(x) &= 0, \quad x \in \partial\Omega, \end{aligned} \tag{2}$$

where $y(u) \in V = H^1(\Omega)$, state space, is state variable, boundary $\partial\Omega = \Gamma_N \cup \Gamma_D$ is smooth, Γ_N and Γ_D are, respectively, Neumann boundary and Dirichlet boundary, and $\Gamma_N \cap \Gamma_D = \emptyset$. Yan [3] explored the adaptive finite element methods for some optimal control problems governed by (2). However, when the adaptive finite methods are used to solve this kind of problems, the calculation will be very large if the calculation area of the problem is large. To overcome this difficulty, Yan [4] introduced iterative nonoverlapping domain decomposition method for the optimal boundary control problem (1) governed by the elliptic equations (2), which can avoid large amounts of calculation produced by traditional numerical methods. Many scholars devoted themselves to this kind of optimal control problem (1) with elliptic PDE (partial differential equation) constraints (2). For more details, we refer to [5] and references therein. Further, Pearson and

Wathen [6] presented a new Schur complement approximation for PDE-constrained optimization and designed preconditioners under some certain optimality properties, derived eigenvalue bounds to verify the effectiveness of the approximation, and presented numerical results that show that these new preconditioners work well in practice. Zeng et al. [7] developed some preconditioning techniques for reduced saddle point systems arising from linear elliptic distributed optimal control problems and obtained the bounds of these eigenvalues with respect to the mesh size. These methods are mainly based on the time required for solution scales linearly with the problem size, and the mesh size in these optimal methods is also hard to choose. Very recently, Xu and Zhang [8] explored the positive solutions for singular positone and semipositone boundary value problems by use of the Leray-Schauder nonlinear alternative and a fixed point theorem on cones.

By using the u_0-positive operator and the fixed point index theorem, Yao et al. [9] investigated the existence and uniqueness of positive solutions of the following boundary value problem:

$$\begin{aligned} -D_{0^+}^{\alpha}x(t)+bx(t)&=a(t)f(t,x(t)), \quad 0<t<1,\\ x(0)&=0, \\ x(1)&=0, \end{aligned} \tag{3}$$

where $D_{0^+}^{\alpha}$ is the Riemann-Liouville fractional derivative, $1<\alpha<2$, $b>0$, $f:[0,1]\times[0,\infty)\longrightarrow[0,\infty)$ is continuous, and $a(t)$ is continuous and may be singular at $t=0$ and $t=1$.

In [10], Lan pointed out that "the time-dependent form of partial differential equations can be rewritten as a time-independent form under some suitable conditions". This Neumann elliptic boundary value problem of (3) has attracted widely attention to the researchers because of its existence in many fields. Ashyralyyev and Dedeturk [11] considered the inverse problem for a multidimensional elliptic equation with mixed boundary conditions and overdetermination. In this paper they constructed the first and second orders of accuracy in t and the second order of accuracy in space variables and analyzed the stability, almost coercive stability, and coercive stability for the approximate solution of this inverse problem. Furthermore, the solvability of perturbed optimal control problems, the uniqueness of their solutions, the asymptotic properties of optimal pairs as the perturbation parameter $\varepsilon>0$ tends to zero, and deriving of optimality conditions for the perturbed optimal control problems were discussed in [12]. The sufficient conditions of the existence of weak solutions to the given class of nonlinear Neumann boundary value problems were established and a way for their approximation was also proposed. Lately, Kashiwabara and Kemmochi [13] established $O(h^2|\log h|)$ and $O(h)$ errors bounds in the L^{∞} and $W^{1,\infty}$-norms for the Neumann boundary value problems in a smooth space by combining the technique of regularized Green's function with local H^1- and L^2-estimates in dyadic annuli. And elliptic variational forms of second-order physician, physicist, and anatomist equation can also be represented by some special cases of the Dirichlet problem (2) (see [14]). As is known to all, many problems in physics and other applications cannot be formulated as equations because of some more complicated structure, usually of a so-called complementarity problem, which is equivalent to a variational inequality. And the applicability of variational inequality theory which was initially developed to cope with equilibrium problems has been extended to involve problems from engineering science, electrodynamics, optimization, economics, finance, mechanics, and game theory. So the variational method is very important in optimal control theory, and such generalization is often needed in optimal control theory of elliptic problems.

On the other hand, iteration methods of Picard, Mann, Ishikawa, and the other associated iterations are the research focus to solve optimal control problem for the constraint of partial differential equation. These iterative processes have been deeply studied and applied by many authors. Such as, Khan [15] introduced a Picard-Mann hybrid iterative process to solve equation systems which converges faster than all of Picard, Mann, and Ishikawa iterative processes for contractions. Based on this, Deng [16] introduced a modified Picard-Mann hybrid iterative algorithm for a sequence of nonexpansive mappings. He also established strong convergence and weak convergence of the iterative sequence generated by the modified hybrid iterative algorithm in a convex Banach space. After that, Picard-Krasnoselskii hybrid iterations which converge faster than Picard, Mann, Krasnoselskii, and Ishikawa iterative processes for contractive nonlinear operators were introduced by Okeke and Abbas [17]. What is more, Jiang et al. [18] studied convergence of Mann iterative sequences for approximating solutions of a higher order nonlinear neutral delay differential equation and proposed advantages of the presented results through three extraordinary examples. Very recently, Li and Lan [19] studied the convergence and stability of a class of new Picard-Mann iterative methods with mixed errors for common fixed points of two different nonexpansive and contraction operators in a normed space X as follows:

$$\begin{aligned} x_{n+1}&=T_1y_n+h_n,\\ y_n&=(1-\alpha_n)x_n+\alpha_nT_2x_n+\alpha_nd_n+e_n, \end{aligned} \tag{4}$$

where $T_1,T_2:X\longrightarrow X$ are two nonlinear operators and $h_n,d_n,e_n\in X$ are errors to take into account a possible inexact computation of the operator points. Further, we explored iterative approximation of solutions for an elliptic boundary value problem in Hilbert spaces by using the new Picard-Mann iterative methods with mixed errors. After the research of [15], Khan [20] approximated common attractive points of further generalized hybrid mappings by using iterative process without closedness assumption to the case of two mappings in Hilbert spaces. Levajković [21] presented an approximation framework for computing the solution of the stochastic linear quadratic control problem on Hilbert spaces. These brought generalizations and improvements of some results in the literature. The more expatiation of Picard, Mann, and Ishikawa process and other iterative approximation methods can be obtained by referring to [22–27] and references therein.

Moreover, as one of the powerful numerical methods for solving ordinary differential equations and differential algebraic equations (see, for example, [28–30] and the references therein), the implicit midpoint rule has been considered as the approximation methods. Luo et al. [28] introduced a viscosity iterative algorithm for the implicit midpoint rule in uniformly smooth spaces and proved some strong convergence theorems under some appropriate conditions on the parameters. Xu et al. [31] established the viscosity technique for the implicit midpoint rule in Hilbert spaces and proved the strong convergence of this technique under certain assumptions. Alghamdi [32] established implicit midpoint rule for nonexpansive mappings and analyzed the weak convergence of the algorithm in Hilbert space. As applications, the authors applied the main results in solving fixed point problems of strict pseudocontractive mappings, variational inequality problems in Banach spaces, and equilibrium problems in Hilbert spaces. Very recently, by using the generalized forward-backward splitting method and implicit midpoint rule, Chang et al. [33] introduced and proved some strong convergence of an iterative algorithm for finding a common element of solutions to quasi variational inclusions with accretive mapping and fixed points for a λ-strict pseudocontractive mapping in Banach spaces. In [34], Tang and Bao introduced a new semi-implicit midpoint rule with the general contraction for monotone mappings in Banach spaces, which converges strongly to a fixed point. To find the fixed point of nonexpansive mapping, using the implicit midpoint rule, Yao et al. [35] established an iteration algorithm, which is formed as

$$y_{n+1} = \left(1-\alpha_n\right) y_n + \alpha_n \left(\frac{T_n y_n + T_{n+1} y_{n+1}}{2}\right), \quad n \geq 0, \quad (5)$$

where $\alpha_n \in (0,1)$ for all $n \geq 0$, $T : C \longrightarrow C$ is nonexpansive operator with a nonempty closed convex bounded subset C of X. The authors proved the weak convergence of this iteration algorithm and proposed three control conditions. Moreover, as applications, the algorithm was applied in hierarchical minimization problem

$$\min_{x \in S_0} \psi_1 (x), \quad (6)$$

where $S_0 := \arg\min_{x\in H} \psi_0(x)$ and $\psi_0(x)$, $\psi_1(x)$ are two lower semicontinuous convex functions from H into $\mathbb{R}$ and their gradients satisfy the Lipschitz continuity conditions. An iteration algorithm based on (5) was presented to (6). The algorithm (5) was also explored in nonlinear time-dependent evolution equation, Fredholm integral equations, and variational inequalities and showed effectiveness in solving these problems. The technic of implicit midpoint rule and iteration algorithm are so powerful in equations that it deserves further studies.

Furthermore, Roussel [36] pointed out that "equilibria are not always stable". Being able to identify equilibrium points based on their stability is useful since stable and unstable equilibria play quite different roles in dynamics. And there are many authors and researchers who discussed stability of the iterative sequence generated by the algorithm for solving the investigated problems. See, for example, [11, 37] and the references therein. Recently, convergence and stability theorems for the Picard-Mann iterative scheme of iteration are attracting more and more attention in researches. Akewe and Okeke [38], perhaps for the first time, gave the stability theorems for the Picard-Mann hybrid iterative scheme for a general class of contractive-like operators. After that, Li and Lan [19] used different methods to analyze the stability and extended the application of the stability in iterations. But the convergence and stability theorems for the Picard-Mann iteration are often based on the iterative scheme of iteration, so the analysis of the convergence and stability is necessary when we apply the implicit midpoint rule in Picard-Mann iteration progress.

On the basis of the above studies, in this paper, it is different from existing methods in the literature that we will introduce and study a class of new Picard-Mann iteration processes with mixed errors for the implicit midpoint rules. Then, we analyze convergence and stability of the new Picard-Mann iterative approximations of the implicit midpoint rules for nonexpansive mappings in normed spaces. Finally, we give some numerical examples and applications to optimal control problems with elliptic boundary value constraints based on the new Picard-Mann iterative approximations. Finally, simulation results are provided to illustrate the effectiveness of the proposed methods.

2. New Picard-Mann Iterative Approximations

In this section, we shall introduce and study a new Picard-Mann iterative methods with mixed errors for the implicit midpoint rule of common fixed points of two different contraction and nonexpansive operators and prove convergence and stability of the new Picard-Mann iterative approximation.

We need the following definitions and lemmas for our main results.

Definition 1. Let K be a nonempty subset of a normed space X. Then a mapping $T : K \longrightarrow K$ is said to be

(i) nonexpansive if

$$\|Tu - Tv\| \leq \|u - v\|, \quad \forall u, v \in K; \quad (7)$$

(ii) contraction if

$$\|Tu - Tv\| \leq \theta \|u - v\|, \quad \forall u, v \in K, \ \theta \in [0,1). \quad (8)$$

Here, we recall the constant θ as Lipschitz constant of T. Thus, contractive mapping is sometimes said to be Lipschitzian mapping. Further, we call the mapping T as nonexpansive when this condition hold instead for $\theta \leq 1$.

Definition 2 (see [19]). Let S be a selfmap of normed space X, $x_0 \in X$, and let $x_{n+1} = h(S, x_n)$ define an iteration procedure which yields a sequence of points $\{x_n\} \subset X$. Suppose that $\{x \in X : Sx = x\} \neq \emptyset$ and $\{x_n\}$ converges to a fixed point x^* of S. Let $\{\omega_n\} \subset X$ and let $\varepsilon_n = \|\omega_{n+1} - h(S, w_n)\|$. If $\varepsilon_n = 0$ implies that $w_n \longrightarrow x^*$, then the iteration procedure defined by $x_{n+1} = h(S, x_n)$ is said to be S-stable or stable with respect to S.

Lemma 3 (see [39]). *Let X be a normed space and C a nonempty closed convex bounded subset of X. Then each nonexpansive mapping $T : C \longrightarrow C$ has a fixed point in C.*

Lemma 4 (see [40]). *Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be three nonnegative real sequences satisfying*

$$a_{n+1} \leq (1 - t_n) a_n + t_n b_n + c_n, \tag{9}$$

where $t_n \in [0, 1]$, $\sum_{n=0}^{\infty} t_n = \infty$, $\min_{n\longrightarrow\infty} b_n = 0$, $\sum_{n=0}^{\infty} c_n < \infty$. *Then* $a_n \longrightarrow 0$ $(n \longrightarrow \infty)$.

Based on new Picard-Mann iterative methods with mixed errors for common fixed point of two different nonexpansive and contraction operators due to Li and Lan [19], now we further consider the following new iteration process x_n (in short, (PMMDI)) with mixed errors for the implicit midpoint rule of two nonlinear mappings T_1 and T_2 in normed space X:

$$\begin{aligned} x_{n+1} &= T_1\left(\frac{x_n + y_n}{2}\right) + h_n, \\ y_n &= (1 - \alpha_n) x_n + \alpha_n T_2\left(\frac{x_n + y_n}{2}\right) + \alpha_n d_n + e_n. \end{aligned} \tag{10}$$

Further, the Picard-Mann iteration process with mixed errors for the implicit midpoint rule of one of nonlinear mappings T_1 and T_2 is respectively defined as follows:

$$\begin{aligned} x_{n+1} &= T_1\left(\frac{x_n + y_n}{2}\right) + h_n, \\ y_n &= (1 - \alpha_n) x_n + \alpha_n T_2 x_n + \alpha_n d_n + e_n, \end{aligned} \tag{11}$$

which is called a new Picard-Mann iteration process with mixed errors for the implicit midpoint rule of Picard mapping (in short, (PMMDIP)) and that of Mann mapping (in short, (PMMDIM)) is

$$\begin{aligned} x_{n+1} &= T_1 y_n + h_n, \\ y_n &= (1 - \alpha_n) x_n + \alpha_n T_2\left(\frac{x_n + y_n}{2}\right) + \alpha_n d_n + e_n, \end{aligned} \tag{12}$$

where $h_n, d_n, e_n \in X$ are errors to take into account a possible inexact computation of the mapping points which satisfy the following conditions **EC**:

(i) $d_n = d'_n + d''_n$;

(ii) $\lim_{n\longrightarrow\infty} \|d'_n\| = \lim_{n\longrightarrow\infty} \|h_n\| = 0$;

(iii) $\sum_{n=0}^{\infty} \|d''_n\| < \infty$, $\sum_{n=0}^{\infty} \|e_n\| < \infty$.

Remark 5. For special choices of the operators T_1 and T_2, the space X, and errors h_n, d_n, and e_n in (10)-(12), one can obtain a large number of Picard iterative process, Mann iterative process, Picard-Mann iterative process, and other related iterations for the implicit midpoint rule. Now we list some special cases as follows.

Special Case I. If $h_n = d_n = e_n = 0$, the iterative process (10) becomes to the Picard-Mann iteration of the implicit midpoint rule in for two different operators (in short, (PMDI)). For any given $x_0 \in X$,

$$\begin{aligned} x_{n+1} &= T_1\left(\frac{x_n + y_n}{2}\right), \\ y_n &= (1 - \alpha_n) x_n + \alpha_n T_2\left(\frac{x_n + y_n}{2}\right). \end{aligned} \tag{13}$$

The iterative process (11) reduces to the following implicit midpoint rule in Picard-Mann iterations for Picard type mapping (in short, (PMDIP)):

$$\begin{aligned} x_{n+1} &= T_1\left(\frac{x_n + y_n}{2}\right), \\ y_n &= (1 - \alpha_n) x_n + \alpha_n T_2 x_n, \end{aligned} \tag{14}$$

and the iterative process (12) becomes as the following implicit midpoint rule in Picard-Mann iterations for Mann type mapping (in short, (PMDIM)):

$$\begin{aligned} x_{n+1} &= T_1 y_n, \\ y_n &= (1 - \alpha_n) x_n + \alpha_n T_2\left(\frac{x_n + y_n}{2}\right). \end{aligned} \tag{15}$$

Special Case II. When $T_1 = T_2 = T$, the iterations (10), (11), and (12), respectively, reduce to the Picard-Mann iteration of implicit midpoint rule with mixed errors for one nonlinear mapping (in short, (PMMI))

$$\begin{aligned} x_{n+1} &= T\left(\frac{x_n + y_n}{2}\right) + h_n, \\ y_n &= (1 - \alpha_n) x_n + \alpha_n T\left(\frac{x_n + y_n}{2}\right) + \alpha_n d_n + e_n, \end{aligned} \tag{16}$$

the implicit midpoint rule in Picard-Mann iteration with mixed errors for Picard type mapping (in short, (PMMIP))

$$\begin{aligned} x_{n+1} &= T\left(\frac{x_n + y_n}{2}\right) + h_n, \\ y_n &= (1 - \alpha_n) x_n + \alpha_n T x_n + \alpha_n d_n + e_n, \end{aligned} \tag{17}$$

and

$$\begin{aligned} x_{n+1} &= T y_n + h_n, \\ y_n &= (1 - \alpha_n) x_n + \alpha_n T\left(\frac{x_n + y_n}{2}\right) + \alpha_n d_n + e_n, \end{aligned} \tag{18}$$

which is called the implicit midpoint rule in Picard-Mann iteration with mixed errors for Mann type mapping (in short, (PMMIM)).

Special Case III. If $T_1 = T_2 = T$, then (13), (14), and (15), respectively, become as the Picard-Mann iterative process of implicit midpoint rule for one nonlinear mapping (in short, (PMI))

$$\begin{aligned} x_{n+1} &= T\left(\frac{x_n + y_n}{2}\right), \\ y_n &= (1 - \alpha_n) x_n + \alpha_n T\left(\frac{x_n + y_n}{2}\right), \end{aligned} \tag{19}$$

that for Picard mapping (in short, (PMIP))

$$x_{n+1} = T\left(\frac{x_n + y_n}{2}\right), \quad y_n = (1-\alpha_n) x_n + \alpha_n T x_n, \tag{20}$$

and that for Mann mapping (in short, (PMIM))

$$x_{n+1} = T y_n, \quad y_n = (1-\alpha_n) x_n + \alpha_n T\left(\frac{x_n + y_n}{2}\right). \tag{21}$$

Special Case IV. When $T_2 = T$ and $T_1 = I$, the identity mapping, for any given $x_0 \in X$, the iterations (PMMDI) (10), (PMMDIP) (11), and (PMMDIM) (12) can be, respectively, written as Picard-Mann iterative process of the explicit and implicit midpoint rule with mixed errors (in short, (MMDI))

$$x_{n+1} = \frac{x_n + y_n}{2} + h_n, \quad y_n = (1-\alpha_n) x_n + \alpha_n T\left(\frac{x_n + y_n}{2}\right) + \alpha_n d_n + e_n, \tag{22}$$

the implicit midpoint rule for Picard iteration in Picard-Mann iterative process (in short, (MMDIP))

$$x_{n+1} = \frac{x_n + y_n}{2} + h_n, \quad y_n = (1-\alpha_n) x_n + \alpha_n T x_n + \alpha_n d_n + e_n, \tag{23}$$

and the implicit midpoint rule in Picard-Mann iteration process for Mann mapping (in short, (MMDIM)):

$$x_{n+1} = y_n + h_n, \quad y_n = (1-\alpha_n) x_n + \alpha_n T\left(\frac{x_n + y_n}{2}\right) + \alpha_n d_n + e_n. \tag{24}$$

Remark 6. We note that the iterations mentioned above are new and not studied in the literature yet.

Based on Lemma 3 and existence of fixed point for nonexpensive mapping, in the sequel, we will prove convergence and stability of the new Picard-Mann iterative process (PMMDI) (10).

Theorem 7. *Let X be a normed space and $C \subset X$ be a nonempty closed convex bounded set. Let $T_1 : C \longrightarrow C$ be nonexpansive and $T_2 : C \longrightarrow C$ be a contraction mapping with constant $\theta \in [0,1)$. Suppose that $F(T_1 \cap T_2) := \{x \in C : T_i x = x, i = 1,2\} \neq \emptyset$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then*

(i) the iterative sequence $\{x_n\}$ generated by (10) converges to $x^* \in F(T_1 \cap T_2)$ with convergence rate

$$\varrho = 1 - \frac{1-\theta}{2-\theta\widehat{\alpha}}\widehat{\alpha} < 1, \tag{25}$$

where $\widehat{\alpha} = \limsup_{n\longrightarrow\infty} \alpha_n \in (0,1]$;

(ii) if, moreover, for any sequence $\{s_n\} \subset X$, there exists an $\alpha > 0$ such that $\alpha_n \geq \alpha$, then

$$\lim_{n\longrightarrow\infty} s_n = x^* \quad \text{if and only if} \quad \lim_{n\longrightarrow\infty} \delta_n = 0, \tag{26}$$

where $\{\delta_n\}$ is defined by

$$\delta_n = \left\| s_{n+1} - \left[T_1\left(\frac{s_n + \eta_n}{2}\right) + h_n \right] \right\|, \quad \eta_n = (1-\alpha_n) s_n + \alpha_n T_2\left(\frac{s_n + \eta_n}{2}\right) + \alpha_n d_n + e_n. \tag{27}$$

Proof. From the proof of [19, Theorem 2.1], iteration process of (10), and the conditions **EC**, it follows that

$$\begin{aligned}
\|x_{n+1} - x^*\| &\leq \left\| \frac{x_n + y_n}{2} - x^* \right\| + \|h_n\| \leq \frac{1}{2} \Big\| (x_n - x^*) \\
&\quad + (1-\alpha_n)(x_n - x^*) + \alpha_n \left[T_2\left(\frac{x_n + y_n}{2}\right) - x^* \right] \Big\| \\
&\quad + \frac{1}{2} \|\alpha_n d_n + e_n\| + \|h_n\| \leq \frac{1}{2} \|(x_n - x^*) \\
&\quad + (1-\alpha_n)(x_n - x^*)\| + \frac{\theta\alpha_n}{2} \left\| \left(\frac{x_n + y_n}{2}\right) - x^* \right\| \\
&\quad + \frac{1}{2} \|\alpha_n d_n + e_n\| + \|h_n\| \leq \frac{2-\alpha_n}{2} \|x_n - x^*\| + \frac{\theta\alpha_n}{2} \\
&\quad \cdot \frac{2-\theta}{2-\theta\alpha_n} \|x_n - x^*\| + \frac{\theta\alpha_n}{2} \cdot \frac{1}{2-\theta\alpha_n} (\|\alpha_n d_n \\
&\quad + \|e_n\|) + \frac{1}{2} (\|\alpha_n d_n\| + \|e_n\|) + \|h_n\| \leq \Big(1 \\
&\quad - \frac{1-\theta}{2-\theta\alpha_n}\alpha_n \Big) \|x_n - x^*\| + \frac{1-\theta}{2-\theta\alpha_n}\alpha_n \cdot \frac{1}{1-\theta} \|d_n'\| \\
&\quad + (\|d_n''\| + \|e_n\| + \|h_n\|)
\end{aligned} \tag{28}$$

Since $\sum_{n=0}^{\infty} \alpha_n = \infty$, by Lemma 4 and (28), now we know that $\|x_n - x^*\| \longrightarrow 0$ as $n \longrightarrow \infty$. That is, the sequence $\{x_n\}$ converges to x^*.

Next, we prove stability of the new method of the implicit midpoint rule for Picard-Mann iteration with mixed errors. In fact, since $0 < \alpha \leq \alpha_n$, it follows from (27) and (10) that

$$\begin{aligned}
&\left\| T_1\left(\frac{s_n + \eta_n}{2}\right) + h_n - x^* \right\| \leq \left\| \frac{s_n + \eta_n}{2} - x^* \right\| + \|h_n\| \\
&\quad \leq \frac{1}{2} \Big\| (s_n - x^*) + (1-\alpha_n)(s_n - x^*) \\
&\quad + \alpha_n \left[T_2\left(\frac{s_n + \eta_n}{2}\right) - x^* \right] \Big\| + \frac{1}{2} \|\alpha_n d_n + e_n\| \\
&\quad + \|h_n\| \leq \frac{2-\alpha_n}{2} \|s_n - x^*\| + \frac{\theta\alpha_n}{2} \left\| \frac{s_n + \eta_n}{2} - x^* \right\| \\
&\quad + \frac{1}{2} (\|\alpha_n d_n\| + \|e_n\|) + \|h_n\| \leq \frac{2-\alpha_n}{2} \|s_n - x^*\| \\
&\quad + \frac{\theta\alpha_n}{2} \cdot \frac{2-\theta}{2-\theta\alpha_n} \|s_n - x^*\| + \frac{\theta\alpha_n}{2}
\end{aligned}$$

$$\cdot \frac{1}{2-\theta\alpha_n}\left(\left\|\alpha_n d_n\right. + \left\|e_n\right\|\right) + \frac{1}{2}\left(\left\|\alpha_n d_n\right\| + \left\|e_n\right\|\right) + \left\|h_n\right\| \leq \left(1 - \frac{1-\theta}{2-\theta\alpha_n}\alpha_n\right)\left\|s_n - x^*\right\| + \frac{1-\theta}{2-\theta\alpha_n}\alpha_n \cdot \frac{1}{1-\theta}\left\|d_n'\right\| + \left(\left\|d_n''\right\| + \left\|e_n\right\| + \left\|h_n\right\|\right). \tag{29}$$

Thus, by (27), we have

$$\begin{aligned}\delta_n &= \left\|\left(s_{n+1} - x^*\right) - \left[T_1\left(\frac{s_n + \eta_n}{2}\right) + h_n - x^*\right]\right\| \\ &\geq \left\|s_{n+1} - x^*\right\| - \left\|T_1\left(\frac{s_n + \eta_n}{2}\right) + h_n - x^*\right\|\end{aligned} \tag{30}$$

and so it follows from (29) that

$$\begin{aligned}\left\|s_{n+1} - x^*\right\| &\leq \left\|T_1\left(\frac{s_n + \eta_n}{2}\right) + h_n - x^*\right\| + \delta_n \\ &\leq \left(1 - \frac{1-\theta}{2-\theta\alpha_n}\alpha_n\right)\left\|s_n - x^*\right\| \\ &\quad + \frac{1-\theta}{2-\theta\alpha_n}\alpha_n \cdot \frac{1}{1-\theta}\left\|d_n'\right\| \\ &\quad + \left(\left\|d_n''\right\| + \left\|e_n\right\| + \left\|h_n\right\|\right) + \delta_n.\end{aligned} \tag{31}$$

Let $\lim_{n\longrightarrow\infty}\delta_n = 0$. Then by $\sum_{n=0}^{\infty}\alpha_n = \infty$, Lemma 4 and (31), we know that $\lim_{n\longrightarrow\infty} s_n = x^*$.

On the contrary, if $\lim_{n\longrightarrow\infty} s_n = x^*$, then it follows from (27), (29), $\alpha_n \leq 1$ for all $n \geq 0$, and Lemma 4 that

$$\begin{aligned}\delta_n &= \left\|s_{n+1} - \left[T_1\left(\frac{s_n + \eta_n}{2}\right) + h_n\right]\right\| \\ &\leq \left\|s_{n+1} - x^*\right\| + \left\|T_1\left(\frac{s_n + \eta_n}{2}\right) + h_n - x^*\right\| \longrightarrow 0\end{aligned} \tag{32}$$

as $n \longrightarrow \infty$. This completes the proof. □

From Theorem 7, similarly we can obtain the following results.

Corollary 8. *Assume that X, C, and T_1 and T_2 are the same as in Theorem 7. If $F(T_1 \cap T_2) \neq \emptyset$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$, then*

(i) the iterative sequence $\{x_n\}$ generated by (13) converges to $x^* \in F(T_1 \cap T_2)$ with convergence rate

$$\varrho = 1 - \frac{1-\theta}{2-\theta\widehat{\alpha}}\widehat{\alpha} < 1, \tag{33}$$

where $\widehat{\alpha} = \limsup_{n\longrightarrow\infty}\alpha_n \in (0,1]$;

(ii) further, if for any sequence $\{u_n\} \subset X$, there exists an $\alpha > 0$ such that $\alpha_n \geq \alpha$ and

$$\lim_{n\longrightarrow\infty} u_n = x^* \quad \text{if and only if} \lim_{n\longrightarrow\infty}\aleph_n = 0, \tag{34}$$

where $\aleph_n$ is defined as follows

$$\begin{aligned}\aleph_n &= \left\|u_{n+1} - T_1\left(\frac{u_n + v_n}{2}\right)\right\|, \\ v_n &= \left(1 - \alpha_n\right)u_n + \alpha_n T_2\left(\frac{u_n + v_n}{2}\right).\end{aligned} \tag{35}$$

Corollary 9. *Suppose that X and C are the same as in Theorem 7. Let $T : C \longrightarrow C$ be a contraction mapping with constant $\theta \in [0,1)$ with $F(T) \coloneqq \{x \in C : Tx = x\} =\neq \emptyset$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$. Then*

(i) the sequence $\{x_n\}$ generated by (16) converges to $x^* \in F(T)$ with convergence rate

$$\kappa = \left(1 - \frac{1-\theta}{2-\theta\widehat{\alpha}}\widehat{\alpha}\right)\theta < 1, \tag{36}$$

where $\widehat{\alpha} = \limsup_{n\longrightarrow\infty}\alpha_n \in (0,1]$;

(ii) moreover, if for any sequence $\{w_n\} \subset X$, there exists an $\alpha > 0$ such that $\alpha_n \geq \alpha$ and

$$\lim_{n\longrightarrow\infty} w_n = x^* \quad \text{if and only if} \lim_{n\longrightarrow\infty}\varsigma_n = 0, \tag{37}$$

where ς_n is defined by

$$\begin{aligned}\varsigma_n &= \left\|w_{n+1} - \left[T\left(\frac{w_n + j_n}{2}\right) + h_n\right]\right\|, \\ j_n &= \left(1 - \alpha_n\right)w_n + \alpha_n T\left(\frac{w_n + j_n}{2}\right) + \alpha_n d_n + e_n.\end{aligned} \tag{38}$$

Corollary 10. *Let T, X, and C be the same as in Corollary 9. Then*

(i) the iterative sequence $\{x_n\}$ generated by (22) converges to $x^* \in F(T)$ with convergence rate

$$\varrho = 1 - \frac{1-\theta}{2-\theta\widehat{\alpha}}\widehat{\alpha} < 1, \tag{39}$$

where $\widehat{\alpha} = \limsup_{n\longrightarrow\infty}\alpha_n \in (0,1]$;

(ii) for any sequence $\{r_n\} \subset X$, if there exists an $\alpha > 0$ such that $\alpha_n \geq \alpha$ and

$$\lim_{n\longrightarrow\infty} r_n = x^* \quad \text{if and only if} \lim_{n\longrightarrow\infty}\hbar_n = 0, \tag{40}$$

where

$$\begin{aligned}\hbar_n &= \left\|r_{n+1} - \left[\left(\frac{r_n + \varphi_n}{2}\right) + h_n\right]\right\|, \\ \varphi_n &= \left(1 - \alpha_n\right)r_n + \alpha_n T_2\left(\frac{r_n + \varphi_n}{2}\right) + \alpha_n d_n + e_n.\end{aligned} \tag{41}$$

Further, to the Picard-Mann iteration processes (11) and (12) with mixed errors for the implicit midpoint rule of one of nonlinear mappings T_1 and T_2, we, respectively, have the following results.

Theorem 11. *Let X be a normed space and $C \subset X$ be a nonempty closed convex bounded set. Let $T_1 : C \longrightarrow C$ be nonexpansive and $T_2 : C \longrightarrow C$ be a contraction mapping with constant $\theta \in [0,1)$. Suppose that $F(T_1 \cap T_2) \coloneqq \{x \in C : T_i x = x,\ i = 1,2\} \neq \emptyset$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$. Then*

(i) the iterative sequence $\{x_n\}$ generated by (PMMDIP) converges to $x^* \in F(T_1 \cap T_2)$ with convergence rate

$$\tau = 1 - \frac{1-\theta}{2}\widehat{\alpha} < 1, \tag{42}$$

where $\widehat{\alpha} = \limsup_{n\longrightarrow\infty}\alpha_n \in (0,1]$;

(ii) if, moreover, for any sequence $\{p_n\} \subset X$, there exists an $\alpha > 0$ such that $\alpha_n \geq \alpha$ and

$$\lim_{n\to\infty} p_n = x^* \quad \text{if and only if} \lim_{n\to\infty} \chi_n = 0, \tag{43}$$

where χ_n is defined as follows

$$\begin{aligned} \chi_n &= \left\| p_{n+1} - \left[T_1\left(\frac{p_n + q_n}{2}\right) + h_n \right] \right\|, \\ q_n &= (1-\alpha_n) p_n + \alpha_n T_2 p + \alpha_n d_n + e_n. \end{aligned} \tag{44}$$

Theorem 12. *Let X be a normed space and $C \subset X$ be a nonempty closed convex bounded set. Let $T_1 : C \longrightarrow C$ be nonexpansive and $T_2 : C \longrightarrow C$ be a contraction mapping with constant $\theta \in [0,1)$. Suppose that $F(T_1 \cap T_2) := \{x \in C : T_i x = x,\ i = 1,2\} \neq \emptyset$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then*

(i) the iterative sequence $\{x_n\}$ generated by (PMMDIM) (12) converges to $x^* \in F(T_1 \cap T_2)$ with convergence rate

$$\rho = 1 - \frac{2-2\theta}{2-\theta\widehat{\alpha}}\widehat{\alpha} < 1, \tag{45}$$

where $\widehat{\alpha} = \limsup_{n\to\infty} \alpha_n \in (0,1]$;

(ii) if for every sequence $\{o_n\} \subset X$, there exists an $\alpha > 0$ such that $\alpha_n \geq \alpha$ and

$$\lim_{n\to\infty} o_n = x^* \quad \text{if and only if} \lim_{n\to\infty} \mu_n = 0, \tag{46}$$

where

$$\begin{aligned} \mu_n &= \left\| o_{n+1} - (T_1 \sigma_n + h_n) \right\|, \\ \sigma_n &= (1-\alpha_n) o_n + \alpha_n T_2\left(\frac{o_n + \sigma_n}{2}\right) + \alpha_n d_n + e_n. \end{aligned} \tag{47}$$

3. Simulations and Applications

In this section, using the new Picard-Mann iterative methods with mixed errors for the implicit midpoint rule presented in the above section, we will give a numerical simulation for approximating the solution of the elliptic boundary value problem (2) and an application to an elliptic optimal control problem.

3.1. Numerical Example. In order to verify Theorem 7, we first give the following examples and their numerical simulations and to display effectiveness of the new Picard-Mann iterative methods.

Example 1. Let $X = R$, $1 < m \leq 39/16$, $C = [-1, 3 + \sqrt{23/(m-1)}]$, $T_1 x = (1/\pi)\sin(\pi x) + 5$, and $T_2 x = \sqrt{x^2 - 6x + 30}$ for all $x \in C$ and $h_n = 1/10^n$, $\alpha_n = 1/2$, $d_n = 1/n + 1/n^2$, and $e_n = -10/n^5$ for $n \geq 1$. It follows that

$$\left\| T_1 x - T_1 y \right\| \leq \frac{1}{\pi} \left\| \pi x - \pi y \right\| = \left\| x - y \right\|, \tag{48}$$

and

$$\begin{aligned} \left\| T_2 x - T_2 y \right\| &= \left\| \frac{(x-3)^2 - (y-3)^2}{\sqrt{(x-3)^2 + 21} + \sqrt{(y-3)^2 + 21}} \right\| \\ &= \left\| \frac{(x-y)\left[(x-3) + (y-3)\right]}{\left\|x-3\right\| + \left\|y-3\right\|} \right\| \\ &\quad \cdot \frac{\left\|x-3\right\| + \left\|y-3\right\|}{\sqrt{(x-3)^2 + 21} + \sqrt{(y-3)^2 + 21}} \\ &\leq \frac{1}{\sqrt{k}} \left\| x - y \right\|. \end{aligned} \tag{49}$$

It is easy to see that T_1 is nonexpansive and T_2 is contraction mapping with constant $1/\sqrt{k}$. And $F(T_1 \cap T_2) = \{5\} \neq \emptyset$. Hence, the conditions of Theorem 7 hold and the sequence $\{x_n\}$ generated by (PMMDI), (PMMDIP), and (PMMDIM) can be rewritten as follows:

(PMMDI)

$$\begin{aligned} x_{n+1} &= \frac{1}{\pi} \sin\left(\frac{x_n + y_n}{2}\pi\right) + 5 + \frac{1}{100^n}, \\ y_n &= \left(1 - \frac{1}{2}\right) x_n \\ &\quad + \sqrt{\left(\frac{x_n + y_n}{2}\right)^2 - 3(x_n + y_n) + 30} \\ &\quad + \frac{1}{2^n}\left(\frac{1}{n} + \frac{1}{n^2}\right) - \frac{10}{n^5}. \end{aligned} \tag{50}$$

(PMMDIP)

$$\begin{aligned} x_{n+1} &= \frac{1}{\pi} \sin\left(\frac{x_n + y_n}{2}\pi\right) + 5 + \frac{1}{100^n}, \\ y_n &= \left(1 - \frac{1}{2}\right) x_n + \frac{1}{2}\sqrt{x_n^2 - 6x_n + 30} \\ &\quad + \frac{1}{2^n}\left(\frac{1}{n} + \frac{1}{n^2}\right) - \frac{10}{n^5}. \end{aligned} \tag{51}$$

(PMMDIP)

$$\begin{aligned} x_{n+1} &= \frac{1}{\pi} \sin(\pi y_n) + 5 + \frac{1}{100^n}, \\ y_n &= \left(1 - \frac{1}{2}\right) x_n \\ &\quad + \frac{1}{2}\sqrt{\left(\frac{x_n + y_n}{2}\right)^2 - 3(x_n + y_n) + 30} \\ &\quad + \frac{1}{2^n}\left(\frac{1}{n} + \frac{1}{n^2}\right) - \frac{10}{n^5}. \end{aligned} \tag{52}$$

The special cases which have been discussed in the second part are listed as follows:

(PMDI)

$$x_{n+1} = \frac{1}{\pi} \sin\left(\frac{x_n + y_n}{2}\pi\right) + 5,$$

$$y_n = \left(1 - \frac{1}{2}\right) x_n + \frac{1}{2}\sqrt{\left(\frac{x_n + y_n}{2}\right)^2 - 3\left(x_n + y_n\right) + 30}. \quad (53)$$

(PMMI)

$$\begin{aligned} x_{n+1} &= \sqrt{\left(\frac{x_n + y_n}{2}\right)^2 - 3\left(x_n + y_n\right) + 30} + \frac{1}{100^n}, \\ y_n &= \left(1 - \frac{1}{2}\right) x_n + \frac{1}{2}\sqrt{\left(\frac{x_n + y_n}{2}\right)^2 - 3\left(x_n + y_n\right) + 30} \\ &\quad + \frac{1}{2^n}\left(\frac{1}{n} + \frac{1}{n^2}\right) - \frac{10}{n^5}. \end{aligned} \quad (54)$$

(MMDI)

$$\begin{aligned} x_{n+1} &= \frac{x_n + y_n}{2} + \frac{1}{100^n}, \\ y_n &= \left(1 - \frac{1}{2}\right) x_n + \frac{1}{2}\sqrt{\left(\frac{x_n + y_n}{2}\right)^2 - 3\left(x_n + y_n\right) + 30} \\ &\quad + \frac{1}{2^n}\left(\frac{1}{n} + \frac{1}{n^2}\right) - \frac{10}{n^5}. \end{aligned} \quad (55)$$

By Theorems 7, 11, and 12 and Corollaries 8, 9, and 10, one can easily know that $\{x_n\}$ generated by (PMMDI), (PMMDIP), (PMMDIM), (PMDI), (PMMI), (PMI), and (MMDI) converges to $x^* = 5$. To show the availability of the new Picard-Mann iterative methods, by using software Matlab 7.0, the numerical simulation results for the sequences $\{x_n\}$ generated by (PMMDI), (PMMDIP), and (PMMDIM) are given in Figure 1 and Table 1, the iteration process generated by (PMMDI), (PMDI), (PMMI), (PMI), and (MMDI) is given in Figure 2 and Table 2, respectively.

Remark 2. From Figure 1 and Table 1, it follows that the iterative process (10), (11), and (12) are effective and the sequences $\{x_n\}$ generated by them is convergent by using the technic of implicit midpoint rule. Further, comparing with the processes (PMMDI) and (PMMDIP), iteration process (PMMDIM) converges much faster, and the convergence rates are in accordance with the analysis in Theorems 7, 11, and 12:

$$\begin{aligned} \tau &= 1 - \frac{1-\theta}{2}\widehat{\alpha} > \varrho = 1 - \frac{1-\theta}{2-\theta\widehat{\alpha}}\widehat{\alpha} > \rho \\ &= 1 - \frac{2-2\theta}{2-\theta\widehat{\alpha}}\widehat{\alpha}. \end{aligned} \quad (56)$$

TABLE 1: A comparison of (PMMDI), (PMMDIP), and (PMMDIM).

Iteration Number	(PMMDI)	(PMMDIP)	(PMMDIM)
0	25.0000	25.0000	25.0000
5	5.1003	4.9073	4.9646
10	4.9740	5.0396	5.0010
15	5.0068	4.9825	5.0000
20	4.9982	5.0078	5.0000
25	5.0005	4.9966	5.0000
30	4.9999	5.0015	5.0000
35	5.0000	4.9993	5.0000
40	5.0000	5.0003	5.0000
45	5.0000	4.9999	5.0000
50	5.0000	5.0001	5.0000
55	5.0000	5.0000	5.0000

Remark 3. From Figure 2 and Table 2, it is easy to see that the iterative process and the special cases (PMMDI), (PMDI), (PMMI), and (MMDI) are effective and the sequences $\{x_n\}$ generated by them are also convergent with the technic of implicit midpoint rule. Iteration process (PMMI) converges much faster than the processes (PMMDI), (PMDI), and (MMDI), and the convergence rates are in accordance with the analysis in Theorem 7 and Corollaries 8, 9, and 10:

$$\varrho = 1 - \frac{1-\theta}{2-\theta\widehat{\alpha}}\widehat{\alpha} > \kappa = \left(1 - \frac{1-\theta}{2-\theta\widehat{\alpha}}\widehat{\alpha}\right)\theta. \quad (57)$$

3.2. An Application to Elliptic Optimal Control Problems. In [19], Li and Lan studied a kind of new iterative approximation of solutions for an elliptic boundary value problem in Hilbert spaces by using the new Picard-Mann iterative methods with mixed errors for contraction operators. Here, we try to explore the new iteration methods to optimal control problem with elliptic boundary value constraint.

One can know that a weak solution of (2) is a solution of the following variational problem:

$$\begin{aligned} &\int_\Omega \nabla u \cdot \nabla v dx - \int_\Omega f(x,u) \cdot v dx = 0, \\ &\qquad \forall v \in H_0^1(\Omega), \\ &u(x) \in H_0^1(\Omega). \end{aligned} \quad (58)$$

Theorem 4. *Let X be a normed space and $C \subset X$ be a nonempty closed convex bounded set. Let $\phi(y) = (1/2)\|y\|^2 - \int_\Omega F(x,y)dx$, $T = I - \phi'$, $F(x,y) = \int_0^y f(x,\zeta)d\zeta$, and $C = [v,w] = u \in H_0^1(\Omega) : v(x) \leq y(x) \leq w(x)$ for all $x \in \Omega$, where $v, w \in H_0^1(\Omega)$ are a subsolution and a supersolution of the problem (58), respectively. Suppose that $F(T) \neq \emptyset$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, then*

(i) the iterative sequence $\{x_n\}$ generated by (PMMI) converges to a weak solution $x^* \in F(T_1 \cap T_2)$ of (2) with convergence rate

$$\iota = \theta\left(1 - \frac{1-\theta}{2-\theta\widehat{\alpha}}\widehat{\alpha}\right) \quad (59)$$

where $\widehat{\alpha} = \limsup_{n\longrightarrow\infty} \alpha_n \in (0,1]$ and $\theta = \sup_{n\longrightarrow\infty}\|I' - \varphi''\|$;

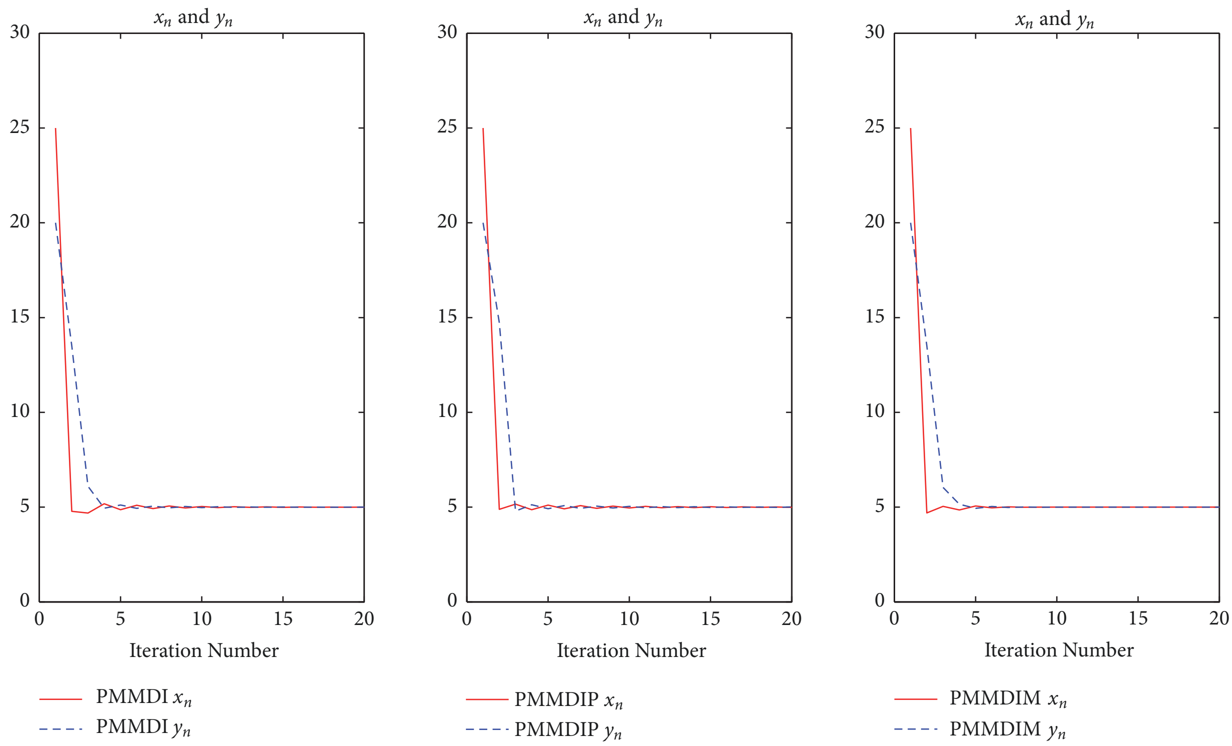

FIGURE 1: Iterative solutions of (PMMDI), (PMMDIP), and (PMMDIM).

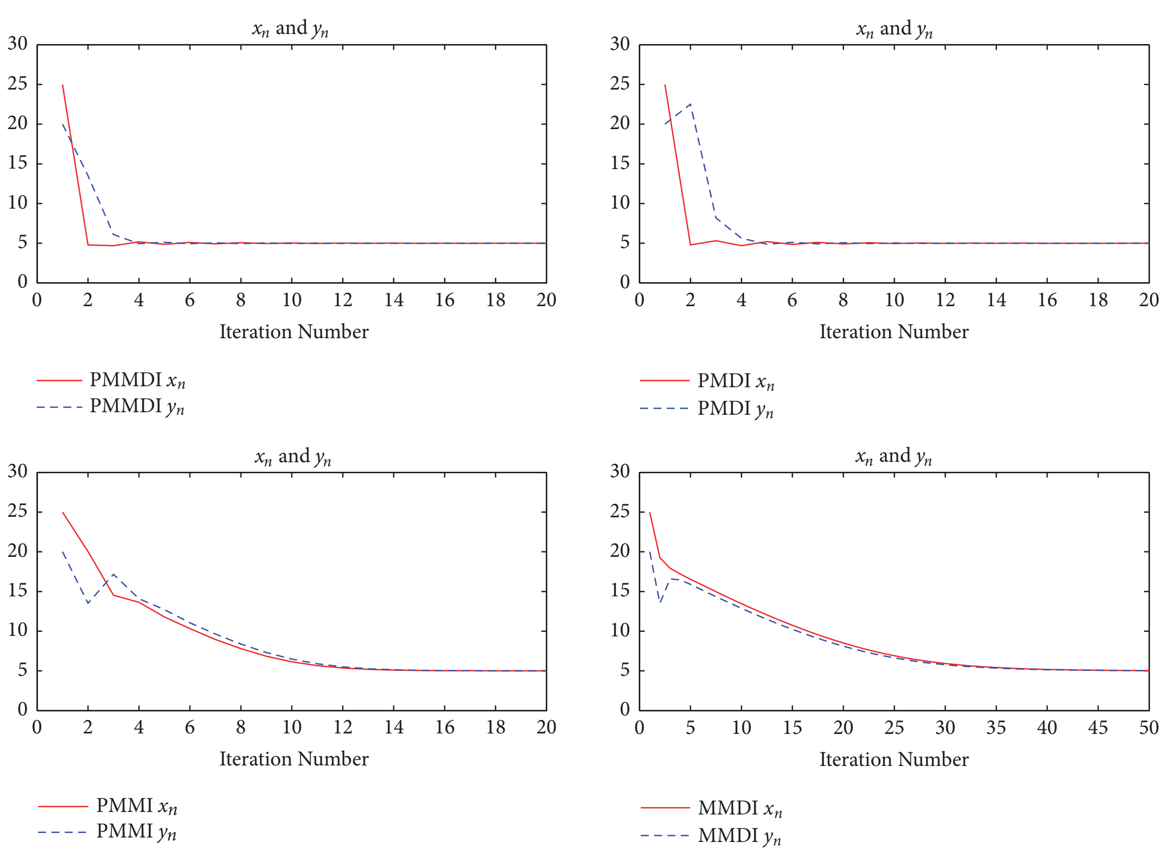

FIGURE 2: Iterative solutions of (PMMDI), (PMDI), (PMMI), and (MMDI).

Table 2: A comparison of (PMMDI), (PMDI), (PMMI), and (MMDI).

Iteration Number	(PMMDI)	(PMDI)	(PMMI)	(MMDI)
0	25.0000	25.0000	25.0000	25.0000
5	5.1003	4.8491	10.3264	15.9082
10	4.9740	5.0376	5.6542	12.8914
15	5.0068	4.9902	5.0246	10.2503
20	4.9982	5.0026	5.0008	8.1355
25	5.0005	4.9993	5.0000	6.6612
30	4.9999	5.0002	5.0000	5.7941
35	5.0000	5.0000	5.0000	5.3551
40	5.0000	5.0000	5.0000	5.1533
45	5.0000	5.0000	5.0000	5.0651
50	5.0000	5.0000	5.0000	5.0275
55	5.0000	5.0000	5.0000	5.0015

(ii) if there exists an $\alpha > 0$ such that $\alpha_n \geq \alpha$ for any sequence $\{z_n\} \subset X$ and all $n \geq 0$, then

$$\lim_{n\longrightarrow\infty} z_n = u^* \quad \text{if and only if} \lim_{n\longrightarrow\infty} \kappa_n = 0, \tag{60}$$

where

$$\begin{aligned} \kappa_n &= \left\| z_{n+1} - \left[\frac{z_n + s_n}{2} - \phi'\left(\frac{z_n + s_n}{2}\right) + h_n \right] \right\|, \\ s_n &= (1-\alpha_n) z_n + \alpha_n \left[\frac{z_n + s_n}{2} - \phi'\left(\frac{z_n + s_n}{2}\right) \right] \\ &\quad + \alpha_n d_n + e_n. \end{aligned} \tag{61}$$

Proof. From the proof of [41, Theorem 6], it follows that $C \subset H_0^1(\Omega)$ is a closed convex and bounded subset and $\|(I' - \phi'')u\| < 1$ for some $u \in C$. By the proof of Theorem 4 in [41], we know that T is a contraction mapping. Since a contraction mapping shows fixed points, the results hold from Theorem 4. This completes the proof. □

Combining (17) and (18) with Theorem 4, we give the following corollaries.

Corollary 5. *Let X be a normed space and $C \subset X$ be a nonempty closed convex bounded set. Let $\phi(y) = (1/2)\|y\|^2 - \int_\Omega F(x,y)dx$, $T = I - \phi'$, $F(x,y) = \int_0^y f(x,\zeta)d\zeta$, and $C = [v,w] = u \in H_0^1(\Omega) : v(x) \leq y(x) \leq w(x)$ for all $x \in \Omega$, where $v, w \in H_0^1(\Omega)$ are a subsolution and a supersolution of the problem (58), respectively. Suppose that $F(T) \neq \emptyset$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, then*

(i) the iterative sequence $\{x_n\}$ generated by (PMMIP) and (PMMIM) converges to a weak solution $x^* \in F(T_1 \cap T_2)$ of (2) with convergence rate

$$\begin{aligned} \jmath &= \theta\left(1 - \frac{1-\theta}{2}\widehat{\alpha}\right), \\ \ell &= \theta\left(1 - \frac{2-2\theta}{2-\theta\widehat{\alpha}}\widehat{\alpha}\right), \end{aligned} \tag{62}$$

respectively;

(ii) if there exists an $\alpha > 0$ such that $\alpha_n \geq \alpha$ for any sequence $\{z_n\} \subset X$ and all $n \geq 0$, then

$$\lim_{n\longrightarrow\infty} z_n = u^* \quad \text{if and only if} \lim_{n\longrightarrow\infty} \lambda_n = 0, \tag{63}$$

where

$$\begin{aligned} \lambda_n &= \left\| z_{n+1} - \left[\frac{z_n + t_n}{2} - \phi'\left(\frac{z_n + t_n}{2}\right) + h_n \right] \right\|, \\ t_n &= (1-\alpha_n) z_n + \alpha_n \left[z_n - \phi'(z_n) \right] + \alpha_n d_n + e_n. \end{aligned} \tag{64}$$

or

$$\begin{aligned} \lambda_n &= \left\| z_{n+1} - \left[t_n - \phi'(t_n) + h_n \right] \right\|, \\ t_n &= (1-\alpha_n) z_n + \alpha_n \left[\frac{z_n + t_n}{2} - \phi'\left(\frac{z_n + t_n}{2}\right) \right] \\ &\quad + \alpha_n d_n + e_n, \end{aligned} \tag{65}$$

which are defined by (17) and (18), respectively.

Corollary 6. *Let $\mathbb{R}^n$ be a n-dimensional bounded space and $\Omega \subset \mathbb{R}^n$ be a nonempty closed convex bounded set. If $\phi(y) = (1/2)\|y\|^2 - \int_\Omega F(x,y)dx$, $T = I - \phi'$, $F(x,y) = \int_0^y f(x,\zeta)d\zeta$,*

(i) then $T(u) = (I - \phi')u = -y'(x)$ is a constant mapping;

(ii) taking the iteration results $y^*(x_p) = p$ of (PMMI), (PMMIP), and (PMMIM) into $u(x)$, then $u(x_p)$ is the solution space of optimal control problem (1) and (2).

4. Concluding Remarks

In this paper, we introduced new Picard-Mann iteration processes with mixed errors for the implicit midpoint rule as follows:

$$\begin{aligned} x_{n+1} &= T_1\left(\frac{x_n + y_n}{2}\right) + h_n, \\ y_n &= (1-\alpha_n) x_n + \alpha_n T_2\left(\frac{x_n + y_n}{2}\right) + \alpha_n d_n + e_n, \end{aligned} \tag{66}$$

and the sequence defined by

$$x_{n+1} = T_1 \left(\frac{x_n + y_n}{2} \right) + h_n,$$
$$y_n = (1 - \alpha_n) x_n + \alpha_n T_2 x_n + \alpha_n d_n + e_n, \tag{67}$$

$$x_{n+1} = T_1 y_n + h_n,$$
$$y_n = (1 - \alpha_n) x_n + \alpha_n T_2 \left(\frac{x_n + y_n}{2} \right) + \alpha_n d_n + e_n, \tag{68}$$

where $T_1, T_2 : X \longrightarrow X$ are two nonlinear operators; $h_n, d_n, e_n \in X$ are errors to take into account a possible inexact computation. The iteration (66) includes Picard-Mann iterative process. What is noteworthy is that the iterative processes are not discussed in the literature.

Then, we gave convergence and stability analysis of the new Picard-Mann iterative approximation for implicit midpoint rule in normed space and proposed numerical examples to show the convergence of different iterative processes. Finally, as applications of new Picard-Mann iteration processes with mixed errors for the implicit midpoint rule of nonexpansive mappings and contractive mappings, we explored iterative approximation of solutions for the following optimal control problem with elliptic boundary value constraint:

$$\min J(u), \tag{69}$$

where state variable $y(u) \in V = H^1(\Omega)$, state space, and control variable $u \in U = L^2(\partial\Omega)$, control space, satisfy

$$-\Delta y = f(x, u), \quad x \in \Omega,$$
$$u(x) = 0, \quad x \in \partial\Omega, \tag{70}$$

$\Omega \subset \mathbb{R}^2$ is a bounded convex region with smooth boundary $\partial\Omega$; $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function.

However, as is pointed out in [19], when T_2 is also nonexpansive in Theorems 7 and 11, what results can be obtained?

Abbreviations

PMMDI: New Picard-Mann iteration with mixed errors for the implicit midpoint rule of two different nonlinear operators
PMMDIP: New Picard-Mann iteration with mixed errors for the implicit midpoint rule of Picard mapping of two different nonlinear operators
PMMDIM: New Picard-Mann iteration with mixed errors for the implicit midpoint rule of Mann mapping of two different nonlinear operators
PMDI: Picard-Mann iterations of the implicit midpoint rule in for two different operators
PMDIP: Picard-Mann iterations of the implicit midpoint rule of Picard mapping for two different operators
PMDIM: Picard-Mann iterations of the implicit midpoint rule of Mann mapping for two different operators
PMMI: Picard-Mann iteration of implicit midpoint rule with mixed errors for one nonlinear mapping
PMMIP: Picard-Mann iteration of implicit midpoint rule of Picard mapping with mixed errors for one nonlinear mapping
PMMIM: Picard-Mann iteration of implicit midpoint rule of Mann mapping with mixed errors for one nonlinear mapping
PMI: Picard-Mann iterative process of implicit midpoint rule for one nonlinear mapping
PMIP: Picard-Mann iterative process of implicit midpoint rule of Picard mapping for one nonlinear mapping
PMIM: Picard-Mann iterative process of implicit midpoint rule of Mann mapping for one nonlinear mapping
MMDI: Picard-Mann iterative process of the explicit and implicit midpoint rule with mixed errors
MMDIP: Picard-Mann iterative process of the explicit and implicit midpoint rule of Picard mapping with mixed errors
MMDIM: Picard-Mann iterative process of the explicit and implicit midpoint rule of Mann mapping with mixed errors.

Data Availability

All data included in this study are available upon request by contact with the corresponding author.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

Acknowledgments

This work was partially supported by Sichuan Science and Technology Program (2019YJ0541) and the Scientific Research Project of Sichuan University of Science & Engineering (2013PY07).

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