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Research Article

Toeplitz Operator and Carleson Measure on Weighted Bloch Spaces

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In this paper, we consider Toeplitz operator acting on weighted Bloch spaces. Meanwhile, the inclusion map from weighted Bloch spaces into tent space is also investigated.

1. Introduction

Denote the open unit disk of the complex plane $\mathbb{C}$ by $\mathbb{D}$ and the boundary of $\mathbb{D}$ by $\partial\mathbb{D}$. Let $H(\mathbb{D})$ denote the space of all functions analytic in $\mathbb{D}$. For any $a \in \mathbb{D}$,

$$\varphi_a(z) = \frac{a-z}{1-\bar{a}z}, \quad z \in \mathbb{D} \tag{1}$$

is the automorphism of $\mathbb{D}$ which exchanges 0 for a. Recall that

$$\beta(z,a) = \frac{1}{2}\log\frac{1+|\varphi_a(z)|}{1-|\varphi_a(z)|} \tag{2}$$

is the Bergman metric. For any $0 < r < \infty, a \in \mathbb{D}$,

$$D(a,r) = \{z \in D : \beta(z,a) < r\} \tag{3}$$

is the Bergman disk. Let $|D(a,r)|$ denote the normalized area of $D(a,r)$. From [1], we see that $|D(a,r)| \approx (1-|a|^2)^2$ when r is fixed.

For $0 < p < \infty$ and $\alpha > -1$, the weighted Bergman space A^p_α is the space of all $f \in H(\mathbb{D})$ such that

$$\|f\|^p_{A^p_\alpha} := \int_{\mathbb{D}} |f(z)|^p \left(1-|z|^2\right)^\alpha dA(z) < \infty. \tag{4}$$

When $\alpha = 0$, A^p_α is the classical Bergman space. We refer the readers to [1, 2] for more results on weighted Bergman spaces.

Let $0 < \alpha < \infty$. An $f \in H(\mathbb{D})$ is said to belong to the weighted Bloch space, denoted by $\mathscr{B}^\alpha$, if

$$\|f\|_{\mathscr{B}^\alpha} = \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha \left|f'(z)\right| < \infty. \tag{5}$$

The space $\mathscr{B}^\alpha$ has been studied extensively in [3]. See [1, 4–8] for the study of some operators on weighted Bloch spaces.

Let $\varphi \in L^\infty(\mathbb{D})$. The Toeplitz operator T_φ with symbol φ is defined by

$$T_\varphi f(z) = \int_{\mathbb{D}} \frac{\varphi(w) f(w)}{(1-\overline{w}z)^{2+\alpha}} dA_\alpha(w), \tag{6}$$

where $dA_\alpha(w) = (1-|w|^2)^\alpha dA(w)$. There are many results related to T_φ, see [1] and the references therein. Especially, some characterizations for the operator T_φ on L^2_α have been obtained by many authors. Since $\mathscr{B}^\alpha \subseteq A^1_\alpha$, it is nature to ask

$$T_\varphi f \in \mathscr{B}^\alpha \Longleftrightarrow ?, \quad f \in \mathscr{B}^\alpha. \tag{7}$$

The following theorem is the first main result in this paper.

Theorem 1. *Let $0 < \alpha < \infty$ and $\varphi \in L^1(\mathbb{D})$ be harmonic. Then the following statements hold.*

(1) $T_\varphi : \mathscr{B}^{\alpha+1} \longrightarrow \mathscr{B}^{\alpha+1}$ is bounded if and only if φ is bounded.

(2) $T_\varphi : \mathscr{B}^{\alpha+1} \longrightarrow \mathscr{B}^{\alpha+1}$ is compact if and only if $\varphi = 0$.

Given a positive Borel measure μ, the Toeplitz operator with the symbol μ is defined by

$$T_\mu f(z) = \int_{\mathbb{D}} \frac{f(w)}{(1-\overline{w}z)^{2+\alpha}} d\mu(w), \quad f \in L^1(dA_\alpha). \tag{8}$$

For the Toeplitz operator T_μ, we have the following result.

Theorem 2. *Let $0 < \alpha, r < \infty$ and μ be a positive Borel measure. Then the following statements hold.*

(1) $T_\mu : \mathscr{B}^{\alpha+1} \longrightarrow \mathscr{B}^{\alpha+1}$ is bounded if and only if

$$\sup_{a\in\mathbb{D}} \frac{\mu(D(a,r))}{\left(1-|a|^2\right)^{2+\alpha}} < \infty. \tag{9}$$

(2) $T_\mu : \mathscr{B}^{\alpha+1} \longrightarrow \mathscr{B}^{\alpha+1}$ is compact if and only if

$$\lim_{|a|\longrightarrow 0} \frac{\mu(D(a,r))}{\left(1-|a|^2\right)^{2+\alpha}} = 0. \tag{10}$$

For $I \subset \partial\mathbb{D}$, $|I| = (1/2\pi)\int_I |d\xi|$ is the normalized length of the subarc I and the corresponding Carleson square for I is defined as follows (see [9]).

$$S(I) = \{r\xi \in \mathbb{D} : r \in [1-|I|, 1), \xi \in I\}. \tag{11}$$

For $0 < p < \infty$, a positive Borel measure μ on $\mathbb{D}$ is said to be a p-Carleson measure if

$$\|\mu\|_p =: \sup_{I\subset\partial\mathbb{D}} \left(\frac{\mu(S(I))}{|I|^p}\right)^{1/2} < \infty. \tag{12}$$

If $p = 1$, p-Carleson measure is the classical Carleson measure. From Lemma 3.1.1 of [10], for $p, q \in (0,\infty)$ we know that μ is a p-Carleson measure if and only if

$$\|\mu\|_{p,q} := \sup_{z\in\mathbb{D}} \left(\int_{\mathbb{D}} \frac{\left(1-|z|^2\right)^q}{|1-\overline{z}w|^{p+q}} d\mu(w)\right)^{1/2} < \infty. \tag{13}$$

Moreover, $\|\mu\|_{p,q} \approx \|\mu\|_p$.

Let $0 < p, q < \infty$ and μ be a positive Borel measure on $\mathbb{D}$. The tent space $T_\mu^{p,q}$ is the class of all $f \in H(\mathbb{D})$ which satisfy

$$\sup_{I\subseteq\partial\mathbb{D}} \frac{1}{|I|^p} \int_{S(I)} |f(z)|^q \, d\mu(z) < \infty. \tag{14}$$

The tent space $T_\mu^{p,2}$ was introduced by J. Xiao [11] to studied Carleson measure for Q_s space. He proved that Q_s space is continuously contained in $T_\mu^{p,2}$ if and only if

$$\sup_{I\subseteq\partial\mathbb{D}} \frac{\mu(S(I))}{|I|^s} \left(\log\frac{2}{|I|}\right)^2 < \infty. \tag{15}$$

J. Pau and R. Zhao [12] generalized the main results in [11]. In [13], J. Liu and Z.Lou studied Morrey spaces. They proved that an equivalent condition for Morrey spaces $L^{2,s}$ continuously contained in $T_\mu^{p,2}$ is that μ is a Carleson measure. See [14, 15] for more information of the Morrey space.

We state the last main result in this paper as follows.

Theorem 3. *Let $0 < \alpha < \infty$ and μ be a positive Borel measure. Then the following statements hold.*

(1) The inclusion map $I_d : \mathscr{B}^{\alpha+1} \longrightarrow T_\mu^{2+\alpha,1}$ is bounded if and only if μ is a $(2+2\alpha)$-Carleson measure.

(2) The inclusion map $I_d : \mathscr{B}^{\alpha+1} \longrightarrow T_\mu^{2+\alpha,1}$ is compact if and only if μ is a vanishing $(2+2\alpha)$-Carleson measure.

Throughout this paper, the letter C will denote constants and may differ from one occurrence to the other. The notation $A \lesssim B$ means that there is a positive constant C such that $A \le CB$. The notation $A \approx B$ means $A \lesssim B$ and $B \lesssim A$.

2. Proofs of Main Results

To prove our main results in this paper, we need some auxiliary results. The following result can be found in [16, Theorem 3.8].

Lemma 4. *Let $p \ge 1$, $\alpha > 0$, $-1 + p\alpha < \eta < \infty$, and $c > 0$. Then $f \in \mathscr{B}^{\alpha+1}$ if and only if*

$$\sup_{z\in\mathbb{D}} \int_{\mathbb{D}} |f(w) - f(z)|^p \frac{\left(1-|z|^2\right)^{c+p\alpha}}{|1-\overline{z}w|^{2+c+\eta}} dA_\eta(w) < \infty. \tag{16}$$

From Lemma 4, we can easily deduce the following result.

Lemma 5. *Let $p \ge 1$, $\alpha > 0$, $-1 + p\alpha < \eta < \infty$, and $c > 0$. Then $f \in \mathscr{B}^{\alpha+1}$ if and only if*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|^p \frac{\left(1-|a|^2\right)^{c+p\alpha}}{|1-\overline{a}z|^{2+c+\eta}} dA_\eta(z) < \infty. \tag{17}$$

Proof. First assume that $f \in \mathscr{B}^{\alpha+1}$. It is clear that

$$|f(z)| \lesssim \frac{\|f\|_{\mathscr{B}^{\alpha+1}}}{\left(1-|z|^2\right)^\alpha}, \quad z \in \mathbb{D}. \tag{18}$$

Thus,

$$\begin{aligned}
&\int_{\mathbb{D}} |f(z)|^p \frac{\left(1-|a|^2\right)^{c+p\alpha}}{|1-\overline{a}z|^{2+c+\eta}} dA_\eta(z) \\
&\lesssim \int_{\mathbb{D}} |f(z)-f(a)|^p \frac{\left(1-|a|^2\right)^{c+p\alpha}}{|1-\overline{a}z|^{2+c+\eta}} dA_\eta(z) \\
&\quad + \int_{\mathbb{D}} |f(a)|^p \frac{\left(1-|a|^2\right)^{c+p\alpha}}{|1-\overline{a}z|^{2+c+\eta}} dA_\eta(z) \\
&\lesssim \|f\|_{\mathscr{B}^{\alpha+1}}^p + \|f\|_{\mathscr{B}^{\alpha+1}}^p \int_{\mathbb{D}} \frac{\left(1-|a|^2\right)^{c}}{|1-\overline{a}z|^{2+c+\eta}} dA_\eta(z) \\
&\lesssim \|f\|_{\mathscr{B}^{\alpha+1}}^p.
\end{aligned} \tag{19}$$

The proof of the inverse direction is similar to the above statements we omit the details. The proof is complete. ☐

Proof of Theorem 1. (1) First assume that $\varphi \in L^\infty(\mathbb{D})$. For $f \in \mathscr{B}^{\alpha+1}$, since

$$\|f\|_{\mathscr{B}^{\alpha+1}} \approx \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha |f(z)|, \tag{20}$$

we obtain

$$\begin{aligned}\left\|T_\varphi f\right\|_{\mathscr{B}^{\alpha+1}} &\approx \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^\alpha \left|T_\varphi f(z)\right| \\ &= \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^\alpha \left|\int_{\mathbb{D}}\frac{\varphi(w) f(w)}{(1-\overline{w}z)^{2+\alpha}}dA_\alpha(w)\right| \\ &\lesssim \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^\alpha \int_{\mathbb{D}}\frac{|\varphi(w)||f(w)|}{|1-\overline{w}z|^{2+\alpha}}dA_\alpha(w) \\ &\lesssim \|\varphi\|_{L^\infty(\mathbb{D})}\|f\|_{\mathscr{B}^{\alpha+1}} \\ &\cdot\left(\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^\alpha \int_{\mathbb{D}}\frac{1}{|1-\overline{w}z|^{2+\alpha}}dA(w)\right) \\ &\lesssim \|\varphi\|_{L^\infty(\mathbb{D})}\|f\|_{\mathscr{B}^{\alpha+1}}.\end{aligned} \tag{21}$$

Hence $T_\varphi : \mathscr{B}^{\alpha+1} \longrightarrow \mathscr{B}^{\alpha+1}$ is bounded.

Conversely, assume that $T_\varphi : \mathscr{B}^{\alpha+1} \longrightarrow \mathscr{B}^{\alpha+1}$ is bounded. For $z \in \mathbb{D}$, set

$$f_z(w) = \frac{\left(1-|z|^2\right)^2}{(1-\overline{z}w)^{2+\alpha}} \in \mathscr{B}^{\alpha+1}. \tag{22}$$

It is easy to check that $\|f_z\|_{\mathscr{B}^{\alpha+1}} \approx 1$. Using Lemma 5 with $p=1$, $\eta=\alpha$ and $c = 2+\alpha$, we get

$$\begin{aligned}\infty > \left\|T_\varphi\right\| &\gtrsim \left\|T_\varphi f_z\right\|_{\mathscr{B}^{1+\alpha}} \\ &\gtrsim \int_{\mathbb{D}}\left|T_\varphi f_z(w)\right|\frac{\left(1-|z|^2\right)^{2+2\alpha}}{|1-\overline{z}w|^{2(2+\alpha)}}dA_\alpha(w) \\ &= \left(1-|z|^2\right)^\alpha \int_{\mathbb{D}}\left|T_\varphi f_z(\varphi_z(w))\right|dA_\alpha(w) \\ &\gtrsim \left(1-|z|^2\right)^\alpha \left|T_\varphi f_z(\varphi_z(0))\right| \gtrsim |\varphi(z)|,\end{aligned} \tag{23}$$

which implies that $\varphi \in L^\infty(\mathbb{D})$, as desired.

(2) Sufficiency. The result is obvious.

Necessity. For any $z_n \in \mathbb{D}$, let

$$f_{z_n}(w) = \frac{\left(1-|z_n|^2\right)^2}{\left(1-\overline{z_n}w\right)^{2+\alpha}}, \quad w\in\mathbb{D}. \tag{24}$$

$f_{z_n} \in \mathscr{B}^{1+\alpha}$. It is easy to check that $f_{z_n} \longrightarrow 0$ uniformly on compact subsets on $\mathbb{D}$ as $|z_n| \longrightarrow 1$. From the fact that T_φ is compact on $\mathscr{B}^{\alpha+1} \longrightarrow \mathscr{B}^{\alpha+1}$ and the proof of (1), we have

$$\begin{aligned}0 &\longleftarrow \left\|T_\varphi f_{z_n}\right\|_{\mathscr{B}^{\alpha+1}} \\ &\gtrsim \int_{\mathbb{D}}\left|T_\varphi f_{z_n}(w)\right|\frac{\left(1-|z_n|^2\right)^{2+2\alpha}}{\left|1-\overline{z_n}w\right|^{2(2+\alpha)}}dA_\alpha(w) \\ &\gtrsim \left(1-|z|^2\right)^\alpha \left|T_\varphi f_{z_n}\left(\varphi_{z_n}(0)\right)\right| \gtrsim |\varphi(z_n)|.\end{aligned} \tag{25}$$

By the arbitrariness of z_n and the Maximal Module Principle, we get $\varphi = 0$. The proof is complete. □

Proof of Theorem 2. (1) First suppose that $T_\mu : \mathscr{B}^{\alpha+1} \longrightarrow \mathscr{B}^{\alpha+1}$ is bounded. For any $a \in \mathbb{D}$, from the proof of Theorem 1, we obtain that $f_a \in \mathscr{B}^{\alpha+1}$ and $\|f_a\|_{\mathscr{B}^{\alpha+1}} \lesssim 1$. Thus,

$$\begin{aligned}\infty > \left\|T_\mu\right\| &\gtrsim \left(1-|a|^2\right)^\alpha \left|T_\mu f_a(a)\right| \\ &= \left(1-|a|^2\right)^\alpha \left|\int_{\mathbb{D}}\frac{f_a(w)}{(1-\overline{w}a)^{2+\alpha}}d\mu(w)\right| \\ &= \left(1-|a|^2\right)^\alpha \int_{\mathbb{D}}\frac{\left(1-|a|^2\right)^2}{|1-\overline{w}a|^{4+2\alpha}}d\mu(w) \\ &\geq \left(1-|a|^2\right)^\alpha \int_{D(a,r)}\frac{\left(1-|a|^2\right)^2}{|1-\overline{w}a|^{4+2\alpha}}d\mu(w) \\ &\approx \frac{\mu(D(a,r))}{\left(1-|a|^2\right)^{2+\alpha}},\end{aligned} \tag{26}$$

as desired.

Conversely, suppose that

$$\sup_{a\in\mathbb{D}}\frac{\mu(D(a,r))}{\left(1-|a|^2\right)^{2+\alpha}} < \infty. \tag{27}$$

Then we can get that μ is a Carleson measure for $L^1(dA_\alpha)$. If $g \in L^1$ and $f \in \mathscr{B}^{\alpha+1}$, we can easily obtain that $fg \in L^1(dA_\alpha)$. Using Fubini's Theorem, we obtain

$$\begin{aligned}&\left|\int_{\mathbb{D}} g(z)\overline{T_\mu f(z)}\left(1-|z|^2\right)^\alpha dA(z)\right| \\ &= \left|\int_{\mathbb{D}} g(z)\left(1-|z|^2\right)^\alpha dA(z)\int_{\mathbb{D}}\frac{\overline{f(w)}}{(1-w\overline{z})^{2+\alpha}}d\mu(w)\right| \\ &= \left|\int_{\mathbb{D}}\overline{f(w)}d\mu(w)\int_{\mathbb{D}}\frac{\left(1-|z|^2\right)^\alpha g(z)}{(1-w\overline{z})^{2+\alpha}}dA(z)\right| \\ &\leq \int_{\mathbb{D}}\left|g(w)\overline{f(w)}\right|d\mu(w) \\ &\lesssim \int_{\mathbb{D}}|g(w)||f(w)|dA_\alpha(w) \lesssim \|g\|_{L^1}\|f\|_{\mathscr{B}^{\alpha+1}}.\end{aligned} \tag{28}$$

Hence $T_\mu : \mathscr{B}^{\alpha+1} \longrightarrow \mathscr{B}^{\alpha+1}$ is bounded.

(2) Suppose that $T_\mu : B^{\alpha+1} \longrightarrow \mathscr{B}^{\alpha+1}$ is compact. Let $a_n \in \mathbb{D}$. Set

$$f_{a_n}(z) = \frac{\left(1-|a_n|^2\right)^2}{\left(1-\overline{a_n}z\right)^{2+\alpha}}, \quad z\in\mathbb{D}. \tag{29}$$

Then $f_{a_n} \in \mathscr{B}^{\alpha+1}$ and $f_{a_n} \longrightarrow 0$ uniformly on compact subset on $\mathbb{D}$ as $|a_n| \longrightarrow 1$. Thus,

$$\begin{aligned}\left\|T_\mu f_{a_n}\right\|_{\mathscr{B}^{\alpha+1}} &\gtrsim \left(1-|a_n|^2\right)^\alpha \left|T_\mu f_{a_n}(a_n)\right| \\ &= \left(1-|a_n|^2\right)^\alpha \left|\int_{\mathbb{D}}\frac{f_{a_n}(w)}{\left(1-\overline{w}a_n\right)^{2+\alpha}}d\mu(w)\right|\end{aligned}$$

$$
\begin{aligned}
&= \left(1-|a_n|^2\right)^{\alpha} \int_{\mathbb{D}} \frac{\left(1-|a_n|^2\right)^2}{|1-\overline{w}a_n|^{4+2\alpha}} d\mu(w) \\
&\geq \left(1-|a_n|^2\right)^{\alpha} \int_{D(a_n,r)} \frac{\left(1-|a_n|^2\right)^2}{|1-\overline{w}a_n|^{4+2\alpha}} d\mu(w) \\
&\approx \frac{\mu(D(a_n,r))}{\left(1-|a_n|^2\right)^{2+\alpha}},
\end{aligned} \tag{30}
$$

which implies the desired result.

Conversely, assume that

$$
\lim_{|a|\longrightarrow 1} \frac{\mu(D(a,r))}{\left(1-|a|^2\right)^{2+\alpha}} = 0. \tag{31}
$$

We know that μ is a vanishing Carleson measure for $L^1(dA_\alpha)$. We want to show that T_μ is compact. Using Fubini's Theorem we have

$$
\int_{\mathbb{D}} \overline{T_\mu f(z)} g(z)\, dA_\alpha(z) = \int_{\mathbb{D}} g(z)\, \overline{f(z)} d\mu(z). \tag{32}
$$

If $g \in L^1$ and $f \in \mathscr{B}^{\alpha+1}$, we can easily obtain $fg \in L^1(dA_\alpha)$. Therefore,

$$
\begin{aligned}
\left| \int_{\mathbb{D}} \overline{T_\mu f(z)} g(z)\, dA_\alpha(z) \right| &\leq \int_{\mathbb{D}} \left| g(z)\, \overline{f(z)} \right| d\mu(z) \\
&\lesssim \int_{\mathbb{D}} |g(z)| \, |f(z)|\, dA_\alpha(z) \lesssim \|g\|_{L^1} \|f\|_{\mathscr{B}^{\alpha+1}}.
\end{aligned} \tag{33}
$$

That is,

$$
\|T_\mu f\|_{\mathscr{B}^{\alpha+1}} \lesssim \|f\|_{\mathscr{B}^{\alpha+1}}. \tag{34}
$$

If $f_n \longrightarrow 0$ weakly in $\mathscr{B}^{\alpha+1}$, it follows that $\|T_\mu f_n\|_{\mathscr{B}^{\alpha+1}} \longrightarrow 0$. The proof of Theorem 2 is complete. □

Proof of Theorem 3. (1) Suppose that $I_d : \mathscr{B}^{\alpha+1} \longrightarrow T_\mu^{2+\alpha,1}$ is bounded. For $a \in \mathbb{D}$, set

$$
f_a(z) = \frac{\left(1-|a|^2\right)^2}{(1-\overline{a}z)^{2+\alpha}}, \quad z \in \mathbb{D}. \tag{35}
$$

Then $f_a \in \mathscr{B}^{\alpha+1}$. For any $I \subseteq \partial\mathbb{D}$, we get

$$
\frac{\mu(S(I))}{|I|^{2+2\alpha}} \lesssim \frac{1}{|I|^{2+\alpha}} \int_{S(I)} |f_a(z)|\, d\mu(z) < \infty, \tag{36}
$$

as desired.

Conversely, assume that μ is a $(2+2\alpha)$-Carleson measure. Let $f \in \mathscr{B}^{\alpha+1}$. Using the well-known fact

$$
|f(b)| \lesssim \frac{1}{\left(1-|b|^2\right)^{\alpha}}, \quad b \in \mathbb{D}, \tag{37}
$$

we have

$$
\begin{aligned}
&\frac{1}{|I|^{2+\alpha}} \int_{S(I)} |f(w)|\, d\mu(w) \\
&\quad \lesssim \frac{1}{|I|^{2+\alpha}} \int_{S(I)} |f(w) - f(b)|\, d\mu(w) \\
&\qquad + \frac{1}{|I|^{2+\alpha}} \int_{S(I)} |f(b)|\, d\mu(w) \\
&\quad \lesssim \frac{1}{|I|^{2+\alpha}} \int_{S(I)} |f(w) - f(b)|\, d\mu(w) + \frac{\mu(S(I))}{|I|^{2+2\alpha}}.
\end{aligned} \tag{38}
$$

Note that

$$
\frac{\mu(S(I))}{|I|^{2+\alpha}} \lesssim \frac{\mu(S(I))}{|I|^{2+2\alpha}}. \tag{39}
$$

Then μ is a Carleson measure for $L^1(dA_\alpha)$. Since $\mathscr{B}^{\alpha+1} \subseteq L^1(dA_\alpha)$, combined with Lemma 4, we obtain

$$
\begin{aligned}
&\frac{1}{|I|^{2+\alpha}} \int_{S(I)} |f(w) - f(b)|\, d\mu(w) \\
&\quad \approx \left(1-|b|^2\right)^{2+\alpha} \int_{S(I)} \left| \frac{f(w) - f(b)}{\left(1-\overline{b}w\right)^{4+2\alpha}} \right| d\mu(w) \\
&\quad \lesssim \left(1-|b|^2\right)^{2+\alpha} \int_{\mathbb{D}} \left| \frac{f(w) - f(b)}{\left(1-\overline{b}w\right)^{4+2\alpha}} \right| d\mu(w) \\
&\quad \lesssim \left(1-|b|^2\right)^{2+\alpha} \int_{\mathbb{D}} \left| \frac{f(w) - f(b)}{\left(1-\overline{b}w\right)^{4+2\alpha}} \right| dA_\alpha(w) \\
&\quad \lesssim \int_{\mathbb{D}} |f(w) - f(b)| \frac{\left(1-|b|^2\right)^{2+\alpha}}{\left|1-\overline{b}w\right|^{4+2\alpha}} dA_\alpha(w) \\
&\quad \lesssim \|f\|_{\mathscr{B}^{\alpha+1}}.
\end{aligned} \tag{40}
$$

Hence $I_d : \mathscr{B}^{\alpha+1} \longrightarrow T_\mu^{2+\alpha,1}$ is bounded.

(2) Suppose that $I_d : \mathscr{B}^{\alpha+1} \longrightarrow T_\mu^{2+\alpha,1}$ is compact. Let $a_n \in \mathbb{D}$ such that $|a_n| \longrightarrow 1$ as $n \longrightarrow \infty$. We know that $f_{a_n} \in \mathscr{B}^{\alpha+1}$ and $f_{a_n} \longrightarrow 0$ uniformly on compact subsets of $\mathbb{D}$ as $n \longrightarrow \infty$. By Theorem 5.15 of [1] it follows that $f_{a_n} \longrightarrow 0$ weakly as $n \longrightarrow \infty$. Hence for the compact operator $I_d : \mathscr{B}^{\alpha+1} \longrightarrow T_\mu^{2+\alpha,1}$, we have $\|f_{a_n}\|_{T_\mu^{2+\alpha,1}} \longrightarrow 0$ as $n \longrightarrow \infty$. Thus,

$$
\begin{aligned}
\frac{\mu(S(I_n))}{|I_n|^{2+2\alpha}} &\lesssim \frac{1}{\left(1-|a_n|^2\right)^{2+\alpha}} \int_{S(I_n)} |f_{a_n}(z)|\, d\mu(z) \\
&\lesssim \|f_{a_n}\|_{T_\mu^{2+\alpha,1}} \longrightarrow 0
\end{aligned} \tag{41}
$$

as $n \longrightarrow \infty$. Hence μ is a vanishing $(2+2\alpha)$-Carleson measure.

Conversely, assume that μ is a vanishing $(2+2\alpha)$-Carleson measure. Let $f_n \in \mathscr{B}^{\alpha+1}$, $\|f_n\|_{\mathscr{B}^{\alpha+1}} \lesssim 1$, and $f_n \longrightarrow 0$

($n \longrightarrow \infty$) uniformly on compact subsets of $\mathbb{D}$. Then it is easy to get that

$$\begin{aligned}
&\frac{1}{|I|^{2+\alpha}} \int_{S(I)} \left|f_n(z)\right| d\mu(z) \\
&\quad \lesssim \frac{1}{|I|^{2+\alpha}} \int_{S(I)} \left|f_n(z)\right| d\mu_r(z) \\
&\qquad + \frac{1}{|I|^{2+\alpha}} \int_{S(I)} \left|f_n(z)\right| d\left(\mu - \mu_r\right)(z) \\
&\quad \lesssim \frac{1}{|I|^{2+\alpha}} \int_{S(I)} \left|f_n(z)\right| d\mu_r(z) \\
&\qquad + \left\|\mu - \mu_r\right\|^2 \left\|f_n\right\|_{\mathscr{B}^{\alpha+1}}.
\end{aligned} \tag{42}$$

Let $r \longrightarrow 1^-$ and $n \longrightarrow \infty$; we get the desired result. The proof is complete. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that they have no conflicts of interest.

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