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Research Article

Lyapunov-Type Inequalities for a Conformable Fractional Boundary Value Problem of Order $3 < \alpha \leq 4$

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We establish new Lyapunov-type inequalities for the following conformable fractional boundary value problem (BVP): $T_\alpha^a u(t) + q(t)u(t) = 0$, $a < t < b$, $u(a) = u'(a) = u''(a) = u''(b) = 0$, where T_α^a is the conformable fractional derivative of order $\alpha \in (3,4]$ and q is a real-valued continuous function. Some applications to the corresponding eigenvalue problem are discussed.

1. Introduction

In [1], Lyapunov proved that if the boundary value problem *BVP*

$$\begin{gathered} u''(t) + q(t)u(t) = 0, \quad a < t < b, \\ u(a) = u(b) = 0, \end{gathered} \tag{1}$$

where $q : [a,b] \longrightarrow \mathbb{R}$, has a nontrivial continuous solution, then

$$\int_a^b |q(r)|\, dr > \frac{4}{b-a}. \tag{2}$$

Moreover, the constant 4 in (2) is sharp (see [2]).

We emphasize that the above inequality has been proved to be very useful in the study of various problems related to differential equations; see, for instance, [2–5] and the references therein.

Many researchers have studied generalizations and extensions of Lyapunov's inequality.

In [6], Wintner improved inequality (2) and obtained the following version:

$$\int_a^b q^+(r)\, dr > \frac{4}{b-a}, \tag{3}$$

where $q^+(r) = \max\{q(r), 0\}$.

In [2], Hartamn generalized inequality (2) as follows:

$$\int_a^b (b-r)(r-a)\, q^+(r)\, dr > b-a. \tag{4}$$

In the frame of fractional differential equations, Ferreira (see [7]) proved a Lyapunov-type inequality for the Caputo fractional *BVP*

$$\begin{gathered} {}^C D_{a^+}^{\alpha} u(t) + q(t)u(t) = 0, \quad a < t < b,\ 1 < \alpha \leq 2 \\ u(a) = u(b) = 0. \end{gathered} \tag{5}$$

where q is a real and continuous function.

He showed that if a nontrivial continuous solution to the above problem exists, then

$$\int_a^b |q(r)|\, dr > \Gamma(\alpha) \left(\frac{4}{b-a}\right)^{\alpha-1}. \tag{6}$$

In [8], the same author investigated a Lyapunov-type inequality for the Riemann-Liouville fractional *BVP*

$$\begin{gathered} D_{a^+}^{\alpha} u(t) + q(t)u(t) = 0, \quad a < t < b,\ 1 < \alpha \leq 2 \\ u(a) = u(b) = 0. \end{gathered} \tag{7}$$

He proved that if (7) has a nontrivial continuous solution, then

$$\int_a^b |q(r)|\, dr > \Gamma(\alpha) \frac{\alpha^\alpha}{[(\alpha-1)(b-a)]^{\alpha-1}}. \tag{8}$$

For definitions and properties of Caputo fractional derivatives and Riemann-Liouville fractional, we refer the reader to [9, 10].

Observe that inequalities (6) and (8) lead to Lyapunov's classical inequality (2) when $\alpha = 2$.

Recently, Khalil et al. [11] introduced a new definition of a fractional derivative called conformable fractional derivative (see Definition 1). This derivative is much easier to handle, is well-behaved, and obeys the Leibniz rule and chain rule [12].

In short time, this new fractional derivative definition has attracted many researchers. In [13], Chung used the conformable fractional derivative and integral to discuss fractional Newtonian mechanics.

In [14], the authors proved a generalized Lyapunov-type inequality for a conformable *BVP* of order $1 < \alpha \le 2$. They have established that if the *BVP*

$$\begin{gathered} T_\alpha^a u(t) + q(t)u(t) = 0, \quad a < t < b,\ 1 < \alpha \le 2 \\ u(a) = u(b) = 0, \end{gathered} \tag{9}$$

where T_α^a is the conformable derivative of order $\alpha \in (1,2]$, has a nontrivial continuous solution, then

$$\int_a^b |q(r)|\,dr > \frac{\alpha^\alpha}{[(\alpha-1)(b-a)]^{\alpha-1}}. \tag{10}$$

For other generalizations and extensions of the classical Lyapunov's inequality, we refer the reader to [2, 5, 15–23] and the references therein.

In this paper, we establish new Hartman-type and Lyapunov-type inequalities for the following conformable fractional *BVP*:

$$\begin{gathered} T_\alpha^a u(t) + q(t)u(t) = 0, \quad a < t < b,\ 3 < \alpha \le 4 \\ u(a) = u'(a) = u''(a) = u''(b) = 0, \end{gathered} \tag{11}$$

where T_α^a is the conformable derivative starting at a of order $3 < \alpha \le 4$ and q is a real-valued continuous function on $[a,b]$. Some applications to the corresponding eigenvalue problem are discussed. The obtained results are new in the context of conformable fractional derivatives.

The outline of the paper is as follows. In Section 2, we recall and collect basic properties on conformable derivatives. This allows us to construct Green's function of the corresponding linear problem. Some properties of this Green's function are established. In Section 3, we state and prove our main results. Some applications are discussed.

2. Preliminaries on Conformable Derivatives

In this section, we recall some basic definitions and lemmas, which will be very useful to state our results.

Definition 1 (see [11, 12]). For a given function $h : [a,\infty) \longrightarrow \mathbb{R}$, the conformable fractional derivative of h of order $\alpha \in (0,1]$ is defined by

$$T_\alpha^a h(t) = \lim_{\varepsilon \longrightarrow 0} \frac{h\left(t + \varepsilon (t-a)^{1-\alpha}\right) - h(t)}{\varepsilon}, \quad \text{for all } t > a. \tag{12}$$

If $a = 0$, we write T_α. If $T_\alpha^a h(t)$ exists on (a,b), then define $T_\alpha^a h(a) = \lim_{t\longrightarrow a^+} T_\alpha^a h(t)$. The geometric and physical interpretation of the conformable fractional derivatives was given in Zhao [24].

Remark 2. (i) Let $0 < \alpha \le 1$ and h be differentiable function at $t > a$, and then

$$T_\alpha^a h(t) = (t-a)^{1-\alpha} h'(t). \tag{13}$$

(ii) For $h(t) = 2\sqrt{t-a}$, we have $T_{1/2}h(t) = 1$, for all $t > a$. Therefore $T_{1/2}h(a) = 1$, but h is not differentiable at a.

Some important properties for the conformable fractional derivative given in [11, 12] are as follows.

Theorem 3. *Let $\alpha \in (0,1]$ and f, g be α-differentiable at a point t, and then*

(i) $T_\alpha^a(\lambda f + \mu g) = \lambda T_\alpha^a(f) + \mu T_\alpha^a(g)$, for all $\lambda, \mu \in \mathbb{R}$.
(ii) $T_\alpha^a((s-a)^\mu)(t) = \mu(t-a)^{\mu-\alpha}$, for all $t > a$ and $\mu \in \mathbb{R}$.
(iii) $T_\alpha^a(fg) = fT_\alpha^a(g) + gT_\alpha^a(f)$.
(iv) $T_\alpha^a(f/g) = (fT_\alpha^a(g) - gT_\alpha^a(f))/g^2$.
(v) Assume further that the function g is defined in the range of f, and then for all t with $t \ne a$ and $g(t) \ne 0$, one has the following Chain Rule:

$$T_\alpha^a (f \circ g)(t) = T_\alpha^a f(g(t)) . T_\alpha^a g(t) . (g(t))^{\alpha-1} \tag{14}$$

The following conformable fractional derivatives of certain functions [11] are worth noting:

(i) $T_\alpha^a((1/\alpha)(t-a)^\alpha) = 1$.
(ii) $T_\alpha^a(\sin(1/\alpha)(t-a)^\alpha) = \cos(1/\alpha)(t-a)^\alpha$.
(iii) $T_\alpha^a(\cos(1/\alpha)(t-a)^\alpha) = -\sin(1/\alpha)(t-a)^\alpha$.
(iv) $T_\alpha^a(e^{(1/\alpha)(t-a)^\alpha}) = e^{(1/\alpha)(t-a)^\alpha}$.

Definition 4 (see [11, 12]). Let $n < \alpha \le n+1$ and $h : [a,\infty) \longrightarrow \mathbb{R}$ be a function such that $h^{(n)}(t)$ exists. The conformable fractional derivative of h of order α is defined by

$$T_\alpha^a h(t) = \left(T_\gamma^a h^{(n)}\right)(t), \quad \text{for which } \gamma = \alpha - n. \tag{15}$$

Definition 5 (see [11, 12]). Let $n < \alpha \le n+1$. The fractional integral of a function $h : [a,\infty) \longrightarrow \mathbb{R}$ of order α is defined by

$$\left(I_\alpha^a h\right)(t) = \frac{1}{n!}\int_a^t (t-s)^n (s-a)^{\alpha-n-1} h(s)\,ds. \tag{16}$$

Lemma 6 (see [11, 12]). *Let $\alpha \in (n, n+1]$.*

(i) *If h is continuous on $[a,\infty)$, then*

$$T_\alpha^a \left(I_\alpha^a h\right)(t) = h(t), \quad \text{for all } t \ge a. \tag{17}$$

(ii) *$T_\alpha^a h(t) = 0$ if and only if $h(t) = \sum_{k=0}^n c_k (t-a)^k$, where $c_k \in \mathbb{R}$, for $k = 0, 1, \ldots, n$.*

(iii) *If $T_\alpha^a h$ is continuous on $[a,\infty)$, then, for $t > a$,*

$$I_\alpha^a (T_\alpha^a h)(t) = h(t) + c_0 + c_1 (t-a) + \cdots + c_n (t-a)^n, \quad (18)$$

where $c_k \in \mathbb{R}$, for $k = 0, 1, \ldots, n$.

Lemma 7. *Let $\alpha \in (3,4]$ and $h \in C([a,b])$. Then the* BVP

$$\begin{gathered} T_\alpha^a u(t) = -h(t), \quad a < t < b, \\ u(a) = u'(a) = u''(a) = u''(b) = 0, \end{gathered} \quad (19)$$

admits a solution $u \in C([a,b], \mathbb{R})$ if and only if

$$u(t) = \int_a^b G(t,s) h(s)\, ds, \quad (20)$$

where $G(t,s)$ is Green's function defined as

$$G(t,s) = \frac{1}{6}(s-a)^{\alpha-4} \cdot \begin{cases} \dfrac{(t-a)^3 (b-s)}{b-a} - (t-s)^3, & a \le s \le t \le b, \\ \dfrac{(t-a)^3 (b-s)}{b-a}, & a \le t \le s \le b. \end{cases} \quad (21)$$

Proof. Using Lemma 6 and Definition 5, we deduce that $u \in C([a,b], \mathbb{R})$ is a solution of problem (19) if and only if

$$\begin{aligned} u(t) &= c_0 + c_1 (t-a) + c_2 (t-a)^2 + c_3 (t-a)^3 \\ &\quad - \frac{1}{6} \int_a^t (t-s)^3 (s-a)^{\alpha-4} h(s)\, ds, \end{aligned} \quad (22)$$

where $(c_0, c_1, c_2, c_3) \in \mathbb{R}^4$.

This, together with the boundary conditions, implies $c_0 = c_1 = c_2 = 0$ and

$$c_3 = \frac{1}{6(b-a)} \int_a^b (b-s)(s-a)^{\alpha-4} h(s)\, ds. \quad (23)$$

Hence

$$\begin{aligned} u(t) &= \frac{1}{6(b-a)} \int_a^b (t-a)^3 (b-s)(s-a)^{\alpha-4} h(s)\, ds \\ &\quad - \frac{1}{6} \int_a^t (t-s)^3 (s-a)^{\alpha-4} h(s)\, ds, \\ &= \int_a^b G(t,s) h(s)\, ds, \end{aligned} \quad (24)$$

where $G(t,s)$ is given in (21). □

Lemma 8. *Let $\alpha \in (3,4]$. The following property is satisfied by Green's function (21): For any (t,s) in $[a,b] \times [a,b]$,*

$$\begin{aligned} 0 &\le G(t,s) \le G(b,s) \\ &= \frac{1}{6}(s-a)^{\alpha-3} (b-s)(2b-a-s). \end{aligned} \quad (25)$$

Proof. Fix s in $[a,b]$. By differentiating $G(t,s)$ with respect to t, we obtain

$$\partial_t G(t,s) = \frac{1}{2}(s-a)^{\alpha-4} \cdot \begin{cases} \dfrac{(t-a)^2 (b-s)}{b-a} - (t-s)^2, & a \le s \le t \le b, \\ \dfrac{(t-a)^2 (b-s)}{b-a}, & a \le t \le s \le b. \end{cases} \quad (26)$$

Hence $a \le t \le s \le b$, and we have

$$\partial_t G(t,s) = \frac{1}{2}(s-a)^{\alpha-4} \frac{(t-a)^2 (b-s)}{b-a} \ge 0, \quad (27)$$

and for $a \le s \le t \le b$, we have

$$\begin{aligned} \partial_t G(t,s) &= \frac{1}{2}(s-a)^{\alpha-4} \left(\frac{(t-a)^2 (b-s)}{b-a} - (t-s)^2 \right) \\ &= \frac{1}{2}(s-a)^{\alpha-4} \frac{(t-a)^2 (b-s)}{b-a} \left[1 \right. \\ &\quad \left. - \left(\frac{b-s}{b-a} \right) \left(\frac{(b-a)(t-s)}{(b-s)(t-a)} \right)^2 \right]. \end{aligned} \quad (28)$$

Using the fact that $((b-s)/(b-a))$ and $(b-a)(t-s)/(b-s)(t-a)$ are in $[0,1]$, we deduce that

$$\partial_t G(t,s) \ge 0, \quad \text{for } a \le s \le t \le b. \quad (29)$$

So, the function $t \longrightarrow G(t,s)$ is nondecreasing on $[a,b]$. This implies that

$$\begin{aligned} 0 = G(a,s) &\le G(t,s) \le G(b,s), \\ &\qquad (t,s) \in [a,b] \times [a,b]. \end{aligned} \quad (30)$$

The proof is completed. □

3. Main Results

Theorem 9 (Hartman-type inequality). *Assume that the* BVP *(11) has a nontrivial continuous solution; then*

$$\int_a^b (s-a)^{\alpha-3} (b-s)(2b-a-s) \left| q(s) \right| ds \ge 6. \quad (31)$$

Proof. Let $\alpha \in (3,4]$, $a, b \in \mathbb{R}$ with $a < b$ and $q \in C([a,b], \mathbb{R})$.

Consider the Banach space $C([a,b], \mathbb{R})$, equipped with the uniform norm $\|u\|_\infty = \sup_{t \in [a,b]} |u(t)|$.

Assume that problem (11) has a nontrivial solution $u \in C([a,b], \mathbb{R})$.

By (20), we have

$$u(t) = \int_a^b G(t,s) q(s) u(s)\, ds, \quad t \in [a,b]. \quad (32)$$

Note that since u is nontrivial, then q cannot be the zero function on $[a,b]$. This, with Lemma 8, implies, for all $t \in [a,b]$,

$$\begin{aligned}|u(t)| &\leq \int_a^b G(t,s)\left|q(s)\right| |u(s)|\, ds \\ &\leq \|u\|_\infty \left(\int_a^b G(b,s)\left|q(s)\right| ds\right).\end{aligned} \tag{33}$$

Therefore

$$\|u\|_\infty \leq \|u\|_\infty \left(\int_a^b G(b,s)\left|q(s)\right| ds\right). \tag{34}$$

Now, since $\|u\|_\infty \neq 0$, then we deduce that

$$1 \leq \left(\int_a^b G(b,s)\left|q(s)\right| ds\right), \tag{35}$$

where

$$G(b,s) = \frac{1}{6}(s-a)^{\alpha-3}(b-s)(2b-a-s), \quad \text{for } s \in [a,b]. \tag{36}$$

The proof is completed. □

Corollary 10. *Assume that the* BVP *(11) has a nontrivial continuous solution; then*

$$\int_a^b (s-a)^{\alpha-3}(b-s)\left|q(s)\right| ds \geq \frac{3}{b-a}. \tag{37}$$

Proof. The property follows from Theorem 9 and the fact

$$(2b-a-s) \leq 2(b-a), \quad \text{for } s \in [a,b]. \tag{38}$$

□

Theorem 11 (Lyapunov-type inequality). *Assume that the* BVP *(11) has a nontrivial continuous solution; then*

$$\int_a^b \left|q(s)\right| ds \geq \frac{3(\alpha-2)^{\alpha-2}}{(\alpha-3)^{\alpha-3}(b-a)^{\alpha-1}}. \tag{39}$$

Proof. From Corollary 10, we have

$$\int_a^b \varphi(s)\left|q(s)\right| ds \geq \frac{3}{b-a}, \tag{40}$$

where $\varphi(s) = (s-a)^{\alpha-3}(b-s)$, $s \in [a,b]$.

By simple computation, one can check that

$$\begin{aligned}\max_{s\in[a,b]} \varphi(s) &= \varphi\left(\frac{a+(\alpha-3)b}{\alpha-2}\right) \\ &= (\alpha-3)^{\alpha-3}\left(\frac{b-a}{\alpha-2}\right)^{\alpha-2}.\end{aligned} \tag{41}$$

This fact with (40) gives the required result. □

Corollary 12. *If λ is an eigenvalue to the fractional* BVP

$$\begin{aligned}&T_\alpha u(t) + \lambda u(t) = 0, \quad 0 < t < 1, \\ &u(0) = u'(0) = u''(0) = u''(1) = 0,\end{aligned} \tag{42}$$

then

$$|\lambda| \geq 6\frac{\alpha(\alpha-1)(\alpha-2)}{\alpha+2}. \tag{43}$$

Proof. By using Theorem 9, we obtain

$$|\lambda| \int_0^1 s^{\alpha-3}(1-s)(2-s)\, ds \geq 6. \tag{44}$$

Now, by simple computation, we have

$$\int_0^1 s^{\alpha-3}(1-s)(2-s)\, ds = \frac{\alpha+2}{\alpha(\alpha-1)(\alpha-2)}. \tag{45}$$

This gives inequality (43). □

Corollary 13. *Let $3 < \alpha \leq 4$ and $q \in C([0,1],\mathbb{R})$, such that*

$$\int_0^1 \left|q(r)\right| dr < \frac{3(\alpha-2)^{\alpha-2}}{(\alpha-3)^{\alpha-3}}. \tag{46}$$

Then the fractional BVP

$$\begin{aligned}&T_\alpha u(t) + q(t)u(t) = 0, \quad 0 < t < 1, \\ &u(0) = u'(0) = u''(0) = u''(1) = 0,\end{aligned} \tag{47}$$

has no nontrivial solution.

Proof. The assertion follows from Theorem 11. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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