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Research Article

Functional Inequalities for Generalized Complete Elliptic Integrals with Two Parameters

Xiangkai Dou,[1] Li Yin,[1] and Xiu-Li Lin[2]

[1]*College of Science, Binzhou University, Binzhou City, Shandong Province, 256603, China*
[2]*College of Mathematics Science, Qufu Normal University, Qufu City, Shandong Province, 273165, China*

Correspondence should be addressed to Li Yin; yinli_79@163.com

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In this paper, we establish some functional inequalities for generalized complete elliptic integrals with two parameters, such as estimation of bounds and mean inequalities. Our main results give (p,q)-analogues to the early results for classical complete elliptic integrals.

1. Introduction

For the given complex numbers a, b, c with $c \neq 0, -1, -2, \cdots$, the *Gaussian hypergeometric function* is defined by

$$F(a,b;c;z) = {}_2F_1(a,b;c;z) = \sum_{n\geqslant 0} \frac{(a,n)(b,n)}{(c,n)} \frac{z^n}{n!}, \quad |z| < 1. \tag{1}$$

Here $(a,0) = 1$ for $(a \neq 0)$, and (a,n) for $n \in \mathbf{N}$ is the shifted factorial or Appell symbol $(a,n) = a(a+1)\cdots(a+n-1)$. The following is the integral representation formula of the Gaussian hypergeometric function:

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt \tag{2}$$

$\mathrm{Re}(c) > \mathrm{Re}(a) > 0, |\arg(1-z)| < \pi$; see [1]. For more information on the history, background, properties, and applications, please refer to [1, 2] and related references. Here, we give the following definitions of some classical functions. For real number $x, y > 0$, the Euler gamma function Γ and its logarithmic derivative ψ, the so-called digamma function, are defined by (cf. [1, 3])

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt,$$
$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \tag{3}$$
$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

respectively.

Recently, Takeuchi [4] studied the (p,q)-trigonometric functions depending on two parameters. For $p = q$, these functions reduce to the so-called p-trigonometric functions introduced by Lindqvist [5]. The following (p,q)-eigenvalue problem with Dirichlét boundary condition was considered by Drábek and Manásevich [3]. Let $\phi_p(x) = |x|^{p-2}x$. For $T, \lambda > 0$ and $p, q > 1$

$$\left(\phi_p\left(u'\right)\right)' + \lambda\phi_q(u) = 0, \quad t \in (0,T), \qquad u(0) = u(T) = 0, \tag{4}$$

They found the complete solution to this problem. The solution to this problem also appears in [4, Thm 2.1]. In

particular, for $T = \pi_{p,q}$ the function $u(t) = \sin_{p,q}(t)$ is a solution to this problem with $\lambda = p/q(p-1)$, where

$$\pi_{p,q} = \int_0^1 \left(1 - t^q\right)^{-1/p} dt = \frac{2}{q} B\left(1 - \frac{1}{p}, \frac{1}{q}\right). \tag{5}$$

For $p = q$, $\pi_{p,q}$ reduces to π_p; see [6]. In order to give the definition of the function $\sin_{p,q}$, first of all we define its inverse function $\arcsin_{p,q}$ and then the function itself. For $x \in [0,1]$, set

$$F_{p,q}(x) = \arcsin_{p,q}(x) = \int_0^x \left(1 - t^q\right)^{-1/p} dt. \tag{6}$$

The function $F_{p,q} : [0,1] \longrightarrow [0, \pi_{p,q}/2]$ is an increasing homeomorphism, and

$$\sin_{p,q} = F_{p,q}^{-1} \tag{7}$$

is defined on the interval $[0, \pi_{p,q}/2]$. The function $\sin_{p,q}$ can be extended to $[0, \pi_{p,q}]$ by

$$\sin_{p,q}(x) = \sin_{p,q}\left(\pi_{p,q} - x\right), \quad x \in \left[\frac{\pi_{p,q}}{2}, \pi_{p,q}\right]. \tag{8}$$

By oddness, the further extension can be made to $[-\pi_{p,q}, \pi_{p,q}]$. Finally, the function $\sin_{p,q}$ is extended to whole $\mathbb{R}$ by $2\pi_{p,q}$-periodicity; see [7]. Similarly, the other generalized inverse trigonometric and hyperbolic functions that appeared in the current paper are defined as follows:

$$\begin{aligned} \arctan_p(x) &= \int_0^x \left(1 + t^p\right)^{-1} dt \\ \arcsin h_p(x) &= \int_0^x \left(1 + t^p\right)^{-1/p} dt \\ \arctan h_p(x) &= \int_0^x \left(1 - t^p\right)^{-1} dt. \end{aligned} \tag{9}$$

For the expression of the above generalized inverse trigonometric and hyperbolic functions in terms of hypergeometric functions, we have the following formula [8, 9]:

$$\begin{aligned} \arcsin_p(x) &= xF\left(\frac{1}{p}, \frac{1}{p}; 1 + \frac{1}{p}; x^p\right) \\ \arctan h_p(x) &= xF\left(1, \frac{1}{p}; 1 + \frac{1}{p}; x^p\right) \\ \arcsin_{p,q}(x) &= xF\left(\frac{1}{p}, \frac{1}{q}; 1 + \frac{1}{q}; x^q\right). \end{aligned} \tag{10}$$

Based on the above definitions, we simply introduce generalized elliptic integrals defined by Takeuchi [10] with two parameters. For all $p, q \in (1, \infty)$ and $r \in (0,1)$, the complete (p,q)-elliptic integrals of the first and second kinds [11] are defined by

$$\begin{aligned} K_{p,q}(r) &:= \int_0^{\pi_{p,q}/2} \left(1 - r^q \sin_{p,q}^q t\right)^{(1/p-1)} dt, \\ K'_{p,q} &= K'_{p,q}(r) = K_{p,q}\left(r'\right) \end{aligned} \tag{11}$$

and

$$\begin{aligned} E_{p,q}(r) &:= \int_0^{\pi_{p,q}/2} \left(1 - r^q \sin_{p,q}^q t\right)^{1/p} dt, \\ E'_{p,q} &= E'_{p,q}(r) = E_{p,q}\left(r'\right), \end{aligned} \tag{12}$$

respectively. Here, $p, q > 1, r \in (0,1)$ and $r' = (1 - r^q)^{1/q}$. The complete (p,q)-elliptic integrals can be expressed in terms of hypergeometric functions as follows:

$$K_{p,q}(r) = \frac{\pi_{p,q}}{2} F\left(\frac{1}{q}, 1 - \frac{1}{p}; 1 - \frac{1}{p} + \frac{1}{q}; r^p\right), \tag{13}$$

and

$$E_{p,q}(r) = \frac{\pi_{p,q}}{2} F\left(\frac{1}{q}, -\frac{1}{p}; 1 - \frac{1}{p} + \frac{1}{q}; r^p\right). \tag{14}$$

If $p = q = 2$, we can obtain the classically complete elliptic integrals as follows: for $r \in (0,1)$,

$$\begin{aligned} K(r) &= \int_0^{\pi/2} \left(1 - r^2 \sin^2 t\right)^{-1/2} dt = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; r^2\right), \\ E(r) &= \int_0^{\pi/2} \left(1 - r^2 \sin^2 t\right)^{1/2} dt \\ &= \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}; 1; r^2\right). \end{aligned} \tag{15}$$

If $p = q$, we can obtain the generalized complete elliptic integrals with single parameters

$$\begin{aligned} K_p(r) &= \int_0^{\pi_p/2} \left(1 - r^p \sin^p t\right)^{-1/p} dt \\ &= \frac{\pi_p}{2} F\left(\frac{1}{p}, 1 - \frac{1}{p}; 1; r^p\right), \\ E_p(r) &= \int_0^{\pi_p/2} \left(1 - r^p \sin^p t\right)^{1/p} dt \\ &= \frac{\pi_p}{2} F\left(-\frac{1}{p}, \frac{1}{p}; 1; r^p\right). \end{aligned} \tag{16}$$

It is worth noting that Bhayo and Yin also gave new (p,q)-complete elliptic integrals in [12]. For more on this topic, the readers can see related references [12–25]. Some recent results regarding the case of semielliptical crack in a cylindrical rod for viscoelastic medium can be found in [26, 27].

2. Lower and Upper Bounds for Generalized Complete Elliptic Integrals

In 1992, Anderson et al. [28] discovered that the complete elliptic integral of the first kind can be approximated by the inverse hyperbolic tangent function:

$$\arctan h(r) = rF\left(\frac{1}{2}, 1; \frac{3}{2}; r^2\right). \tag{17}$$

For $r \in (0,1)$, they proved

$$\frac{\pi}{2}\left(\frac{\arctan h(r)}{r}\right)^{1/2} < K(r) < \frac{\pi}{2}\frac{\arctan h(r)}{r}. \tag{18}$$

Later, Qi and Huang [29] obtained the following inequality by using Chebyshev inequality:

$$\frac{\pi}{2}\frac{\arcsin(r)}{r} < K(r) < \frac{\pi}{2}\frac{\arctan h(r)}{r}. \tag{19}$$

In 2004, Alzer and Qiu [30] proved that, for $r \in (0,1)$, we have

$$\frac{\pi}{2}\left(\frac{\arctan h(r)}{r}\right)^{\alpha} < K(r) < \frac{\pi}{2}\left(\frac{\arctan h(r)}{r}\right)^{\beta} \tag{20}$$

with the best possible constants $\alpha = 3/4$ and $\beta = 1$. Here, we shall show some new inequalities for the generalized complete elliptic integrals.

Lemma 1 (see [31, Lemma 1]). *Consider the power series $f(x) = \sum_{n\geq 0} a_n x^n$ and $g(x) = \sum_{n\geq 0} b_n x^n$, where $a_n \in \mathbf{R}$ and $b_n > 0$ for all $n \in \mathbf{N}\backslash\{0\}$, and suppose that both converge on $(-r,r), r > 0$. If the sequence $\{a_n/b_n\}_{n\geq 0}$ is increasing (decreasing), then the function $x \longmapsto f(x)/g(x)$ is increasing (decreasing) on $(0,r)$.*

Theorem 2. *Let $p,q > 1, r \in (0,1)$ and let $K_{p,q}, E_{p,q}$ be as (11) and (12). Then we have the following:*

(1) The function $rE_{p,q}(r)/\arcsin_{p,q}(r)$ is strictly decreasing from $(0,1)$ onto

$$\left(\frac{\pi_{p,q}}{2}\left[\frac{\Gamma(1+1/p+1/q)}{\Gamma(1+1/q)\Gamma(1-1/p)}\right]^2, \frac{\pi_{p,q}}{2}\right) \tag{21}$$

(2) The function $rK_{p,q}(r)/\arctan h_q(r)$ is strictly decreasing from $(0,1)$ onto $(1,\pi_{p,q}/2)$

(3) The function $rE_{p,q}(r)/\arctan h_q(r)$ is strictly decreasing from $(0,1)$ onto $(0,\pi_{p,q}/2)$

(4) The function $E_{p,q}(r)/(2/(1+\sqrt{1-x^q}))^{1/2}$ is strictly decreasing from $(0,1)$ onto

$$\left(\frac{\Gamma(1-1/p+1/q)}{\sqrt{2}\Gamma(1+1/q)\Gamma(1-1/p)}, 1\right) \tag{22}$$

Proof. The proofs of assertion (1)-(4) are similar to each other. Here we mainly prove (1) for the sake of simplicity. First of all, we consider the function $\alpha_{p,q} : (0,1) \longrightarrow (0,\infty)$ defined by

$$\begin{aligned}\alpha_{p,q}(r) &= \frac{\pi_{p,q}}{2}\frac{F(-1/p,1/q;1-1/p+1/q;r^q)}{F(1/p,1/q;1+1/q;r^q)} = \frac{\pi_{p,q}}{2} \\ &\cdot \frac{\sum_{n=0}^{\infty}((-1/p)_n(1/q)_n/(1+1/q-1/p)_n)(r^{qn}/n!)}{\sum_{n=0}^{\infty}((1/p)_n(1/q)_n/(1+1/q)_n)(r^{qn}/n!)}.\end{aligned} \tag{23}$$

Considering Lemma 1, we only need to discuss the monotonicity of the sequence $\{\omega_n\}_{n\geq 0}$ defined by

$$\omega_n = \frac{(-1/p)_n(1+1/q)_n}{(1/p)_n(1+1/q-1/p)_n}. \tag{24}$$

Since

$$\begin{aligned}&\frac{\omega_{n+1}}{\omega_n} = \frac{(-1/p+n)(1+1/q+n)}{(1/p+n)(1-1/p+1/q+n)} \leq 1 \Longleftrightarrow \\ &\left(2-\frac{1}{p}\right)\frac{1}{p} + \frac{2}{pq} + \frac{n}{p} > 0,\end{aligned} \tag{25}$$

we get that the sequence $\{\omega_n\}_{n\geq 0}$ is strictly decreasing. On the other hand, making use of Gauss formula

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{26}$$

we may conclude that

$$\begin{aligned}\alpha_{p,q}(0^+) &= \frac{\pi_{p,q}}{2}, \\ \alpha_{p,q}(1^-) &= \frac{\pi_{p,q}}{2}\left[\frac{\Gamma(1+1/p+1/q)}{\Gamma(1+1/q)\Gamma(1-1/p)}\right]^2.\end{aligned} \tag{27}$$

The proof of assertion (1) is complete. Due to (4), we only note the known identity

$$\left(\frac{2}{(1+\sqrt{1-x^q})}\right)^{1/2} = F\left(\frac{1}{4},\frac{3}{4};\frac{3}{2},x^q\right). \tag{28}$$

Similar method can complete the proof. Here we omit the details. □

As a direct result of Theorem 2, we have the following Corollary 3.

Corollary 3. *For $p,q > 1$ and $r \in (0,1)$, we have*

(1) $(\pi_{p,q}/2)[\Gamma(1+1/p+1/q)/\Gamma(1+1/q)\Gamma(1-1/p)]^2((\arcsin_{p,q}(r))/r) < E_{p,q}(r) < (\pi_{p,q}/2)((\arcsin_{p,q}(r))/r)$

(2) $(\arctan h_q(r))/r < K_{p,q}(r) < (\pi_{p,q}/2)((\arctan h_q(r))/r)$

(3) $E_{p,q}(r) < (\pi_{p,q}/2)((\arctan h_q(r))/r)$

(4) $\Gamma(1-1/p+1/q)/\Gamma(1+1/q)\Gamma(1-1/p)(1/\sqrt{(1+\sqrt{1-r^q})}) < E_{p,q}(r) < \sqrt{2}/\sqrt{(1+\sqrt{1-r^q})}$

where the constants in the above inequalities are the best possible.

Theorem 4. *Let $p \geq 2, q > 1$, and $r \in (0,1)$. The function $rK_{p,q}(r)/\arcsin_{p,q}(r)$ is strictly increasing from $(0,1)$ onto $(\pi_{p,q}/2,\infty)$. As a result, we have the following inequality:*

$$K_{p,q}(r) > \frac{\pi_{p,q}}{2}\frac{\arcsin_{p,q}(r)}{r}. \tag{29}$$

Proof. Define

$$\begin{aligned}\beta_{p,q}(r) &= \frac{\pi_{p,q}}{2}\frac{F(1-1/p,1/q;1-1/p+1/q;r^p)}{F(1/p,1/q;1+1/q;r^p)} \\ &= \frac{\pi_{p,q}}{2} \\ &\cdot \frac{\sum_{n=0}^{\infty}((1-1/p)_n(1/q)_n/(1+1/q-1/p)_n)(r^n/n!)}{\sum_{n=0}^{\infty}((1/p)_n(1/q)_n/(1+1/q)_n)(r^n/n!)}.\end{aligned} \tag{30}$$

And $\{\Omega_n\}_{n\geq 0}$ is defined by

$$\Omega_n = \frac{(1-1/p)_n (1+1/q)_n}{(1/p)_n (1+1/q-1/p)_n}. \tag{31}$$

Simple computation yields

$$\begin{gathered}\frac{\Omega_{n+1}}{\Omega_n} = \frac{(1-1/p+n)(1+1/q+n)}{(1/p+n)(1-1/p+1/q+n)} \geq 1 \Longleftrightarrow \\ \left(1-\frac{1}{p}\right)n + \left(1+\frac{1}{q}\right)\left(1-\frac{2}{p}\right) + \frac{1}{p^2} > 0.\end{gathered} \tag{32}$$

This implies that the sequence $\{\Omega_n\}_{n\geq 0}$ is strictly increasing. Using Lemma 1, we obtain that the function $\beta_{p,q}$ is strictly increasing on $(0,1)$. The limiting value follows easily. This completes the proof. □

Remark 5. Corollary 3 and Theorem 4 give (p,q)-analogues to the right of (18), (20), and inequality (19). Considering the left of (20), we naturally pose the following open question: for $p,q \geq 2$ and $r \in (0,1)$, the following inequality holds true:

$$K_{p,q}(r) > \frac{\pi_{p,q}}{2}\left(\frac{\arctan h_q(r)}{r}\right)^{(p+q-1)/pq}. \tag{33}$$

Theorem 6. *Let $p,q > 1$ and $r \in (0,1)$. Then*

$$\begin{gathered}1 - \frac{2\log(1-r^q)}{q\pi_{p,q}} < K_{p,q}(r) \\ < 1 - \frac{(p-1)\log(1-r^q)}{pq+p-q}.\end{gathered} \tag{34}$$

Proof. Using the known fact [32], for $a,b > 0$, the function $(1-F(a,b;a+b;x))/\log(1-x)$ is strictly increasing from $(0,1)$ onto $(ab/(a+b), 1/B(a,b))$. We easily complete the proof by taking $a = 1-1/p, b = 1/q, x = r^q$. □

Theorem 7. *Let $1 < p \leq 4/3, q > 1$ and $r \in (0,1)$. Then*

$$\frac{\pi_{p,q}}{2} < K_{p,q}(r) < \frac{\pi_{p,q}}{2(1-r^q)^{1/4}}. \tag{35}$$

Proof. We apply the following fact [32]: for $a,b > 0$, the function $(1-x)^{1/4}F(a,b;a+b;x)$ is strictly decreasing on $(0,1)$ if and only if $4ab \leq a+b$. By taking $a = 1-1/p, b = 1/q, x = r^q$, we easily verify the condition based on $1 < p \leq 4/3, q > 1$. Due to limiting value, we have

$$\begin{gathered}\lim_{r\longrightarrow 0^+}(1-r^q)^{1/4}K_{p,q}(r) = \frac{\pi_{p,q}}{2}, \\ \lim_{r\longrightarrow 1^-}(1-r^q)^{1/4}K_{p,q}(r) = 0,\end{gathered} \tag{36}$$

by using the formula $F(a,b;a+b;x) \sim -(1/B(a,b))\log(1-x), x \longrightarrow 1$. So, the proof is complete. □

3. Mean Inequalities for Generalized Complete Elliptic Integrals

For two distinct positive real numbers x and y, the arithmetic mean, geometric mean, logarithmic mean, and identric mean are, respectively, defined by

$$\begin{aligned}A(x,y) &= \frac{x+y}{2}, \\ G(x,y) &= \sqrt{xy}, \\ L(x,y) &= \frac{x-y}{\log(x)-\log(y)}, \quad x \neq y, \\ H(x,y) &= \frac{1}{A(1/x,1/y)}, \\ I(x,y) &= \frac{1}{e}\left(\frac{x^x}{y^y}\right)^{1/(x-y)}, \quad x \neq y.\end{aligned} \tag{37}$$

Let $f : I \longrightarrow (0,\infty)$ be continuous, where I is a subinterval of $(0,\infty)$. Let M and N be the means defined above; then we call that the function f is MN-convex (concave) if

$$f(M(x,y)) \leq (\geq) N(f(x), f(y)) \tag{38}$$

for all $x,y \in I$. Recently, generalized convexity/concavity with respect to general mean values has been studied by Anderson et al. in [28]. In [33], Baricz studied that if the function f is differentiable, then it is (a,b)-convex (concave) on I if and only if $x^{1-a}f'(x)/f(x)^{1-b}$ is increasing (decreasing). In [34], Bhayo and Vuorinen established all kinds of mean inequalities for the generalized trigonometric functions. In this section, we shall show logarithmic and identic means inequalities for the generalized complete elliptic integrals by using Chebyshev inequality.

Lemma 8 (see [29]). *Let $f,g : [a,b] \longrightarrow R$ be integrable functions, both increasing or both decreasing. Furthermore, let $p : [a,b] \longrightarrow R$ be a positive, integrable function. Then*

$$\begin{gathered}\int_a^b p(x)f(x)\,dx \cdot \int_a^b p(x)g(x)\,dx \\ \leq \int_a^b p(x)\,dx \cdot \int_a^b p(x)f(x)g(x)\,dx.\end{gathered} \tag{39}$$

If one of the functions f or g is nonincreasing and the other is nondecreasing, then the inequality in (39) is reversed.

Lemma 9 (see [35]). *For $a \leq y < x \leq b$, then*

$$\varphi\left(\frac{x+y}{2}\right) \leq \frac{1}{x-y}\int_y^x \varphi(u)\,du \leq \frac{\varphi(x)+\varphi(y)}{2} \tag{40}$$

if the function $\varphi(x)$ is convex on $[a,b]$.

Theorem 10. *Let $p,q > 1$ and $r,s \in (0,1)$. Then*

$$K_{p,q}(L(r,s)) \leq L\left(K_{p,q}(r), K_{p,q}(s)\right). \tag{41}$$

Proof. Let us suppose $r \le s$ without loss of generality. Define

$$f(r) = \frac{1}{\left(1 - r^q \left(\sin_{p,q} x\right)^q\right)^{1-1/p}}. \tag{42}$$

Simple computation yields

$$\begin{aligned}
(\log f(r))' &= \left(1 - \frac{1}{p}\right) \frac{q r^{q-1} \left(\sin_{p,q} x\right)^q}{1 - r^q \left(\sin_{p,q} x\right)^q}, \\
(\log f(r))'' &= \left(1 - \frac{1}{p}\right) q \left(\sin_{p,q} x\right)^q \\
&\quad \cdot \frac{(q-1) r^{q-2} \left(1 - r^q \left(\sin_{p,q} x\right)^q\right) + q r^{2q-2} \left(\sin_{p,q} x\right)^q}{\left(1 - r^q \left(\sin_{p,q} x\right)^q\right)^2} \\
&> 0.
\end{aligned} \tag{43}$$

This implies that the function $f(r)$ is strictly log-convex on $(0,1)$. By using the fact that the integral preserves monotonicity and log-convexity, we get that the function $K_{p,q}(r)$ is strictly increasing and log-convex on $(0,1)$. The substitution $t = K_{p,q}(u)$ results in

$$\begin{aligned}
L\left(K_{p,q}(r), K_{p,q}(s)\right) &= \frac{K_{p,q}(r) - K_{p,q}(s)}{\log K_{p,q}(r) - \log K_{p,q}(s)} \\
&= \frac{\int_{K_{p,q}(r)}^{K_{p,q}(s)} 1\, dt}{\int_{K_{p,q}(r)}^{K_{p,q}(s)} 1/t\, dt} \\
&= \frac{\int_r^s \left(K_{p,q}(u)\right)' du}{\int_r^s \left(K_{p,q}(u)\right)' / K_{p,q}(u)\, du}.
\end{aligned} \tag{44}$$

Since the functions $K_{p,q}(u)$ and $(K_{p,q}(u))'/K_{p,q}(u)$ are strictly increasing, we may obtain

$$\begin{aligned}
&\int_r^s K_{p,q}(u)\, du \int_r^s \frac{\left(K_{p,q}(u)\right)'}{K_{p,q}(u)} du \\
&\quad \le \int_r^s 1\, du \int_r^s \left(K_{p,q}(u)\right)' du
\end{aligned} \tag{45}$$

by taking $p(u) = 1, f(u) = K_{p,q}(u), g(u) = (K_{p,q}(u))'/K_{p,q}(u)$ in Lemma 8. This is equivalent to

$$\frac{\int_r^s \left(K_{p,q}(u)\right)' du}{\int_r^s \left(K_{p,q}(u)\right)' / K_{p,q}(u)\, du} \ge \frac{\int_r^s K_{p,q}(u)\, du}{s - r}. \tag{46}$$

Noting that log-convexity implies the convexity and Lemma 9, we have

$$\begin{aligned}
\frac{\int_r^s K_{p,q}(u)\, du}{s - r} &\ge K_{p,q}\left(\frac{\int_r^s u\, du}{s - r}\right) = K_{p,q}(A(r,s)) \\
&\ge K_{p,q}(L(r,s))
\end{aligned} \tag{47}$$

where we apply the known inequality $L(r,s) \le A(r,s)$. □

Theorem 11. *Let $p, q > 1$ and $r, s \in (0,1)$. Then*

$$K_{p,q}(I(r,s)) \le I\left(K_{p,q}(r), K_{p,q}(s)\right). \tag{48}$$

Proof. This proof is similar to Theorem 10. We still suppose $r \le s$. Direct calculation yields

$$\begin{aligned}
&\ln I\left(K_{p,q}(r), K_{p,q}(s)\right) \\
&\quad = \frac{K_{p,q}(r) \log K_{p,q}(r) - K_{p,q}(s) \log K_{p,q}(s) - 1}{K_{p,q}(r) - K_{p,q}(s)} \\
&\quad = \frac{\int_{K_{p,q}(r)}^{K_{p,q}(s)} \ln t\, dt}{\int_{K_{p,q}(r)}^{K_{p,q}(s)} 1\, dt} = \frac{\int_r^s \log K_{p,q}(u) \left(K_{p,q}(u)\right)' du}{\int_r^s \left(K_{p,q}(u)\right)' du}
\end{aligned} \tag{49}$$

Since the functions $\log K_{p,q}(u)$ and $(K_{p,q}(u))'$ are strictly increasing, we have

$$\begin{aligned}
&\int_r^s \left(K_{p,q}(u)\right)' du \int_r^s \log K_{p,q}(u)\, du \\
&\quad \le \int_r^s 1\, du \int_r^s \log K_{p,q}(u) \left(K_{p,q}(u)\right)' du
\end{aligned} \tag{50}$$

by taking $p(u) = 1, f(u) = (K_{p,q}(u))', g(u) = \log K_{p,q}(u)$ in Chebyshev inequality of Lemma 8. Applying Lemma 9 and the known inequality $I(r,s) \le A(r,s)$, we easily prove

$$\begin{aligned}
\log I\left(K_{p,q}(r), K_{p,q}(s)\right) &\ge \frac{\int_r^s \log K_{p,q}(u)\, du}{s - r} \\
&\ge \log K_{p,q}\left(\frac{\int_r^s u\, du}{s - r}\right) = \log K_{p,q}(A(r,s)) \\
&\ge \log K_{p,q}(I(r,s)).
\end{aligned} \tag{51}$$

The proof is complete. □

Remark 12. Our method may be an effective way to deal with Baricz's conjecture: if m_1, m_2 are two-variable means, i.e., for $i = 1, 2$ and for all $x, y, \alpha > 0$, we have

$$\begin{aligned}
m_i(x,y) &= m_i(y,x), \\
m_i(x,x) &= x, \\
m_i(\alpha x, \alpha y) &= \alpha m_i(x,y)
\end{aligned} \tag{52}$$

and $x < m_i(x,y) < y$ whenever $x < y$; then find conditions on $a_1, a_2 > 0$ and $c > 0$ for which the inequality

$$m_1\left(F_{a_1}(r), F_{a_2}(r)\right) \le (\ge) F_{m_2(a_1,a_2)}(r) \tag{53}$$

holds true for all $r \in (0,1)$, where $F_a(r) = F(a, c-a; c; r)$.

4. Grünbaum Type Inequalities for Generalized Complete Elliptic Integrals

Lemma 13 (see [9, Lemma 3, p246]). *Let us consider the function $f : (a, \infty) \longrightarrow \mathbf{R}$, where $a \ge 0$. If the function g,*

defined by $g(x) = (1/x)(f(x) - 1)$, is increasing on (a, ∞), then, for the function h, defined by $h(x) = f(x^2)$, we have the following Grünbaum type inequality:

$$1 + h(z) \geq h(x) + h(y), \tag{54}$$

where $x, y \geq a$ and $z^2 = x^2 + y^2$. If the function g is decreasing, then inequality (54) is reversed.

Lemma 14. *As a function of $r \in (0, 1)$, the function $K_{p,q}(r) + c \log r$ is increasing if and only if $c \geq 0$. In particular, the function $K_{p,q}(r)$ is increasing on $r \in (0, 1)$.*

Proof. Simple calculation yields

$$\begin{aligned} &\frac{d}{dr}\left(K_{p,q}(r) + c\log r\right) = \frac{E_{p,q}(r) - (r')^q K_{p,q}(r)}{r(r')^q} + \frac{c}{r} \\ &\geq 0 \Longleftrightarrow \\ &\inf\left\{\frac{E_{p,q}(r) - (r')^q K_{p,q}(r)}{(r')^q}\right\} = 0 \geq -c \Longleftrightarrow \\ &c \geq 0. \end{aligned} \tag{55}$$

This completes the proof. □

Theorem 15. *Let $x, y, c > 0$ and $x^2 + y^2 = z^2$. Then*

$$\begin{aligned} &z^2 K_{p,q}(z^2) + cx^2 \log\left(\frac{z}{x}\right)^2 + cy^2 \log\left(\frac{z}{y}\right)^2 \\ &\geq x^2 K_{p,q}(x^2) + y^2 K_{p,q}(y^2). \end{aligned} \tag{56}$$

Proof. By defining $f(x) = x[K_{p,q}(x) + c\log x] + 1$ and applying Lemmas 13 and 14, the proof can easily be completed. □

Remark 16. Since $K_{p,q}(r)$ is increasing on (0,1) and $E_{p,q}(r)$ is decreasing on (0,1), we also obtain the following Grünbaum type inequalities by applying Lemma 13:

$$\begin{aligned} z^2 K_{p,q}(z^2) &\geq x^2 K_{p,q}(x^2) + y^2 K_{p,q}(y^2), \\ z^2 E_{p,q}(z^2) &\leq x^2 E_{p,q}(x^2) + y^2 E_{p,q}(y^2). \end{aligned} \tag{57}$$

5. Conclusion

We mainly established some functional inequalities for generalized complete elliptic integrals with two parameters. First of all, our results give (p, q)-analogues to the early results for classical complete elliptic integrals. Moreover, we show logarithmic and identic means inequalities by using Chebyshev inequality. Furthermore, we show a Grünbaum type inequality.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the manuscript and read and approved the final manuscript.

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