Research Article

Weighted Estimates for Rough Maximal Functions with Applications to Angular Integrability

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In this note we establish certain weighted estimates for a class of maximal functions with rough kernels along “polynomial curves” on $\mathbb{R}^n$. As applications, we obtain the bounds of the above operators on the mixed radial-angular spaces, on the vector-valued mixed radial-angular spaces, and on the vector-valued function spaces. Particularly, the above bounds are independent of the coefficients of the polynomials in the definition of the operators.

1. Introduction

During the last several years, a considerable amount of attention has been given to the investigation of the boundedness for various kinds of integral operators on the Lebesgue spaces and other more general function spaces (see [1–8], for examples). The primary aim of this article is to establish the boundedness for maximal operators related to singular integrals on the mixed radial-angular spaces.

Let $\mathbb{R}^n$ ($n \geq 2$) be the n -dimensional Euclidean space and S^{n-1} denote the unit sphere in $\mathbb{R}^n$ equipped with the induced Lebesgue measure $d\sigma$. Assume that $\Omega \in L^1(S^{n-1})$ is a homogeneous function of degree zero and satisfies

$$\int_{S^{n-1}} \Omega(u) d\sigma(u) = 0. \quad (1)$$

Let $P_N(t)$ be a real polynomial on $\mathbb{R}$ of degree N satisfying $P_N(0) = 0$ and h be a suitable function defined on $\mathbb{R}$; we define the singular integral operator T_{h,Ω,P_N} along the “polynomial curve” P_N on $\mathbb{R}^n$ by

$$\begin{aligned} & T_{h,\Omega,P_N} f(x) \\ &= \text{p.v.} \int_{\mathbb{R}^n} f(x - P_N(|y|)y') \frac{h(|y|)\Omega(y')}{|y|^n} dy, \end{aligned} \quad (2)$$

where $y' = y/|y|$ for $y \neq 0$. Let $\mathbb{R}_+ := (0, \infty)$ and $\mathcal{H}_\gamma(\mathbb{R}_+)$ ($\gamma > 0$) denote the set of all measurable functions $h: \mathbb{R}_+ \rightarrow \mathbb{R}$ satisfying

$$\|h\|_{\mathcal{H}_\gamma(\mathbb{R}_+)} := \left(\int_0^\infty |h(t)|^\gamma \frac{dt}{t} \right)^{1/\gamma} < \infty. \quad (3)$$

Let T_{h,Ω,P_N} be defined as in (2) and let $\bar{U}(\mathbb{R}_+)$ denote the closed unit ball in $\mathcal{H}_2(\mathbb{R}_+)$. Define the maximal function $\mathcal{S}_{\Omega,P_N}$ by

$$\mathcal{S}_{\Omega,P_N} f(x) = \sup_{h \in \bar{U}(\mathbb{R}_+)} |T_{h,\Omega,P_N} f(x)|. \quad (4)$$

If $P_N(t) = t$ and $h(t) \equiv 1$, we denote $T_{h,\Omega,P_N} = T_\Omega$ and $\mathcal{S}_{\Omega,P_N} = \mathcal{S}_\Omega$.

As a formal extension of the Lebesgue space L^p , the mixed radial-angular space $L^p_{|x|} L^{\tilde{p}}_\theta(\mathbb{R}^n)$ has already been successfully used in studying Strichartz estimates and dispersive equations (see [9–16]) over the last several years. Recall that the mixed radial-angular spaces $L^p_{|x|} L^{\tilde{p}}_\theta(\mathbb{R}^n)$, $1 \leq p, \tilde{p} \leq \infty$, consist of all functions u satisfying $\|u\|_{L^p_{|x|} L^{\tilde{p}}_\theta(\mathbb{R}^n)} < \infty$, where

$$\|u\|_{L^p_{|x|} L^{\tilde{p}}_\theta(\mathbb{R}^n)} := \left(\int_0^\infty \|u(\rho \cdot)\|_{L^{\tilde{p}}(S^{n-1})}^p \rho^{n-1} d\rho \right)^{1/p} \quad (5)$$

and

$$\|u\|_{L^\infty_{|x|} L^\tilde{p}_\theta(\mathbb{R}^n)} := \sup_{\rho>0} \|u(\rho \cdot)\|_{L^{\tilde{p}}(S^{n-1})}. \quad (6)$$

It is clear that the spaces $L^p_{|x|} L^\tilde{p}_\theta(\mathbb{R}^n)$ have the following easy properties.

(i) If $1 \leq p \leq \infty$ and $\tilde{p} = p$, then

$$\|u\|_{L^p_{|x|} L^\tilde{p}_\theta(\mathbb{R}^n)} = \|u\|_{L^p(\mathbb{R}^n)}. \quad (7)$$

(ii) If u is a radial function on $\mathbb{R}^n$ and $1 \leq p \leq \infty$, $1 \leq \tilde{p} \leq \infty$, then

$$\|u\|_{L^p_{|x|} L^\tilde{p}_\theta(\mathbb{R}^n)} \simeq \|u\|_{L^p(\mathbb{R}^n)}. \quad (8)$$

(iii) If $1 \leq p \leq \infty$ and $1 \leq \tilde{p}_1 \leq \tilde{p}_2 \leq \infty$, then

$$\|u\|_{L^p_{|x|} L^{\tilde{p}_1}_\theta(\mathbb{R}^n)} \leq C_{n,p,\tilde{p}_1,\tilde{p}_2} \|u\|_{L^p_{|x|} L^{\tilde{p}_2}_\theta(\mathbb{R}^n)}. \quad (9)$$

Here the notation $A \simeq B$ means that there are two positive constants C, C' such that $A \leq CB$ and $B \leq C'A$. Throughout this paper, we use $C_{\alpha,\beta,\dots}$ to denote positive constants that depend on parameters $\alpha, \beta, \dots$

Recently, the mixed radial-angular space $L^p_{|x|} L^\tilde{p}_\theta(\mathbb{R}^n)$ is also playing active roles in the theory of singular integral operator (see [17–19]). In [18], among other things, Córdoba proved that T_Ω is bounded on $L^p_{|x|} L^\tilde{p}_\theta(\mathbb{R}^n)$ for all $1 < p < \infty$ and $\tilde{p} = 2$, provided that $\Omega \in \mathcal{C}^1(S^{n-1})$. Later on, D'Ancona and Lucà [19] extended the above index $\tilde{p} = 2$ to the range $1 < \tilde{p} < \infty$ by applying the same argument in [18, Theorem 2.1] (see also [17, Theorem 1.1] for the weighted case). Recently, Liu and Fan [20] improved the above result to the case $\Omega \in L^q(S^{n-1})$ and extended the above results to the singular integral operators along polynomial curves. Precisely, let $\Delta_\gamma(\mathbb{R}_+)$ ($\gamma > 0$) be the set of all measurable functions h defined on $\mathbb{R}_+$ satisfying

$$\|h\|_{\Delta_\gamma(\mathbb{R}_+)} := \sup_{R>0} \left(R^{-1} \int_0^R |h(t)|^\gamma dt \right)^{1/\gamma} < \infty. \quad (10)$$

Liu and Fan [20] proved the following result. Our main results can be formulated as follows.

Theorem A (see [20]). *Let $P_N(t)$ be a real polynomial on $\mathbb{R}$ of degree N and satisfy $P_N(0) = 0$. Suppose that $\Omega \in L^q(S^{n-1})$*

satisfies (1) and $h \in \Delta_\gamma(\mathbb{R}_+)$ for some $1 < q, \gamma \leq \infty$. Then for $1 < p, \tilde{p} < \infty$, the following inequalities hold:

$$\begin{aligned} \|T_{h,\Omega,P_N} f\|_{L^p_{|x|} L^\tilde{p}_\theta(\mathbb{R}^n)} &\leq C_{h,\Omega,q,\gamma,p,\tilde{p},N} \|f\|_{L^p_{|x|} L^\tilde{p}_\theta(\mathbb{R}^n)}; \\ \left\| \left(\sum_{j \in \mathbb{Z}} |T_{h,\Omega,P_N} f_j|^{\tilde{p}} \right)^{1/\tilde{p}} \right\|_{L^p_{|x|} L^\tilde{p}_\theta(\mathbb{R}^n)} &\leq C_{h,\Omega,q,\gamma,p,\tilde{p},N} \left\| \left(\sum_{j \in \mathbb{Z}} |f_j|^{\tilde{p}} \right)^{1/\tilde{p}} \right\|_{L^p_{|x|} L^\tilde{p}_\theta(\mathbb{R}^n)}; \\ \left\| \left(\sum_{j \in \mathbb{Z}} |T_{h,\Omega,P_N} f_j|^{\tilde{p}} \right)^{1/\tilde{p}} \right\|_{L^p(\mathbb{R}^n)} &\leq C_{h,\Omega,q,\gamma,p,\tilde{p},N} \left\| \left(\sum_{j \in \mathbb{Z}} |f_j|^{\tilde{p}} \right)^{1/\tilde{p}} \right\|_{L^p(\mathbb{R}^n)}. \end{aligned} \quad (11)$$

Here the above constants $C_{h,\Omega,q,\gamma,p,\tilde{p},N} > 0$ are independent of the coefficients of P_N .

On the other hand, the classical maximal operator $\mathcal{S}_\Omega$ was originally introduced by Chen and Lin [21] who proved that if $\Omega \in \mathcal{C}^1(S^{n-1})$, then $\mathcal{S}_\Omega$ is of type (p, p) for any $p > 2n/(2n-1)$ and the range of p is best possible. Subsequently, the L^p mapping properties of $\mathcal{S}_\Omega$ have been discussed extensively by many authors (see [22–26], for example). Particularly, Al-Salman [23] proved the following result.

Theorem B (see [23]). *Let Ω satisfy (1) and $\Omega \in L(\log L)^{1/2}(S^{n-1})$. Then*

$$\|\mathcal{S}_{\Omega,P_N} f\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)} \quad (12)$$

for all $2 \leq p < \infty$. Here $C_p > 0$ is independent of the coefficients of P_N .

It is well known that the following are valid.

$$\mathcal{H}_\infty(\mathbb{R}_+) = \Delta_\infty(\mathbb{R}_+) = L^\infty(\mathbb{R}_+);$$

$$\Delta_{\gamma_1}(\mathbb{R}_+) \subsetneq \Delta_{\gamma_2}(\mathbb{R}_+), \quad \forall 1 < \gamma_2 < \gamma_1 < \infty; \quad (13)$$

$$\mathcal{H}_\gamma(\mathbb{R}_+) \subsetneq \Delta_\gamma(\mathbb{R}_+), \quad \forall 1 < \gamma < \infty;$$

$$\mathcal{C}^1(S^{n-1}) \subsetneq \bigcup_{1 < q < \infty} L^q(S^{n-1}) \subsetneq L(\log L)^{1/2}(S^{n-1}). \quad (14)$$

It follows from (14) and Theorem B that the condition $\Omega \in L^q(S^{n-1})$ implies the L^p -boundedness of $\mathcal{S}_{\Omega,P_N}$ for $2 \leq p < \infty$. A question that arises naturally is the following.

Question A. Is the operator $\mathcal{S}_{\Omega,P_N}$ bounded on $L^p_{|x|} L^\tilde{p}_\theta(\mathbb{R}^n)$ for $1 < p, \tilde{p} < \infty$ ($p \neq \tilde{p}$) if $\Omega \in L^q(S^{n-1})$ for some $q > 1$?

In this paper we will give an affirmative answer to Question A. In order to obtain the $L^p_{|x|} L^\tilde{p}_\theta(\mathbb{R}^n)$ boundedness

for $\mathcal{S}_{\Omega, P_N}$, we shall establish the following weighted estimates for $\mathcal{S}_{\Omega, P_N}$.

Theorem 1. *Let $P_N(t)$ be a real polynomial on $\mathbb{R}$ of degree N and satisfy $P_N(0) = 0$. Suppose that Ω satisfies (1) and $\Omega \in L^q(S^{n-1})$ for some $1 < q \leq \infty$. Then, for any nonnegative measurable function u on $\mathbb{R}^n$, it holds that*

$$\|\mathcal{S}_{\Omega, P_N} f\|_{L^p(u)} \leq C_{\Omega, q, p, s, N} \|f\|_{L^p(\Lambda_{N, s} u)}, \quad (15)$$

provided that one of the following conditions holds:

- (i) $1 < p < 2$ and $s > 2/p$;
- (ii) $2 \leq p < \infty$ and $s > 1$.

Here $\Lambda_{N, s} u = \sum_{i=0}^N M_s^{N+1-i} M_{i, s}^{\bar{\sigma}} M_s u$ and $M_{\lambda, s}^{\bar{\sigma}} u = (M_{\lambda}^{\bar{\sigma}}(u^s))^{1/s}$, $M_s^k u = (M^k u^s)^{1/s}$ for any $k \in \mathbb{N}$. M^k is the Hardy-Littlewood maximal operator M iterated k times for all $k = 1, 2, \dots$. Specially, $M^k = M$ when $k = 1$. $M_{\lambda}^{\bar{\sigma}}$ is a maximal operator defined as in Section 2.

As applications of Theorem 1, we can obtain the following result.

Theorem 2. *Let $\mathcal{S}_{\Omega, P_N}$ be the maximal operator defined by (4). Let $P_N(t)$ be a real polynomial on $\mathbb{R}$ of degree N and satisfy $P_N(0) = 0$. Suppose that Ω satisfies (1) and $\Omega \in L^q(S^{n-1})$ for some $1 < q \leq \infty$. Then, for $1 < \tilde{p} < 2$ and $\tilde{p} < p < 2\tilde{p}/(2 - \tilde{p})$ or $2 \leq \tilde{p} \leq p < \infty$, the following inequalities hold:*

$$\|\mathcal{S}_{\Omega, P_N} f\|_{L_{|\lambda|}^p L_{\theta}^{\tilde{p}}(\mathbb{R}^n)} \leq C_{\Omega, q, p, \tilde{p}, N} \|f\|_{L_{|\lambda|}^p L_{\theta}^{\tilde{p}}(\mathbb{R}^n)}; \quad (16)$$

$$\left\| \left(\sum_{j \in \mathbb{Z}} |\mathcal{S}_{\Omega, P_N} f_j|^{\tilde{p}} \right)^{1/\tilde{p}} \right\|_{L_{|\lambda|}^p L_{\theta}^{\tilde{p}}(\mathbb{R}^n)} \quad (17)$$

$$\leq C_{\Omega, q, p, \tilde{p}, N} \left\| \left(\sum_{j \in \mathbb{Z}} |f_j|^{\tilde{p}} \right)^{1/\tilde{p}} \right\|_{L_{|\lambda|}^p L_{\theta}^{\tilde{p}}(\mathbb{R}^n)};$$

$$\left\| \left(\sum_{j \in \mathbb{Z}} |\mathcal{S}_{\Omega, P_N} f_j|^{\tilde{p}} \right)^{1/\tilde{p}} \right\|_{L^p(\mathbb{R}^n)} \quad (18)$$

$$\leq C_{\Omega, q, p, \tilde{p}, N} \left\| \left(\sum_{j \in \mathbb{Z}} |f_j|^{\tilde{p}} \right)^{1/\tilde{p}} \right\|_{L^p(\mathbb{R}^n)}.$$

Here the constants $C_{\Omega, q, p, \tilde{p}, N} > 0$ are independent of the coefficients of P_N .

Remark 3. We remark that our main results are new even in the special case $P_N(t) = t$.

The rest of this paper is organized as follows. In Section 2 we shall present some notations and auxiliary lemmas. The proofs of Theorems 1 and 2 will be given in Section 3. It should be pointed out that the main idea in the proof of Theorem 1 is

a combination of ideas and arguments from [18–20, 27–29]. The proof of Theorem 2 is based on Theorem 1 and a criterion established in Section 2 (see Proposition 7). Throughout this note, for any $p \in (1, \infty)$, we let p' denote the dual exponent to p defined as $1/p + 1/p' = 1$. We also use the convention $\prod_{j \in \emptyset} a_j = 1$. In what follows, for any function f , we define $\tilde{f}$ by $\tilde{f}(x) = f(-x)$. For $f \in L^p(w)$, we set

$$\|f\|_{L^p(w)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p}. \quad (19)$$

2. Preliminary Lemmas

In this section we shall give some notations and necessary lemmas, which will play key roles in the proof of our main results. In what follows, we assume that $P_N(t) = \sum_{i=1}^N a_i t^i$ with $a_i \neq 0$. Let $P_{\lambda}(t) = \sum_{i=1}^{\lambda} a_i t^i$ for $\lambda = 1, 2, \dots, N$ and $P_0(t) = 0$. For $\lambda \in \{1, 2, \dots, N\}$, we define a family of measures $\{\sigma_{\lambda, r}\}_{r>0}$ and the related maximal operator M_{λ}^{σ} by

$$\widehat{\sigma_{\lambda, r}}(x) = \int_{S^{n-1}} e^{-2\pi i P_{\lambda}(r)x \cdot y'} \Omega(y') d\sigma(y'), \quad (20)$$

$$M_{\lambda}^{\sigma} f(x) = \sup_{k \in \mathbb{Z}} \int_{2^k}^{2^{k+1}} |\sigma_{\lambda, r}| * f(x) \frac{dr}{r},$$

where $|\sigma_{\lambda, r}|$ is defined in the same way as $\sigma_{\lambda, r}$, but with Ω replaced by $|\Omega|$. We also define the maximal operator $M_{\lambda}^{\bar{\sigma}}$ by

$$M_{\lambda}^{\bar{\sigma}} f(x) = \sup_{k \in \mathbb{Z}} \int_{2^k}^{2^{k+1}} |\bar{\sigma}_{\lambda, r}| * f(x) \frac{dr}{r}, \quad (21)$$

where $\bar{\sigma}_{\lambda, r}$ is defined in the following way

$$\int_{\mathbb{R}^n} f(x) d|\bar{\sigma}_{\lambda, r}|(x) = \int_{\mathbb{R}^n} f(-x) d|\sigma_{\lambda, r}|(x). \quad (22)$$

Lemma 4 ([30, p.186, corollary]). *Let $\Phi(t) = t^{\alpha_1} + \mu_2 t^{\alpha_2} + \dots + \mu_n t^{\alpha_n}$ and $\Psi \in C^1[a, b]$, where $\mu_2, \dots, \mu_n$ are real parameters and $\alpha_1, \dots, \alpha_n$ are distinct positive (not necessarily integer) exponents. Then*

$$\left| \int_a^b e^{-i\lambda \Phi(t)} \Psi(t) dt \right| \quad (23)$$

$$\leq C \lambda^{-\epsilon} \left\{ \sup_{t \in [a, b]} |\Psi(t)| + \int_a^b |\Psi'(t)| dt \right\},$$

with $\epsilon = \min\{1/\alpha_1, 1/n\}$ and C does not depend on $\mu_2, \dots, \mu_n$ as long as $0 \leq a < b \leq 1$.

Lemma 5. Let Ω satisfy (1) and $\Omega \in L^q(S^{n-1})$ for some $q > 1$. Then for $\lambda \in \{1, 2, \dots, N\}$ and $k \in \mathbb{Z}$, the following estimates hold:

$$\sup_{r>0} \max \left\{ \|\sigma_{\lambda,r}\|, \left| \widehat{\sigma_{\lambda,r}}(\xi) \right|, \left| \widehat{|\sigma_{\lambda,r}|}(\xi) \right| \right\} \leq \|\Omega\|_{L^1(S^{n-1})}; \quad (24)$$

$$\int_{2^k}^{2^{k+1}} \left| \widehat{\sigma_{\lambda,r}}(\xi) \right|^2 \frac{dr}{r} \leq C_{\Omega,q} \min \left\{ 1, \left| 2^{k\lambda} a_\lambda \xi \right|^{-1/(2q'\lambda)} \right\}; \quad (25)$$

$$\begin{aligned} & \int_{2^k}^{2^{k+1}} \left| \widehat{|\sigma_{\lambda,r}|}(\xi) \right|^2 \frac{dr}{r} \\ & \leq C_{\Omega,q} \min \left\{ 1, \left| 2^{k\lambda} a_\lambda \xi \right|^{-1/(2q'\lambda)} \right\}; \end{aligned} \quad (26)$$

$$\int_{2^k}^{2^{k+1}} \left| \widehat{\sigma_{\lambda,r}}(\xi) - \widehat{\sigma_{\lambda-1,r}}(\xi) \right|^2 \frac{dr}{r} \leq C_{\Omega,q} \left(\min \left\{ 1, \left| 2^{(k+1)\lambda} a_\lambda \xi \right| \right\} \right)^2; \quad (27)$$

$$\begin{aligned} & \int_{2^k}^{2^{k+1}} \left| \widehat{|\sigma_{\lambda,r}|}(\xi) - \widehat{|\sigma_{\lambda-1,r}|}(\xi) \right|^2 \frac{dr}{r} \\ & \leq C_{\Omega,q} \left(\min \left\{ 1, \left| 2^{(k+1)\lambda} a_\lambda \xi \right| \right\} \right)^2. \end{aligned} \quad (28)$$

Proof. Estimate (24) is trivial. By Lemma 4, Hölder's inequality, and the changes of variable, we have

$$\begin{aligned} & \int_{2^k}^{2^{k+1}} \left| \widehat{\sigma_{\lambda,r}}(\xi) \right|^2 \frac{dr}{r} \\ & = \int_{2^k}^{2^{k+1}} \iint_{(S^{n-1})^2} \Omega(u) \cdot \overline{\Omega(y')} e^{-2\pi i P_\lambda(r)\xi \cdot (u-y')} d\sigma(u) d\sigma(y') \frac{dt}{t} \\ & \leq \iint_{(S^{n-1})^2} \left| \Omega(u) \overline{\Omega(y')} \right| \left| \int_{2^k}^{2^{k+1}} e^{-2\pi i P_\lambda(r)\xi \cdot (u-y')} \frac{dt}{t} \right| d\sigma(u) d\sigma(y') \\ & \leq \iint_{(S^{n-1})^2} \left| \Omega(u) \overline{\Omega(y')} \right| \left| \int_1^2 e^{-2\pi i P_\lambda(2^k r)\xi \cdot (u-y')} \frac{dt}{t} \right| d\sigma(u) d\sigma(y') \\ & \leq \iint_{(S^{n-1})^2} \left| \Omega(u) \overline{\Omega(y')} \right| \min \left\{ 1, \left| 2^{k\lambda} a_\lambda \xi \cdot (u-y') \right|^{-1/\lambda} \right\} d\sigma(u) d\sigma(y') \\ & \leq C \|\Omega\|_{L^q(S^{n-1})}^2 \left| 2^{k\lambda} a_\lambda \xi \right|^{-1/\epsilon}, \end{aligned} \quad (29)$$

for any $\epsilon \in (0, \min\{1/q', 1/\lambda\})$. This together with (24) yields (25). Similarly, we can prove (26). It is clear that

$$\begin{aligned} & \max \left\{ \left| \widehat{\sigma_{\lambda,r}}(\xi) - \widehat{\sigma_{\lambda-1,r}}(\xi) \right|, \left| \widehat{|\sigma_{\lambda,r}|}(\xi) - \widehat{|\sigma_{\lambda-1,r}|}(\xi) \right| \right\} \\ & \leq \int_{S^{n-1}} \left| \Omega(y') \right| \left| e^{-2\pi i P_\lambda(r)\xi \cdot y'} - e^{-2\pi i P_{\lambda-1}(r)\xi \cdot y'} \right| d\sigma(y') \quad (30) \\ & \leq 2\pi \|\Omega\|_{L^1(S^{n-1})} \left| r^\lambda a_\lambda \xi \right| \leq C_{\Omega,q} \left| r^\lambda a_\lambda \xi \right|, \end{aligned}$$

which together with (24) and the fact that $\sigma_{0,r}(\xi) = 0$ yields (27)-(28) and completes the proof. $\square$

Lemma 6. Let Ω satisfy (1) and $\Omega \in L^q(S^{n-1})$ for some $q > 1$. Then for $\lambda \in \{0, 1, \dots, N\}$, we have

$$\|M_\lambda^\sigma\|_{L^p(\mathbb{R}^n)} \leq C_{\Omega,q,p} \|f\|_{L^p(\mathbb{R}^n)} \quad (31)$$

for all $1 < p < \infty$.

Proof. We shall prove (31) by induction on λ . It is easy to see that

$$M_0^\sigma f(x) \leq \|\Omega\|_{L^1(S^{n-1})} |f(x)|. \quad (32)$$

This yields (31) for $\lambda = 0$. Suppose that (31) holds for $\lambda = l - 1$ with $l \in \{1, 2, \dots, N\}$. We shall prove (31) for $\lambda = l$. Let ϕ be a nonnegative Schwartz function supported in $\{|t| \leq$

1} satisfying $\phi(t) = 1$ when $|t| < 1/2$. Define the measures $\{\tau_k\}_{k \in \mathbb{Z}}$ by

$$\widehat{\tau_k}(\xi) = \int_{2^k}^{2^{k+1}} \widehat{|\sigma_{l,r}|}(\xi) \frac{dr}{r} - \int_{2^k}^{2^{k+1}} \widehat{|\sigma_{l-1,r}|}(\xi) \frac{dr}{r} \phi(|2^{kl} a_l \xi|). \quad (33)$$

By Lemma 5 and the definition of τ_k , we have

$$|\widehat{\tau_k}(\xi)| \leq C_{\Omega,q} \min \left\{ 1, |2^{k\lambda} a_\lambda \xi|, |2^{k\lambda} a_\lambda \xi|^{-1/(2q'\lambda)} \right\}; \quad (34)$$

$$\|M_l^\sigma f\|_{L^p(\mathbb{R}^n)} \leq \|G(f)\|_{L^p(\mathbb{R}^n)} + C \|M_{l-1}^\sigma f\|_{L^p(\mathbb{R}^n)}, \quad (35)$$

$$1 < p < \infty;$$

$$\|\tau^*(f)\|_{L^p(\mathbb{R}^n)} \leq \|G(f)\|_{L^p(\mathbb{R}^n)} + C \|M_{l-1}^\sigma f\|_{L^p(\mathbb{R}^n)}, \quad (36)$$

$$1 < p < \infty,$$

where

$$G(f)(x) = \left(\sum_{k \in \mathbb{Z}} |\tau_k * f(x)|^2 \right)^{1/2} \quad (37)$$

$$\text{and } \tau^*(f)(x) = \sup_{k \in \mathbb{Z}} |\tau_k * f(x)|.$$

By (34) and Plancherel's theorem, we get

$$\|G(f)\|_{L^2(\mathbb{R}^n)} \leq C_{\Omega,q} \|f\|_{L^2(\mathbb{R}^n)}. \quad (38)$$

From (36)-(38) and our assumption, we have

$$\|\tau^*(f)\|_{L^2(\mathbb{R}^n)} \leq C_{\Omega,q} \|f\|_{L^2(\mathbb{R}^n)}. \quad (39)$$

By the lemma on page 544 of [27] ($q = 2, p_0 = 4$), we have

$$\|G(f)\|_{L^p(\mathbb{R}^n)} \leq C_{\Omega,q,p} \|f\|_{L^p(\mathbb{R}^n)}, \quad \frac{4}{3} < p < 4. \quad (40)$$

From our assumption, (35) and (40), we get

$$\|\tau^*(f)\|_{L^p(\mathbb{R}^n)} \leq C_{\Omega,q,p} \|f\|_{L^p(\mathbb{R}^n)}, \quad \frac{4}{3} < p < 4. \quad (41)$$

By the lemma on page 544 of [27] ($q = 4/3, p_0 = 8$), we have

$$\|G(f)\|_{L^p(\mathbb{R}^n)} \leq C_{\Omega,q,p} \|f\|_{L^p(\mathbb{R}^n)}, \quad \frac{8}{7} < p < 8. \quad (42)$$

By using this argument repeatedly, one can obtain ultimately

$$\|G(f)\|_{L^p(\mathbb{R}^n)} \leq C_{\Omega,q,p} \|f\|_{L^p(\mathbb{R}^n)}, \quad 1 < p < \infty. \quad (43)$$

Combining (43) with (35) and assumption yields that

$$\|M_l^\sigma f\|_{L^p(\mathbb{R}^n)} \leq C_{\Omega,q,p} \|f\|_{L^p(\mathbb{R}^n)}, \quad 1 < p < \infty. \quad (44)$$

This completes the proof of Lemma 6. $\square$

To prove Theorem 1, we need the following proposition, which is of interest in its own right.

Proposition 7. Let $1 < p < \infty$ and $s_0 \geq 1$. Suppose that T is a linear or sublinear operator such that

$$\|Tf\|_{L^p(u)} \leq C_{p,s,s_0} \|f\|_{L^p(\mathcal{G}_s(u))}, \quad \forall s > s_0 \quad (45)$$

for any nonnegative measurable function u on $\mathbb{R}^n$, where $\mathcal{G}_s$ is a bounded operator from $L^r(\mathbb{R}^n)$ to itself for all $r \in (s, \infty)$. Then for any $q \in (p, ps_0/(s_0 - 1))$, the following inequalities hold:

$$\|Tf\|_{L^q_{|x|} L^p_\theta(\mathbb{R}^n)} \leq C_{p,q} \|f\|_{L^q_{|x|} L^p_\theta(\mathbb{R}^n)}; \quad (46)$$

$$\left\| \left(\sum_{j \in \mathbb{Z}} |Tf_j|^p \right)^{1/p} \right\|_{L^q_{|x|} L^p_\theta(\mathbb{R}^n)} \quad (47)$$

$$\leq C_{p,q} \left\| \left(\sum_{j \in \mathbb{Z}} |f_j|^p \right)^{1/p} \right\|_{L^q_{|x|} L^p_\theta(\mathbb{R}^n)};$$

$$\left\| \left(\sum_{j \in \mathbb{Z}} |Tf_j|^p \right)^{1/p} \right\|_{L^q(\mathbb{R}^n)} \quad (48)$$

$$\leq C_{p,q} \left\| \left(\sum_{j \in \mathbb{Z}} |f_j|^p \right)^{1/p} \right\|_{L^q(\mathbb{R}^n)}.$$

Proof. We only prove (46) since (47) and (48) can be proved similarly. The argument of the proof for (46) is similar to those of the proof in [19, Theorem 2.6] essentially. Fix $q \in (p, ps_0/(s_0 - 1))$ and write $r = q/(q - p)$. We can choose a number s such that $s_0 < s < r$. Let X denote the set of all $g \in \mathcal{S}(\mathbb{R})$ with $\int_0^\infty g^r(\rho) \rho^{n-1} d\rho \leq 1$. By changes of variables, one has

$$\begin{aligned} & \|Tf\|_{L^q_{|x|} L^p_\theta(\mathbb{R}^n)}^p \\ &= \left(\int_0^\infty \left(\int_{S^{n-1}} |Tf(\rho\theta)|^p d\sigma(\theta) \right)^{q/p} \rho^{n-1} d\rho \right)^{p/q} \\ &= \sup_{g \in X} \int_0^\infty \int_{S^{n-1}} |Tf(\rho\theta)|^p g(\rho) \rho^{n-1} d\sigma(\theta) d\rho \\ &= \sup_{g \in X} \int_{\mathbb{R}^n} |Tf(x)|^p g(|x|) dx. \end{aligned} \quad (49)$$

Fix $g \in X$. Let $I(g) := \int_{\mathbb{R}^n} |Tf(x)|^p g(|x|) dx$ and $h(x) = g(|x|)$. By (45), Hölder's inequality, and changes of variables,

$$\begin{aligned} I(g) &\leq C_{p,s,s_0} \int_{\mathbb{R}^n} |f(x)|^p \mathcal{G}_s(h)(x) dx \\ &\leq C_{p,s,s_0} \int_0^\infty \int_{S^{n-1}} |f(\rho\theta)|^p d\sigma(\theta) \mathcal{G}_s(g)(\rho) \rho^{n-1} d\rho \\ &\leq C_{p,s,s_0} \int_0^\infty \left(\int_{S^{n-1}} |f(\rho\theta)|^p d\sigma(\theta) \right)^{q/p} \rho^{n-1} d\rho \Big)^{p/q} \\ &\quad \times \left(\int_0^\infty (\mathcal{G}_s(g)(\rho))^r \rho^{n-1} d\rho \right)^{1/r} \\ &\leq C_{p,q} \|f\|_{L_{|x|}^q L_\theta^p(\mathbb{R}^n)}^p \|\mathcal{G}_s(h)\|_{L^r(\mathbb{R}^n)} \\ &\leq C_{p,q} \|f\|_{L_{|x|}^q L_\theta^p(\mathbb{R}^n)}^p, \end{aligned} \quad (50)$$

which together with (49) leads to (46). $\square$

3. Proofs of Theorems 1 and 2

We begin with the proof of Theorem 1.

Proof of Theorem 1. By the duality we can write

$$\mathcal{S}_{\Omega, P_N} f(x) = \left(\int_0^\infty |\sigma_{N,r} * f(x)|^2 \frac{dr}{r} \right)^{1/2}. \quad (51)$$

Let ϕ be a nonnegative Schwartz function supported in $\{|t| \leq 1\}$ satisfying $\phi(t) = 1$ when $|t| < 1/2$. For $1 \leq \lambda \leq N$, we define the Borel measures $\{\mu_{\lambda,r}\}_{r>0}$ on $\mathbb{R}^n$ by

$$\begin{aligned} \widehat{\mu_{\lambda,r}}(\xi) &= \widehat{\sigma_{\lambda,r}}(\xi) \prod_{j=\lambda+1}^N \phi(r^j a_j \xi) \\ &\quad - \widehat{\sigma_{\lambda-1,r}}(\xi) \prod_{j=\lambda}^N \phi(r^j a_j \xi), \quad \forall \xi \in \mathbb{R}^n. \end{aligned} \quad (52)$$

One can verify that

$$\sigma_{N,r} = \sum_{\lambda=1}^N \mu_{\lambda,r}; \quad (53)$$

$$\begin{aligned} \int_{2^k}^{2^{k+1}} |\widehat{\mu_{\lambda,r}}(\xi)|^2 \frac{dr}{r} \\ \leq C_{\Omega,q} \left(\min \left\{ 1, |2^{k\lambda} a_\lambda \xi|, |2^{k\lambda} a_\lambda \xi|^{-1} \right\} \right)^{1/(2q'\lambda)}; \end{aligned} \quad (54)$$

$$M_\lambda^\mu f(x) \leq M^{N-\lambda} M_\lambda^\sigma |f|(x) + M^{N-\lambda+1} M_{\lambda-1}^\sigma |f|(x). \quad (55)$$

Equation (51) together with (53) and Minkowski's inequality yields that

$$\begin{aligned} \mathcal{S}_{\Omega, P_N} f(x) &\leq \sum_{\lambda=1}^N \left(\int_0^\infty |\mu_{\lambda,r} * f(x)|^2 \frac{dr}{r} \right)^{1/2} \\ &= \sum_{\lambda=1}^N T_\lambda f(x). \end{aligned} \quad (56)$$

It follows from [31] that $u \leq M_s u$ and $M_s u \in A_1$ for $s > 1$. From (55) we have

$$\begin{aligned} \sum_{\lambda=1}^N M_s M_{\lambda,s}^{\tilde{\mu}} M_s u \\ \leq \sum_{\lambda=1}^N \left(M_s^{N+1-\lambda} M_{\lambda,s}^{\tilde{\sigma}} M_s u + M_s^{N+2-\lambda} M_{\lambda-1,s}^{\tilde{\sigma}} M_s u \right) \\ \leq \Lambda_{N,s} u. \end{aligned} \quad (57)$$

By (56) and (57), to prove Theorem 1, it suffices to show that

$$\|T_\lambda f\|_{L^p(u)} \leq C_{\Omega,q,p,\lambda,s} \|f\|_{L^p(M_s M_{\lambda,s}^{\tilde{\mu}} u)} \quad (58)$$

holds for all $1 \leq \lambda \leq N$ and any nonnegative measurable function u on $\mathbb{R}^n$, provided that one of the following conditions holds:

- (i) $1 < p < 2$ and $s > 2/p$;
- (ii) $2 \leq p < \infty$ and $s > 1$.

We now prove (58) for the case $1 < p < 2$ and $s > 2/p$. For $1 \leq \lambda \leq N$, let $\Psi_\lambda(t) \in \mathcal{C}_c^\infty((1/4, 1))$ such that $0 \leq \Psi_\lambda \leq 1$ and $\sum_{k \in \mathbb{Z}} (\Psi_\lambda(2^{k\lambda} |a_\lambda \xi|))^3 = 1$. Define the Fourier multiplier operators $\{\Gamma_{k,\lambda}\}_{k \in \mathbb{Z}}$ by $\Gamma_{k,\lambda} f(x) = \Theta_{k,\lambda} * f(x)$, where $\widehat{\Theta_{k,\lambda}}(\xi) = \Psi_\lambda(2^{k\lambda} |a_\lambda \xi|)$. It was shown in [28] that

$$\left\| \left(\sum_{k \in \mathbb{Z}} |\Gamma_{k,\lambda} f|^2 \right)^{1/2} \right\|_{L^p(w)} \leq C_{p,w,\lambda} \|f\|_{L^p(w)} \quad (59)$$

and

$$\left\| \sum_{k \in \mathbb{Z}} \Gamma_{k,\lambda} f_k \right\|_{L^p(w)} \leq C_{p,w,\lambda} \left\| \left(\sum_{k \in \mathbb{Z}} |f_k|^2 \right)^{1/2} \right\|_{L^p(w)} \quad (60)$$

for all $1 < p < \infty$ and $w \in A_p$.

By the changes of variables and Minkowski's inequality, we can write

$$\begin{aligned} T_\lambda f(x) &= \left(\sum_{k \in \mathbb{Z}} \int_{2^k}^{2^{k+1}} |\mu_{\lambda,r} * f(x)|^2 \frac{dr}{r} \right)^{1/2} \\ &= \left(\sum_{k \in \mathbb{Z}} \int_1^2 |\mu_{\lambda,2^k r} * f(x)|^2 \frac{dr}{r} \right)^{1/2} \\ &= \left(\sum_{k \in \mathbb{Z}} \int_1^2 \left| \sum_{j \in \mathbb{Z}} \Gamma_{j+k,\lambda}^3 (\mu_{\lambda,2^k r} * f)(x) \right|^2 \frac{dr}{r} \right)^{1/2} \\ &\leq \sum_{j \in \mathbb{Z}} \left(\sum_{k \in \mathbb{Z}} \int_1^2 |\Gamma_{j+k,\lambda}^3 (\mu_{\lambda,2^k r} * f)(x)|^2 \frac{dr}{r} \right)^{1/2} \\ &=: \sum_{j \in \mathbb{Z}} T_{\lambda,j} f(x). \end{aligned} \quad (61)$$

Hence, by (61), to prove (58) for the case $1 < p < 2$ and $s > 2/p$, it suffices to show that for any $1 < p < 2$ and $s > 2/p$, there exists $\alpha(p, q, \gamma) > 0$ independent of j such that

$$\|T_{\lambda,j} f\|_{L^p(u)} \leq C_{\Omega,q,p,\lambda} 2^{-\alpha(p,q,\gamma)|j|} \|f\|_{L^p(M_s M_{\lambda,s}^{\tilde{\mu}} u)}. \quad (62)$$

We now prove (62). Fix a nonnegative measurable function u on $\mathbb{R}^n$. By (54) and Plancherel's theorem, we have

$$\begin{aligned} & \int_{\mathbb{R}^n} \int_1^2 |\mu_{\lambda, 2^k r} * \Gamma_{j+k, \lambda} w(x)|^2 \frac{dr}{r} dx \\ & \leq C_{\Omega, q} 2^{-|j|/2q' \gamma'} \int_{\mathbb{R}^n} |w(x)|^2 dx \end{aligned} \quad (63)$$

for arbitrary function w on $\mathbb{R}^n$. One can easily check that

$$\begin{aligned} & \int_{\mathbb{R}^n} \int_1^2 |\mu_{\lambda, 2^k r} * \Gamma_{j+k, \lambda} w(x)|^2 \frac{dr}{r} u^s(x) dx \\ & \leq \sup_{r>0} \|\mu_{\lambda, r}\| \|\Theta_{j+k, \lambda}\|_{L^1(\mathbb{R}^n)} \\ & \cdot \int_1^2 \int_{\mathbb{R}^n} |\mu_{\lambda, 2^k r}| * |\Theta_{j+k, \lambda}| * |w|^2(x) u^s(x) dx \frac{dr}{r} \\ & \leq C_{\Omega, q} \int_{\mathbb{R}^n} |w(x)|^2 M M_{\lambda}^{\tilde{\mu}} u^s(x) dx \end{aligned} \quad (64)$$

for any $s > 1$. By (63)-(64) and the interpolation of L^2 -spaces with change of measure ([32, Theorem 5.4.1]), we obtain

$$\begin{aligned} & \int_{\mathbb{R}^n} \int_1^2 |\mu_{\lambda, 2^k r} * \Gamma_{j+k, \lambda} w(x)|^2 \frac{dr}{r} u(x) dx \\ & \leq C_{\Omega, q, s} 2^{-((1-1/s)/2q' \gamma')|j|} \int_{\mathbb{R}^n} |w(x)|^2 M_s M_{\lambda, s}^{\tilde{\mu}} u(x) dx \end{aligned} \quad (65)$$

for any $s > 1$. By (65) with $w = \Gamma_{j+k, \lambda} f$, (60), and the well-known property of the Rademacher's function, we obtain

$$\begin{aligned} & \|T_{\lambda, j} f\|_{L^2(u)}^2 \\ & = \int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}} \int_1^2 |\Gamma_{j+k, \lambda}^3 \mu_{\lambda, 2^k r} * f(x)|^2 \frac{dr}{r} u(x) dx \\ & = \sum_{k \in \mathbb{Z}} \int_1^2 \int_{\mathbb{R}^n} |\Gamma_{j+k, \lambda}^3 \mu_{\lambda, 2^k r} * f(x)|^2 u(x) dx \frac{dr}{r} \\ & \leq C_{\lambda} \sum_{k \in \mathbb{Z}} \int_1^2 \int_{\mathbb{R}^n} |\mu_{\lambda, 2^k r} * \Gamma_{j+k, \lambda}^2 f(x)|^2 u(x) dx \frac{dr}{r} \quad (66) \\ & \leq C_{\Omega, q, \lambda, s} 2^{-((1-1/s)/2q' \gamma')|j|} \int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}} |\Gamma_{j+k, \lambda} f(x)|^2 \\ & \cdot M_s M_{\lambda, s}^{\tilde{\mu}} u(x) dx \\ & \leq C_{\Omega, q, \lambda, s} 2^{-((1-1/s)/2q' \gamma')|j|} \|f\|_{L^2(M_s M_{\lambda, s}^{\tilde{\mu}} u)}^2. \end{aligned}$$

It follows that

$$\|T_{\lambda, j} f\|_{L^2(u)} \leq C_{\Omega, q, \lambda, s} 2^{-((1-1/s)/4q' \gamma')|j|} \|f\|_{L^2(M_s M_{\lambda, s}^{\tilde{\mu}} u)} \quad (67)$$

for any $s > 1$.

On the other hand, fix $k \in \mathbb{Z}$, and it is easy to see that

$$\sup_{r>0} \|\mu_{\lambda, 2^k r} * f\|_{L^\infty(\mathbb{R}^n)} \leq C_{\Omega, q} \|f\|_{L^\infty(\mathbb{R}^n)}; \quad (68)$$

$$\int_{\mathbb{R}^n} \int_1^2 |\mu_{\lambda, 2^k r} * f(x)| \frac{dr}{r} u(x) dx \leq \|f\|_{L^1(M_{\lambda}^{\tilde{\mu}} u)}. \quad (69)$$

The interpolation between (68) and (69) implies that

$$\begin{aligned} & \int_{\mathbb{R}^n} \int_1^2 |\mu_{\lambda, 2^k r} * f_k(x)|^p \frac{dr}{r} u(x) dx \\ & \leq C_{\Omega, q, p} \int_{\mathbb{R}^n} |f_k(x)|^p M_{\lambda}^{\tilde{\mu}} u(x) dx \end{aligned} \quad (70)$$

for all $1 < p < 2$. It follows from (70) that

$$\begin{aligned} & \int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}} \int_1^2 |\mu_{\lambda, 2^k r} * f_k(x)|^p \frac{dr}{r} u(x) dx \\ & \leq C_{\Omega, q, p} \int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}} |f_k(x)|^p M_{\lambda}^{\tilde{\mu}} u(x) dx \end{aligned} \quad (71)$$

for all $1 < p < 2$. On the other hand, we have

$$\begin{aligned} & \int_{\mathbb{R}^n} \left(\sup_{k \in \mathbb{Z}} \sup_{r \in [1, 2]} |\mu_{\lambda, 2^k r} * f_k(x)| \right)^p dx \\ & \leq C_{\Omega, q, p} \int_{\mathbb{R}^n} \left(\sup_{k \in \mathbb{Z}} |f_k(x)| \right)^p dx \end{aligned} \quad (72)$$

for all $1 < p < 2$. By interpolating between (71) and (72),

$$\begin{aligned} & \int_{\mathbb{R}^n} \left(\sum_{k \in \mathbb{Z}} \int_1^2 |\mu_{\lambda, 2^k r} * f_k(x)|^2 \frac{dr}{r} \right)^{p/2} u^{1/t_1}(x) dx \\ & \leq C_{\Omega, q, p, t_1} \int_{\mathbb{R}^n} \left(\sum_{k \in \mathbb{Z}} |f_k(x)|^2 \right)^{p/2} (M_{\lambda}^{\tilde{\mu}} u)^{1/t_1}(x) dx \end{aligned} \quad (73)$$

for all $1 < p < 2$, where $t_1 = 2/p$. Substituting u^{t_1} for u in (73), we obtain

$$\begin{aligned} & \int_{\mathbb{R}^n} \left(\sum_{k \in \mathbb{Z}} \int_1^2 |\mu_{\lambda, 2^k r} * f_k(x)|^2 \frac{dr}{r} \right)^{p/2} u(x) dx \\ & \leq C_{\Omega, q, p, t_1} \int_{\mathbb{R}^n} \left(\sum_{k \in \mathbb{Z}} |f_k(x)|^2 \right)^{p/2} (M_{\lambda, t_1}^{\tilde{\mu}} u)(x) dx. \end{aligned} \quad (74)$$

Since $M_{\lambda, t_1}^{\tilde{\mu}} u \in A_1$, by the weighted Littlewood-Paley theory and (74),

$$\begin{aligned} & \|T_{\lambda, j} f\|_{L^p(u)} \\ & = \left\| \left(\sum_{k \in \mathbb{Z}} \int_1^2 |\mu_{\lambda, 2^k r} * \Gamma_{j+k, \lambda}^3 f|^2 \frac{dr}{r} \right)^{1/2} \right\|_{L^p(u)} \\ & \leq C_{\Omega, q, p, t_1} \left\| \left(\sum_{k \in \mathbb{Z}} |\Gamma_{j+k, \lambda}^3 f|^2 \right)^{1/2} \right\|_{L^p(M_{\lambda, t_1}^{\tilde{\mu}} u)} \\ & \leq C_{\Omega, q, p, \lambda, t_1} \|f\|_{L^p(M_{\lambda, t_1}^{\tilde{\mu}} u)} \end{aligned} \quad (75)$$

for all $1 < p < 2$. Letting $s = t_1$, (67) gives us that

$$\begin{aligned} & \|T_{\lambda,j}f\|_{L^2(u)} \\ & \leq C_{\Omega,q,\lambda,t_1} 2^{-(1-1/t_1)/4q'\gamma'|j|} \|f\|_{L^2(M_{t_1}M_{\lambda,t_1}^{\tilde{\mu}}u)}. \end{aligned} \quad (76)$$

By an interpolation between (75) and (76), one has

$$\|T_{\lambda,j}f\|_{L^p(u)} \leq C_{\Omega,q,p,\lambda} 2^{-\alpha(p,q,\gamma)|j|} \|f\|_{L^p(M_{t_1}M_{\lambda,t_1}^{\tilde{\mu}}u)} \quad (77)$$

for all $1 < p < 2$, where $\alpha(p,q,\gamma) > 0$. Note that $M_{t_1}M_{\lambda,t_1}^{\tilde{\mu}}u \leq CM_sM_{\lambda,s}^{\tilde{\mu}}u$ for $s > t_1$ by Hölder's inequality. This together with (61) yields (58).

It remains to show (58) for the case $2 \leq p < \infty$ and $s > 1$. We want to show that

$$\|T_{\lambda,j}f\|_{L^p(u)} \leq C_{\Omega,q,p,\lambda,s} \|f\|_{L^p(M_sM_{\lambda,s}^{\tilde{\mu}}u)} \quad (78)$$

for all $2 < p < \infty$. Actually, by (67), (78), and an interpolation (see [32, Corollary 5.5.4]), one has

$$\|T_{\lambda,j}f\|_{L^p(u)} \leq C_{\Omega,q,p,\lambda,s} 2^{-\beta(p,q,\gamma,s)|j|} \|f\|_{L^p(M_sM_{\lambda,s}^{\tilde{\mu}}u)} \quad (79)$$

for $2 \leq p < \infty$ and $s > 1$. Here $\beta(p,q,\gamma,s) > 0$ depends only on p, q, γ , and s . Combining (79) with (61) yields (58) for the case $2 \leq p < \infty$ and $s > 1$.

Next we shall prove (78). Fix $2 < p < \infty$. By duality we can choose a function $v \in L^{(p/2)'}(u)$ with unit norm such that

$$\begin{aligned} & \left\| \left(\sum_{k \in \mathbb{Z}} \int_1^2 |\mu_{\lambda,2^k r} * g_k|^2 \frac{dr}{r} \right)^{1/2} \right\|_{L^p(u)}^2 \\ & = \int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}} \int_1^2 |\mu_{\lambda,2^k r} * g_k(x)|^2 \frac{dr}{r} \cdot v(x) u(x) dx. \end{aligned} \quad (80)$$

This together with the fact that $\|\mu_{k,t}^\lambda\| \leq C_{\Omega,q}$ shows that

$$\begin{aligned} & \left\| \left(\sum_{k \in \mathbb{Z}} \int_1^2 |\mu_{\lambda,2^k r} * g_k|^2 \frac{dr}{r} \right)^{1/2} \right\|_{L^p(u)}^2 \\ & \leq C_{\Omega,q} \int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}} |g_k(x)|^2 \int_1^2 |\tilde{\mu}_{\lambda,2^k r}| * (vu)(x) \frac{dr}{r} dx. \end{aligned} \quad (81)$$

Fix $s > 1$ and let $t = ps/2$. By Hölder's inequality,

$$\begin{aligned} & |\tilde{\mu}_{\lambda,2^k r}| * (vu) \\ & \leq (|\tilde{\mu}_{\lambda,2^k r}| * u^s)^{1/t} \left(|\tilde{\mu}_{\lambda,2^k r}| * \left(u^{t'/(p/2)'} v^{t'} \right) \right)^{1/t'}. \end{aligned} \quad (82)$$

By Hölder's inequality with exponents $p/2$ and $(p/2)'$ again and (81)-(82),

$$\begin{aligned} & \left\| \left(\sum_{k \in \mathbb{Z}} \int_1^2 |\mu_{\lambda,2^k r} * g_k|^2 \frac{dr}{r} \right)^{1/2} \right\|_{L^p(u)}^2 \\ & \leq C_{h,\Omega,q,\gamma} \int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}} |g_k(x)|^2 (M_{\lambda}^{\tilde{\mu}} u^s)^{1/t} \\ & \quad \cdot M_{\lambda}^{\tilde{\mu}} \left(u^{t'/(p/2)'} v^{t'} \right)^{1/t'}(x) dx \\ & \leq C_{h,\Omega,q,\gamma} \left\| \left(\sum_{k \in \mathbb{Z}} |g_k|^2 \right)^{1/2} \right\|_{L^p(M_{\lambda,s}^{\tilde{\mu}}u)}^2 \\ & \quad \cdot \left\| M_{\lambda}^{\tilde{\mu}} \left(u^{t'/(p/2)'} v^{t'} \right) \right\|_{L^{(p/2)'/t'}(\mathbb{R}^n)}^{1/t'}. \end{aligned} \quad (83)$$

On the other hand, by Lemma 6 and (55), we have

$$\|M_{\lambda}^{\tilde{\mu}} f\|_{L^r(\mathbb{R}^n)} \leq C_{h,\Omega,q,\gamma,r} \|f\|_{L^r(\mathbb{R}^n)}, \quad \forall 1 < r < \infty. \quad (84)$$

Since $p/2 = t/s < t$, then $(p/2)' > t'$. (84) leads to

$$\begin{aligned} & \left\| M_{\lambda}^{\tilde{\mu}} \left(u^{t'/(p/2)'} v^{t'} \right) \right\|_{L^{(p/2)'/t'}(\mathbb{R}^n)}^{1/t'} \\ & \leq C_{h,\Omega,q,\gamma,p,s} \left\| u^{t'/(p/2)'} v^{t'} \right\|_{L^{(p/2)'/t'}(\mathbb{R}^n)}^{1/t'} \leq C_{\Omega,q,p,s}. \end{aligned} \quad (85)$$

This together with (84) yields that

$$\begin{aligned} & \left\| \left(\sum_{k \in \mathbb{Z}} \int_1^2 |\mu_{\lambda,2^k r} * g_k|^2 \frac{dr}{r} \right)^{1/2} \right\|_{L^p(u)} \\ & \leq C_{\Omega,q,p,s} \left\| \left(\sum_{k \in \mathbb{Z}} |g_k|^2 \right)^{1/2} \right\|_{L^p(M_{\lambda,s}^{\tilde{\mu}}u)} \end{aligned} \quad (86)$$

for all $2 < p < \infty$ and any $s > 1$.

It follows from (59), (60), (86), and the fact that $M_{\lambda,s}^{\tilde{\mu}}u \leq M_s M_{\lambda,s}^{\tilde{\mu}}u$ that

$$\begin{aligned} & \|T_{\lambda,j}f\|_{L^p(u)} \\ & = \left\| \left(\sum_{k \in \mathbb{Z}} \int_1^2 |\mu_{\lambda,2^k r} * \Gamma_{j+k,\lambda}^3 f|^2 \frac{dr}{r} \right)^{1/2} \right\|_{L^p(u)} \\ & \leq C_{\Omega,q,p,\lambda,s} \left\| \left(\sum_{k \in \mathbb{Z}} |\Gamma_{j+k,\lambda}^3 f|^2 \right)^{1/2} \right\|_{L^p(M_{\lambda,s}^{\tilde{\mu}}u)} \\ & \leq C_{\Omega,q,p,\lambda,s} \|f\|_{L^p(M_s M_{\lambda,s}^{\tilde{\mu}}u)} \end{aligned} \quad (87)$$

for all $2 < p < \infty$ and any $s > 1$. This gives (78) and completes the proof of Theorem 1. $\square$

Proof of Theorem 2. By Lemma 6, one has

$$\|\Lambda_{N,s}u\|_{L^r(\mathbb{R}^n)} \leq C_{h,\Omega,q,\gamma} \|u\|_{L^r(\mathbb{R}^n)} \quad (88)$$

for any $r > s$. By (88), Theorem 1, and Proposition 7, we obtain (16)-(18) for the case $1 < \tilde{p} < 2$ and $\tilde{p} < p < 2\tilde{p}/(2 - \tilde{p})$ or $2 \leq \tilde{p} < p < \infty$. It was known that M_{Ω,p_N} is bounded on $L^p(\mathbb{R}^n)$ for $2 \leq p < \infty$. This together with (7) yields (16)-(18) for the case of $2 \leq p = \tilde{p} < \infty$. $\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

The authors read and approved the final manuscript.

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