Hindawi
Journal of Function Spaces
Volume 2019, Article ID 5193862, 8 pages
https://doi.org/10.1155/2019/5193862

Research Article

Integral Type F-Contractions in Partial Metric Spaces

Vildan Ozturk

Department of Mathematics, Faculty of Polatli Science and Arts, University of Ankara Haci Bayram Veli, Ankara, Turkey

Correspondence should be addressed to Vildan Ozturk; vildan_ozturk@hotmail.com

Received 24 December 2018; Accepted 1 February 2019; Published 25 March 2019

Academic Editor: Henryk Hudzik

Partial metric spaces were introduced as a generalization of usual metric spaces where the self-distance for any point need not be equal to zero. In this work, we defined generalized integral type F-contractions and proved common fixed point theorems for four mappings satisfying this type (Branciari type) of contractions in partial metric spaces.

1. Introduction and Preliminaries

Let X be a nonempty set and $p : X \times X \longrightarrow [0, \infty)$ satisfy

(PM1): $x = y \Longleftrightarrow p(x,x) = p(y,y) = p(x,y)$,

(PM2): $p(x,x) \leq p(x,y)$,

(PM3): $p(x,y) = p(y,x)$,

(PM4): $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$

for all x, y and $z \in X$. Then the pair (X, p) is called a partial metric space (in short PMS) and p is called a partial metric on X ([1]).

Let (X, p) be a PMS. Then, the functions $d_p, d_m : X \times X \longrightarrow [0, \infty)$ given by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y), \quad (1)$$

$$d_m(x,y) = \max\{p(x,y) - p(x,x), p(x,y) - p(y,y)\} \quad (2)$$

are (usual) metrics on X. It is clear that d_p and d_m are equivalent ([1]).

Definition 1 (see [1]).

(i) A sequence $\{x_n\}$ in a PMS (X, p) converges to $x \in X$ if and only if $p(x,x) = \lim_{n\longrightarrow\infty} p(x, x_n)$.

(ii) A sequence $\{x_n\}$ in a PMS (X, p) is called a Cauchy sequence if and only if $\lim_{n,m\longrightarrow\infty} p(x_n, x_m)$ exists (and finite).

(iii) A PMS (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\longrightarrow\infty} p(x_n, x_m)$.

(iv) A mapping $f : X \longrightarrow X$ is said to be continuous at $x_0 \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B(x_0, \delta)) \subset B(f(x_0), \varepsilon)$.

Lemma 2 (see [1]).

(i) *A sequence $\{x_n\}$ is Cauchy in a PMS (X, p) if and only if $\{x_n\}$ is Cauchy in a metric space (X, d_p).*

(ii) *A PMS (X, p) is complete if and only if the metric space (X, d_p) is complete. Moreover,*

$$\lim_{n\longrightarrow\infty} d_p(x, x_n) = 0 \Longleftrightarrow p(x,x) = \lim_{n\longrightarrow\infty} p(x, x_n) = \lim_{n,m\longrightarrow\infty} p(x_n, x_m) \quad (3)$$

where x is a limit of $\{x_n\}$ in (X, d_p).

Remark 3 (see [2]). Let (X, p) be a PMS. Therefore,

(i) if $p(x,y) = 0$, then $x = y$;

(ii) if $x \neq y$, then $p(x,y) > 0$.

Lemma 4 (see [3]). *Assume $x_n \longrightarrow z$ as $n \longrightarrow \infty$ in a PMS (X, p) such that $p(z,z) = 0$. Then $\lim_{n\longrightarrow\infty} p(x_n, y) = p(z, y)$ for every $y \in X$.*

In literature, there are many generalizations of Banach contraction principle in metric and generalized metric spaces. One of them is integral type contraction which was defined by Brianciari ([4]). On the other hand, Wardowski [5] introduced F−contraction in metric spaces as a generalization Banach contraction principle. For more details, you can see [5–9]. In this work, we will introduce generalized integral type F−contraction in partial metric spaces and prove common fixed point theorems.

Definition 5 (see [5]). Let a mapping $F:(0,\infty)\longrightarrow\mathbb{R}$ satisfy the following:

(F1) F is strictly increasing, i.e., for all $\alpha,\beta\in(0,\infty)$ such that $\alpha<\beta$, $F(\alpha)<F(\beta)$;

(F2) for each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$, $\lim_{n\longrightarrow\infty}\alpha_n = 0 \Longleftrightarrow \lim_{n\longrightarrow\infty}F(\alpha_n)=-\infty$;

(F3) there exists $k\in(0,1)$ such that $\lim_{\alpha\longrightarrow 0^+}\alpha^k F(a)=0$.

Definition 6 (see [5]). A mapping $T:X\longrightarrow X$ is said to be F−contraction if there exists $\tau>0$ such that

$$\begin{gathered}\forall x,y\in X,\\ d(Tx,Ty)>0\Longrightarrow\\ \tau+F(d(Tx,Ty))\le F(d(x,y)).\end{gathered}\tag{4}$$

Theorem 7 (see [5]). *Let (X,d) be a complete metric space and let $T:X\longrightarrow X$ be an F−contraction. Then T has a unique fixed point in X.*

Example 8 (see [5]). Let $F:\mathbb{R}^+\longrightarrow\mathbb{R}$ be given by $F(\alpha)=\ln\alpha$. F satisfies (F1), (F2), and (F3). Each mapping $T:X\longrightarrow X$ is an F-contraction such that, for all x,y in X and $Tx\neq Ty$,

$$d(Tx,Ty)\le e^{-\tau}d(x,y)\tag{5}$$

It is clear that for $x,y\in X$ such that $Tx=Ty$ the inequality $d(Tx,Ty)\le e^{-\tau}d(x,y)$ also holds; i.e., T is a Banach contraction.

Definition 9 (see [10]). The mappings $f,g:X\longrightarrow X$ are said to be weakly compatible if f and g commute at each coincidence point; i.e., $fx=gx$ for some $x\in X$.

2. Main Results

Theorem 10. *Let (X,p) be a complete partial metric space and $f,g,S,T:X\longrightarrow X$ are mappings satisfying $f(X)\subseteq T(X)$ and $g(X)\subseteq S(X)$. Suppose there exist $F\in\mathcal{F}$ and $\tau>0$ such that for all $x,y\in X$ satisfying $p(fx,gy)>0$*

$$\tau+F\left(\int_0^{p(fx,gy)}\varphi(t)\,dt\right)\le F\left(\int_0^{M(x,y)}\varphi(t)\,dt\right)\tag{6}$$

where

$$\begin{aligned}M(x,y)=\max\Big\{&p(Sx,Ty),p(fx,Sx),p(gy,Ty),\\ &\frac{p(Sx,gy)+p(fx,Ty)}{2}\Big\}\end{aligned}\tag{7}$$

and $\varphi:[0,\infty)\longrightarrow[0,\infty)$ is a Lebesgue integrable mapping which is summable, nonnegative and for each $\mu>0$

$$\int_0^{\mu}\varphi(t)\,dt>0.\tag{8}$$

If

(i) *$f(X)$, $g(X)$, $S(X)$, or $T(X)$ is closed,*

(ii) *F is continuous,*

(iii) *$\{f,S\}$ and $\{g,T\}$ are weakly compatible,*

then the pairs f,g,S, and T have a unique common fixed point.

Proof. Let $x_0\in X$ be arbitrary. Define a sequence $\{x_n\}$ for $n\ge 0$ by

$$\begin{gathered}y_{2n+1}=fx_{2n}=Tx_{2n+1}\\ \text{and }\ y_{2n+2}=gx_{2n+1}=Sx_{2n+2}.\end{gathered}\tag{9}$$

Step I. Prove that $p(y_n,y_{n+1})\longrightarrow 0$ as $n\longrightarrow\infty$.

By (6),

$$\begin{aligned}\tau+F&\left(\int_0^{p(y_{2n+1},y_{2n+2})}\varphi(t)\,dt\right)\\ &=\tau+F\left(\int_0^{p(fx_{2n},gx_{2n+1})}\varphi(t)\,dt\right)\\ &\le F\left(\int_0^{M(x_{2n},x_{2n+1})}\varphi(t)\,dt\right)\end{aligned}\tag{10}$$

where

$$\begin{aligned}M(x_{2n},x_{2n+1})&=\max\left\{\begin{array}{c}p(Sx_{2n},fx_{2n}),p(Tx_{2n+1},gx_{2n+1}),\\ \rho(Sx_{2n},Tx_{2n+1}),\\ \dfrac{p(Tx_{2n+1},fx_{2n})+p(Sx_{2n},gx_{2n+1})}{2}\end{array}\right\}\\ &=\max\left\{\begin{array}{c}p(y_{2n},y_{2n+1}),p(y_{2n+1},y_{2n+2}),p(y_{2n},y_{2n+1}),\\ \dfrac{p(y_{2n+1},y_{2n+1})+p(y_{2n},y_{2n+2})}{2}\end{array}\right\}\end{aligned}$$

$$\leq \max\left\{\begin{array}{c} p(y_{2n}, y_{2n+1}), p(y_{2n+1}, y_{2n+2}), \\ \dfrac{p(y_{2n+1}, y_{2n+1}) + p(y_{2n}, y_{2n+1}) + p(y_{2n+1}, y_{2n+2}) - p(y_{2n+1}, y_{2n+1})}{2} \end{array}\right\}$$
$$= \max\{p(y_{2n}, y_{2n+1}), p(y_{2n+1}, y_{2n+2})\}. \tag{11}$$

If $\max\{p(y_{2n}, y_{2n+1}), p(y_{2n+1}, y_{2n+2})\} = p(y_{2n+1}, y_{2n+2})$, then it follows from (10)

$$\tau + F\left(\int_0^{p(y_{2n+1},y_{2n+2})} \varphi(t)\,dt\right) \leq F\left(\int_0^{p(y_{2n+1},y_{2n+2})} \varphi(t)\,dt\right) \tag{12}$$

which is a contradiction (as $\tau > 0$). Thus

$$\max\{p(y_{2n}, y_{2n+1}), p(y_{2n+1}, y_{2n+2})\} = p(y_{2n}, y_{2n+1}). \tag{13}$$

From (10),

$$F\left(\int_0^{p(y_{2n+1},y_{2n+2})} \varphi(t)\,dt\right) \leq F\left(\int_0^{p(y_{2n},y_{2n+1})} \varphi(t)\,dt\right) - \tau. \tag{14}$$

Continuing this way, we have

$$F\left(\int_0^{p(y_{2n},y_{2n+1})} \varphi(t)\,dt\right) \leq F\left(\int_0^{p(y_{2n-1},y_{2n})} \varphi(t)\,dt\right) - \tau. \tag{15}$$

Using (14) and (15),

$$\begin{aligned} F\left(\int_0^{p(y_{2n+1},y_{2n+2})} \varphi(t)\,dt\right) &\leq F\left(\int_0^{p(y_{2n},y_{2n+1})} \varphi(t)\,dt\right) - \tau \\ &\leq F\left(\int_0^{p(y_{2n-1},y_{2n})} \varphi(t)\,dt\right) - 2\tau \leq \cdots \\ &\leq F\left(\int_0^{p(y_0,y_1)} \varphi(t)\,dt\right) - (2n+1)\tau. \end{aligned} \tag{16}$$

And

$$\begin{aligned} F\left(\int_0^{p(y_{2n},y_{2n+1})} \varphi(t)\,dt\right) &\leq F\left(\int_0^{p(y_{2n-1},y_{2n})} \varphi(t)\,dt\right) - \tau \\ &\leq F\left(\int_0^{p(y_{2n-2},y_{2n-1})} \varphi(t)\,dt\right) - 2\tau \leq \cdots \\ &\leq F\left(\int_0^{p(y_0,y_1)} \varphi(t)\,dt\right) - (2n)\tau \end{aligned} \tag{17}$$

Then, it follows $\lim_{n\to\infty} F(\int_0^{p(y_n,y_{n+1})} \varphi(t)dt) = -\infty$. By $F \in \mathscr{F}$ and (F2), we have

$$\lim_{n\to\infty} p(y_n, y_{n+1}) = 0. \tag{18}$$

Step II. Now, we prove that $\{y_n\}$ is p–Cauchy sequence. By $F \in \mathscr{F}$ and (F3), there exits $k \in (0,1)$ such that

$$\lim_{n\to\infty} (p(y_n, y_{n+1}))^k F(p(y_n, y_{n+1})) = 0. \tag{19}$$

By (16) and (17),

$$\begin{aligned} &(p(y_{2n+1}, y_{2n+2}))^k \left(F\left(\int_0^{p(y_{2n+1},y_{2n+2})} \varphi(t)\,dt\right) - F\left(\int_0^{p(y_0,y_1)} \varphi(t)\,dt\right)\right) \leq -(2n+1) \\ &\cdot (p(y_{2n+1}, y_{2n+2}))^k \tau \leq 0 \end{aligned} \tag{20}$$

and

$$\begin{aligned} &(p(y_{2n}, y_{2n+1}))^k \left(F\left(\int_0^{p(y_{2n},y_{2n+1})} \varphi(t)\,dt\right) - F\left(\int_0^{p(y_0,y_1)} \varphi(t)\,dt\right)\right) \leq -(2n)(p(y_{2n}, y_{2n+1}))^k \\ &\cdot \tau \leq 0 \end{aligned} \tag{21}$$

Using the above inequalities and (19),

$$\lim_{n\to\infty} n(p(y_n, y_{n+1}))^k = 0. \tag{22}$$

Therefore, there exists $n_1 \in \mathbb{N}$ such that $n(p(y_n, y_{n+1}))^k < 1$ for all $n > n_1$, or

$$p(y_n, y_{n+1}) < \frac{1}{n^{1/k}}. \tag{23}$$

Let $m, n \in \mathbb{N}$ with $m > n > n_1$; using triangular inequality, we have

$$\begin{aligned} p(y_n, y_m) &= p(y_n, y_{n+1}) + p(y_{n+1}, y_{n+2}) + \cdots \\ &\quad + p(y_{m-1}, y_m) - [p(y_{n+1}, y_{n+1}) + p(y_{n+2}, y_{n+2}) \\ &\quad + \cdots + p(y_{m-1}, y_{m-1})] \leq p(y_n, y_{n+1}) \\ &\quad + p(y_{n+1}, y_{n+2}) + \cdots + p(y_{m-1}, y_m) \\ &= \sum_{i=n}^{m-1} p(y_i, y_{i+1}) \leq \sum_{i=n}^{\infty} p(y_i, y_{i+1}) \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}. \end{aligned} \tag{24}$$

As $k \in (0,1)$, the series $\sum_{i=n}^{\infty}(1/i^{1/k})$ converges, so

$$\lim_{n,m \longrightarrow \infty} p(y_n, y_m) = 0. \tag{25}$$

Thus y_n is a Cauchy sequence in (X, p). Therefore, y_n is a Cauchy sequence in (X, d_p). Since (X, p) is complete partial metric space, then (X, d_p) is complete metric space. Then, there exists a $u \in X$ such that $\lim_{n \longrightarrow \infty} d_p(y_n, u) = 0$. Moreover

$$p(u, u) = \lim_{n \longrightarrow \infty} p(y_n, u) = \lim_{n,m \longrightarrow \infty} p(y_n, y_m) = 0. \tag{26}$$

Since $y_n \longrightarrow u$, then $fx_{2n}, Tx_{2n+1}, gx_{2n+1}, Sx_{2n+2}$ converge to u.

Step III. We will prove that f, g, S, and T have a coincidence point.

Suppose $T(X)$ is closed, there exists $v \in X$ such that $Tv = u$. We shall show that $gv = u$. Then from (6),

$$\tau + F\left(\int_0^{p(fx_{2n}, gv)} \varphi(t)\, dt\right) \leq F\left(\int_0^{M(x_{2n}, v)} \varphi(t)\, dt\right) \tag{27}$$

where

$$\begin{aligned} &M(x_{2n}, v) \\ &= \max \left\{ \begin{matrix} p(Sx_{2n}, fx_{2n}), p(Tv, gv), p(Sx_{2n}, Tv), \\ \dfrac{p(Tv, fx_{2n}) + p(Sx_{2n}, gv)}{2} \end{matrix} \right\} \\ &= \max \left\{ \begin{matrix} p(Sx_{2n}, fx_{2n}), p(u, gv), p(Sx_{2n}, u), \\ \dfrac{p(u, fx_{2n})) + p(Sx_{2n}, gv)}{2} \end{matrix} \right\}. \end{aligned} \tag{28}$$

Passing to limit as $n \longrightarrow \infty$,

$$\tau + F\left(\int_0^{p(u, gv)} \varphi(t)\, dt\right) \leq F\left(\int_0^{p(u, gv)} \varphi(t)\, dt\right). \tag{29}$$

This is a contradiction with $\tau > 0$. Thus we have $gv = u$. Therefore $Tv = gv = u$. Since g and T are weakly compatible $gu = gTv = Tgv = Tu$.

Now we show that $gu = u$.

$$\tau + F\left(\int_0^{p(fx_{2n}, gu)} \varphi(t)\, dt\right) \leq F\left(\int_0^{M(x_{2n}, u)} \varphi(t)\, dt\right) \tag{30}$$

where

$$\begin{aligned} &M(x_{2n}, u) \\ &= \max \left\{ \begin{matrix} p(Sx_{2n}, fx_{2n}), p(Tu, gu), p(Sx_{2n}, Tu), \\ \dfrac{p(Tu, fx_{2n}) + p(Sx_{2n}, gu)}{2} \end{matrix} \right\} \\ &= \max \left\{ \begin{matrix} p(Sx_{2n}, fx_{2n}), p(gu, gu), p(Sx_{2n}, gu), \\ \dfrac{p(gu, fx_{2n})) + \rho(Sx_{2n}, gu)}{2} \end{matrix} \right\}. \end{aligned} \tag{31}$$

Passing to the limit as $n \longrightarrow \infty$ and using continuity of F, we have

$$\tau + F\left(\int_0^{p(u, gu)} \varphi(t)\, dt\right) \leq F\left(\int_0^{p(u, gu)} \varphi(t)\, dt\right), \tag{32}$$

which is a contradiction. Therefore $p(u, gu) = 0$; that is, u is a fixed point of g and T.

Now we show that u is a fixed point of f and S. Since $g(X) \subseteq S(X)$, there exists a point $z \in X$ such that $gu = Sz$. Suppose that $fz \neq Sz$, then

$$\tau + F\left(\int_0^{p(fz, gu)} \varphi(t)\, dt\right) \leq F\left(\int_0^{M(z, u)} \varphi(t)\, dt\right) \tag{33}$$

where

$$\begin{aligned} &M(z, u) \\ &= \max \left\{ \begin{matrix} p(Sz, fz), p(Tu, gu), p(Sz, Tu), \\ \dfrac{p(Tu, fz) + p(Sz, gu)}{2} \end{matrix} \right\} \\ &= \max \left\{ p(Sz, fz), \frac{p(Tu, fz) + p(Sz, gu)}{2} \right\} \\ &= p(Sz, fz) = p(gu, fz). \end{aligned} \tag{34}$$

Thus

$$\tau + F\left(\int_0^{p(fz, gu)} \varphi(t)\, dt\right) \leq F\left(\int_0^{p(fz, gu)} \varphi(t)\, dt\right), \tag{35}$$

which is a contradiction. Thus $fz = gu = Sz$. By weak compatibility of f and S, $Su = Sfz = fSz = fu$. Finally we show that $fu = u$. From (6),

$$\begin{aligned} \tau + F\left(\int_0^{p(fu, u)} \varphi(t)\, dt\right) &= \tau + F\left(\int_0^{p(fu, gu)} \varphi(t)\, dt\right) \\ &\leq F\left(\int_0^{M(u, u)} \varphi(t)\, dt\right) \end{aligned} \tag{36}$$

where

$$\begin{aligned} M(u,u) &= \max\left\{ \begin{array}{c} p(Su,fu),p(Tu,gu),p(Su,Tu), \\ \dfrac{p(Tu,fu)+p(Su,gu)}{2} \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} p(fu,fu),p(u,u),p(fu,u), \\ \dfrac{p(u,fu))+p(fu,u)}{2} \end{array} \right\} \\ &= p(fu,u). \end{aligned} \tag{37}$$

Thus,

$$\tau + F\left(\int_0^{p(fu,u)} \varphi(t)\,dt\right) \le F\left(\int_0^{p(fu,u)} \varphi(t)\,dt\right) \tag{38}$$

and we have $fu = Su = u$.

So u is a common fixed point of f, g, S, and T.

Step IV. We show uniqueness of common fixed point. Let w be another common fixed point of f and g and $u \neq w$.

From (6), we have

$$\begin{aligned} \tau + F\left(\int_0^{p(u,w)} \varphi(t)\,dt\right) &= \tau + F\left(\int_0^{p(fu,fw)} \varphi(t)\,dt\right) \\ &\le F\left(\int_0^{M(u,w)} \varphi(t)\,dt\right) \end{aligned} \tag{39}$$

where

$$\begin{aligned} M(u,w) &= \max\left\{ \begin{array}{c} p(Su,fu),p(Tw,gw),p(Su,Tw), \\ \dfrac{p(Tw,fu))+p(Su,gw))}{2} \end{array} \right\}, \\ &= \max\left\{ p(u,u),p(w,w),p(u,w), \right. \\ &\quad \left. \frac{p(w,u))+p(u,w))}{2} \right\}. \end{aligned} \tag{40}$$

Hence,

$$\tau + F\left(\int_0^{p(u,w)} \varphi(t)\,dt\right) \le F\left(\int_0^{p(u,w)} \varphi(t)\,dt\right) \tag{41}$$

which is a contradiction. So $u = w$. □

Corollary 11. *Let (X,p) be a complete partial metric space and $f,g : X \longrightarrow X$ are two mappings. Suppose there exist $F \in F$ and $\tau > 0$ such that for all $x,y \in X$ satisfying $p(fx,gy) > 0$*

$$\tau + F\left(\int_0^{p(fx,gy)} \varphi(t)\,dt\right) \le F\left(\int_0^{M(x,y)} \varphi(t)\,dt\right) \tag{42}$$

where

$$M(x,y) = \max\left\{ \begin{array}{c} p(x,y),p(x,fx),p(y,gy), \\ \dfrac{p(y,fx))+\rho(x,gy)}{2} \end{array} \right\}. \tag{43}$$

and $\varphi : [0,\infty) \longrightarrow [0,\infty)$ is a Lebesgue integrable mapping which is summable, nonnegative and for each $\mu > 0$

$$\int_0^{\mu} \varphi(t)\,dt > 0. \tag{44}$$

If

(i) *$f(X)$ or $g(X)$ is closed,*

(ii) *F is continuous,*

then the pairs f and g have a unique common fixed point.

Corollary 12. *Let (X,p) be a complete partial metric space and $f,g : X \longrightarrow X$ are two mappings. Suppose there exist $F \in F$ and $\tau > 0$ such that for all $x,y \in X$ satisfying $p(fx,gy) > 0$*

$$\tau + F\left(\int_0^{p(fx,gy)} \varphi(t)\,dt\right) \le F\left(\int_0^{p(x,y)} \varphi(t)\,dt\right). \tag{45}$$

And $\varphi : [0,\infty) \longrightarrow [0,\infty)$ is a Lebesgue integrable mapping which is summable, nonnegative and for each $\mu > 0$

$$\int_0^{\mu} \varphi(t)\,dt > 0. \tag{46}$$

If

(i) *$f(X)$ or $g(X)$ is closed,*

(ii) *F is continuous,*

then the pairs f and g have a unique common fixed point.

Theorem 13. *Let (X,p) be a complete partial metric space and $f,g : X \longrightarrow X$ be mappings. Suppose there exist $F \in F$ and $\tau > 0$ such that for all $x,y \in X$ satisfying $p(fx,gy) > 0$*

$$\tau + F\left(\int_0^{p(fx,gy)} \varphi(t)\,dt\right) \le F\left(\int_0^{M(x,y)} \varphi(t)\,dt\right) \tag{47}$$

where

$$\begin{aligned} M(x,y) = \max\left\{ p(x,y),p(x,fx),p(y,gy), \right. \\ \left. \frac{p(y,fx)+p(x,gy)}{2} \right\}. \end{aligned} \tag{48}$$

And $\varphi : [0,\infty) \longrightarrow [0,\infty)$ is a Lebesgue integrable mapping which is summable, nonnegative and for each $\mu > 0$

$$\int_0^{\mu} \varphi(t)\,dt > 0. \tag{49}$$

If

(i) f *or* g *is continuous, or*

(ii) F *is continuous,*

then the pairs f *and* g *have a unique common fixed point.*

Proof. Let $x_0 \in X$ be arbitrary. Define a sequence $\{x_n\}$ for $n \geq 0$ by

$$x_{2n+1} = fx_{2n} \quad \text{and } x_{2n+2} = gx_{2n+1}. \tag{50}$$

Step I. Prove that $p(x_n, x_{n+1}) \longrightarrow 0$ as $n \longrightarrow \infty$.

By (47),

$$\begin{aligned} \tau + F\left(\int_0^{p(x_{2n+1},x_{2n+2})} \varphi(t)\,dt\right) &= \tau + F\left(\int_0^{p(fx_{2n},gx_{2n+1})} \varphi(t)\,dt\right) \\ &\leq F\left(\int_0^{M(x_{2n},x_{2n+1})} \varphi(t)\,dt\right) \end{aligned} \tag{51}$$

where

$$\begin{aligned} &M(x_{2n}, x_{2n+1}) \\ &= \max\left\{\begin{matrix} p(x_{2n}, x_{2n+1}), p(x_{2n}, fx_{2n}), p(x_{2n+1}, gx_{2n+1}), \\ \dfrac{p(x_{2n+1}, fx_{2n}) + p(x_{2n} gx_{2n+1})}{2} \end{matrix}\right\} \\ &= \max\left\{\begin{matrix} p(x_{2n}, x_{2n+1}), p(x_{2n}, x_{2n+1}), p(x_{2n+1}, x_{2n+2}), \\ \dfrac{p(x_{2n+1}, x_{2n+1}) + p(x_{2n}, x_{2n+2})}{2} \end{matrix}\right\} \\ &\leq \max\Big\{p(x_{2n}, x_{2n+1}), p(x_{2n+1}, x_{2n+2}), \\ &\quad \frac{p(x_{2n}, x_{2n+1}) + p(x_{2n+1}, x_{2n+2})}{2}\Big\} = \max\{p(x_{2n}, x_{2n+1}), \\ &\quad p(x_{2n+1}, x_{2n+2})\}. \end{aligned} \tag{52}$$

Then the proof is similar proof of Theorem 10.

We will prove that f and g have common fixed point. Since (X, p) is complete partial metric space, then (X, d_p) is complete metric space. Then, there exists $u \in X$ such that $\lim_{n\longrightarrow\infty} d_p(y_n, u) = 0$. Moreover

$$p(u, u) = \lim_{n\longrightarrow\infty} p(y_n, u) = \lim_{n,m\longrightarrow\infty} p(y_n, y_m) = 0. \tag{53}$$

We consider two cases.

Case 1. Suppose f is continuous. Then, $u = \lim_{n\longrightarrow\infty} x_n = \lim_{n\longrightarrow\infty} x_{2n} = \lim_{n\longrightarrow\infty} x_{2n+1} = \lim_{n\longrightarrow\infty} fx_{2n} = fu$. Thus u is a fixed point of f.

Now we prove u is a fixed point of g. On the contrary, we assume $gu \neq u$. From (47),

$$\tau + F\left(\int_0^{p(fx_{2n},gu)} \varphi(t)\,dt\right) \leq F\left(\int_0^{M(x_{2n},u)} \varphi(t)\,dt\right) \tag{54}$$

where

$$\begin{aligned} M(x_{2n}, u) &= \max\Big\{p(x_{2n}, u), p(x_{2n}, fx_{2n}), p(u, gu), \\ &\quad \frac{p(u, fx_{2n}) + p(x_{2n}, gu)}{2}\Big\} = \max\Big\{p(x_{2n}, u), \\ &\quad p(x_{2n}, x_{2n+1}), p(u, gu), \\ &\quad \frac{p(u, x_{2n+1}) + p(x_{2n}, gu)}{2}\Big\}. \end{aligned} \tag{55}$$

Letting $n \longrightarrow \infty$,

$$\begin{gathered} \lim_{n\longrightarrow\infty} M(x_{2n}, u) = p(u, gu). \\ \tau + F\left(\int_0^{p(u,gu)} \varphi(t)\,dt\right) \leq F\left(\int_0^{p(u,gu)} \varphi(t)\,dt\right). \end{gathered} \tag{56}$$

This is a contradiction with $\tau > 0$. Thus we have $gu = u$.

Similarly, we have the same results when g is continuous.

Case 2. Now, we suppose that F is continuous. We can assume there exists $n_1 \in \mathbb{N}$ such that $fx_{n+1} \neq u$ (i.e., $p(x_n, u) > 0$) for all $n \geq n_1$. Then from (47) we have

$$\begin{aligned} \tau + F\left(\int_0^{p(fu,gx_{2n+1})} \varphi(t)\,dt\right) \\ \leq F\left(\int_0^{M(u,x_{2n+1})} \varphi(t)\,dt\right) \end{aligned} \tag{57}$$

where

$$\begin{aligned} M(u, x_{2n+1}) &= \max\Big\{p(u, x_{2n+1}), p(u, fu), \\ &\quad p(x_{2n+1}, gx_{2n+1}), \frac{p(x_{2n+1}, fu) + p(u, gx_{2n+1})}{2}\Big\} \\ &= \max\Big\{p(u, x_{2n+1}), p(u, fu), p(x_{2n+1}, x_{2n+2}), \\ &\quad \frac{p(x_{2n+1}, fu) + p(u, x_{2n+2})}{2}\Big\}. \end{aligned} \tag{58}$$

Then there exists $n_2 \in \mathbb{N}$ such that, for all $n \geq n_2$, we have

$$\begin{aligned} &\max\left\{\begin{matrix} p(u, x_{2n+1}), p(u, fu), p(x_{2n+1}, x_{2n+2}), \\ \dfrac{p(x_{2n+1}, u) + p(u, fu) + p(u, x_{2n+2}) - p(u, u)}{2} \end{matrix}\right\} \\ &= p(u, fu). \end{aligned} \tag{59}$$

Thus, we have

$$\tau + F\left(\int_0^{p(fu,gx_{2n+1})} \varphi(t)\,dt\right) \leq F\left(\int_0^{p(u,fu)} \varphi(t)\,dt\right), \tag{60}$$

for all $n \geq \max\{n_1, n_2\}$. Since F is continuous, taking the limit as $n \longrightarrow \infty$, we get

$$\tau + F\left(\int_0^{p(fu,u)} \varphi(t)\,dt\right) \leq F\left(\int_0^{p(u,fu)} \varphi(t)\,dt\right) \quad (61)$$

which is a contradiction. Therefore $p(fu, u) = 0$ and u is a fixed point of f.

Now we show that u is a fixed point of g.

$$\begin{aligned}\tau + F\left(\int_0^{p(u,gu)} \varphi(t)\,dt\right) &= \tau + F\left(\int_0^{p(fu,gu)} \varphi(t)\,dt\right) \\ &\leq F\left(\int_0^{M(u,u)} \varphi(t)\,dt\right)\end{aligned} \quad (62)$$

where

$$\begin{aligned}M(u,u) = \max\Big\{&p(u,u), p(u,fu), p(u,gu), \\ &\frac{p(u,fu)+p(u,gu)}{2}\Big\} = \max\Big\{p(u,u), p(u,u), \\ &p(u,gu), \frac{p(u,u)+p(u,gu)}{2}\Big\} = p(u,gu).\end{aligned} \quad (63)$$

Thus

$$\tau + F\left(\int_0^{p(u,gu)} \varphi(t)\,dt\right) \leq F\left(\int_0^{p(u,gu)} \varphi(t)\,dt\right). \quad (64)$$

Hence, $u = gu$.

So u is a common fixed point of f and g.

Now we prove uniqueness of common fixed point. We assume that v is another common fixed point of f and g and $u \neq v$.

From (47), we have

$$\begin{aligned}\tau + F\left(\int_0^{p(u,v)} \varphi(t)\,dt\right) &= \tau + F\left(\int_0^{p(fu,gv)} \varphi(t)\,dt\right) \\ &\leq F\left(\int_0^{M(u,v)} \varphi(t)\,dt\right)\end{aligned} \quad (65)$$

where

$$\begin{aligned}M(u,v) = \max\Big\{&p(u,v), p(u,fu), p(v,gv), \\ &\frac{p(v,fu)+p(u,gv)}{2}\Big\}, = \max\Big\{p(u,v), p(u,u), \\ &p(v,v), \frac{p(v,u)+p(u,v)}{2}\Big\}.\end{aligned} \quad (66)$$

Hence,

$$\tau + F\left(\int_0^{p(u,v)} \varphi(t)\,dt\right) \leq F\left(\int_0^{p(u,v)} \varphi(t)\,dt\right), \quad (67)$$

which is a contradiction with $\tau > 0$. So $u = v$. □

Corollary 14. *Let (X, p) be a complete partial metric space and $f, g : X \longrightarrow X$ two mappings. Suppose there exist $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in X$ satisfying $p(fx, gy) > 0$*

$$\tau + F\left(\int_0^{p(fx,gy)} \varphi(t)\,dt\right) \leq F\left(\int_0^{p(x,y)} \varphi(t)\,dt\right) \quad (68)$$

and $\varphi : [0,\infty) \longrightarrow [0,\infty)$ is a Lebesgue integrable mapping which is summable, nonnegative and for each $\mu > 0$

$$\int_0^{\mu} \varphi(t)\,dt > 0. \quad (69)$$

If

(i) *f or g is continuous, or*

(ii) *F is continuous, then the pairs f and g have a unique common fixed point.*

Example 15. Let $X = [0, 1]$, and $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. Then (X, p) is complete partial metric space. Let $f, g, S, T : X \longrightarrow X$ and $\varphi : (0,\infty) \longrightarrow (0,\infty)$

$$\begin{aligned}f(x) &= \frac{x}{8}, \\ g(x) &= 0, \\ S(x) &= \frac{3x}{4} \\ \text{and } T(x) &= x, \\ \varphi(t) &= 2t.\end{aligned} \quad (70)$$

Consider F in Example 8. Then all conditions of Theorem 10 and the contractive condition (6) are satisfied for some $\tau > 0$ and for $p(x, y) > 0$.

If $3x/4 > y$,

$$\begin{aligned}\tau + F\left(\int_0^{p(fx,gy)} \varphi(t)\,dt\right) &= \tau + \ln\left(\frac{x^2}{128}\right) \\ \leq \ln\left(\frac{9x^2}{16}\right) &= F\left(\int_0^{M(x,y)} \varphi(t)\,dt\right).\end{aligned} \quad (71)$$

If $3x/4 < y$,

$$\begin{aligned}\tau + F\left(\int_0^{p(fx,gy)} \varphi(t)\,dt\right) &= \tau + \ln\left(\frac{x^2}{128}\right) \\ &< \tau + \ln\left(\frac{16y^2}{3.128}\right) \leq \ln\left(\frac{y^2}{2}\right) \\ &= F\left(\int_0^{M(x,y)} \varphi(t)\,dt\right).\end{aligned} \quad (72)$$

Therefore 0 is a fixed point of f, g, S, and T.

Data Availability

The author did not use any data set.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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