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Research Article

Weighted Endpoint Estimates for Commutators of Singular Integral Operators on Orlicz-Morrey Spaces

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In this paper, we obtain the weighted endpoint estimates for the commutators of the singular integral operators with the *BMO* functions and the associated maximal operators on Orlicz-Morrey Spaces. We also get the similar results for the commutators of the fractional integral operators with the *BMO* functions and the associated maximal operators.

1. Introduction and Main Results

The Morrey spaces were introduced by Morrey in [1] to investigate the local behavior of solutions to second-order elliptic partial differential equations. Chiarenza and Frasca [2] showed the boundedness of the Hardy-Littlewood maximal operator, singular integral operators, and the fractional integral operators on the Morrey spaces. Komori and Shirai [3] introduced the weighted Morrey spaces and proved that, for $1 < p < \infty$ and $w \in A_p$, T and $[b,T]$ are bounded on $L^{p,\kappa}(w)$, and if $p = 1$ and $w \in A_1$, then for all $t > 0$ and any cube Q,

$$w\left(\left\{x \in Q : \left|Tf(x)\right| > t\right\}\right) \leq \frac{C}{t} \|f\|_{L^{1,\kappa}(w)} w(Q)^{\kappa}. \quad (1)$$

In this paper, we obtain the weighted endpoint estimates for the commutators of the singular integral operators with *BMO* functions and associated maximal operators. We also obtain the similar results for the commutators of the fractional integral operators with *BMO* functions and associated maximal operators.

Let f be a measurable function on $\mathbb{R}^n$ and $1 \leq p < \infty$, $0 \leq \kappa < 1$, for two weights w and u, and the weighted Morrey space is defined by

$$L^{p,\kappa}(w,u) = \left\{f \in L_{loc}{}^{p}(w) : \|f\|_{L^{p,\kappa}(w,u)} < \infty\right\}, \quad (2)$$

where

$$\|f\|_{L^{p,\kappa}(w,u)} = \sup_Q \left(\frac{1}{u(Q)^{\kappa}} \int_Q \left|f(x)\right|^p w(x)\, dx\right)^{1/p}, \quad (3)$$

and the supremum is taken over all cubes Q in $\mathbb{R}^n$. When $w = u$, we write $L^{p,\kappa}(w,u)$ as $L^{p,\kappa}(w)$.

We say that T is a singular integral operator if there exists a function K which satisfies the following conditions:

$$\begin{gathered} Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x-y) f(y)\, dy, \\ |K(x)| \leq \frac{C}{|x|^n}, \\ |\nabla K(x)| \leq \frac{C}{|x|^{n+1}}, \\ x \neq 0. \end{gathered} \quad (4)$$

The $BMO(\mathbb{R}^n)$ space is defined by

$$BMO(\mathbb{R}^n) = \left\{b \in L_{loc}(\mathbb{R}^n) : \|b\|_{BMO} = \sup_Q \frac{1}{|Q|} \int_Q \left|b(x) - b_Q\right| dx < \infty\right\}, \quad (5)$$

where $b_Q = (1/|Q|) \int_Q b(y) dy$.

For the singular integral operator T and $b \in BMO$, the commutator $[b,T]$ is defined by

$$[b,T]\, f(x) = \int_{\mathbb{R}^n} (b(x) - b(y)) K(x-y) f(y)\, dy. \quad (6)$$

In order to state our results, we need to recall some notations and facts about the Young functions and Orlicz spaces; for further information, see [4]. A function $\Phi : [0,\infty) \longrightarrow [0,\infty)$ is a Young function if it is convex and increasing, and if $\Phi(0) = 0$ and $\Phi(t) \longrightarrow \infty$ as $t \longrightarrow \infty$.

Let Φ be a Young function, $0 < \kappa < 1$ and two weights w and u, and the weighted Orlicz-Morrey Class $L^{\Phi,\kappa}(w,u)$ is defined as

$$L^{\Phi,\kappa}(w,u) = \left\{ f : \|f\|_{L^{\Phi,\kappa}(w,u)} < \infty \right\}, \quad (7)$$

where

$$\|f\|_{L^{\Phi,\kappa}(w,u)} = \sup_{Q} \frac{1}{u(Q)^{\kappa}} \int_Q \Phi(|f(x)|)\, w(x)\, dx. \quad (8)$$

When $w = u$, we write $L^{\Phi,\kappa}(w,u)$ as $L^{\Phi,\kappa}(w)$.

Given a locally integrable function f and a Young function Φ, define the mean Luxemburg norm of f on a cube Q by

$$\|f\|_{\Phi,Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \Phi\left(\frac{|f(x)|}{\lambda} \right) dx \le 1 \right\}. \quad (9)$$

For $\alpha, 0 \le \alpha < n$, and a Young function Φ, we define Orlicz maximal operator

$$M_{\alpha,\Phi} f(x) = \sup_{Q \ni x} |Q|^{\alpha/n} \|f\|_{\Phi,Q}. \quad (10)$$

If $\alpha = 0$, we write $M_{\alpha,\Phi}$ simply as M_{Φ}. If $\alpha = 0$ and $\Phi(t) = t$, $M_{\alpha,\Phi}$ is the Hardy-Littlewood maximal operator M. If $\Phi_{\varepsilon}(t) = t\log(e+t)^{\varepsilon}, \varepsilon \ge 0$, we write $M_{\Phi_{\varepsilon}}$ simply as $M_{L(\log L)^{\varepsilon}}$.

If $0 < \alpha < n$ and $\Phi(t) = t$, $M_{\alpha,\Phi}$ is the fractional maximal operator of order α and we write it as M_{α}. If $\Phi_{\varepsilon}(t) = t\log(e+t)^{\varepsilon}$, we write $M_{\alpha,\Phi}$ simply as $M_{\alpha,L(\log L)^{\varepsilon}}$.

Take $w \in A_1$, which means $Mw(x) \le Cw(x)$ for a.e. $x \in \mathbb{R}^n$.

Given $\alpha, 0 < \alpha < n$, for an appropriate function f on $\mathbb{R}^n$, the fractional integral operator (or the Riesz potential) of order α is defined by

$$I_{\alpha} f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy. \quad (11)$$

For $b \in BMO(\mathbb{R}^n)$, we define the commutators of the operator I_{α} and b by

$$[b, I_{\alpha}]\, f(x) = \int_{\mathbb{R}^n} \frac{(b(x) - b(y)) f(y)}{|x-y|^{n-\alpha}} dy. \quad (12)$$

The following theorems are our main results.

Theorem 1. *Let $w \in A_1$ and $\Phi(t) = t\log(e+t)$, then there exists a positive constant C such that, for any cube Q and any $t > 0$,*

$$w\left(\left\{ x \in Q : M_{L(\log L)} f(x) \mid > t \right\}\right) \le C \left\| \frac{|f|}{t} \right\|_{L^{\Phi,\kappa}(w)} w(Q)^{\kappa}. \quad (13)$$

Theorem 2. *Let T be any singular integral operator, $w \in A_1$, $\Phi(t) = t\log(e+t)$, and $b \in BMO$. Then there exists a positive constant C such that, for any cube Q and any $t > 0$,*

$$w\left(\left\{ x \in Q : \left|[b,T]\, f(x)\right| > t \right\}\right) \le C \left\| \frac{|f|}{t} \right\|_{L^{\Phi,\kappa}(w)} w(Q)^{\kappa}. \quad (14)$$

Theorem 3. *Let $0 < \alpha < n$, $w \in A_1$, $1/q = 1 - \alpha/n$, $0 < \kappa < 1/q$, $\Phi(t) = t\log(e+t)$, $\Psi(t) = t^{1/q}\log(e+t)^{-1}$, and $\Theta(t) = t^{1/q}\log(e+t^{-1})$. Then there exists a positive constant C such that, for any cube Q and any $t > 0$,*

$$\Psi\left(w\left(\left\{ x \in Q : M_{\alpha,L(\log L)} f(x) > t \right\}\right)\right) \le C \left\| \frac{|f|}{t} \right\|_{L^{\Phi,\kappa}(w,\Theta(w))} w(Q)^{\kappa}. \quad (15)$$

Theorem 4. *Let $0 < \alpha < n$, $w \in A_1$, $b \in BMO$, $1/q = 1 - \alpha/n$, $0 < \kappa < 1/q$, $\Phi(t) = t\log(e+t)$, $\Psi(t) = t^{1/q}\log(e+t)^{-1}$, and $\Theta(t) = t^{1/q}\log(e+t^{-1})$. Then there exists a positive constant C such that, for any cube Q and any $t > 0$,*

$$\Psi\left(w\left(\left\{ x \in Q : \left|[b, I_{\alpha}]\, f(x)\right| > t \right\}\right)\right) \le C \left\| \frac{|f|}{t} \right\|_{L^{\Phi,\kappa}(w,\Theta(w))} w(Q)^{\kappa}. \quad (16)$$

2. Proof of Theorems 1 and 2

Lemma 5 (see [5]). *Let $\Phi(t) = t\log(e+t)$, then there exists a positive constant C such that, for any weight w and all $t > 0$,*

$$w\left(\left\{ x \in \mathbb{R}^n : M_{L(\log L)} f(x) > t \right\}\right) \le C \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{t} \right) Mw(x)\, dx \quad (17)$$

for every locally integrable function f.

Lemma 6 (see [6]). *Let $w \in A_1$, then there exist a constant $C > 0$ and $\eta > 0$ such that, for any cube Q and a measurable subset $E \subset Q$,*

$$\frac{w(E)}{w(Q)} \le C \left(\frac{|E|}{|Q|} \right)^{\eta}. \quad (18)$$

Proof of Theorem 1. Fix a cube Q centered at x_0. By Lemma 5, we have

$$\begin{aligned} w\left(\left\{x \in Q : M_{L(\log L)} f(x) > t\right\}\right) &= \int_{\{x\in\mathbb{R}^n : M_{L(\log L)} f(x) > t\}} \chi_Q w(x)\, dx \\ &\le C \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{t}\right) M(\chi_Q w)(x)\, dx \\ &\le C\left(\int_{3Q} + \int_{(3Q)^c}\right) \Phi\left(\frac{|f(x)|}{t}\right) M(\chi_Q w)(x)\, dx \\ &\le \mathrm{I} + \mathrm{II}. \end{aligned} \tag{19}$$

To estimate term I, since $w \in A_1$, we have

$$\begin{aligned} \mathrm{I} &\le C \int_{3Q} \Phi\left(\frac{|f(x)|}{t}\right) w(x)\, dx \\ &\le C \left\|\frac{|f|}{t}\right\|_{L^{\Phi,\kappa}(w)} w(Q)^\kappa. \end{aligned} \tag{20}$$

For term II, observe that, for $x \in (3Q)^c$, $x \in R$ and $R \cap Q \neq \emptyset$. We have

$$\begin{aligned} \frac{1}{|R|}\int_R \chi_Q(y) w(y)\, dy &= \frac{1}{|R|}\int_{R\cap Q} w(y)\, dy \\ &\le \frac{C}{|x - x_0|^n}\int_Q w(y)\, dy = \frac{C}{|x - x_0|^n} w(Q). \end{aligned} \tag{21}$$

Therefore we obtain

$$M(\chi_Q w)(x) \le C|x - x_0|^{-n} w(Q). \tag{22}$$

Since $w \in A_1$, using Lemma 6, we get

$$\begin{aligned} \mathrm{II} &\le C\int_{(3Q)^c} \Phi\left(\frac{|f(x)|}{t}\right)|x - x_0|^{-n} w(Q)\, dx \\ &\le Cw(Q)\sum_{j=1}^{\infty}\int_{3^{j+1}Q\setminus 3^j Q} \Phi\left(\frac{|f(x)|}{t}\right)|x - x_0|^{-n}\, dx \\ &\le Cw(Q)\sum_{j=1}^{\infty}\frac{1}{|3^j Q|}\int_{3^{j+1}Q} \Phi\left(\frac{|f(x)|}{t}\right) dx \\ &\le Cw(Q)^\kappa \\ &\quad\cdot \sum_{j=1}^{\infty}\frac{w(Q)^{1-\kappa}}{w(3^{j+1}Q)^{1-\kappa}}\frac{1}{w(3^{j+1}Q)^\kappa}\int_{3^{j+1}Q} \Phi\left(\frac{|f(x)|}{t}\right) \\ &\quad\cdot w(x)\, dx \le Cw(Q)^\kappa \left\|\frac{|f|}{t}\right\|_{L^{\Phi,\kappa}(w)} \sum_{j=1}^{\infty}\frac{1}{3^{jn\eta(1-\kappa)}} \\ &\le Cw(Q)^\kappa \left\|\frac{|f|}{t}\right\|_{L^{\Phi,\kappa}(w)}. \end{aligned} \tag{23}$$

This ends the proof. □

Lemma 7 (see [7]). *Let T be any Calderón-Zygmund singular integral operator, $\Phi(t) = t\log(e + t)$, $\varepsilon > 0$, and $b \in BMO$. Then there exists a positive constant C such that, for all weights w,*

$$\begin{aligned} w(\{x \in \mathbb{R}^n : |[b,T] f(x)| > t\}) &\le C\int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{t}\right) M_{L(\log L)^{1+\varepsilon}} w(x)\, dx. \end{aligned} \tag{24}$$

Lemma 8 (see [6]). *Let $w \in A_1$, then there exist a constant $C > 0$ and $\theta > 0$ such that, for any cube Q,*

$$\left(\frac{1}{|Q|}\int_Q w(y)^{1+\theta}\, dy\right)^{1/(1+\theta)} \le C\frac{1}{|Q|}\int_Q w(y)\, dy. \tag{25}$$

Proof of Theorem 2. Fix a cube Q centered at x_0. By Lemma 7, we have

$$\begin{aligned} w(\{x \in Q : |[b,T] f(x)| > t\}) &= \int_{\{x\in\mathbb{R}^n : [b,T] f(x) > t\}} w(x)\chi_Q(x)\, dx \\ &\le C\int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{t}\right) M_{L(\log L)^{1+\varepsilon}}(w\chi_Q)(x)\, dx \\ &\le C\left(\int_{3Q} + \int_{(3Q)^c}\right)\Phi\left(\frac{|f(x)|}{t}\right) M_{L(\log L)^{1+\varepsilon}}(w\chi_Q) \\ &\quad\cdot (x)\, dx \le \mathrm{I} + \mathrm{II}. \end{aligned} \tag{26}$$

To estimate term I, since $w \in A_1$, it is easy to prove that $M_{L(\log L)^{1+\varepsilon}}(w\chi_Q)(x) \le Cw(x)$, $x \in 3Q$, and we have

$$\begin{aligned} \mathrm{I} &\le C\int_{3Q} \Phi\left(\frac{|f(x)|}{t}\right) w(x)\, dx \\ &\le C\left\|\frac{|f|}{t}\right\|_{L^{\Phi,\kappa}(w)} w(Q)^\kappa. \end{aligned} \tag{27}$$

For term II, observe that, for $x \in (3Q)^c$, $x \in R$, R is a cube, and $R \cap Q \neq \emptyset$, by Lemma 8, for any $\delta : 0 < \delta \le \theta$, we have

$$\begin{aligned} &\left(\frac{1}{|R|}\int_R (w(y)\chi_Q(y))^{1+\delta}\, dy\right)^{1/(1+\delta)} \\ &\quad\le \left(\frac{1}{|R|}\int_Q w(y)^{1+\delta}\, dy\right)^{1/(1+\delta)} \\ &\quad= \left(\frac{|Q|}{|R|}\right)^{1/(1+\delta)}\left(\frac{1}{|Q|}\int_Q w(y)^{1+\delta}\, dy\right)^{1/(1+\delta)} \\ &\quad\le C\left(\frac{|Q|}{|R|}\right)^{1/(1+\delta)}\left(\frac{1}{|Q|}\int_Q w(y)\, dy\right) \\ &\quad\le C\left(\frac{|Q|}{|R|}\right)^{1/(1+\delta)}\frac{w(Q)}{|Q|}. \end{aligned} \tag{28}$$

Noticing the definition of the maximal function M, we obtain

$$\begin{aligned} M_{L(\log L)^{1+\epsilon}}(w\chi_Q)(x) &\le \left(M\left(w^{1+\delta}\chi_Q\right)(x)\right)^{1/(1+\delta)} \\ &\le C\left(\frac{|Q|}{|x-x_0|^n}\right)^{1/(1+\delta)}\frac{w(Q)}{|Q|}. \end{aligned} \tag{29}$$

By Lemma 6, we get

$$\begin{aligned} \mathrm{II} &\le C\int_{(3Q)^c}\Phi\left(\frac{|f(x)|}{t}\right)\left(\frac{|Q|}{|x-x_0|^n}\right)^{1/(1+\delta)} \\ &\cdot\frac{w(Q)}{|Q|}dx \le C\sum_{j=1}^{\infty}\int_{3^{j+1}Q\setminus 3^jQ}\Phi\left(\frac{|f(x)|}{t}\right) \\ &\cdot\left(\frac{|Q|}{|x-x_0|^n}\right)^{1/(1+\delta)}\frac{w(Q)}{|Q|}dx \le Cw(Q)^{\kappa} \\ &\cdot\sum_{j=1}^{\infty}\left(\frac{w(Q)}{w(3^{j+1}Q)}\right)^{1-\kappa}\left(\frac{|Q|}{|3^{j+1}Q|}\right)^{-\delta/(1+\delta)} \\ &\cdot\frac{1}{(w(3^{j+1}Q))^{\kappa}}\int_{3^{j+1}Q}\Phi\left(\frac{|f(x)|}{t}\right)w(x)\,dx \\ &\le Cw(Q)^{\kappa}\sum_{j=1}^{\infty}\left(\frac{|Q|}{|3^{j+1}Q|}\right)^{\eta(1-\kappa)-\delta/(1+\delta)}\left\|\frac{|f|}{t}\right\|_{L^{\Phi,\kappa}(w)} \\ &\le Cw(Q)^{\kappa}\left\|\frac{|f|}{t}\right\|_{L^{\Phi,\kappa}(w)}, \end{aligned} \tag{30}$$

in which we take $\delta > 0$ small enough such that $\eta(1-\kappa) - \delta/(1+\delta) > 0$. This ends the proof. □

3. Proof of Theorems 3 and 4

Given an increasing function $\varphi : [0,\infty) \longrightarrow [0,\infty)$, as in [8], we define the function h_φ by

$$h_\varphi(s) = \sup_{t>0}\frac{\varphi(st)}{\varphi(t)}, \quad 0 \le s < \infty. \tag{31}$$

If φ is submultiplicative, then $h_\varphi \approx \varphi$. Also, for all $s, t > 0$, $\varphi(st) \le h_\varphi(s)\varphi(t)$.

In this section, we set $\Phi(t) = t\log(e+t)$, it is submultiplicative, and so $h_\Phi \approx \Phi$. Let $0 < \alpha < n$, and q be a number $1/q = 1 - \alpha/n$. Denote

$$\Psi(t) = \begin{cases} 0, & t = 0, \\ \dfrac{t}{\Phi(t^{\alpha/n})}, & t > 0. \end{cases} \tag{32}$$

So

$$\Psi(t) \approx t^{1/q}\log(e+t)^{-1}. \tag{33}$$

The function Ψ is invertible with

$$\Psi^{-1}(t) \approx \Gamma(t) = [t\log(e+t)]^q = \Phi(t)^q. \tag{34}$$

Lemma 9 (see [8]). *If $\varphi(t)/t$ is decreasing, then, for any positive sequence $\{t_j\}$,*

$$\varphi\left(\sum_j t_j\right) \le \sum_j \varphi(t_j). \tag{35}$$

Lemma 10. *Let $0 < \alpha < n$, $1/q = 1 - \alpha/n$. Then there exists a constant $C > 0$ such that, for any $t > 0$, for any weight w, we have*

$$\begin{aligned} &\Psi\left(w\left(\left\{x \in \mathbb{R}^n : M_{\alpha,L\log L}(f)(x) > t\right\}\right)\right) \\ &\quad\le C\int_{\mathbb{R}^n}\Phi\left(\frac{|f(y)|}{t}\right)h_\Psi(Mw(y))\,dy. \end{aligned} \tag{36}$$

Proof. By homogeneity, we may assume that $t = 1$. Define the set

$$\Omega = \left\{x \in \mathbb{R}^n : M_{\alpha,L(\log L)}(f)(x) > 1\right\}. \tag{37}$$

It is easy to see that Ω is open and we may assume that it is not empty. To estimate the size of Ω, it is enough to estimate the size of every compact set F contained in Ω. We can cover F by a finite family of cubes $\{Q_j\}$ for which

$$|Q_j|^{\alpha/n}\|f\|_{L(\log L),Q_j} > 1. \tag{38}$$

Using Vitali's covering lemma, we can extract a subfamily of disjoint cubes $\{Q_k\}$ such that

$$F \subset \bigcup_k 3Q_k. \tag{39}$$

For each k, by homogeneity and the properties of the norm $\|\cdot\|_{\Phi,Q}$, we have

$$\begin{aligned} 1 &< \frac{1}{|Q_k|}\int_{Q_k}\Phi\left(f(y)|Q_k|^{\alpha/n}\right)dy \\ &\le C\frac{\Phi\left(|Q_k|^{\alpha/n}\right)}{|Q_k|}\int_{Q_k}\Phi(f(y))\,dy \\ &\le \frac{C}{\Psi(|Q_k|)}\int_{Q_k}\Phi(f(y))\,dy. \end{aligned} \tag{40}$$

For each k, we have

$$\begin{aligned} \Psi(w(Q_k)) &\le C\frac{\Psi(w(Q_k))}{\Psi(|Q_k|)}\int_{Q_k}\Phi(f(y))\,dy \\ &\le Ch_\Psi\left(\frac{w(Q_k)}{|Q_k|}\right)\int_{Q_k}\Phi(f(y))\,dy \\ &\le C\int_{Q_k}\Phi(f(y))\,h_\Psi(Mw(y))\,dy. \end{aligned} \tag{41}$$

It is easy to see that $\Psi(t)/t$ is decreasing; by Lemma 9, we have

$$\begin{aligned}\Psi(w(F)) &\leq \sum_k \Psi(w(Q_k)) \\ &\leq C\sum_k \int_{Q_k} \Phi(f(y))\, h_\Psi(Mw(y))\, dy \\ &\leq C\int_{\mathbb{R}^n} \Phi(f(y))\, h_\Psi(Mw(y))\, dy.\end{aligned} \tag{42}$$

This ends the proof. □

Proof of Theorem 3. Fix a cube Q centered at x_0. By Lemma 10, we have

$$\begin{aligned}&\Psi\left(w\left(\left\{x\in Q: M_{\alpha,L(\log L)}f(x)>t\right\}\right)\right) \\ &\quad= \Psi\left(\int_{\{x\in\mathbb{R}^n: M_{\alpha,L(\log L)}f(x)>t\}} w(x)\,\chi_Q(x)\,dx\right) \\ &\quad\leq C\int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{t}\right) h_\Psi(M(w\chi_Q))(x)\,dx \\ &\quad\leq C\left(\int_{3Q}+\int_{(3Q)^c}\right)\Phi\left(\frac{|f(x)|}{t}\right) h_\Psi(M(w\chi_Q)) \\ &\quad\cdot (x)\,dx \leq \mathrm{I}+\mathrm{II}.\end{aligned} \tag{43}$$

Now we estimate term I. Noticing that, for $s>0$, we have

$$h_\Psi(s) = \sup_{t>0}\frac{\Psi(st)}{\Psi(t)} = s\sup_{t>0}\frac{\Phi\left(t^{\alpha/n}\right)}{\Phi(\left((st)^{\alpha/n}\right)} \leq C\Theta(s). \tag{44}$$

Since $w\in A_1$, we get

$$\begin{aligned}\mathrm{I} &\leq C\int_{3Q}\Phi\left(\frac{|f(x)|}{t}\right) h_\Psi(w(x))\,dx \\ &\leq C\int_{3Q}\Phi\left(\frac{|f(x)|}{t}\right)\Theta(w(x))\,dx \\ &\leq C\left\|\frac{|f|}{t}\right\|_{L^{\Phi,\kappa}(w,\Theta(w))} w(Q)^\kappa.\end{aligned} \tag{45}$$

For term II, observe that, for $x\in(3Q)^c$, $x\in R$ and $R\cap Q\neq\emptyset$. As in the proof of Theorem 1, we have

$$M(\chi_Q w)(x) \leq C\left|x-x_0\right|^{-n} w(Q). \tag{46}$$

Since $w\in A_1$, Θ is submultiplicative, and using Lemma 6, we get

$$\begin{aligned}\mathrm{II} &\leq C\int_{(3Q)^c}\Phi\left(\frac{|f(x)|}{t}\right) h_\Psi\left(\left|x-x_0\right|^{-n} w(Q)\right)dx \\ &\leq C\sum_{j=1}^\infty \int_{3^{j+1}Q\setminus 3^jQ}\Phi\left(\frac{|f(x)|}{t}\right)\Theta\left(\frac{w(Q)}{\left|3^{j+1}Q\right|}\right)dx \\ &\leq C\sum_{j=1}^\infty \int_{3^{j+1}Q}\Phi\left(\frac{|f(x)|}{t}\right) \\ &\quad\cdot\Theta\left(\frac{w\left(3^{j+1}Q\right)}{\left|3^{j+1}Q\right|}\frac{w(Q)}{w\left(3^{j+1}Q\right)}\right)dx \\ &\leq C\sum_{j=1}^\infty \int_{3^{j+1}Q}\Phi\left(\frac{|f(x)|}{t}\right) \\ &\quad\cdot\Theta\left(w(x)\frac{w(Q)}{w\left(3^{j+1}Q\right)}\right)dx \\ &\leq C\sum_{j=1}^\infty \frac{w(Q)^{1/q}}{w\left(3^{j+1}Q\right)^{1/q}}\log\left(e+\frac{w\left(3^{j+1}Q\right)}{w(Q)}\right) \\ &\quad\cdot\int_{3^{j+1}Q}\Phi\left(\frac{|f(x)|}{t}\right)\Theta(w(x))\,dx \leq Cw(Q)^\kappa \\ &\quad\cdot\left\|\frac{|f|}{t}\right\|_{L^{\Phi,\kappa}(w,\Theta(w))}\sum_{j=1}^\infty\frac{1}{3^{jn\eta(1/q-\kappa)}}\log\left(e+3^{jn\eta}\right) \\ &\leq Cw(Q)^\kappa\left\|\frac{|f|}{t}\right\|_{L^{\Phi,\kappa}(w,\Theta(w))}.\end{aligned} \tag{47}$$

This ends the proof. □

Lemma 11 (see [9]). *Let $0<\alpha<n$, $1/q=1-\alpha/n$, $w\in A_1$, and $b\in BMO$. Then there exists a constant $C>0$ such that, for any $t>0$,*

$$\begin{aligned}&\Psi\left(w\left(\left\{x\in\mathbb{R}^n:\left[b,I_\alpha\right](f)(x)>t\right\}\right)\right) \\ &\quad\leq C\int_{\mathbb{R}^n}\Phi\left(\frac{|f(y)|}{t}\right)\Theta(w(y))\,dy.\end{aligned} \tag{48}$$

Lemma 12 (see [6]). *Let $f(x)\geq 0$, $f\in L^1_{loc}(\mathbb{R}^n)$, and $0<\delta<1$, then $M(f)^\delta\in A_1$.*

Proof of Theorem 4. Fix a cube Q centered at x_0, for any $w\in A_1$ and $\delta: 0<\delta\leq\theta$, and by Lemma 12, we have $M(w^{1+\delta}\chi_Q)^{1/(1+\delta)}\in A_1$. By Lemma 11, we obtain

$$\begin{aligned}&\Psi\left(w\left(\left\{x\in Q:\left[b,I_\alpha\right]f(x)\mid\ >t\right\}\right)\right) \\ &\quad=\Psi\left(\int_{\{x\in\mathbb{R}^n:[b,I_\alpha]f(x)>t\}} w(x)\,\chi_Q(x)\,dx\right) \\ &\quad\leq C\Psi\left(\int_{\{x\in\mathbb{R}^n:[b,I_\alpha]f(x)>t\}} M(w\chi_Q)(x)\,dx\right) \\ &\quad\leq C\Psi\left(\int_{\{x\in\mathbb{R}^n:[b,I_\alpha]f(x)>t\}}\left(M\left(w^{1+\delta}\chi_Q\right)\right.\right. \\ &\quad\left.\left.\cdot(x)\right)^{1/(1+\delta)}dx\right) \leq C\int_{\mathbb{R}^n}\Phi\left(\frac{|f(x)|}{t}\right) \\ &\quad\cdot\Theta\left(\left(M\left(w^{1+\delta}\chi_Q\right)(x)\right)^{1/(1+\delta)}\right)dx\end{aligned}$$

$$\leq C\left(\int_{3Q}+\int_{(3Q)^c}\right)\Phi\left(\frac{|f(x)|}{t}\right) \cdot \Theta\left(\left(M\left(w^{1+\delta}\chi_Q\right)(x)\right)^{1/(1+\delta)}\right)dx \leq \mathrm{I}+\mathrm{II}. \tag{49}$$

Now we estimate term I. Noticing that $w \in A_1$, Lemma 8, we have $\Theta((M(w^{1+\delta}\chi_Q)(x))^{1/(1+\delta)} \leq C\Theta(Mw(x)) \leq C\Theta(w(x))$. Then

$$\mathrm{I} \leq C\int_{3Q}\Phi\left(\frac{|f(x)|}{t}\right)\Theta(w(x))\,dx \leq C\left\|\frac{|f|}{t}\right\|_{L^{\Phi,\kappa}(w,\Theta(w))} w(Q)^{\kappa}. \tag{50}$$

For term II, as the proof of Theorem 2, for $x \in (3Q)^c$,

$$\left(M\left(w^{1+\delta}\chi_Q\right)(x)\right)^{1/(1+\delta)} \leq C\left(\frac{|Q|}{|x-x_0|^n}\right)^{1/(1+\delta)}\frac{w(Q)}{|Q|}. \tag{51}$$

By Lemma 6, we get

$$\begin{aligned}\mathrm{II} &\leq C\int_{(3Q)^c}\Phi\left(\frac{|f(x)|}{t}\right)\Theta\left(\left(\frac{|Q|}{|x-x_0|^n}\right)^{1/(1+\delta)} \cdot \frac{w(Q)}{|Q|}\right)dx \leq C\sum_{j=1}^{\infty}\int_{3^{j+1}Q\setminus 3^jQ}\Phi\left(\frac{|f(x)|}{t}\right) \\ &\cdot \Theta\left(\left(\frac{|Q|}{|3^{j+1}Q|}\right)^{\eta-\delta/(1+\delta)} w(x)\right)dx \leq Cw(Q)^{\kappa} \cdot \sum_{j=1}^{\infty}\left(\frac{|Q|}{|3^{j+1}Q|}\right)^{\eta(1/q-\kappa)-\delta/q(1+\delta)} \\ &\cdot \log\left(e+\left(\frac{|3^{j+1}Q|}{|Q|}\right)^{\eta-\delta/(1+\delta)}\right)\cdot\frac{1}{(w(3^{j+1}Q))^{\kappa}} \cdot \int_{3^{j+1}Q}\Phi\left(\frac{|f(x)|}{t}\right)\Theta(w(x))\,dx \leq Cw(Q)^{\kappa} \\ &\cdot \left\|\frac{|f|}{t}\right\|_{L^{\Phi,\kappa}(w,\Theta(w))}\sum_{j=1}^{\infty}\left(\frac{1}{3^{jn}}\right)^{\eta(1/q-\kappa)-\delta/q(1+\delta)} \cdot \log\left(e+3^{jn(\eta-\delta/(1+\delta))}\right) \leq Cw(Q)^{\kappa}\left\|\frac{|f|}{t}\right\|_{L^{\Phi,\kappa}(w,\Theta(w))},\end{aligned} \tag{52}$$

in which we take $\delta > 0$ small enough such that $\eta(1/q-\kappa) - \delta/q(1+\delta) > 0$ and $\eta - \delta/(1+\delta) > 0$. This ends the proof. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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