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Research Article

Structure Properties for Binomial Operators

Chungou Zhang and Shifen Wang

School of Mathematical Sciences, Capital Normal University, Beijing 100048, China

Correspondence should be addressed to Chungou Zhang; 3773@mail.cnu.edu.cn

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In this paper, we discuss binomial operators structure properties, such as moments representation, derivatives representation, and binary representation and introduce some applications in preservation.

1. Introduction

As an extension to the well-known Bernstein operators, binomial operators are defined as follows (see [1], [2], or [3]):

$$\left(L_n^Q f\right)(x)=\frac{1}{b_n(1)}\sum_{k=0}^{n}\binom{n}{k}b_k(x)\,b_{n-k}(1-x)\,f\left(\frac{k}{n}\right),\quad f\in C[0,1]; \tag{1}$$

if $b_n(1)=0$, then those operators are substituted by

$$\left(L_n^Q f\right)(x)=\frac{1}{b_n(n)}\sum_{k=0}^{n}\binom{n}{k}b_k(nx)\,b_{n-k}(n-nx)\,f\left(\frac{k}{n}\right),\quad f\in C[0,1], \tag{2}$$

where $(b_n)_{n\geq 0}$ is a sequence of binomial polynomials; i.e., $b_n(x)$ is a polynomial of n degree satisfying

$$b_n(x+y)=\sum_{k=0}^{n}\binom{n}{k}b_k(x)\,b_{n-k}(y),\quad n=0,1,2,\cdots. \tag{3}$$

Q is a delta operator (see Definition 2 below), which is determined by the sequence of binomial polynomials $(b_n)_{n\geq 0}$ uniquely.

In order to explore the binomial operators approximation and preservation, in present paper we are to investigate some structural properties, since behaviors of an operator are strongly dependent on its structure. In view of the Bernstein operators

$$(B_n f)(x)=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}f\left(\frac{k}{n}\right),\quad f\in C[0,1], \tag{4}$$

with the following structural properties: endpoints interpolation, moments representation, derivatives representation, difference representation, and binary representation (see [4–10]), so our attention to the binomial operators will also focus on these aspects. On the study of Bernstein type operators, here we also want to refer to literatures [11–16].

In the next section, we introduce some primary concepts and results involved in this paper, which can be found in [1, 2, 17].

2. Notations and Preliminaries

Let Π_n be the linear space of polynomials of degree at most n and Π the space of all polynomials; i.e., $\Pi=\bigcup_{n\geq 0}\Pi_n$. We denote by I the identity operator and D the derivative. For real number a, the shift operator $E^a:\Pi\longrightarrow\Pi$ is defined by $E^a p(x)=p(x+a)$, $x\in(-\infty,+\infty)$.

Definition 1. If a linear operator $T:\Pi\longrightarrow\Pi$ commutes with all shift operators, then it is called a shift-invariant operator; i.e., $TE^a=E^aT$ for any a real number a.

From [17], we can find that if T_1 and T_2 are shift-invariant operators, then $T_1 T_2 = T_2 T_1$.

Definition 2. A shift-invariant operator Q is called a delta operator iff $Qe_1 = const \neq 0$, where $e_i = e_i(t) = t^i$, $i = 0, 1, \cdots$.

For delta operators, we have the following assertion.

Theorem 3. *The following statements are equivalent:*

(i) Q is a delta operator.

(ii) There exists a reversible shift-invariant operator P such that Q = DP.

(iii) There exists a power series $\phi(t) = \Sigma_{k=0}^{\infty} c_k (t^k/k!)$ *with* $c_0 = 0, c_1 \neq 0$ *such that* $Q = \phi(D)$.

Every shift-invariant operator can always be represented by any one delta operator that is so-called "First Expansion Theorem" as below.

Theorem 4. *Let T be a shift-invariant operator, and let Q be a delta operator with basic polynomials* $b_n(x)$*; then*

$$T = \sum_{k=0}^{\infty} \frac{a_k}{k!} Q^k \tag{5}$$

with $a_k = [Tb_k(x)]_{x=0}$.

Definition 5. Let $K : \Pi \longrightarrow \Pi$ be defined as $(Kh)(t) = th(t)$. For an operator $T : \Pi \longrightarrow \Pi$, its Pincherle derivative T' is defined by $T' = TK - KT$.

From [17], it is known that if T is a shift-invariant operator, then also is T'.

Definition 6. Let Q be a delta operator. A polynomial sequence $(b_n)_{n\geq 0}$ is called the sequence of basic polynomials associated with Q iff

$$\begin{aligned} b_0(x) &= 1, \\ b_n(0) &= 0, \\ Qb_n(x) &= nb_{n-1}(x), \end{aligned} \tag{6}$$
$$n \geq 1.$$

It has been proved that every delta operator has a unique sequence of basic polynomials (see [17], Proposition 3]). Moreover, if $(b_n)_{n\geq 0}$ is the basic sequence of a delta operator Q, then $(b_n)_{n\geq 0}$ is a sequence of binomial polynomials, and the converse is also right.

In this paper, we also need the following assertions. If Q is a delta operator, then Q'^{-1} exists and the binomial operator L_n^Q has the following representation.

Theorem 7. *The operator* L_n^Q *can be represented in the form*

$$\left(L_n^Q f\right)(x) = \sum_{k=0}^{n} \frac{k!}{n^k} \binom{n}{k} \left[0, \frac{1}{n}, \cdots \frac{k}{n}; f\right] d_{k,n}(x), \tag{7}$$

where

$$d_{k,n}(x) = \frac{1}{b_n(1)} \left(\Theta^k E^{1-x} b_{n-k}\right)(x), \quad \Theta = KQ'^{-1} \tag{8}$$

and $[0, 1/n, \cdots k/n; f]$ *is divided difference of the function* f. *Using this theorem, we can get the moments of* L_n^Q *immediately*

$$\begin{aligned} &\left(L_n^Q e_r\right)(x) \\ &= \frac{1}{b_n(1)} \sum_{k=0}^{r} \frac{k!}{n^k} \left[0, \frac{1}{n} \cdots \frac{k}{n}; e_r\right] \left(\Theta^k E^{1-x} b_{n-k}\right)(x), \end{aligned} \tag{9}$$
$$r = 0.1.2 \cdots.$$

In particular, when $r = 0, 1, 2$, *we have*

$$\begin{aligned} &\left(L_n^Q e_0\right)(x) = 1, \\ &\left(L_n^Q e_1\right)(x) = x, \\ &\left(L_n^Q e_2\right)(x) \\ &= x^2 + \left(1 - \frac{n-1}{n} \frac{\left(Q'^{-2} b_{n-2}\right)(1)}{b_n(1)}\right) x(1-x). \end{aligned} \tag{10}$$

According to these moments and Korovkin theorem, it is easy to see that if L_n^Q *is positive and* $((n-1)/n)((Q'^{-2}b_{n-2})(1)/b_n(1)) \longrightarrow 1$, *then* L_n^Q *uniformly converges to the continuous function* f.

For convenience, we denote all the positive operators L_n^Q *by* $\mathscr{B}$. *Let* $\mathscr{F}$ *be the set of all formal power series* $\phi(t) = \sum_{k=1}^{\infty} d_k (t^k/k!)$ *with* $d_1 > 0, d_k \geq 0$*; then the positivity of* L_n^Q *can be characterized as follows.*

Theorem 8. *Let* L_n^Q *be defined as before, and* $Q = \phi(D)$*; then* $L_n^Q \in \mathscr{B}$ *iff*

$$\phi^{-1}(t) := \sum_{k=0}^{\infty} c_k \frac{t^k}{k!} \in \mathscr{F}. \tag{11}$$

Therefore, if $L_n^Q \in \mathscr{B}$, *then* $b_n(x)$ *has nonnegative coefficients, where* $(b_n)_{n\geq 0}$ *is the sequence of basic polynomials associated with Q.*

When $Q = D$, which is the simplest delta operator with the basic polynomials $b_n(x) = x^n, n = 0, 1, \cdots$, its corresponding binomial operator L_n^Q is Bernstein operator B_n; when $Q = I - E^{-1}$, which is also a delta operator and called backward difference operator with the basic polynomials $b_n(x) = x(x+1)\cdots(x+n-1)$, $n = 0, 1, \cdots$, its corresponding binomial operator is so-called Stancu operator (see [1] or [2]).

3. Some Structure Properties of the Operator L_n^Q

Similar to Bernstein operators, the binomial operators L_n^Q, on which some researches can be found in [18–24], also have the property of endpoint interpolation; i.e., $(L_n^Q f)(0) = f(0)$ and $(L_n^Q f)(1) = f(1)$.

Applying $[x_0, x_1, \ldots, x_k; f] = \triangle_h^k / k! h^k$ to Theorem 7, we have the difference representation of L_n^Q as follows:

$$\left(L_n^Q f\right)(x) = \frac{1}{b_n(1)} \sum_{k=0}^{n} \binom{n}{k} \triangle_{1/n}^k f(0) \left(\Theta^k E^{1-x} b_{n-k}\right)(x). \tag{12}$$

As is known to all, the derivative of Bernstein operators can be expressed by difference operators (see [4]):

$$B_n^{(r)}(f, x) = n(n-1)\cdots(n-r+1) \sum_{k=0}^{n-r} \triangle_{1/n}^r f\left(\frac{k}{n}\right) b_{(n-r)k}(x), \tag{13}$$

but the derivatives of L_n^Q can not be expressed in this form; actually when $r = 1, 2$ they have the following representation.

Proposition 9. *Let Q be a delta operator and $(b_n)_{n \geq 0}$ be the sequence of basic polynomial associated with Q; then*

$$\begin{aligned} D\left(L_n^Q f\right)(x) &= \frac{1}{b_n(1)} \sum_{k=0}^{n-1} \binom{n}{k} b'_{n-k}(0) \cdot \sum_{l=0}^{k} \binom{k}{l} b_l(x) b_{k-l}(1-x) \cdot \left[f\left(\frac{n-k+l}{n}\right) - f\left(\frac{l}{n}\right)\right]; \\ D^2\left(L_n^Q f\right)(x) &= \frac{1}{b_n(1)} \sum_{k=1}^{n-1} \binom{n}{k} b'_{n-k}(0) \sum_{l=0}^{k-1} \binom{k}{l} b'_{k-l}(0) \cdot \sum_{j=0}^{l} \binom{l}{j} b_j(x) b_{l-j}(1-x) \cdot \left[f\left(\frac{n-l+j}{n}\right) - f\left(\frac{k-l+j}{n}\right) - f\left(\frac{n-k+j}{n}\right) + f\left(\frac{j}{n}\right)\right]. \end{aligned} \tag{14}$$

Proof. By Theorem 4, we have

$$D = \sum_{l=0}^{\infty} \frac{b'_l(0)}{l!} Q^l; \tag{15}$$

it follows that

$$b'_k(x) = D b_k(x) = \sum_{l=0}^{k} \binom{k}{l} b'_l(0) b_{k-l}(x), \tag{16}$$

so that

$$\begin{aligned} &\sum_{k=0}^{n} \binom{n}{k} b'_k(x) b_{n-k}(1-x) f\left(\frac{k}{n}\right) = \sum_{k=1}^{n} \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} \cdot b'_l(0) b_{k-l}(x) b_{n-k}(1-x) f\left(\frac{k}{n}\right) = \sum_{k=1}^{n} b'_k(0) \\ &\cdot \sum_{l=k}^{n} \binom{n}{l} \binom{l}{k} b_{l-k}(x) b_{n-l}(1-x) f\left(\frac{l}{n}\right) = \sum_{k=1}^{n} \binom{n}{k} \cdot b'_k(0) \sum_{l=0}^{n-k} \binom{n-k}{l} b_l(x) b_{n-k-l}(1-x) f\left(\frac{k+l}{n}\right). \end{aligned} \tag{17}$$

In the same way, we have

$$\sum_{k=0}^{n} \binom{n}{k} b_k(x) b'_{n-k}(1-x) f\left(\frac{k}{n}\right) = \sum_{k=1}^{n} \binom{n}{k} b'_k(0) \cdot \sum_{l=0}^{n-k} \binom{n-k}{l} b_l(x) b_{n-k-l}(1-x) f\left(\frac{l}{n}\right). \tag{18}$$

Therefore, we get

$$\begin{aligned} D\left(L_n^Q f\right)(x) &= \frac{1}{b_n(1)} \sum_{k=1}^{n} \binom{n}{k} b'_k(x) b_{n-k}(1-x) - \frac{1}{b_n(1)} \sum_{k=1}^{n} \binom{n}{k} b_k(x) b'_{n-k}(1-x) = \frac{1}{b_n(1)} \\ &\cdot \sum_{k=1}^{n} \binom{n}{k} b'_k(0) \sum_{l=0}^{n-k} \binom{n-k}{l} b_l(x) b_{n-k-l}(1-x) \cdot \left[f\left(\frac{k+l}{n}\right) - f\left(\frac{l}{n}\right)\right] = \frac{1}{b_n(1)} \sum_{k=0}^{n-1} \binom{n}{k} b'_{n-k}(0) \\ &\cdot \sum_{l=0}^{k} \binom{k}{l} b_l(x) b_{k-l}(1-x) \cdot \left[f\left(\frac{n-k+l}{n}\right) - f\left(\frac{l}{n}\right)\right]. \end{aligned} \tag{19}$$

An argument similar to the above one, it is no difficult to obtain the expression for the second derivative of the operators $D^2(L_n^Q f)(x)$. The proof is completed. □

This proposition supplies us a new proof for the next results known (see [1]).

Theorem 10. *Let $L_n^Q \in \mathscr{B}$.*

(i) If f is increasing (decreasing) on $[0, 1]$, then it also is $L_n^Q f$.

(ii) If f is convex (concave) on $[0, 1]$, then it also is $L_n^Q f$.

Proof. (i) It is trivial.

(ii) Without loss of generality, we may assume f is convex. Recall that if f is convex on $[0,1]$, then for any $x_1, x_2, x_3, x_4 \in [0,1]$ and $x_1 < x_2, x_3 < x_4$ the following holds:

$$\frac{f(x_1)-f(x_2)}{x_1-x_2} \le \frac{f(x_3)-f(x_4)}{x_3-x_4}. \quad (20)$$

Obviously,

$$\begin{gathered}\frac{j}{n} < \frac{k-l+j}{n}, \\ \frac{n-k+j}{n} < \frac{n-l+j}{n}, \\ j<k<l<n,\end{gathered} \quad (21)$$

and it follows that

$$\begin{aligned}&\frac{f((n-k+j)/n)-f(j/n)}{1-k/n} \\ &\quad \le \frac{f((n-l+j)/n)-f((k-l+j)/n)}{1-k/n},\end{aligned} \quad (22)$$

so that

$$\begin{aligned}&f\left(\frac{n-l+j}{n}\right)-f\left(\frac{k-l+j}{n}\right)-f\left(\frac{n-k+j}{n}\right) \\ &\quad + f\left(\frac{j}{n}\right) = \left(1-\frac{k}{n}\right) \\ &\quad \cdot \left[\frac{f((n-l+j)/n)-f((k-l+j)/n)}{1-k/n}\right. \\ &\quad \left. - \frac{f((n-k+j)/n)-f(j/n)}{1-k/n}\right] \ge 0.\end{aligned} \quad (23)$$

According to Theorem 8, it follows that $(b_n)_{n\ge 0}$ has nonnegative coefficients, which mean $b_i(x) \ge 0$ for any $x \in [0,1]$ and $b_i'(0) \ge 0, i = 0, 1, \cdots$. These lead to $D^2(L_n^Q f) \ge 0$, which complete the proof. □

The binomial operators have binary representation similar to Bernstein operators as follows.

Proposition 11. *If L_n^Q is defined as before, then*

$$\begin{aligned}\left(L_n^Q f\right)(x) &= \frac{1}{b_n(1)}\sum_{k=0}^{n}\binom{n}{k}\sum_{l=0}^{n-k}B_{nkl}(x,y)\, f\left(\frac{k}{n}\right); \\ \left(L_n^Q f\right)(y) &= \frac{1}{b_n(1)}\sum_{k=0}^{n}\binom{n}{k}\sum_{l=0}^{n-k}B_{nkl}(x,y)\, f\left(\frac{k+l}{n}\right),\end{aligned} \quad (24)$$

where $B_{nkl} = (n!/k!l!(n-k-l)!)b_k(x)b_l(y-x)b_{n-k-l}(1-y)$, $x < y$.

Proof. By definition of sequence of binomial polynomials, we have

$$\begin{aligned}b_{n-k}(1-x) &= \sum_{l=0}^{n-k}\binom{n-k}{l}b_l(y-x)\,b_{n-k-l}(1-y), \\ b_k(x) &= \sum_{l=0}^{k}\binom{k}{l}b_l(y-x)\,b_{k-l}(x),\end{aligned} \quad (25)$$

and thus

$$\begin{aligned}\left(L_n^Q f\right)(x) &= \frac{1}{b_n(1)}\sum_{k=0}^{n}\binom{n}{k}b_k(x)\,b_{n-k}(1-x)\, f\left(\frac{k}{n}\right) \\ &= \frac{1}{b_n(1)}\sum_{k=0}^{n}\sum_{l=0}^{n-k}\frac{n!}{k!l!(n-k-l)!}b_k(x)\,b_l(y-x) \\ &\quad \cdot b_{n-k-l}(1-y)\, f\left(\frac{k}{n}\right) = \frac{1}{b_n(1)}\sum_{k=0}^{n}\sum_{l=0}^{n-k}B_{nkl}(x,y) \\ &\quad \cdot f\left(\frac{k}{n}\right),\end{aligned} \quad (26)$$

and

$$\begin{aligned}\left(L_n^Q f\right)(y) &= \frac{1}{b_n(1)}\sum_{k=0}^{n}\binom{n}{k}b_k(y)\,b_{n-k}(1-y)\, f\left(\frac{k}{n}\right) \\ &= \frac{1}{b_n(1)}\sum_{k=0}^{n}\sum_{l=0}^{k}\frac{n!}{k!l!(n-k-l)!}b_l(y-x)\,b_{k-l}(x) \\ &\quad \cdot b_{n-k}(1-y)\, f\left(\frac{k}{n}\right) = \frac{1}{b_n(1)} \\ &\quad \cdot \sum_{k=0}^{n}\sum_{l=0}^{n-k}\frac{n!}{k!l!(n-k-l)!}b_k(x)\,b_l(y-x) \\ &\quad \cdot b_{n-k-l}(1-y)\, f\left(\frac{k+l}{n}\right) = \frac{1}{b_n(1)} \\ &\quad \cdot \sum_{k=0}^{n}\sum_{l=0}^{n-k}B_{nkl}(x,y)\, f\left(\frac{k+l}{n}\right),\end{aligned} \quad (27)$$

where

$$B_{nkl} = \frac{n!}{k!l!(n-k-l)!}b_k(x)\,b_l(y-x)\,b_{n-k-l}(1-y). \quad (28)$$

This completes the proof. □

Using this proposition, we not only can prove Theorem 2.2(i) but also show the following result, which can be found in [3].

Theorem 12. *If $L_n^Q \in \mathscr{B}$ and $f \in Lip_M\alpha (0 < \alpha \le 1)$, then $L_n^Q f \in Lip_M\alpha$, where*

$$\begin{aligned}Lip_M\alpha = \{&f \in C[0,1] : \forall x, y \in [0,1], \exists M \\ &> 0,\ s.t. \left|f(y)-f(x)\right| \le M(y-x)^{\alpha}\}.\end{aligned} \quad (29)$$

Proof. First, we need the following two identities:

$$\frac{1}{b_n(1)}\sum_{k=0}^{n}\binom{n}{k}\sum_{l=0}^{n-k}B_{nkl}(x,y)=1,$$
$$\frac{1}{b_n(1)}\sum_{k=0}^{n}\binom{n}{k}\sum_{l=0}^{n-k}B_{nkl}(x,y)\frac{l}{n}=y-x. \tag{30}$$

The first one can be derived easily by the definition of the sequence of binomial polynomials; the rest only need to prove the second one. In fact, by the definition of $B_{nkl}(x,y)$ and exchange the order of sum, we have

$$\begin{aligned}&\frac{1}{b_n(1)}\sum_{k=0}^{n}\binom{n}{k}\sum_{l=0}^{n-k}B_{nkl}(x,y)\frac{l}{n}=\frac{1}{b_n(1)}\\&\cdot\sum_{l=0}^{n}\binom{n}{l}b_l(y-x)\frac{l}{n}\\&\cdot\sum_{k=0}^{n-l}\binom{n-l}{k}b_k(x)b_{n-l-k}(1-y)=\frac{1}{b_n(1)}\\&\cdot\sum_{l=0}^{n}\binom{n}{l}b_l(y-x)b_{n-l}(x+1-y)\frac{l}{n}=y-x.\end{aligned} \tag{31}$$

Now, we turn to the proof of this theorem. Since, for any $x<y$ and $0<\alpha\le 1$, there is

$$\left|f(y)-f(x)\right|\le M(y-x)^{\alpha}, \tag{32}$$

by the identities above and $L_n^Q\in\mathscr{B}$; we obtain

$$\begin{aligned}&\left|\left(L_n^Qf\right)(y)-\left(L_n^Qf\right)(x)\right|\\&\le\frac{1}{b_n(1)}\sum_{k=0}^{n}\sum_{l=0}^{n-k}B_{nkl}(x,y)\left|f\left(\frac{k+l}{n}\right)-f\left(\frac{k}{n}\right)\right|\\&\le M\frac{1}{b_n(1)}\sum_{k=0}^{n}\sum_{l=0}^{n-k}B_{nkl}(x,y)\left(\frac{l}{n}\right)^{\alpha}\\&\le M\left[\frac{1}{b_n(1)}\sum_{k=0}^{n}\sum_{l=0}^{n-k}B_{nkl}(x,y)\frac{l}{n}\right]^{\alpha}=M(y-x)^{\alpha},\end{aligned} \tag{33}$$

which means that $L_n^Qf\in Lip_M\alpha$. The proof of the theorem is now complete. □

In this paper, we discuss structure properties of the binomial operators, present the moments representation, the derivatives representation, and the binary representation of these operators, and introduce some applications in preservation. Here authors also wish to express their gratitude to the reviewers for the careful review and suggestions for improvement of this manuscript.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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