Hindawi
Journal of Function Spaces
Volume 2019, Article ID 6082413, 7 pages
https://doi.org/10.1155/2019/6082413

Research Article

On Approximating the Toader Mean by Other Bivariate Means

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Received 6 January 2019; Accepted 5 February 2019; Published 3 March 2019

Guest Editor: Mirella Cappelletti Montano

In the article, we provide several sharp bounds for the Toader mean by use of certain combinations of the arithmetic, quadratic, contraharmonic, and Gaussian arithmetic-geometric means.

1. Introduction

Let q be a real number, $0 < u < 1$ and $x, y \in \mathbb{R}^+$ with $x \neq y$. Then the complete elliptic integrals $\mathscr{K}(u)$ and $\mathscr{E}(u)$ [1–22] of the first and second kinds, geometric mean $G(x,y)$, logarithmic mean $L(x,y)$, arithmetic mean $A(x,y)$, quadratic mean $Q(x,y)$, contraharmonic mean $C(x,y)$, second contraharmonic mean $\overline{C}(x,y)$, qth Hölder mean $H_q(x,y)$, and Toader mean $T(x,y)$ [23–27] of x and y are given by

$$\begin{aligned}
\mathscr{K}(u) &= \int_0^{\pi/2} \left(1-u^2\sin^2(v)\right)^{-1/2} dv,\\
\mathscr{E}(u) &= \int_0^{\pi/2} \sqrt{1-u^2\sin^2(v)}\,dv,\\
G(x,y) &= \sqrt{xy},\\
L(x,y) &= \frac{x-y}{\log x-\log y},\\
A(x,y) &= \frac{x+y}{2},\\
Q(x,y) &= \sqrt{\frac{x^2+y^2}{2}},\\
C(x,y) &= \frac{x^2+y^2}{x+y},\\
\overline{C}(x,y) &= \frac{x^3+y^3}{x^2+y^2},\\
H_q(x,y) &= \left(\frac{x^q+y^q}{2}\right)^{1/q} \quad (q\neq 0),\\
H_0(x,y) &= \sqrt{xy}
\end{aligned} \tag{1}$$

and

$$\begin{aligned}
T(x,y) &= \frac{2}{\pi}\int_0^{\pi/2} \sqrt{x^2\cos^2(v)+y^2\sin^2(v)}\,dv\\
&= \frac{2x}{\pi}\mathscr{E}\left(\sqrt{1-\left(\frac{y}{x}\right)^2}\right), \quad x>y,\\
&\quad \frac{2y}{\pi}\mathscr{E}\left(\sqrt{1-\left(\frac{x}{y}\right)^2}\right), \quad x<y,
\end{aligned} \tag{2}$$

respectively.

The classical Gaussian arithmetic-geometric mean $AGM(u,v)$ of two positive real numbers u and v is defined by the common limit of the sequences $\{u_n\}$ and $\{v_n\}$, which are given by

$$u_0 = u,$$

$$v_0 = v,$$

$$u_{n+1} = \frac{u_n + v_n}{2},$$
$$v_{n+1} = \sqrt{u_n v_n}. \tag{3}$$

The well-known Gaussian identity [10] and (2) show that

$$T(1,u) = \frac{2\mathscr{E}(u')}{\pi},$$
$$AGM(1,u) = \frac{\pi}{2\mathscr{K}(u')} \tag{4}$$

for all $0 < u < 1$, where $u' = (1-u^2)^{1/2}$.

If $x \neq y$, then it is well known that the function $q \longmapsto H_q(x,y)$ is strictly increasing on the interval $(-\infty,\infty)$ and the inequalities

$$G(x,y) = H_0(x,y) < L(x,y) < A(x,y) = H_1(x,y)$$
$$< Q(x,y) = H_2(x,y) < C(x,y) < \overline{C}(x,y) \tag{5}$$

are valid.

Recently, the bounds for the Toader mean $T(x,y)$ and Gaussian arithmetic-geometric means $AGM(x,y)$ have attracted the interest of many mathematicians. The following inequalities

$$L(x,y) < AGM(x,y) < A(x,y)$$
$$< \frac{\pi}{2}L(x,y), \tag{6}$$

$$A^{1/2}(x,y)G^{1/2}(x,y) < AGM(x,y)$$
$$< \left(\frac{\sqrt{A(x,y)} + \sqrt{G(x,y)}}{2}\right)^2 \tag{7}$$

for all $x, y \in \mathbb{R}^+$ with $x \neq y$ were established in [28–32].

Alzer and Qiu [12] and Barnard, Pearce, and Richards [33] proved that the double inequalities

$$\frac{2}{\pi}\frac{1}{L(x,y)} + \left(1-\frac{2}{\pi}\right)\frac{1}{A(x,y)} < \frac{1}{AGM(x,y)}$$
$$< \frac{3}{4}\frac{1}{L(x,y)} + \frac{1}{4}\frac{1}{A(x,y)}, \tag{8}$$
$$H_{3/2}(x,y) < T(x,y) < H_{\log 2/(\log\pi - \log 2)}(x,y)$$

hold for all $x, y \in \mathbb{R}^+$ with $x \neq y$.

In [23, 34], the authors proved that $\alpha = 1/2$, $\beta = (4-\pi)/[(\sqrt{2}-1)\pi] = 0.6597\cdots$, $\lambda = 1/4$ and $\mu = 4/\pi - 1 = 0.2732\cdots$ are the best possible parameters such that the double inequalities

$$\alpha Q(x,y) + (1-\alpha)A(x,y) < T(x,y)$$
$$< \beta Q(x,y) + (1-\beta)A(x,y), \tag{9}$$

$$\lambda C(x,y) + (1-\lambda)A(x,y) < T(x,y)$$
$$< \mu C(x,y) + (1-\mu)A(x,y) \tag{10}$$

hold for all $x, y \in \mathbb{R}^+$ with $x \neq y$.

Qian, Song, Zhang, and Chu [35] proved that the two-sided inequalities

$$\lambda_1\overline{C}(x,y) + (1-\lambda_1)A(x,y) < T(x,y)$$
$$< \mu_1\overline{C}(x,y) + (1-\mu_1)A(x,y),$$
$$\overline{C}[\lambda_2 x + (1-\lambda_2)y, \lambda_2 y + (1-\lambda_2)x] < T(x,y)$$
$$< \overline{C}[\mu_2 x + (1-\mu_2)y, \mu_2 y + (1-\mu_2)x] \tag{11}$$

are valid for all $x, y \in \mathbb{R}^+$ with $x \neq y$ if and only if $\lambda_1 \leq 1/8$, $\mu_1 \geq 4/\pi - 1 = 0.2732\cdots$, $\lambda_2 \leq 1/2 + \sqrt{2}/8 = 0.6767\cdots$ and $\mu_2 \geq 1/2 + \sqrt{(4-\pi)/(3\pi-4)}/2 = 0.6988\cdots$ if $\lambda_1, \mu_1 \in (0, 1/2)$ and $\lambda_2, \mu_2 \in (1/2, 1)$.

Inequalities (5), (6), and (9) and the identity $Q(x,y) = \sqrt{A(x,y)C(x,y)}$ lead to

$$AGM(x,y) < A(x,y) < T(x,y) < Q(x,y)$$
$$< C(x,y), \tag{12}$$

$$T(x,y) < Q(x,y) = \sqrt{A(x,y)C(x,y)}, \tag{13}$$

$$T(x,y) > \frac{Q(x,y) + A(x,y)}{2}$$
$$> \sqrt{Q(x,y)A(x,y)}$$
$$> \sqrt{Q(x,y)AGM(x,y)} \tag{14}$$

for all $x, y \in \mathbb{R}^+$ with $x \neq y$.

Motivated by inequalities (12)–(14), in this article we deal with the optimality of the parameters α_i and β_i $(i = 1,2,3,4)$ on the interval $(0,1)$ such that

$$\alpha_1 C(x,y) + (1-\alpha_1)AGM(x,y) < T(x,y)$$
$$< \beta_1 C(x,y) + (1-\beta_1)AGM(x,y),$$
$$\alpha_2 Q(x,y) + (1-\alpha_2)AGM(x,y) < T(x,y)$$
$$< \beta_2 Q(x,y) + (1-\beta_2)AGM(x,y),$$
$$\alpha_3 C(x,y) + (1-\alpha_3)AGM(x,y) < \frac{T^2(x,y)}{A(x,y)}$$
$$< \beta_3 C(x,y) + (1-\beta_3)AGM(x,y),$$
$$\alpha_4 Q(x,y) + (1-\alpha_4)AGM(x,y) < \frac{T^2(x,y)}{Q(x,y)}$$
$$< \beta_4 Q(x,y) + (1-\beta_4)AGM(x,y) \tag{15}$$

for all $x, y \in \mathbb{R}^+$ with $x \neq y$.

2. Lemmas

It is well known that $\mathscr{K}(u)$ and $\mathscr{E}(u)$ satisfy the following formulas (see [10]):

$$\frac{d\mathscr{K}(u)}{du}=\frac{\mathscr{E}(u)-u'^2\mathscr{K}(u)}{uu'^2},$$
$$\frac{d\mathscr{E}(u)}{du}=\frac{\mathscr{E}(u)-\mathscr{K}(u)}{u},$$
$$\frac{d\left[\mathscr{E}(u)-u'^2\mathscr{K}(u)\right]}{du}=u\mathscr{K}(u),$$
$$\mathscr{E}\left(\frac{2\sqrt{u}}{1+u}\right)=\frac{2\mathscr{E}(u)-u'^2\mathscr{K}(u)}{1+u}, \quad (16)$$
$$\mathscr{K}\left(\frac{2\sqrt{u}}{1+u}\right)=(1+u)\mathscr{K}(u),$$
$$\lim_{u\to 0^+}\mathscr{K}(u)=\lim_{u\to 0^+}\mathscr{E}(u)=\frac{\pi}{2},$$
$$\lim_{u\to 1^-}\mathscr{K}(u)=\infty,$$
$$\lim_{u\to 1^-}\mathscr{E}(u)=1.$$

Lemma 1 (See [10, Theorem 1.25]). *Let $a, b \in \mathbb{R}$ with $a < b$, $f, g : [a, b] \longmapsto \mathbb{R}$ be continuous real-valued functions and differentiable on (a, b) with $g'(x) \neq 0$. Then both the functions $(f(x) - f(a))/(g(x) - g(a))$ and $(f(x) - f(b))/(g(x) - g(b))$ are (strictly) increasing (decreasing) on (a, b) if the function $f'(x)/g'(x)$ is (strictly) increasing (decreasing) on (a, b).*

Lemma 2. *For the complete elliptic integrals $\mathscr{K}(u)$ and $\mathscr{E}(u)$, we have the following monotonicity results:*

(1) The function $u \longmapsto [\mathscr{E}(u) - u'^2\mathscr{K}(u)]/u^2$ is strictly increasing from $(0, 1)$ onto $(\pi/4, 1)$.

(2) The function $u \longmapsto \mathscr{K}(u)$ is strictly increasing from $(0, 1)$ onto $(\pi/2, \infty)$ and the function $u \longmapsto \mathscr{E}(u)$ is strictly decreasing from $(0, 1)$ onto $(1, \pi/2)$.

(3) The function $u \longmapsto r'^{\lambda}\mathscr{K}(u)$ is strictly decreasing from $(0, 1)$ onto $(0, \pi/2)$ if $\lambda \geq 1/2$.

(4) The function $u \longmapsto (1 + u'^2)\mathscr{K}(u) - 2\mathscr{E}(u)$ is strictly increasing from $(0, 1)$ onto $(0, \infty)$.

(5) The function $u \longmapsto [\mathscr{K}(u) - \mathscr{E}(u)]/u^2$ is strictly increasing from $(0, 1)$ onto $(\pi/4, \infty)$.

(6) The function $u \longmapsto 2\mathscr{E}(u) - u'^2\mathscr{K}(u)$ is strictly increasing from $(0, 1)$ onto $(\pi/2, 2)$.

(7) The function $u \longmapsto \varphi_1(u) = u'^2/\mathscr{E}(u)$ is strictly decreasing from $(0, 1)$ onto $(0, 2/\pi)$.

(8) The function $u \longmapsto \varphi_2(u) = \sqrt{1+u^2}[\mathscr{E}(u)-u'^2\mathscr{K}(u)]/u^2$ is strictly increasing from $(0, 1)$ onto $(\pi/4, \sqrt{2})$.

(9) The function $u \longmapsto \varphi_3(u) = [2\mathscr{E}(u) - u'^2\mathscr{K}(u)]/\sqrt{1+u^2}$ is strictly decreasing from $(0, 1)$ onto $(\sqrt{2}, \pi/2)$.

(10) The function $u \longmapsto \varphi_4(u) = [u'^2\mathscr{K}^2(u)(\mathscr{K}(u) - \mathscr{E}(u))]/u^2$ is strictly decreasing from $(0, 1)$ onto $(0, \pi^3/16)$.

(11) The function $u \longmapsto \varphi_5(u) = [(1 + u^2)\mathscr{E}(u) - u'^2\mathscr{K}(u)]/(u'^2u^2)$ is strictly increasing from $(0, 1)$ onto $(3\pi/4, \infty)$.

Proof. Parts (1)–(6) can be found in [10, Theorem 3.21(1), (2), (7) and (8), and Exercise 3.43(4), (11) and (13)]. For part (7), it is not difficult to verify that

$$\varphi_1(0^+)=\frac{2}{\pi}, \quad \varphi_1(1^-)=0, \quad (17)$$

$$\varphi_1'(u)=-\frac{u^2\mathscr{E}(u)+\mathscr{E}(u)-u'^2\mathscr{K}(u)}{u\mathscr{E}^2(u)}=-\frac{u}{\mathscr{E}^2(u)}\left[\mathscr{E}(u)+\frac{\mathscr{E}(u)-u'^2\mathscr{K}(u)}{u^2}\right]. \quad (18)$$

It follows from parts (1) and (2) together with (18) that

$$\varphi_1'(u)<0 \quad (19)$$

for $0 < u < 1$.

Therefore, part (7) follows from (17) and (19).

For part (8), simple computations lead to

$$\varphi_2(0^+)=\frac{\pi}{4}, \quad \varphi_2(1^-)=\sqrt{2}, \quad (20)$$

$$\varphi_2'(u)=\frac{2\mathscr{K}(u)-2\mathscr{E}(u)-u^2\mathscr{E}(u)}{u^3\sqrt{1+u^2}}=\frac{\left(1+u'^2\right)\mathscr{K}(u)-2\mathscr{E}(u)}{u^3\sqrt{1+u^2}}+\frac{u}{\sqrt{1+u^2}}\frac{\mathscr{K}(u)-\mathscr{E}(u)}{u^2}. \quad (21)$$

From parts (4) and (5) together with (21) we clearly see that

$$\varphi_2'(u)>0 \quad (22)$$

for $0 < u < 1$.

Therefore, part (8) follows from (20) and (22).

For part (9), we clearly see that

$$\varphi_3(0^+)=\frac{\pi}{2}, \quad \varphi_3(1^-)=\sqrt{2}. \quad (23)$$

Differentiating $\varphi_3(u)$ and making use of part (5) we get

$$\varphi_3'(u)=-\frac{uu'^2}{\left(1+u^2\right)^{3/2}}\left[\frac{\mathscr{K}(u)-\mathscr{E}(u)}{u^2}\right]<0 \quad (24)$$

for $0 < u < 1$.

Therefore, part (9) follows from (23) and (24).

For part (10), elaborated computation gives

$$\varphi_4(0^+)=\frac{\pi^3}{16}, \quad \varphi_4(1^-)=0, \quad (25)$$

$$\varphi_4'(u)=-\frac{\mathscr{K}(u)}{u}\left[\frac{\mathscr{K}(u)}{u^2}+\frac{\mathscr{K}(u)-\mathscr{E}(u)}{u^2}\right]\cdot\left[\left(1+u'^2\right)\mathscr{K}(u)-2\mathscr{E}(u)\right]. \quad (26)$$

It follows from parts (2), (4), and (5) together with (26) that

$$\varphi_4'(u) < 0 \tag{27}$$

for $0 < u < 1$.

Therefore, part (10) follows from (25) and (27).

For part (11), it is not difficult to verify that

$$\varphi_5(0^+) = \frac{3\pi}{4}, \quad \varphi_5(1^-) = \infty, \tag{28}$$

$$\varphi_5'(u) = \frac{(u^4 + 5u^2 - 2)\mathscr{E}(u) + 2(2u^4 - 3u^2 + 1)\mathscr{K}(u)}{u^3 u'^4}. \tag{29}$$

Let

$$\varphi_6(u) = (u^4 + 5u^2 - 2)\mathscr{E}(u) + 2(2u^4 - 3u^2 + 1)\mathscr{K}(u). \tag{30}$$

Then we clearly see that that

$$\varphi_6(0^+) = 0, \tag{31}$$

$$\varphi_6'(u) = 11u^3\left[\frac{5}{11}\mathscr{E}(u) + \frac{\mathscr{E}(u) - u'^2\mathscr{K}(u)}{u^2}\right]. \tag{32}$$

Therefore, part (11) follows easily from parts (1) and (2) together with (28)–(32). □

3. Main Results

Theorem 3. *The double inequality*

$$\alpha_1 C(x,y) + (1-\alpha_1) AGM(x,y) < T(x,y) < \beta_1 C(x,y) + (1-\beta_1) AGM(x,y) \tag{33}$$

holds for all $x, y \in \mathbb{R}^+$ with $x \neq y$ if and only if $\alpha_1 \leq 2/5$ and $\beta_1 \geq 2/\pi = 0.6366\cdots$.

Proof. We clearly see that $C(x,y)$, $AGM(x,y)$ and $T(x,y)$ are symmetric and homogenous of degree 1. Without loss of generality, we assume that $x > y > 0$. Let $u = (x-y)/(x+y)$. Then $0 < u < 1$ and (2) and (4) lead to

$$AGM(x,y) = \frac{\pi}{2}\frac{A(x,y)}{\mathscr{K}(u)}, \tag{34}$$

$$T(x,y) = \frac{2}{\pi}A(x,y)\left[2\mathscr{E}(u) - u'^2\mathscr{K}(u)\right]. \tag{35}$$

It follows from (34) and (35) together with $C(x,y) = A(x,y)(1+u^2)$ that

$$\frac{T(x,y) - AGM(x,y)}{C(x,y) - AGM(x,y)} = \frac{(2/\pi)\left[2\mathscr{E}(u) - u'^2\mathscr{K}(u)\right] - \pi/2\mathscr{K}(u)}{(1+u^2) - \pi/2\mathscr{K}(u)} = \frac{(2/\pi)\mathscr{K}(u)\left[2\mathscr{E}(u) - u'^2\mathscr{K}(u)\right] - \pi/2}{(1+u^2)\mathscr{K}(u) - \pi/2} := F(u). \tag{36}$$

Let $f_1(u) = 2\mathscr{K}(u)[2\mathscr{E}(u) - u'^2\mathscr{K}(u)]/\pi - \pi/2$ and $g_1(u) = (1+u^2)\mathscr{K}(u) - \pi/2$. Then elaborated computations lead to

$$f_1(0^+) = g_1(0^+) = 0, \quad F(u) = \frac{f_1(u)}{g_1(u)}, \tag{37}$$

$$\frac{f_1'(u)}{g_1'(u)} = \frac{4}{\pi}\frac{\mathscr{E}(u)\left[\mathscr{E}(u) - u'^2\mathscr{K}(u)\right]}{2u^2\mathscr{E}(u) + u'^2\left[\mathscr{E}(u) - u'^2\mathscr{K}(u)\right]} = \frac{4}{\pi}\frac{1}{2\left[u^2/(\mathscr{E}(u) - u'^2\mathscr{K}(u))\right] + u'^2/\mathscr{E}(u)}. \tag{38}$$

It follows from Lemma 2(1) and (7) together with (38) that $f_1'(u)/g_1'(u)$ is strictly increasing on $(0,1)$. Then from Lemma 1 and (37) we know that $F(u)$ is strictly increasing on $(0,1)$. Moreover,

$$F(0^+) = \frac{2}{5}, \quad F(1^-) = \frac{2}{\pi}. \tag{39}$$

Therefore, Theorem 3 follows from (36) and (39) together with the monotonicity of $F(u)$. □

Theorem 4. *The double inequality*

$$\alpha_2 Q(x,y) + (1-\alpha_2) AGM(x,y) < T(x,y) < \beta_2 Q(x,y) + (1-\beta_2) AGM(x,y) \tag{40}$$

holds for all $x, y \in \mathbb{R}^+$ with $x \neq y$ if and only if $\alpha_2 \leq 2/3$ and $\beta_2 \geq 2\sqrt{2}/\pi = 0.9003\cdots$.

Proof. Since $Q(x,y)$, $AGM(x,y)$, and $T(x,y)$ are symmetric and homogenous of degree 1, without loss of generality, we assume that $x > y > 0$. Let $u = (x-y)/(x+y)$. Then $u \in (0,1)$, and (34) and (35) together with $Q(x,y) = A(x,y)\sqrt{1+u^2}$ lead to

$$\frac{T(x,y) - AGM(x,y)}{Q(x,y) - AGM(x,y)} = \frac{(2/\pi)\left[2\mathscr{E}(u) - u'^2\mathscr{K}(u)\right] - \pi/2\mathscr{K}(u)}{\sqrt{1+u^2} - \pi/2\mathscr{K}(u)} := G(u). \tag{41}$$

Let $f_2(u) = 2[2\mathscr{E}(u) - u'^2\mathscr{K}(u)]/\pi - \pi/[2\mathscr{K}(u)]$ and $g_2(u) = \sqrt{1+u^2} - \pi/[2\mathscr{K}(u)]$. Then simple computations lead to

$$f_2(0^+) = g_2(0^+) = 0,$$
$$G(u) = \frac{f_2(u)}{g_2(u)}, \tag{42}$$

$$\frac{f_2'(u)}{g_2'(u)} = 1 - \frac{1 - (2/\pi)\left(\sqrt{1+u^2}\left[\mathscr{E}(u) - u'^2\mathscr{K}(u)\right]/u^2\right)}{1 + (\pi/2)\left(\sqrt{1+u^2}\left[\mathscr{E}(u) - u'^2\mathscr{K}(u)\right]/u^2\right)\left(1/\left[u'\mathscr{K}(u)\right]^2\right)} \tag{43}$$

It follows from Lemma 2(3) and (8) together with (43) that $f_2'(u)/g_2'(u)$ is strictly increasing on $(0,1)$. Then from Lemma 1 and (42) we know that $G(u)$ is strictly increasing on $(0,1)$. Moreover,

$$G(0^+) = \frac{2}{3},$$
$$G(1^-) = \frac{2\sqrt{2}}{\pi}. \tag{44}$$

Therefore, Theorem 4 follows from (41) and (44) together with the monotonicity of $G(u)$. □

Theorem 5. *The double inequality*

$$\alpha_3 C(x,y) + (1-\alpha_3)\, AGM(x,y) < \frac{T^2(x,y)}{A(x,y)}$$
$$< \beta_3 C(x,y) + (1-\beta_3)\, AGM(x,y) \tag{45}$$

holds for all $x, y \in \mathbb{R}^+$ with $x \neq y$ if and only if $\alpha_3 \leq 3/5$ and $\beta_3 \geq 8/\pi^2 = 0.8105\cdots$.

Proof. Without loss of generality, we assume that $x > y > 0$. Let $u = (x-y)/(x+y) \in (0,1)$. Then it follows from (34), (35) and $C(x,y) = A(x,y)(1+u^2)$ that

$$\frac{T^2(x,y)/A(x,y) - AGM(x,y)}{C(x,y) - AGM(x,y)}$$
$$= \frac{\left(4/\pi^2\right)\left[2\mathscr{E}(u) - u'^2\mathscr{K}(u)\right]^2 - \pi/2\mathscr{K}(u)}{\left(1+u^2\right) - \pi/2\mathscr{K}(u)} \tag{46}$$
$$:= H(u).$$

Let $f_3(u) = 4[2\mathscr{E}(u) - u'^2\mathscr{K}(u)]^2/\pi^2 - \pi/[2\mathscr{K}(u)]$ and $g_3(u) = (1+u^2) - \pi/[2\mathscr{K}(u)]$. Then elaborated computations lead to

$$f_3(0^+) = g_3(0^+) = 0,$$
$$H(u) = \frac{f_3(u)}{g_3(u)}, \tag{47}$$

$$\frac{f_3'(u)}{g_3'(u)} = 1$$
$$- \frac{1 - \left(4/\pi^2\right)\left[2\mathscr{E}(u) - u'^2\mathscr{K}(u)\right]\left[\left(\mathscr{E}(u) - u'^2\mathscr{K}(u)\right)/u^2\right]}{1 + (\pi/4)\left[\left(\mathscr{E}(u) - u'^2\mathscr{K}(u)\right)/u^2\right]\left(1/\left[u'\mathscr{K}(u)\right]^2\right)}. \tag{48}$$

□

It follows from Lemma 2(1), (3), and (6) together with (48) that $f_3'(u)/g_3'(u)$ is strictly increasing on $(0,1)$. Then from Lemma 1 and (47) we know that $H(u)$ is strictly increasing on $(0,1)$. Moreover,

$$H(0^+) = \frac{3}{5},$$
$$H(1^-) = \frac{8}{\pi^2}. \tag{49}$$

Therefore, Theorem 5 follows from (46) and (49) together with the monotonicity of $H(u)$.

Theorem 6. *The double inequality*

$$\alpha_4 Q(x,y) + (1-\alpha_4)\, AGM(x,y) < \frac{T^2(x,y)}{Q(x,y)}$$
$$< \beta_4 Q(x,y) + (1-\beta_4)\, AGM(x,y) \tag{50}$$

holds for all $x, y \in \mathbb{R}^+$ with $x \neq y$ if and only if $\alpha_4 \leq 1/3$ and $\beta_4 \geq 8/\pi^2 = 0.8105\cdots$.

Proof. Without loss of generality, we assume that $x > y > 0$. Let $u = (x-y)/(x+y) \in (0,1)$. Then it follows from (34), (35) and $Q(x,y) = A(x,y)\sqrt{1+u^2}$ that

$$\frac{T^2(x,y)/Q(x,y) - AGM(x,y)}{Q(x,y) - AGM(x,y)}$$
$$= \frac{\left(4/\pi^2\right)\left[2\mathscr{E}(u) - u'^2\mathscr{K}(u)\right]^2 - \pi\sqrt{1+u^2}/2\mathscr{K}(u)}{\left(1+u^2\right) - \pi\sqrt{1+u^2}/2\mathscr{K}(u)}$$
$$= 1 - \frac{1 - \left(4/\pi^2\right)\left[\left(2\mathscr{E}(u) - u'^2\mathscr{K}(u)\right)/\sqrt{1+u^2}\right]^2}{1 - \pi/\left(2\sqrt{1+u^2}\mathscr{K}(u)\right)} \tag{51}$$
$$:= 1 - J(u).$$

Let $f_4(u) = 1 - 4[(2\mathscr{E}(u) - u'^2\mathscr{K}(u))/\sqrt{1+u^2}]^2/\pi^2$ and $g_4(u) = 1 - \pi/[2\sqrt{1+u^2}\mathscr{K}(u)]$. Then simple computations lead to

$$f_4(0^+) = g_4(0^+) = 0,$$
$$J(u) = \frac{f_4(u)}{g_4(u)}, \tag{52}$$

$$\begin{aligned}\frac{f_4'(u)}{g_4'(u)} &= \frac{16}{\pi^3}\left[\frac{2\mathscr{E}(u)-u'^2\mathscr{K}(u)}{\sqrt{1+u^2}}\right] \\ &\quad\cdot\frac{\left[u'^2\mathscr{K}(u)\right]^2\left[\mathscr{K}(u)-\mathscr{E}(u)\right]}{\left[\left(1+u^2\right)\mathscr{E}(u)-u'^2\mathscr{K}(u)\right]} \\ &= \frac{16}{\pi^3}\left[\frac{2\mathscr{E}(u)-u'^2\mathscr{K}(u)}{\sqrt{1+u^2}}\right] \\ &\quad\cdot\frac{\left[u'^2\mathscr{K}^2(u)\left(\mathscr{K}(u)-\mathscr{E}(u)\right)\right]/u^2}{\left[\left(1+u^2\right)\mathscr{E}(u)-u'^2\mathscr{K}(u)\right]/\left(u'^2u^2\right)}.\end{aligned} \tag{53}$$

It follows from Lemma 2(9)–(11) and (53) that $f_4'(u)/g_4'(u)$ is strictly increasing on $(0,1)$. Then from Lemma 1 and (52) we know that $J(u)$ is strictly increasing on $(0,1)$. Moreover,

$$\begin{aligned}J\left(0^+\right) &= \frac{2}{3}, \\ J\left(1^-\right) &= 1-\frac{8}{\pi^2}.\end{aligned} \tag{54}$$

Therefore, Theorem 6 follows from (51) and (54) together with the monotonicity of $J(u)$. □

Let $x = 1$ and $y = u'$. Then (2), (4) and Theorems 3–6 lead to Corollary 7 immediately.

Corollary 7. *The double inequalities*

$$\begin{aligned}&\frac{\pi\left[4\left(1+u'^2\right)\mathscr{K}(u)+3\pi\left(1+u'\right)\right]}{20\left(1+u'\right)\mathscr{K}(u)} < \mathscr{E}(u) \\ &\quad< \frac{4\left(1+u'^2\right)\mathscr{K}(u)+\pi(\pi-2)\left(1+u'\right)}{4\left(1+u'\right)\mathscr{K}(u)}, \\ &\frac{\pi\left[2\sqrt{2\left(1+u'^2\right)}\mathscr{K}(u)+\pi\right]}{12\mathscr{K}(u)} < \mathscr{E}(u) \\ &\quad< \frac{4\sqrt{1+u'^2}\mathscr{K}(u)+\pi\left(\pi-2\sqrt{2}\right)}{4\mathscr{K}(u)}, \\ &\frac{\pi}{2}\sqrt{\frac{3\left(1+u'^2\right)\mathscr{K}(u)+\pi\left(1+u'\right)}{10\mathscr{K}(u)}} < \mathscr{E}(u) \\ &\quad< \frac{1}{4}\sqrt{\frac{16\left(1+u'^2\right)\mathscr{K}(u)+\pi\left(\pi^2-8\right)\left(1+u'\right)}{\mathscr{K}(u)}}, \\ &\frac{\pi}{2}\sqrt{\frac{\left(1+u'^2\right)\mathscr{K}(u)+\pi\sqrt{2\left(1+u'^2\right)}}{6\mathscr{K}(u)}} < \mathscr{E}(u) \\ &\quad< \frac{1}{4}\sqrt{\frac{16\left(1+u'^2\right)\mathscr{K}(u)+\pi\left(\pi^2-8\right)\sqrt{2\left(1+u'^2\right)}}{\mathscr{K}(u)}}\end{aligned} \tag{55}$$

hold for all $0 < u < 1$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This work was supported by the Natural Science Foundation of China (Grant No. 61673169) and the Natural Science Foundation of Huzhou City (Grant No. 2018YZ07).

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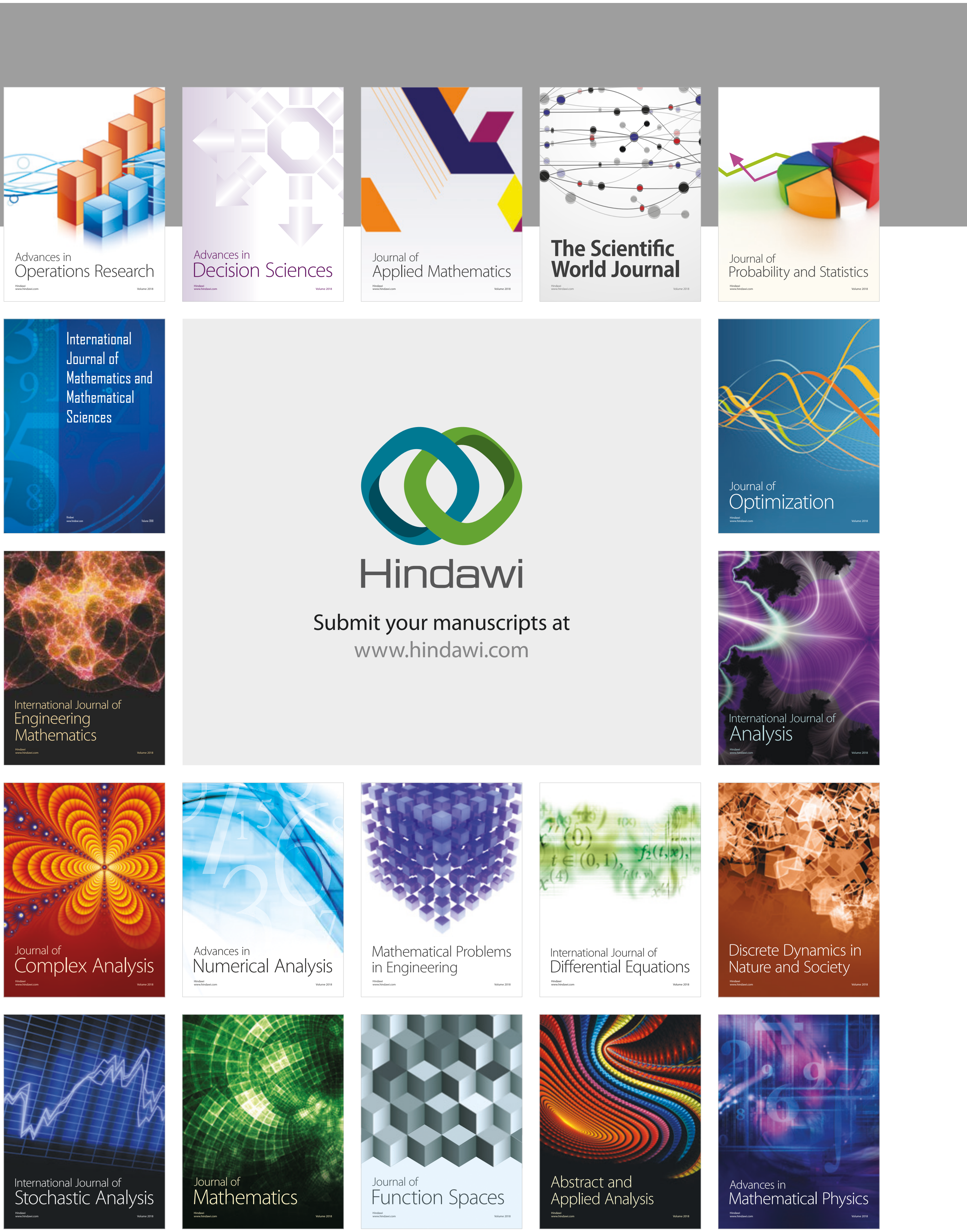
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