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Research Article

Gateaux Differentiability of Convex Functions and Weak Dentable Set in Nonseparable Banach Spaces

Shaoqiang Shang [1] and Yunan Cui [2]

[1] *Academy of Mathematical Sciences, Harbin Engineering University, Harbin 150001, China*
[2] *Department of Mathematics, Harbin University of Science and Technology University, Harbin 150080, China*

Correspondence should be addressed to Shaoqiang Shang; sqshang@163.com

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In this paper, we prove that if C^{**} is a ε-separable bounded subset of X^{**}, then every convex function $g \leq \sigma_C$ is Gâteaux differentiable at a dense G_δ subset G of X^* if and only if every subset of $\partial\sigma_C(0) \cap X$ is weakly dentable. Moreover, we also prove that if C is a closed convex set, then $d\sigma_C(x^*) = x$ if and only if x is a weakly exposed point of C exposed by x^*. Finally, we prove that X is an Asplund space if and only if, for every bounded closed convex set C^* of X^*, there exists a dense subset G of X^{**} such that σ_{C^*} is Gâteaux differentiable on G and $d\sigma_{C^*}(G) \subset C^*$. We also prove that X is an Asplund space if and only if, for every w^*-lower semicontinuous convex function f, there exists a dense subset G of X^{**} such that f is Gâteaux differentiable on G and $df(G) \subset X^*$.

1. Introduction and Preliminaries

Let $(X, \|\cdot\|)$ denote a real Banach space. $B(X)$ and $S(X)$ denote the unit ball and unit sphere of X, respectively. Let X^* denote the dual space of X. Let N, R, and R^+ denote the sets of natural number, reals, and nonnegative reals, respectively. Let $B(x, r)$ denote the closed ball centered at x and of radius $r > 0$. Let $x_n \xrightarrow{w} x$ denote that $\{x_n\}_{n=1}^{\infty}$ is weakly convergent to x.

Let D be a nonempty open convex subset of X and f a continuous convex function on D. We called that f is said to be Gâteaux differentiable at the point x in D if the limit

$$df(x)(y) = \lim_{t \longrightarrow 0} \frac{f(x+ty) - f(x)}{t} \qquad (*)$$

exists for all $y \in X$. Moreover, if the difference quotient in $(*)$ converges to $df(x)(y)$ uniformly for y in the unit ball, then f is said to be Frechet differentiable at x.

Definition 1 (see [1]). X is called a weak Asplund space [Asplund space] if, for every f and D as above, there exists a dense G_δ subset G of D such that f is Gâteaux [Frechet] differentiable at each point of G.

It is well known that l^1 is weak Asplund space, but not Asplund space. Moreover, it is well known that X is an Asplund space if and only if X^* has the Radon-Nikodym property (see [1]). In 1933, Mazur proved that separable Banach spaces have the weak Asplund property (see [1]).

Definition 2 (see [1]). A Banach space is said to be a Gâteaux differentiable space if every convex continuous function on it is Gâteaux differentiable at the points of a dense set.

In 2006, Waren B. Moors and Sivajah Somasundaram proved that there exists a Gâteaux differentiable space that is not a weak Asplund space (see [2]). In 1979, D.G. Larman and R.R.Phelps proved that if X^* is a strictly convex space, then X is a weak Asplund space (see [3]). In 1997, Cheng proved that if f is a continuous convex function on a Banach space X, then every proper convex function g on X with $g \leq f$ is generically Frechet differentiable if and only if the image of the subdifferential map ∂f has the Radon-Nikodym property (see [4]).

Definition 3 (see [1]). A point $x_0^* \in C^*$ is said to be weak* exposed point of C^* if there exists $x \in S(X)$ such that $x_0^*(x) > x^*(x)$ whenever $x^* \in C^* \backslash \{x_0^*\}$.

Definition 4 (see [1]). Suppose that f is a convex function on X, then the set-valued mapping $\partial f(x) = \{x^* \in X^* : (x^*, y - x) \leq f(y) - f(x)$ for all $y \in X\}$ is said to be subdifferential mapping.

Remark 5. It is well known that if f is a continuous convex function, then the set-valued mapping ∂f is norm- weak* upper semicontinuous (see [1]). Moreover, it is well known that ∂f is a sington at x if and if f is Gâteaux differentiable at x (see [1]).

Let C be a bounded subset of X. Let $C^* = \{x^* \in X^* : x^*(x) \leq 1, x \in C\}$ and $C^{**} = \{x^{**} \in X^{**} : x^{**}(x^*) \leq 1, x^* \in C^*\}$. Then it is easy to see that C^{**} is a weak* closed convex set. Define the sublinear functional

$$\begin{aligned} \sigma_C(x^*) &= \sup\{x^*(x) : x \in C\} \\ &= \sup\{x^*(x^{**}) : x^{**} \in C^{**}\}. \end{aligned} \tag{1}$$

Then σ_C is a continuous sublinear functional. It is well known that $\partial\sigma_C(0) = C^{**}$, $\overline{co^{w^*}}(C) = C^{**}$, and $C^* = \{x^* \in X^* : \sigma_C(x^*) \leq 1\}$. Let p be a continuous Minkowski functional on X and $C = \{x \in X : p(x) \leq 1\}$. Then $p(x) = \inf\{\lambda \geq 0 : \lambda^{-1}x \in C\}$ whenever $x \in X$. In this case, we called that p is generated by C. Let $C^* = \{x^* \in X^* : x^*(x) \leq 1, x \in C\}$. Then

$$p(x) = \sigma_{C^*}(x) = \sup\{x^*(x) : x^* \in C^*\}. \tag{2}$$

Moreover, it is well known that

$$\begin{aligned} &(1) \quad \partial p(0) = C^*; \\ &(2) \quad x^* \in \partial p(x) \Longleftrightarrow \\ &\qquad x^* \in C^* \\ &\qquad \text{and } x^*(x) = p(x) \end{aligned} \tag{3}$$

It is well known that if C is a bounded subset of X, then $\sigma_{\overline{co}(C\cup(-C))}(x^*) = \inf\{\lambda : \lambda^{-1}x^* \in C^*\}$. If f is a convex function, then the set

$$epif = \{(x, r) \in X \times R : f(x) \leq r\}. \tag{4}$$

denotes epigraph of f. It is well known that $epif$ is closed if and only if f is lower semicontinuous.

Lemma 6 (see [4]). *Let f be a continuous convex function on X and $f(0) = -1$. Let $p : E \times R \longrightarrow R$ be the Minkowski functional generated by $epif$. Then $x^* \in \partial f(x)$ if and only if $(y^*, r) \in \partial p(x, f(x))$ with $y^* = [x^*(x) - f(x)]^{-1}x^*$ and $r = -[x^*(x) - f(x)]^{-1}$.*

Definition 7. A set $D \subset X$ is said to be weak dentable set if for any weak neighborhood U of origin, there exists $z \in D$ such that $z \notin \overline{co}(D\backslash(z + U))$.

Definition 8 (see [5]). A set $D \subset X$ is said to be dentable set if, for any $\varepsilon > 0$, there exists $z \in D$ such that $z \notin \overline{co}(D\backslash B(z, \varepsilon))$.

Definition 9 (see [5]). A Banach space X is said to have the Radon-Nikodym property (see [1]) if (T, Σ, μ) is a nonatomic measure space and ν is a vector measure on Σ with values in X which is absolutely continuous with respect to μ and has a bounded variation, then there exists $f \in L_1(X)$ such that, for any $A \in \Sigma$,

$$\nu(A) = \int_A f(t)\, dt. \tag{5}$$

It is well known that a Banach space X has the Radon-Nikodym property if and only if every bounded subset of X is dentable. By Definitions 8 and 9, it is easy to see that if C is dentable, then C is weak dentable. Moreover, there exists a weak dentable set such that it is not dentable. We will give two examples in Sections 3 and 4.

Proposition 10. *The weak neighborhood $A = \{x \in X : \bigcap_{i=1}^k |(x_i^*, x)| \leq r\}$ is not a weak dentable set, where $\{x_1^*, x_2^*, \ldots, x_k^*\} \subset S(X^*)$ and $r \in (0, 1)$.*

Proof. Let $A = \{x \in X : \bigcap_{i=1}^k |(x_i^*, x)| \leq r\}$, where $\{x_1^*, x_2^*, \ldots, x_k^*\} \subset S(X^*)$ and $r \in (0, 1)$. Pick $x_0 \in S(X)$ and

$$x_0 \in \bigcap_{i=1}^k \{x \in X : (x_i^*, x) = 0\}. \tag{6}$$

Then, by the Hahn-Banach theorem, there exists $x_0^* \in S(X^*)$ such that $x_0^*(x_0) = 1$. Define a weak neighborhood

$$V = \left\{x \in X : |(x_0^*, x)| < \frac{1}{4}\right\} \tag{7}$$

of origin. Then, for any $x \in A$, we have

$$\begin{aligned} |(x_0^*, x + x_0 - x)| &= |(x_0^*, x_0)| = 1 \\ \text{and } |(x_0^*, x - x_0 - x)| &= |(x_0^*, -x_0)| = 1. \end{aligned} \tag{8}$$

Therefore, by formula (7), we have

$$\begin{aligned} x + x_0 &\notin x + V \\ \text{and } x - x_0 &\notin x + V. \end{aligned} \tag{9}$$

Moreover, for any $i \in \{1, 2, \ldots, k\}$, we have

$$\begin{aligned} |(x_i^*, x + x_0)| &= |(x_i^*, x)| \leq r \\ \text{and } |(x_i^*, x - x_0)| &= |(x_i^*, x)| \leq r. \end{aligned} \tag{10}$$

Therefore, by formula (6), we have

$$\begin{aligned} x + x_0 &\in A\backslash(x + V) \\ \text{and } x - x_0 &\in A\backslash(x + V). \end{aligned} \tag{11}$$

Hence we obtain that

$$\begin{aligned} x &= \frac{1}{2}(x + x_0) + \frac{1}{2}(x - x_0) \in co(A\backslash(x + V)) \\ &\subset \overline{co}(A\backslash(x + V)). \end{aligned} \tag{12}$$

This implies that A is not weak dentable, which finishes the proof. □

Definition 11. A set $A \subset X^*$ is said to be ε-separable if there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ such that $\sup_{n\geq 1} x^*(x_n) > 0$ for any $x^* \in A\backslash\{0\}$.

It is well known that if X is a separable space, then every subset of X^* is ε-separable.

Proposition 12. *Suppose that $A \subset X^*$ is separable and $A \neq \{0\}$. Then A is ε-separable.*

Proof. Since A is a separable subset of X^*, there exists a sequence $\{x_n^*\}_{n=1}^{\infty} \subset A$ such that $\overline{\{x_n^*\}_{n=1}^{\infty}} = A$. Then we may assume without loss of generality that $x_n^* \neq 0$ for any $n \in N$. Hence there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset S(X)$ such that $(x_n^*, x_n) \geq \|x_n^*\|/2$.

Pick $y^* \in A\backslash\{0\}$. We will prove that $\sup_{n\geq 1} y^*(x_n) > 0$. In fact, suppose that $\sup_{n\geq 1} y^*(x_n) \leq 0$. Then $(x_n, y^*) \leq 0$ for all $n \in N$. Since $\overline{\{x_n^*\}_{n=1}^{\infty}} = A$, there exists a natural number n_0 such that $\|x_{n_0}^* - y^*\| < (1/8)\|y^*\|$. Then $\|y^*\| - \|x_{n_0}^*\| < (1/8)\|y^*\|$. Hence we obtain that $\|y^*\| < (8/7)\|x_{n_0}^*\|$. Therefore, by $(x_n, y^*) \leq 0$ for any $n \in N$, we have

$$\begin{aligned}\left(x_{n_0}, x_{n_0}^*\right) &= \left(x_{n_0}, x_{n_0}^* - y^*\right) + \left(x_{n_0}, y^*\right) \\ &\leq \left(x_{n_0}, x_{n_0}^* - y^*\right) + 0 \leq \left\|x_{n_0}^* - y^*\right\| \\ &< \frac{1}{8}\left\|y^*\right\| < \frac{1}{8}\cdot\frac{8}{7}\left\|x_{n_0}^*\right\| = \frac{1}{7}\left\|x_{n_0}^*\right\|,\end{aligned} \tag{13}$$

which contradicts $(x_n^*, x_n) \geq \|x_n^*\|/2$ for every $n \in N$. This implies that $\sup_{n\geq 1} y^*(x_n) > 0$. Hence A is ε-separable, which finishes the proof. □

Example 13. Let $X = c_0$. Then $X^* = l^1$ is separable and $X^{**} = l^{\infty}$ is not a separable space. Let Y be a Banach space and Y be not a separable space. Define $C = B(X) \times \{0\} \subset X \times Y$. Then

$$\begin{aligned} C^* &= B\left(X^*\right) \times Y^* \subset X^* \times Y^* \\ \text{and } C^{**} &= B\left(X^{**}\right) \times \{0\} \subset X^{**} \times Y^{**}. \end{aligned} \tag{14}$$

Let $(x^{**}, 0) \in C^{**}\backslash\{(0,0)\}$. Since X^* is separable, there exists $\{x_n^*\}_{n=1}^{\infty} \subset S(X^*)$ such that $\overline{\{x_n^*\}_{n=1}^{\infty}} = S(X^*)$. Then $\sup_{n\geq 1} x^{**}(x_n^*) > 0$. Hence

$$\sup_{n\geq 1}\left(x^{**}, 0\right)\left(x_n^*, 0\right) = \sup_{n\geq 1}\left(x^{**}, x_n^*\right) > 0. \tag{15}$$

This implies that C^{**} is ε-separable and bounded. Moreover, it is easy to see that C^* and C^{**} are not separable and C is not dentable.

The paper is organized as follows. In Section 1 some necessary definitions and notations are collected. In Section 2 we prove that if C^{**} is a ε-separable bounded subset of X^{**}, then every convex function $g \leq \sigma_C$ is Gâteaux differentiable at a dense G_δ subset G of X^* if and only if every subset of $\partial\sigma_C(0) \cap X$ is weakly dentable. In Section 3 we prove that if C is a closed convex set, then $d\sigma_C(x^*) = x$ if and only if x is a weakly exposed point of C exposed by x^*. Moreover, we also prove that X is an Asplund space if and only if for every bounded closed convex set C^* of X^*, there exists a dense subset G of X^{**} such that σ_{C^*} is Gâteaux differentiable on G and $d\sigma_{C^*}(G) \subset C^*$. We also prove that X is an Asplund space if and only if for every w^*-lower semicontinuous convex function f, there exists a dense subset G of X^{**} such that f is Gâteaux differentiable on G and $df(G) \subset X^*$. In Section 4 we prove that there exists an exposed point such that it is not a weak exposed point in Orlicz function spaces. The topic of this paper is related to the topic of [5–12].

2. Gateaux Differentiability, Weakly Dentable Set, and ε-Separable Set

Theorem 14. *Suppose that C^{**} is a ε-separable bounded subset of X^{**}. Then the following statements are equivalent.*

(1) Every w^*-lower semicontinuous convex function $g \leq \sigma_C$ is Gâteaux differentiable at a dense G_δ subset G of X^*.

(2) Every convex subset of $\partial\sigma_C(0) \cap X$ is a weakly dentable set of X.

(3) Every weak* closed convex subset of $\partial\sigma_C(0)$ is the weak* closed convex hull of its weak* exposed points.

In order to prove the theorem, we give some lemmas.

Lemma 15. *Suppose that*

(1) C^{**} is a ε-separable bounded subset of X^{**} and C is a closed convex set;

(2) f is a continuous convex function and $\partial f(X^*) \subset C^{**}$;

(3) for any $D \subset C$ and weak neighborhood U of origin, there exists a slice $S(x^*, C, \alpha)$ such that $S(x^*, D, \alpha) - S(x^*, D, \alpha) \subset U$. Then f is Gâteaux differentiable on a dense G_δ subset G of X^*.

Proof. We claim that if C^{**} is ε-separable, then the set $C^{**} - C^{**}$ is ε-separable. In fact, since C^{**} is ε-separable, there exists a sequence $\{x_n^*\}_{n=1}^{\infty} \subset X^*$ such that $\sup_{n\geq 1} x^{**}(x_n^*) > 0$ for any $x^{**} \in C^{**}\backslash\{0\}$. Let $x_0^{**} \in C^{**}$, $y_0^{**} \in C^{**}$, and $x_0^{**} - y_0^{**} \neq 0$. Then, for any $x^* \in C^*$, we have

$$\begin{aligned}&\left|\left(\frac{1}{2}x_0^{**} - \frac{1}{2}y_0^{**}, x^*\right)\right| \\ &\quad\leq \left|\left(\frac{1}{2}x_0^{**}, x^*\right)\right| + \left|\left(\frac{1}{2}y_0^{**}, x^*\right)\right| \leq \frac{1}{2} + \frac{1}{2} = 1.\end{aligned} \tag{16}$$

This implies that $(x_0^{**} - y_0^{**})/2 \in C^{**}$. Then $\sup_{n\geq 1}(x_0^{**} - y_0^{**}, x_n^*) > 0$. Hence we obtain that $C^{**} - C^{**}$ is ε-separable.

Since $C^{**} - C^{**}$ is a ε-separable bounded subset of X^{**}, there exists a sequence $\{x_n^*\}_{n=1}^{\infty} \subset X^*$ such that $\sup_{n\geq 1} x^{**}(x_n^*) > 0$ for any $x^{**} \in (C^{**} - C^{**})\backslash\{0\}$. Hence, for every natural number $n \in N$, we define a weak* neighborhood

$$W_n = \left\{x^{**} \in X^{**} : \bigcap_{i=1}^{n}\left|\left(x_i^*, x^{**}\right)\right| \leq \frac{1}{n}\right\} \tag{17}$$

of origin in X^{**}. Moreover, for every natural number $n \in N$, we define a weak neighborhood

$$U_n = \left\{ x \in X : \bigcap_{i=1}^{n} \left|(x_i^*, x)\right| \le \frac{1}{n} \right\} \tag{18}$$

of origin in X. Hence, if $y^{**} \in \bigcap_{n=1}^{\infty} W_n$ and $y^{**} \in C^{**} - C^{**}$, then $(x_n^*, y^{**}) = 0$ for all $n \in N$. Since $\sup_{n\ge1} x^{**}(x_n^*) > 0$ for every $x^{**} \in C^{**}\backslash\{0\}$, we have $y^{**} = 0$. Hence, for each $n > 1$, let G_n be the set of all $x^* \in X^*$ for which there exists a norm neighborhood V_n of x^* such that $\partial f(V_n) - \partial f(V_n) \subset W_n$. Let $x^* \in \bigcap_{n=1}^{\infty} G_n$. Pick $x^{**} \in \partial f(x^*)$ and $y^{**} \in \partial f(x^*)$. Then $x^{**} \in \partial f(V_n)$ and $y^{**} \in \partial f(V_n)$ for every $n \in N$. Hence we obtain that

$$x^{**} - y^{**} \subset \partial f(V_n) - \partial f(V_n) \subset W_n \tag{19}$$

for every $n \in N$. This implies that $(x^{**} - y^{**}, x^*) = 0$ for every $x^* \in \{x_n^*\}_{n=1}^{\infty}$. Since $\partial f(X^*) \subset C^{**}$, we have $x^{**} - y^{**} \in C^{**} - C^{**}$. Therefore, by the previous proof, we have $x^{**} = y^{**}$. This implies that

$$\bigcap_{n=1}^{\infty} G_n = \{x^* \in X^* : \partial f(x^*) \text{ is a singleton}\}. \tag{20}$$

Hence we obtain that f is Gâteaux differentiable at each point of $G = \bigcap_{n=1}^{\infty} G_n$.

Since X^* is a Baire space, we next will prove that, for any $n \in N$, the set G_n is open and dense in X^*. It is easy to see that G_n is an open set. We next will prove that G_n is dense in X^*. Let $x^* \in X^*$ and let U be a neighborhood of x^* in X^*. We claim that $\partial f(U) \cap X \neq \emptyset$. In fact, since f is a w^*-lower semicontinuous function on X^*, we obtain that the set $epif$ is a weak* closed set of X^*. Moreover, we may assume without loss of generality that $f(0) = -1$. Let $C^* = epif$ and $\mu_{C^*}(x^*, r) = \inf\{\lambda \in R^+ : \lambda^{-1}(x^*, r) \in C^*\}$. Since $(0,0) \in \operatorname{int} C^*$, we obtain that μ_{C^*} is continuous. Pick $(x^*, f(x^*)-1) \notin epif$. Since $epif$ is weak* closed, by the separation theorem, there exists $(x, r) \in X \times R$ such that

$$\begin{aligned} &x^*(x) + rf(x^*) - r \\ &\quad \ge \sup\{z^*(x) + rh : (z^*, h) \in epif\}. \end{aligned} \tag{21}$$

Hence we may assume without loss of generality that $\sup\{z^*(x) + rh : (z^*, h) \in epif\} = 1$. This implies that the set

$$\begin{aligned} &C \\ &\quad = \{(x, r) \in X \times R : x^*(x) + rh \le 1,\ (x^*, h) \in epif\} \end{aligned} \tag{22}$$

is a nonempty bounded closed convex subset of $X \times R$. Therefore, by the Bishop-Phelps Theorem, we obtain that

$$\begin{aligned} &\{(x^*, r) \in X^* \times R : x^*(x) + rh \\ &\quad = \sup\{x^*(z) + rl : (z, l) \in C\},\ (x, h) \in C\} \end{aligned} \tag{23}$$

is a dense set of $X^* \times R$. Hence

$$\{(x^*, r) \in X^* \times R : \partial\mu_{C^*}(x^*, r) \cap (X \times R) \neq \emptyset\} \tag{24}$$

is a dense set of $X^* \times R$. Therefore, by Lemma 6, it is easy to see that $\partial f(U) \cap X \neq \emptyset$. Therefore, by formulas $\partial f(U) \subset C^{**}$ and $C^{**} \cap X = C$, we obtain that $\partial f(U) \cap X \subset C$. Pick $n \in N$. Then, by hypothesis, there exist a slice

$$\begin{aligned} &S(z^*, \partial f(U) \cap X, \alpha) \\ &\quad = \left\{x \in \partial\sigma_C(U) \cap X : (z^*, x) > \sigma_{\partial f(U)\cap X}(z^*) - \alpha\right\} \end{aligned} \tag{25}$$

and $x_0 \in S(z^*, \partial f(U) \cap X, \alpha)$ such that $S(z^*, \partial f(U) \cap X, \alpha) \subset x_0 + U_n$. Moreover, if $x \in S(z^*, \partial f(U) \cap X, \alpha)$, then $x \in \partial f(x_1^*) \cap X$ for some point $x_1^* \in U$ and $x_0^* = x_1^* + rz^*$ is in U for sufficiently small $r > 0$. We claim that

$$\begin{aligned} &\partial f(x_0^*) \\ &\quad \subset \left\{x^{**} \in X^{**} : (z^*, x^{**}) > \sigma_{\partial f(U)\cap X}(z^*) - \alpha\right\}. \end{aligned} \tag{26}$$

Indeed, if $y^{**} \in \partial f(x_0^*)$, then we have

$$0 \le (y^{**} - x, x_0^* - x_1^*) = r(y^{**} - x, z^*). \tag{27}$$

This implies that

$$y^{**} \in \left\{x^{**} \in X^{**} : (z^*, x^{**}) > \sigma_{\partial f(U)\cap X}(z^*) - \alpha\right\}. \tag{28}$$

Since the set $\{x^{**} \in X^{**} : (z^*, x^{**}) > \sigma_{\partial f(U)\cap X}(z^*) - \alpha\}$ is a weak* open set in X^{**} and since ∂f is norm-to-weak* upper semicontinuous, there exists $\delta > 0$ such that $B(x_0^*, \delta) \subset U$ and

$$\begin{aligned} &\partial f(y^*) \\ &\quad \subset \left\{x^{**} \in X^{**} : (z^*, x^{**}) > \sigma_{\partial f(U)\cap X}(z^*) - \alpha\right\} \end{aligned} \tag{29}$$

for any point $y^* \in B(x_0^*, \delta)$. Moreover, since $\partial f(y^*) \subset \partial f(U)$, we obtain that

$$\begin{aligned} &\partial f(y^*) \\ &\quad \subset \left\{x^{**} \in \partial f(U) : (z^*, x^{**}) > \sigma_{\partial f(U)\cap X}(z^*) - \alpha\right\}. \end{aligned} \tag{30}$$

Pick

$$\begin{aligned} &z_0^{**} \\ &\quad \in \left\{x^{**} \in \partial f(U) : (z^*, x^{**}) > \sigma_{\partial f(U)\cap X}(z^*) - \alpha\right\}. \end{aligned} \tag{31}$$

Since $\overline{C^{w^*}} = C^{**}$, there exists a net $\{z_\beta\}_{\beta\in\Delta} \subset C$ such that $z_\beta \xrightarrow{w^*} z_0^{**}$. Therefore, by formula (31), we obtain that $(z^*, z_0^{**}) > \sigma_{\partial f(U)\cap X}(z^*) - \alpha$. Hence we may assume that $(z^*, z_\beta) \ge \sigma_{\partial f(U)\cap X}(z^*) - \alpha$. Moreover, by formula (31), there exists $z_0^* \in U$ such that

$$(z_0^{**}, x^* - z_0^*) \le f(x^*) - f(z_0^*) \tag{32}$$

for any $x^* \in X^*$. Therefore, by $z_\beta \xrightarrow{w^*} z_0^{**}$, we obtain that

$$(z_\beta, x^* - z_0^*) \le f(x^*) - f(z_0^*) \tag{33}$$

for any $x^* \in X^*$. This implies that $z_\beta \in \partial f(U) \cap X$. Therefore, by formula $(z^*, z_\beta) \geq \sigma_{\partial f(U)\cap X}(z^*) - \alpha$, we have

$$\begin{aligned} z_\beta &\in \left\{x \in \partial f(U) \cap X : (z^*, x) \geq \sigma_{\partial f(U)\cap X}(z^*) - \alpha\right\} \\ &= S(z^*, \partial f(U) \cap X, \alpha). \end{aligned} \tag{34}$$

Therefore, by the previous proof, we obtain that

$$z_\beta \in S(z^*, \partial f(U) \cap X, \alpha) \subset x_0^* + U_n. \tag{35}$$

We claim that $\overline{U_n}^{w^*} \subset W_n$ for all $n \in N$. In fact, let $z^{**} \in \overline{U_n}^{w^*}$. Then there exists a net $\{z_\alpha\}_{\alpha\in\Delta} \subset U_n$ such that $z_\alpha \xrightarrow{w^*} z^{**}$. Hence, for any $i \in \{1, 2, \ldots, k\}$, we obtain that $x_i^*(z_\alpha) \longrightarrow x_i^*(z^{**})$. Since $|x_i^*(z_\alpha)| \leq 1/n$, we have $|x_i^*(z^{**})| \leq 1/n$. This implies that $z^{**} \in W_n$. Hence $\overline{U_n}^{w^*} \subset W_n$. Since $z_\alpha \xrightarrow{w^*} z^{**}$, by formulas (31) and (35), we have

$$z_0^{**} \in \overline{(x_0^* + U_n)}^{w^*} = x_0^* + \overline{U_n}^{w^*} \subset x_0^* + W_n. \tag{36}$$

Since z_0^{**} is arbitrary, we have $\partial f(y^*) \subset x_0^* + W_n$. It follows that $\partial f(B(x_0^*, \delta)) \subset x_0^* + W_n$. This implies that $x_0^* \in G_n \cap U$. Hence G_n is a dense open subset, which finishes the proof. □

Lemma 16. *Suppose that X is a Banach space and C is a bounded convex subset of X. Then (1) $\Longrightarrow$ (2) is true, where*

(1) for any continuous convex function f on X^*, if $\partial f(X^*) \subset \overline{C}^{w^*}$, then f has the Gâteaux differentiable points on X^*;

(2) for any weak neighborhood U of origin and $D \subset C$, there exist a slice $S(x^*, D, \alpha)$ and $x \in S(x^*, D, \alpha)$ such that $S(x^*, D, \alpha) \subset x + U$.

Proof. Suppose that there exist $D \subset C$ and a weak neighborhood U of origin such that, for any weak slice $S(x^*, D, \alpha)$ and $x \in S(x^*, D, \alpha)$, we have $S(x^*, D, \alpha) \not\subset x + U$. Since

$$\begin{aligned} \sigma_D(x^*) &= \sup\{(x, x^*) : x \in D\} \\ &= \sup\left\{(x^{**}, x^*) : x \in \overline{co^{w^*}}(D)\right\}, \end{aligned} \tag{37}$$

by formula (3) and convexity of C, we have

$$\partial\sigma_D(X^*) = \partial\sigma_D(0) \subset \overline{co^{w^*}}(D) \subset \overline{C}^{w^*}. \tag{38}$$

Hence the sublinear functional σ_D has the Gâteaux differentiable points on X^*. Since C is a bounded subset of X, we obtain that D is a bounded subset of X. Hence there exists $M > 0$ such that $\|x\| < M$ whenever $x \in D$. This implies that $\sigma_D(x^*) \leq M\|x^*\|$ for every $x \in D$. Hence σ_D is a continuous sublinear functional. Moreover, since U is a weak neighborhood of origin, there exist $\varepsilon > 0$ and $\{x_1^*, \ldots, x_k^*\} \subset X^*$ such that

$$\left\{x \in X : \bigcap_{i=1}^{k} |(x_i^*, x)| < \varepsilon\right\} \subset U. \tag{39}$$

We will show that the function σ_D is nowhere Gâteaux differentiable. Indeed, given any $x^* \in X^*$, for each slices $S(x^*, D, \varepsilon/3n)$, there exists $x_n \in S(x^*, D, \varepsilon/3n)$ such that $S(x^*, D, \varepsilon/3n) \not\subset x_n + U$. Hence there exist $y_n \in S(x^*, D, \varepsilon/3n)$ and $i \in \{1, \ldots, k\}$ such that $|(x_i^*, x_n - y_n)| \geq \varepsilon$. Otherwise, for any $y \in S(x^*, D, \varepsilon/3n)$ and $i \in \{1, \ldots, k\}$, we have $|(x_i^*, x_n - y_n)| < \varepsilon$. Hence we have

$$\begin{aligned} S\left(x^*, D, \frac{1}{3n}\varepsilon\right) &\subset \left\{x \in X : \bigcap_{i=1}^{k} |(x_i^*, x - x_n)| < \varepsilon\right\} \\ &\subset x_n + U, \end{aligned} \tag{40}$$

a contradiction. Hence we may assume without loss of generality that there exists a subsequence $\{n\}$ denoted again by $\{n\}$, such that $|(x_1^*, x_n - y_n)| \geq \varepsilon$. Moreover, we may assume without loss of generality that there exists a subsequence $\{n\}$ denoted again by $\{n\}$, such that $(x_1^*, x_n - y_n) \geq \varepsilon$. Therefore, by $x_n \in S(x^*, D, \varepsilon/3n)$ and $y_n \in S(x^*, D, \varepsilon/3n)$, we obtain that

$$\begin{aligned} (x^*, x_n) &> \sigma_D(x^*) - \frac{1}{3n}\varepsilon \\ \text{and } (x^*, x_n) &> \sigma_D(x^*) - \frac{1}{3n}\varepsilon. \end{aligned} \tag{41}$$

This implies that

$$\begin{aligned} 2\sigma_D(x^*) &< (x^*, x_n) + \frac{1}{3n}\varepsilon + (x^*, y_n) + \frac{1}{3n}\varepsilon \\ &= (x^*, x_n + y_n) + \frac{2}{3n}\varepsilon. \end{aligned} \tag{42}$$

Therefore, by formula (42), we have

$$\begin{aligned} &\frac{1}{n}\left[\sigma_D\left(x^* + \frac{1}{n}x_1^*\right) + \sigma_D\left(x^* - \frac{1}{n}x_1^*\right) - 2\sigma_D(x^*)\right] \\ &\quad \geq \frac{1}{n}\left[\left(x^* + \frac{1}{n}x_1^*, x_n\right) + \left(x^* - \frac{1}{n}x_1^*, y_n\right)\right. \\ &\quad \left. - (x^*, x_n + y_n) - \frac{2}{3n}\varepsilon\right] \geq \frac{1}{n}\left[\frac{1}{n}(x_1^*, x_n - y_n)\right. \\ &\quad \left. - \frac{2}{3n}\varepsilon\right] \geq \frac{1}{n}\left[\frac{\varepsilon}{n} - \frac{2}{3n}\varepsilon\right] = \frac{1}{3}\varepsilon. \end{aligned} \tag{43}$$

This implies that the sublinear functional σ_D is nowhere Gâteaux differentiable, a contradiction, which finishes the proof. □

Lemma 17. *Let C be a bounded subset of X. Then the following statements are equivalent.*

(1) For any weak neighborhood U of origin, there exist a slice $S(x^*, \alpha, C)$ and $x_0 \in S(x^*, \alpha, C)$ such that $S(x^*, \alpha, C) \subset x_0 + U$.

(2) For any weak neighborhood U of origin, there exists a point $x_0 \in C$ such that $x_0 \notin \overline{co}(C \backslash (x_0 + U))$.

(3) For any weak neighborhood U of origin, there exists a slice $S(x^*, \alpha, C)$ such that $x_1 - x_2 \in U$ for any $x_1, x_2 \in S(x^*, \alpha, C)$.

Proof. (3)$\Longrightarrow$(2). Let $\sigma_C(x^*) = \sup\{(x^*, y) : y \in C\}$. Then, by condition (3), it is easy to see that, for any weak neighborhood U of origin, there exists an open slice $S(x^*, \alpha, C) = \{x \in C : x^*(x) > \sigma_C(x^*) - \alpha\}$ such that $x_1 - x_2 \in U$ for any $x_1 \in S(x^*, \alpha, C)$ and $x_2 \in S(x^*, \alpha, C)$. Hence, for any weak neighborhood U of origin, there exists $x_0 \in S(x^*, \alpha, C)$ such that $S(x^*, \alpha, C) \subset x_0 + U$, it follows that $x^*(x_0) > \sigma_C(x^*) - \alpha$. Since

$$\begin{aligned} C\backslash(x_0 + U) &\subset C\backslash S(x^*, \alpha, C) \\ &\subset \{x \in C : x^*(x) \le \sigma_C(x^*) - \alpha\} \end{aligned} \tag{44}$$

and

$$\begin{aligned} &\{x \in C : x^*(x) \le \sigma_C(x^*) - \alpha\} \\ &\quad = C \cap \{x \in X : x^*(x) \le \sigma_C(x^*) - \alpha\}, \end{aligned} \tag{45}$$

we obtain that $C\backslash(x_0 + U) \subset \{x \in C : x^*(x) \le \sigma_C(x^*) - \alpha\}$. This implies that

$$\begin{aligned} \sigma_C(x^*) - \alpha &\ge \sup\{x^*(x) : x \in C\backslash(x_0 + U)\} \\ &= \sup\{x^*(x) : x \in co(C\backslash(x_0 + U))\} \\ &= \sup\{x^*(x) : x \in \overline{co}(C\backslash(x_0 + U))\}. \end{aligned} \tag{46}$$

Therefore, by $x^*(x_0) > \sigma_C(x^*) - \alpha$, we obtain that $x_0 \notin \overline{co}(C\backslash(x_0 + U))$.

(2)$\Longrightarrow$(1). For any weak neighborhood U of origin, there exists $x_0 \in C$ such that $x_0 \notin \overline{co}(C\backslash(x_0 + U))$. Therefore, by the separation theorem, there exist $x^* \in X^*$ and $r > 0$ such that

$$x^*(x_0) - r > \sup\{x^*(x) : x^* \in \overline{co}(C\backslash(x_0 + U))\}. \tag{47}$$

Let $\alpha = \sigma_C(x^*) - x^*(x_0) + r$. Then $x^*(x_0) = \sigma_C(x^*) - \alpha + r > \sigma_C(x^*) - \alpha$. This implies that $x_0 \in S(x^*, C^*, \alpha)$. Hence, for any $y \in S(x^*, C, \alpha)$, we obtain that

$$\begin{aligned} x^*(y) &> \sigma_C(x^*) - \alpha \\ &= \sigma_C(x^*) - \sigma_C(x^*) + x^*(x_0) - r \\ &= x^*(x_0) - r. \end{aligned} \tag{48}$$

Therefore, by $x^*(y) > x^*(x_0) - r$ and formula (47), we obtain that $y \in x_0 + U$.

(1)$\Longrightarrow$(3). For any weak neighborhood U of origin, there exists a weak neighborhood V of origin such that $V - V \subset U$ and there exists a slice $S(x^*, C, \alpha)$ and $x_0 \in S(x^*, C, \alpha)$ such that $S(x^*, C, \alpha) \subset x_0 + V$. Hence, if $x_1, x_2 \in S(x^*, C, \alpha)$, then $x_1 \in x_0 + V$ and $x_2 \in x_0 + V$. This implies that

$$x_1 - x_2 = (x_0 + V) - (x_0 + V) = V - V \subset U. \tag{49}$$

which finishes the proof. □

Lemma 18. *Suppose that C^{**} is a ε-separable bounded subset of X^{**}. Then the following statements are equivalent.*

(1) Every w^*-lower convex function $g \le \sigma_C$ is Gâteaux differentiable at a dense G_δ subset G of X^*.

(2) For any weak neighborhood U of origin and $D \subset \partial\sigma_C(0) \cap X$, there exist a slice $S(x^*, D, \alpha)$ and $x \in S(x^*, D, \alpha)$ such that $S(x^*, D, \alpha) \subset x + U$.

Proof. (1)$\Longrightarrow$(2). Let $p = \sigma_C$. Suppose that there exist a set $D \subset \partial p(0) \cap X$ and a weak neighborhood U of origin such that for any weak slice $S(x^*, D, \alpha)$ and $x \in S(x^*, D, \alpha)$, we obtain that $S(x^*, D, \alpha) \not\subset x + U$. Since $p = \sigma_C$, we obtain that $C^* = \{x^* \in X^* : p(x^*) \le 1\}$. Then

$$\begin{aligned} &\partial p(0) = \{x^{**} \in X^{**} : x^{**}(x^*) \le 1, x^* \in C^*\} \\ &\text{and } \overline{\partial p(0) \cap X}^{w^*} = \partial p(0). \end{aligned} \tag{50}$$

Hence

$$\begin{aligned} &\partial p(0) \cap X = \{x \in X : x^*(x) \le 1, x^* \in C^*\} \\ &\text{and } p(x^*) = \sigma_{\partial p(0)}(x^*). \end{aligned} \tag{51}$$

Moreover, by formula

$$\begin{aligned} &\overline{\{x \in X : x^*(x) \le 1, x^* \in C^*\}}^{w^*} \\ &\quad = \{x^{**} \in X^{**} : x^{**}(x^*) \le 1, x^* \in C^*\}, \end{aligned} \tag{52}$$

we obtain that

$$\begin{aligned} p(x^*) &= \sigma_{\partial p(0)}(x^*) = \sigma_{\partial p(0) \cap X}(x^*) \\ &= \sup\{x^*(x) : x \in \partial p(0) \cap X\}. \end{aligned} \tag{53}$$

Since $D \subset \partial p(0) \cap X$, we obtain that

$$\begin{aligned} \sigma_{\partial p(0) \cap X}(x^*) &= \sup\{x^*(x) : x \in \partial p(0) \cap X\} \\ &\ge \sup\{x^*(x) : x \in D\} = \sigma_D(x^*). \end{aligned} \tag{54}$$

Therefore, by formula $\sigma_C = \sigma_{\partial p(0) \cap X} \ge \sigma_D$, we obtain that σ_D is Gâteaux differentiable at a dense G_δ subset G of X^*. However, by the proof of Lemma 16, we obtain that σ_D is nowhere Gâteaux differentiable, a contradiction.

(2)$\Longrightarrow$(1). Let g be a w^*-lower semicontinuous convex function on X^* and $g \le p$. Then g is a continuous function. We claim that $\partial g(X^*) \subset \partial p(0)$. In fact, suppose that there exists a point $x^{**} \in \partial g(x^*)$ such that $x^{**} \notin \partial p(0)$. Then, by the separation theorem, there exists a point $x^* \in X^*$ and a real number $r > 0$ such that

$$\begin{aligned} x^{**}(z^*) &> r + \sup\{z^*(z^{**}) : z^{**} \in \partial p(0)\} \\ &\ge r + p(z^*). \end{aligned} \tag{55}$$

Since $x^{**} \in \partial g(x^*)$, we obtain that $g(y^*) - g(x^*) \ge x^{**}(y^* - x^*)$ for all $y^* \in X^*$. Let $y^* = kz^*$. Then

$$\begin{aligned} g(kz^*) - g(x^*) &\ge x^{**}(kz^*) - x^{**}(x^*) \\ &> \frac{1}{2}kr + kp(z^*) - x^{**}(x^*). \end{aligned} \tag{56}$$

This implies that $g(kz^*) > p(kz^*)$ for all sufficiently large $k > 0$, which contradicts formula $g \le p$. Therefore, by Lemma 16, we obtain that g is Gâteaux differentiable at a dense G_δ subset G of X^*, which finishes the proof. □

Proof of Theorem 14. By Lemmas 15–18, we obtain that (1) $\Longleftrightarrow$ (2) is true. (1) $\Longrightarrow$ (3). Let $p = \sigma_C$ and D be a weak* closed convex subset of $\partial p(0)$. Then $p \geq \sigma_D$. This implies that $\sigma_{D'}$ is Gâteaux differentiable at a dense G_δ subset for any $D' \subset D$. Therefore, by Theorem 2 of [3], we obtain that D is the weak* closed convex hull of its weak* exposed points.

(3) $\Longrightarrow$ (2). Let D be a closed convex subset of $\partial\sigma_C(0) \cap X$ and $V \subset X$ be a weak neighborhood of origin. Since C^{**} is bounded, by the definition of C^{**}, we obtain that C is bounded and C^{**} is a weak* bounded closed subset of X^{**}. Hence D is bounded and D^{**} is a weak* bounded closed subset of X^{**}. Since V is a weak neighborhood of origin, we may assume that there exist $\{x_1^*, x_2^*, \ldots, x_k^*\} \subset S(X^*)$ and $\varepsilon > 0$ such that

$$V = \bigcap_{i=1}^{k} \left\{ x \in X : \left|\left(x_i^*, x\right)\right| < \frac{1}{2}\varepsilon \right\}. \tag{57}$$

Let

$$V_0 = \bigcap_{i=1}^{k} \left\{ x^{**} \in X^{**} : \left|\left(x_i^*, x^{**}\right)\right| < \frac{1}{2}\varepsilon \right\}. \tag{58}$$

Then V_0 is a weak* neighborhood of origin in X^{**}. Since D^{**} is a weak* bounded closed subset of X^{**} and $D^{**} \subset \partial\sigma_C(0)$, there exists $z_0^{**} \in D^{**}$ such that z_0^{**} is a weak* exposed point of D^{**}. Hence there exists $x_0^* \in S(X^*)$ such that

$$\begin{aligned} z_0^{**}\left(x_0^*\right) &= \sup\left\{x^{**}\left(x_0^*\right) : x^{**} \in D^{**}\right\} \\ \text{and } y^{**}\left(x_0^*\right) &< \sup\left\{x^{**}\left(x_0^*\right) : x^{**} \in D^{**}\right\} \end{aligned} \tag{59}$$

for each $y^{**} \in D^{**}\backslash\{z_0^{**}\}$. We claim that for any weak* neighborhood U of origin, there exists $k > 0$ such that

$$\begin{aligned} &z_0^{**}\left(x_0^*\right) - 4k \\ &\quad \geq \sup\left\{x^{**}\left(x_0^*\right) : x^{**} \in D^{**}\backslash\left(z_0^{**} + U\right)\right\}. \end{aligned} \tag{60}$$

In fact, suppose that there exists a sequence $\{x_n^{**}\}_{n=1}^{\infty} \subset D^{**}\backslash(z_0^{**} + U)$ such that $x_n^{**}(x_0^*) \longrightarrow \sup\{x^{**}(x_0^*) : x^{**} \in D^{**}\}$ as $n \longrightarrow \infty$. Then we may assume without loss of generality that $x_n^{**} \neq x_m^{**}$ for any $m \neq n$. Since D^{**} is a weak* bounded closed convex set of X^{**}, we obtain that D^{**} is weak* compact. Then there exists a point $x_0^{**} \in D^{**}$ such that x_0^{**} is a weak* accumulation point of $\{x_n^{**}\}_{n=1}^{\infty}$. Put

$$\Delta = \left\{U_{x_0^{**}} : U_{x_0^{**}} \text{ is weak}^* \text{ neighborhood of } x_0^{**}\right\}. \tag{61}$$

Hence we define an order by the containing relations, i.e., $U_{x_0^{**}} \supset V_{x_0^{**}}$ if and only if $V_{x_0^{**}} > U_{x_0^{**}}$. This implies that Δ is an order set. Hence

$$\begin{aligned} \Omega = \Big\{ &U_{x_0^{**}} \cap \{x_n^{**}\}_{n=1}^{\infty} : \\ &U_{x_0^{**}} \text{ is weak}^* \text{ neighbourhood of } x_0^{**} \Big\} \end{aligned} \tag{62}$$

is an order set whenever $V_{x_0^{**}} \cap \{x_n^{**}\}_{n=1}^{\infty} > U_{x_0^{**}} \cap \{x_n^{**}\}_{n=1}^{\infty}$ if and only if $U_{x_0^{**}} \cap \{x_n^{**}\}_{n=1}^{\infty} \supset V_{x_0^{**}} \cap \{x_n^{**}\}_{n=1}^{\infty}$. Therefore, by the Zermelo Lemma, we obtain that there exists a mapping f on Ω such that $f(U_{x_0^{**}} \cap \{x_n^{**}\}_{n=1}^{\infty}) \in U_{x_0^{**}} \cap \{x_n^{**}\}_{n=1}^{\infty}$. Put $x_\alpha^{**} = f(U_{x_0^{**}} \cap \{x_n^{**}\}_{n=1}^{\infty})$. Hence we define a net $\{x_\alpha^{**}\}_{\alpha\in\Delta} \subset \{x_n^{**}\}_{n=1}^{\infty}$. Therefore, by the definition of net $\{x_\alpha^{**}\}_{\alpha\in\Delta}$, we have

$$\begin{aligned} &x_0^{**}\left(x_0^*\right) = \sup\left\{x^{**}\left(x_0^*\right) : x^{**} \in D^{**}\right\} \\ &\text{and } x_\alpha^{**} \xrightarrow{w^*} x_0^{**}, \end{aligned} \tag{63}$$

which contradicts $\{x_\alpha^{**}\}_{\alpha\in\Delta} \subset \{x_n^{**}\}_{n=1}^{\infty} \subset D^{**}\backslash(z_0^{**} + U)$. Moreover, since V_0 is a weak* neighborhood of origin in X^{**}, there exists weak* neighborhood W of origin, such that $W + W \subset V_0$. Since there exists $k > 0$ such that

$$\begin{aligned} &z_0^{**}\left(x_0^*\right) - 4k \\ &\quad \geq \sup\left\{x^{**}\left(x_0^*\right) : x^{**} \in D^{**}\backslash\left(z_0^{**} + W\right)\right\}, \end{aligned} \tag{64}$$

by formula $\overline{D}^{w^*, X^{**}} = D^{**} \subset X^{**}$, there exists $x \in D$ such that $z_0^{**} \in x + W$ and $|z_0^{**}(x_0^*) - x(x_0^*)| < k$. Then

$$\begin{aligned} &x\left(x_0^*\right) - 3k \\ &\quad \geq \sup\left\{x^{**}\left(x_0^*\right) : x^{**} \in D^{**}\backslash\left(z_0^{**} + W\right)\right\}. \end{aligned} \tag{65}$$

Therefore, by $z_0^{**} \in x + W$, we have $z_0^{**} + W \subset x + W + W \subset x + V_0$. Moreover, it is easy to see that $D\backslash(x+V) \subset D^{**}\backslash(x+V_0)$. Then

$$\begin{aligned} x\left(x_0^*\right) - 3k &\geq \sup\left\{x^{**}\left(x_0^*\right) : x^{**} \in D^{**}\backslash\left(z_0^{**} + W\right)\right\} \\ &\geq \sup\left\{x^{**}\left(x_0^*\right) : x^{**} \in D^{**}\backslash\left(x + V_0\right)\right\} \\ &\geq \sup\left\{z\left(x_0^*\right) : z \in D\backslash\left(x + V\right)\right\}. \end{aligned} \tag{66}$$

This implies that

$$\begin{aligned} x\left(x_0^*\right) - 3k &\geq \sup\left\{z\left(x_0^*\right) : z \in D\backslash\left(x + V\right)\right\} \\ &= \sup\left\{z\left(x_0^*\right) : z \in co\left(D\backslash\left(y + V\right)\right)\right\} \\ &= \sup\left\{z\left(x_0^*\right) : z \in \overline{co}\left(D\backslash\left(x + V\right)\right)\right\}. \end{aligned} \tag{67}$$

Hence we obtain that $x \notin \overline{co}(D\backslash(x + V))$. This implies that D is weak dentable, which finishes the proof. □

Theorem 19. *Suppose that X^* is a Gâteaux differentiable space. Then every bounded subset of X is weak dentable.*

Proof. By the proof of Theorem 14, we obtain that every closed convex subset of X is weak dentable. Let C be a bounded subset of X. Suppose that C is not weak dentable. Then, by Lemma 17, there exists a weak neighborhood U of origin such that $S(x^*, \alpha, C) - S(x^*, \alpha, C) \not\subset U$ for any $x^* \in X^*$ and $\alpha > 0$. Since $\sigma_C(x^*) = \sigma_{\overline{co}(C)}(x^*)$ for any $x^* \in X^*$, we have $S(x^*, \alpha, C) \subset S(x^*, \alpha, \overline{co}(C))$. This implies that

$$\begin{aligned} &S\left(x^*, \alpha, \overline{co}\left(C\right)\right) - S\left(x^*, \alpha, \overline{co}\left(C\right)\right) \\ &\quad \supset S\left(x^*, \alpha, C\right) - S\left(x^*, \alpha, C\right) \not\supset U. \end{aligned} \tag{68}$$

Therefore, by Lemma 17, we obtain that $\overline{co}(C)$ is not weak dentable, a contradiction, which finishes the proof. □

Example 20. Let $X = c_0$. Then $X^* = l^1$. Since l^1 is separable, by Theorem 19, we obtain that every bounded subset of c_0 is weak dentable. Moreover, it is well known that c_0 has not the Radon-Nikodym property. Hence there exists a bounded subset D of c_0 such that D is not dentable.

Example 21. Let $X = c_0$. Then $X^* = l^1$ is separable and $X^{**} = l^\infty$ is not a separable space. Let Y be a Banach space and Y be not a separable space. Define $C = B(X) \times \{0\} \subset X \times Y$. Then

$$\begin{aligned} C^* &= B(X^*) \times Y^* \subset X^* \times Y^* \\ \text{and } C^{**} &= B(X^{**}) \times \{0\} \subset X^{**} \times Y^{**}. \end{aligned} \tag{69}$$

By Theorem 19, we obtain that C is weak dentable. By Example 13, we obtain that C^{**} is ε-separable and bounded. Therefore, by Theorem 14, we obtain that every convex function $g \le \sigma_C$ is Gâteaux differentiable at a dense G_δ subset G of X^*.

3. Gateaux Differentiability and Weakly Exposed Point

Definition 22. A point $x_0 \in C$ is said to be weakly exposed point of C if there exist $x^* \in S(X^*)$ and $\{x_n\}_{n=1}^\infty \subset C$ such that $x^*(x_n) \longrightarrow \sigma_C(x^*)$; then $x_n \xrightarrow{w} x_0$.

Definition 23. A point $x_0 \in C$ is said to be exposed point of C if there exists $x^* \in S(X^*)$ such that $x^*(x_0) > x^*(x)$ whenever $x \in C \backslash \{x_0\}$.

Definition 24. A point $x_0 \in C$ is said to be strongly exposed point of C if there exist $x^* \in S(X^*)$ and $\{x_n\}_{n=1}^\infty \subset C$ such that $x^*(x_n) \longrightarrow \sigma_C(x^*)$; then $x_n \longrightarrow x_0$.

Definition 25. A point $x \in A$ is said to be an extreme point of A if $2x = y + z$ and $y, z \in A$ imply $y = z$. The set of all extreme points of A is denoted by $ExtA$. If $ExtB(X) = S(X)$, then X is said to be a strictly convex space.

It is easy to see that if x is a strongly exposed point of C, then x is a weakly exposed point of C and if x is a weakly exposed point of C, then x is a exposed point of C. Moreover, weakly exposed point, exposed point, and strongly exposed point are different. We will give two examples in Sections 3 and 4. A Banach space X is said to have the Krein-Milman property if every bounded closed convex subset of X is the closed convex hull of its extreme points. It is well known that if X has the Radon-Nikodym property, then X has the Krein-Milman property. Moreover, we know that X^* has the Krein-Milman property if and only if X^* has the Radon-Nikodym property (see [12]).

Theorem 26. *Suppose that C is a bounded closed convex set. Then $d\sigma_C(x^*) = x$ if and only if x is a weakly exposed point of C exposed by x^*.*

Proof. Necessity. Let $p(x^*) = \sigma_C(x^*)$ and $d\sigma_C(x^*) = x$. Then we have $x \in C^{**} = \partial p(0)$. Therefore, by Lemma 1 of [1], we obtain that x^* exposes $\partial p(0)$ at x; i.e., $x^*(x) = \sigma_C(x^*)$. Let $\{x_n\}_{n=1}^\infty \subset C$ and $x^*(x_n) \longrightarrow \sigma_C(x^*)$ as $n \longrightarrow \infty$. We next will prove that $x_n \xrightarrow{w} x$ as $n \longrightarrow \infty$. In fact, we may assume without loss of generality that $x_n \ne x_m$ for any $m \ne n$. Since C is a bounded closed convex set, we obtain that C^{**} is a bounded set. Hence we obtain that $\partial p(0) = C^{**}$ is a bounded set. This implies that $\partial p(0)$ is a weak* compact set in X^{**}. Hence there exists $x_0^{**} \in \partial p(0)$ such that x_0^{**} is a weak* accumulation point of $\{x_n\}_{n=1}^\infty$. Hence there exists a net $\{x_\alpha\}_{\alpha\in\Delta} \subset \{x_n\}_{n=1}^\infty$ such that

$$\begin{aligned} x_0^{**}(x^*) &= \sup\{x^{**}(x^*) : x^{**} \in \partial p(0)\} \\ \text{and } x_\alpha &\xrightarrow{w^*} x_0^{**}. \end{aligned} \tag{70}$$

This implies that $x_0^{**} \in \partial p(x^*)$. Since $dp(x^*) = x$, we have $x_0^{**} = x$. Moreover, by $x_\alpha \xrightarrow{w^*} x_0^{**}$, we obtain that $x_\alpha \xrightarrow{w} x$.

Suppose that $\{x_n\}_{n=1}^\infty$ does not converge weakly to x. Then there exist a weak neighbourhood V of x and a subsequence $\{n_k\}$ of $\{n\}$ such that $x_{n_k} \notin V$. Repeat the previous proof; there exists a net $\{x_\beta\}_{\beta\in\Delta} \subset \{x_{n_k}\}_{k=1}^\infty$ such that $x_\beta \xrightarrow{w} x$, a contradiction. Hence we have $x_n \xrightarrow{w} x$ as $n \longrightarrow \infty$. This implies that x is a weakly exposed point of C.

Sufficiency. Suppose that there exists $x^{**} \in C^{**}$ such that $x^{**}(x^*) = x(x^*)$. Since x is a weakly exposed point of C and $\overline{C^{w^*}} = C^{**}$, we have

$$\begin{aligned} x^{**}(x^*) = x(x^*) &= \sup\{y(x^*) : y \in C\} \\ &= \sup\{y^{**}(y^*) : y^{**} \in C^{**}\}. \end{aligned} \tag{71}$$

We next will prove that $x^{**} = x$. Suppose that $x^{**} \ne x$. Then there exists a weak* neighborhood V of origin such that $(x^{**} + V) \cap (x + V) = \emptyset$. Let

$$U_n = \left\{ y^{**} \in X^{**} : \left| y^{**}(x^*) - x^{**}(x^*) \right| < \frac{1}{n} \right\}. \tag{72}$$

Then, by $(x^{**} + V) \cap (x + V) = \emptyset$, we obtain that $(x^{**} + (V \cap U_n)) \cap (x+V) = \emptyset$. Moreover, by $x^{**} \in C^{**}$ and $\overline{C^{w^*}} = C^{**}$, we obtain that there exists $x_n \in x^{**} + (V \cap U_n)$ such that $x_n \in C$. Hence we have $x^*(x_n) \longrightarrow x^*(x) = \sup\{x^*(y) : y \in C\}$. Since x is a weakly exposed point of C and exposed by x^*, by formula

$$\lim_{n\longrightarrow\infty} x^*(x_n) = x^*(x) = \sup\{x^*(y) : y \in C\}, \tag{73}$$

we obtain that $x_n \xrightarrow{w} x$, which contradicts $(x^{**} + (V \cap U_n)) \cap (x + V) = \emptyset$. Hence $x^{**} = x$. This implies that $x \in C^{**}$ is a weak* exposed point of C^{**} exposed by x^*. Therefore, by Lemma 1 of [3], we obtain that $d\sigma_C(x^*) = x$, which completes the proof. □

Lemma 27 (see [1]). *Suppose that $x^* \in S(X^*)$, $y^* \in S(X^*)$, and $\varepsilon > 0$. If $|y^*(x)| \le 1$ whenever $x \in X$ satisfies*

$$\begin{aligned} x^*(x) &= 0 \\ \text{and } \|x\| &\le 2\varepsilon^{-1}, \end{aligned} \tag{74}$$

then either $\|x^* - y^*\| \le \varepsilon$ or $\|x^* + y^*\| \le \varepsilon$.

Theorem 28. *The following statements are equivalent:*

(1) X is an Asplund space.

(2) X^* has the Radon-Nikodym property.

(3) Every bounded closed convex subset of X^* is the closed convex hull of its weakly exposed points.

(4) For every bounded closed convex set C^* of X^*, there exists $x^{**} \in X^{**}$ such that σ_{C^*} is Gâteaux differentiable at x^{**} and $d\sigma_{C^*}(x^{**}) \in C^*$.

(5) For every bounded closed convex set C^* of X^*, there exists a dense subset G of X^{**} such that σ_{C^*} is Gâteaux differentiable on G and $d\sigma_{C^*}(G) \subset C^*$.

Proof. It is well known that (1)$\Longleftrightarrow$(2) and (2)$\Longrightarrow$(3) are true. Moreover, by Theorem 26, it is easy to see that (3)$\Longrightarrow$(4) is true. (3)$\Longrightarrow$(2). Suppose that every bounded closed convex subset of X^* is the closed convex hull of its weakly exposed points. Then every bounded closed convex subset of X^* is the closed convex hull of its extreme points. Hence X^* has the Krein-Milman property. This implies that X^* has the Radon-Nikodym property.

(4)$\Longrightarrow$(2). Let C^* be a bounded closed convex set of X^* and J be the closed convex hull of the extreme points of C^*. Suppose that $J \neq C^*$. Then, by the separation theorem and the Bishop-Phelps Theorem, there exist $x_0^{**} \in X^{**}$ and $x_0^* \in X^*$ such that

$$\begin{aligned} &\sup\{x_0^{**}(x^*) : x^* \in C^*\} = x_0^{**}(x_0^*) \\ &\quad > \sup\{x_0^{**}(x^*) : x^* \in J\}. \end{aligned} \tag{75}$$

Then

$$D = \{x \in X : x_0^{**}(x^*) = \sup\{x_0^{**}(y^*) : y^* \in C^*\}\} \tag{76}$$

is a nonempty bounded closed convex set; by Theorem 26, there exists $x_1^* \in D^*$ such that x_1^* is a weakly exposed point of D^*. Then x_1^* is an extreme point of D^*. Let $2x_1^* = y_1^* + z_1^*$, $y_1^* \in C^*$, and $z_1^* \in C^*$. Then

$$\begin{aligned} x_0^{**}(2x_1^*) &= x_0^{**}(y_1^* + z_1^*) = x_0^{**}(y_1^*) + x_0^{**}(z_1^*) \\ &= 2\sup\{x_0^{**}(y^*) : y \in C^*\}. \end{aligned} \tag{77}$$

This implies that

$$x_0^{**}(y_1^*) = x_0^{**}(z_1^*) = \sup\{x_0^{**}(y^{**}) : y^{**} \in C^{**}\}. \tag{78}$$

Hence $y_1^* \in D^*$ and $z_1^* \in D^*$. Since x_1^* is an extreme point of D^*, we have $y_1^* = z_1^*$. This implies that x_1^* is an extreme point of C^*, which contradicts $J \neq C^*$. Hence every bounded closed convex subset of X^* is the closed convex hull of its extreme points. This implies that X^* has the Krein-Milman property. Hence X^* has the Radon-Nikodym property.

It is easy to see that (5)$\Longrightarrow$(4) is true. We next will prove that (4)$\Longrightarrow$(5) is true. Let C^* be a bounded closed convex subset of X^* and C^* be not a singleton. Then we may assume without loss of generality that $C^* \subset B(X^*)$ and $0 \in C^*$. Let $\|x^{**}\| = 1$ and $\varepsilon \in (0, 1/8)$. Then we define the bounded closed convex set $C_1^* = \overline{co}^w(C^* \cup N)$, where

$$N = \left\{x^* \in X^* : (x^*, x^{**}) = 0,\ \|x^*\| \le 2\varepsilon^{-1}\right\}. \tag{79}$$

Then, by $C^* \subset B(X^*)$ and $\varepsilon \in (0, 1/8)$, we have $C_1^* \subset B(X^*)$. Moreover, we may assume without loss of generality that $\sigma_{C_1^*}(x^{**}) - \varepsilon > 0$. Let $\alpha = \sigma_{C_1^*}(x^{**}) - \varepsilon$. By hypothesis, we obtain that C_1^* is the closed convex hull of its weak exposed points, so any slice of C_1^* contains a weakly exposed point of C^*. Hence there exist $y^* \in S(x^{**}, C_1^*, \alpha)$ and $y^{**} \in S(X^{**})$ such that y^* is a weakly exposed point of C_1^* and y^{**} is an exposed functional.

We claim that $y^* \in C^* \cup N$. In fact, suppose that $y^* \notin C^* \cup N$. Since C^* and N are weak closed, we obtain that $C^* \cup N$ is weak closed. Therefore, by $y^* \notin C^* \cup N$, there exists a weak neighborhood U of origin such that $(y^* + U) \cap (C^* \cup N) = \emptyset$. Moreover, by $0 \in C^*$, we may assume without loss of generality that $y^{**}(y^*) > 0$. Therefore, by $C_1^* = \overline{co}^w(C^* \cup N)$, there exists a sequence $\{y_n^*\}_{n=1}^{\infty} \subset (C^* \cup N)$ such that $y^{**}(y_n^*) \longrightarrow \sigma_{C_1^*}(y^{**})$ as $n \longrightarrow \infty$. Since y^* is a weakly exposed point of C_1^* and y^{**} is an exposed functional, we have $y_n^* \xrightarrow{w} y^*$ as $n \longrightarrow \infty$, which contradicts $(y^* + U) \cap (C^* \cup N) = \emptyset$. Hence we have $y^* \in C^* \cup N$. Since $y^* \in S(x^{**}, C_1^*, \alpha)$ and $\alpha = \sigma_{C_1^*}(x^{**}) - \varepsilon$, we have

$$\begin{aligned} x^{**}(y^*) &\ge \sigma_{C_1^*}(x^{**}) - \alpha \\ &= \sigma_{C_1^*}(x^{**}) - \left[\sigma_{C_1^*}(x^{**}) - \varepsilon\right] = \varepsilon. \end{aligned} \tag{80}$$

This implies that $y^* \notin N$. Therefore, by formula $y^* \in C^* \cup N$, we have $y^* \in C^*$. Let $x^* \in N \subset C_1^*$. Then, by $C_1^* \subset B(X^*)$, we have

$$(x^*, y^{**}) \le \sigma_{C_1^*}(y^{**}) = (y^{**}, y^*) \le \|y^{**}\|\,\|y^*\| \le 1. \tag{81}$$

Therefore, by symmetry of N, we obtain that $|(x^*, y^{**})| \le 1$ for all $x^* \subset N \subset C_1^*$. Therefore, by Lemma 27, we obtain that either $\|x^{**} - y^{**}\| \le \varepsilon$ or $\|x^{**} + y^{**}\| \le \varepsilon$. Since $\|y^*\| \le 1$, we have

$$\|x^{**} + y^{**}\| \ge (y^*, x^{**} + y^{**}) > (y^*, x^{**}) \ge \varepsilon. \tag{82}$$

This implies that $\|x^{**} - y^{**}\| \le \varepsilon$. Therefore, by Theorem 26, there exists a dense subset G of X^{**} such that σ_{C^*} is Gâteaux differentiable on G and $d\sigma_{C^*}(G) \subset C^*$, which finishes the proof. □

Moreover, by the proof of Theorems 26 and 28, it is easy to see the following.

Corollary 29. *A Banach space X is an Asplund space if and only if for every bounded closed convex set C^* of X^*, there exists a dense subset G of X^{**} such that σ_{C^*} is Gâteaux differentiable on G and $d\sigma_{C^*}(G) \subset X^*$.*

Theorem 30. *A Banach space X is an Asplund space if and only if, for every w^*-lower semicontinuous convex function f, there exists a dense subset G of X^{**} such that f is Gâteaux differentiable on G and $df(G) \subset X^*$.*

Proof. Noticing the proof of Theorem 2.1 of [9] and Corollary 29, it is easy to see that Theorem 30 is true, which finishes the proof. □

Theorem 31. *Let X^* be a weak Asplund space and continuous convex function f be above bounded in weak* neighborhood U. Then f is Gâteaux differentiable at a dense G_δ subset G of X^* and $df(G) \subset X$.*

In order to prove the theorem, we give a lemma.

Lemma 32. *Suppose that graph of convex function f has an interior point in $(X^* \times R, w^*)$. Then for any $x^* \in X^*$, we have $\partial f(x^*) \subset X$.*

Proof. Since continuous convex function f has an interior point in $(X^* \times R, w^*)$, we obtain that $w^* - \operatorname{int}(epif) \neq \emptyset$. Pick $x^* \in X^*$. Then it is easy to see that $(x^*, f(x^*)) \notin w^* - \operatorname{int}(epif)$. Therefore, by the separation theorem, there exists $(x, r) \in (X \times R)\backslash(0,0)$ such that

$$\begin{aligned} &x^*(x) + rf(x^*) \\ &\quad \geq \sup\{z^*(x) + r\xi : (z^*, \xi) \in w^* - \operatorname{int}(epif)\}. \end{aligned} \tag{83}$$

Hence, for any $y^* \in X^*$, we have $x^*(x) + rf(x^*) \geq y^*(x) + rf(y^*)$ and $x^*(x) + rf(x^*) \geq y^*(x) + r(f(y^*) + 1)$. Let $y^* = x^*$. Then $rf(x^*) \leq r(f(x^*) + 1)$. This implies that $r \leq 0$. Suppose that $r = 0$. Then, for any $y^* \in X^*$, we obtain that $x^*(x) \geq y^*(x)$. This implies that $x = 0$. Hence we have $(x, r) = (0,0)$, a contradiction. This implies that $r < 0$. Hence we may assume without loss of generality that $r = -1$. This implies that $x^*(x) - f(x^*) \geq y^*(x) - f(y^*)$. Hence $x \in \partial f(x^*)$. This implies that $\partial f(x^*) \neq \emptyset$ for any $x^* \in X^*$.

Let $x^{**} \in \partial f(x^*)$. We next will prove that $x^{**} \in X$. Since $x^{**} \in \partial f(x^*)$, we obtain that $(x^{**}, y^* - x^*) \leq f(y^*) - f(x^*)$. This implies that $x^{**}(y^*) - f(y^*) \leq x^{**}(x^*) - f(x^*)$. Hence, for any $(y^*, t) \in w^* - \operatorname{int}(epif)$, we have

$$\begin{aligned} x^{**}(y^*) - t &\leq x^{**}(y^*) - f(y^*) \\ &\leq x^{**}(x^*) - f(x^*). \end{aligned} \tag{84}$$

This implies that

$$\begin{aligned} &x^{**}(x^*) - f(x^*) \\ &\quad \geq \sup\{x^{**}(y^*) - t : (y^*, t) \in w^* - \operatorname{int}(epif)\}. \end{aligned} \tag{85}$$

We next will prove that the functional x^{**} is a continuous functional of (X^*, w^*). Suppose that x^{**} is not continuous at origin. Then there exist a net $\{x_\alpha^*\}_{\alpha\in\Delta} \subset X^*$ and $r > 0$ such that $x_\alpha^* \xrightarrow{w^*} 0$ and $|x^{**}(x_\alpha^*)| > r$. Pick $x_0^* \in \{x_\alpha^*\}_{\alpha\in\Delta}$. Then,

$$\begin{aligned} &\frac{x_\alpha^*}{x^{**}(x_\alpha^*)} - \frac{x_0^*}{x^{**}(x_0^*)} \xrightarrow{w^*} \left(-\frac{x_0^*}{x^{**}(x_0^*)}\right), \\ &\left(x^{**}, -\frac{x_0^*}{x^{**}(x_0^*)}\right) = -1 \end{aligned} \tag{86}$$

and

$$\left(x^{**}, \frac{x_\alpha^*}{x^{**}(x_\alpha^*)} - \frac{x_0^*}{x^{**}(x_0^*)}\right) = 0. \tag{87}$$

This implies that the hyperplane $\{h^* \in X^* : x^{**}(h^*) = 0\}$ is not a weak* closed set. Pick

$$\begin{aligned} z^* \in \overline{\{h^* \in X^* : x^{**}(h^*) = 0\}}^{w^*} \backslash \{h^* \in X^* : x^{**}(h^*) \\ = 0\}. \end{aligned} \tag{88}$$

Then

$$\{\lambda z^* : \lambda \in R\} \subset \overline{\{h^* \in X^* : x^{**}(h^*) = 0\}}^{w^*}. \tag{89}$$

Hence we have

$$\begin{aligned} X^* &= \{\lambda z^* : \lambda \in R\} \oplus \{h^* \in X^* : x^{**}(h^*) = 0\} \\ &\subset \overline{\{h^* \in X^* : x^{**}(h^*) = 0\}}^{w^*}. \end{aligned} \tag{90}$$

This implies that

$$X^* = \overline{\{h^* \in X^* : x^{**}(h^*) = 0\}}^{w^*}. \tag{91}$$

Moreover, there exist a weak* open set U of X^* and an open interval (a, b) such that $U \times (a, b) \subset w^* - \operatorname{int}(epif)$. Pick $h_0^* \in \{h^* \in X^* : x^{**}(h^*) = 0\}$. Then

$$\begin{aligned} X^* &= h_0^* + X^* = h_0^* + \overline{\{h^* \in X^* : x^{**}(h^*) = 0\}}^{w^*} \\ &= \overline{h_0^* + \{h^* \in X^* : x^{**}(h^*) = 0\}}^{w^*} \\ &= \overline{\{h^* \in X^* : x^{**}(h^*) = x^{**}(x^*) - f(x^*) + b\}}^{w^*}. \end{aligned} \tag{92}$$

Therefore, by formula (92), there exists $x_0^* \in U$ such that $x^{**}(x_0^*) = x^{**}(x^*) - f(x^*) + b$. Pick $r \in (a, b)$. Then $(x_0^*, r) \in U \times (a, b) \subset w^* - \operatorname{int}(epif)$. Therefore, by formula (85), we have

$$\begin{aligned} x^{**}(x^*) - f(x^*) &\geq x^{**}(x_0^*) - r \\ &= x^{**}(x^*) - f(x^*) + (b - r). \end{aligned} \tag{93}$$

This implies that $r \geq b$, a contradiction. Hence we obtain that x^{**} is continuous at origin. This implies that x^{**} is a continuous functional of (X^*, w^*). Since $(X^*, w^*)^* = X$, we have $x^{**} \in X$, which finishes the proof. □

Proof of Theorem 31. Let U be a weak* neighbourhood and f is above bounded on U. Then we may assume without loss of generality that $f(x^*) < 0$ whenever $x^* \in U$. Then $U \times (0,1) \subset epif$. Therefore, by Lemma 32, we have $\partial f(x^*) \subset X$ for any $x^* \in X^*$. Since X^* is a weak Asplund space, we obtain that f is Gâteaux differentiable at a dense G_δ subset G of X^*. Hence convex function f is Gâteaux differentiable at a dense G_δ subset G of X^* and $df(G) \subset X$, which finishes the proof. □

Definition 33. A point $x \in S(X)$ is called a smooth point if it has an unique supporting functional f_x. If every $x \in S(X)$ is a smooth point, then X is called a smooth space.

Definition 34. A Banach space X is said to have the H-property if $\{x_n\}_{n=1}^\infty \subset S(X)$, $x \in S(X)$, and $x_n \xrightarrow{w} x$; then $x_n \longrightarrow x$ as $n \longrightarrow \infty$.

Example 35. It is well known that there exists a Banach space X such that X is reflexive and strictly convex and does not have the H-property. Then it is easy to see that there exists $x \in B(X)$ such that x is a weakly exposed point of $B(X)$ and not a strongly exposed point of $B(X)$.

4. Some Examples in Orlicz Function Spaces

Definition 36. $M : R \longrightarrow R$ is called a N-function if it has the following properties:

(1) M is even, continuous, convex and $M(0) = 0$.

(2) $M(u) > 0$ for all $u \neq 0$.

(3) $\lim_{u \longrightarrow 0} M(u)/u = 0$ and $\lim_{u \longrightarrow \infty} M(u)/u = \infty$.

Let M be a N−function and (G, Σ, μ) be a finite nonatomic measure space. Let $p(u)$ denote the right derivative of $M(u)$ and $q(v)$ be the generalized inverse function of $p(u)$ by

$$q(v) = \sup_{u \geq 0} \{u \geq 0 : p(u) \leq v\}. \tag{94}$$

Then the function $N(v)$ defined by $N(v) = \int_0^{|v|} q(s)ds$ for any $v \in R$ is called the complementary function to M in the sense of Young. We define the modular of x by

$$\rho_M(x) = \int_G M(x(t))\, dt. \tag{95}$$

Let us define the Orlicz function space L_M by

$$\begin{aligned} L_M &= \{x(t) : \rho_M(\lambda x) < \infty \text{ for some } \lambda > 0\} \\ E_M &= \{x(t) : \rho_M(\lambda x) < \infty \text{ for all } \lambda > 0\}. \end{aligned} \tag{96}$$

It is well known that L_M and E_M are Banach spaces when it is equipped with the Luxemburg norm

$$\|x\| = \inf\left\{\lambda > 0 : \rho_M\left(\frac{x}{\lambda}\right) \leq 1\right\} \tag{97}$$

or equipped with the Orlicz norm

$$\|x\|^0 = \inf_{k>0} \frac{1}{k}\left[1 + \rho_M(kx)\right]. \tag{98}$$

L_M,E_M denote Orlicz spaces equipped with the Luxemburg norm. L_M^0,E_M^0 denote Orlicz spaces equipped with the Orlicz norm.

Definition 37 (see [12]). We say that an N-function M satisfies condition Δ_2 if there exist $K > 2$ and $u_0 \geq 0$ such that $M(2u) \leq KM(u)$ whenever $u \geq u_0$.

It is well known that $(E_N)^* = L_M^0$ and $(E_N^0)^* = L_M$ (see [12]). Moreover, it is well known that $E_M = L_M$ if and only if $M \in \Delta_2$.

Theorem 38 (see [12]). *Orlicz space E_M is smooth if and only if p is continuous.*

Theorem 39 (see [12]). *Orlicz space L_M is strictly convex if and only if M is strictly convex and $M \in \Delta_2$.*

Theorem 40 (see [12]). *Orlicz space $E_M(E_M^0)$ has the Radon-Nikodym property if and only if $M \in \Delta_2$.*

Example 41. There exist a bounded set $C \subset X$ and $x \in C$ such that x is a exposed point of C and is not a weakly exposed point. Let $X = L_N^0$, where q is continuous, N is strictly convex, $N \in \Delta_2$, and $M \notin \Delta_2$. Since N is strictly convex and $N \in \Delta_2$, we obtain that L_N^0 is strictly convex. Since q is continuous and $N \in \Delta_2$, we obtain that L_N^0 is smooth. Since q is continuous and N is strictly convex, we obtain that p is continuous. Since $M \notin \Delta_2$, we obtain that $L_M \backslash E_M \neq \emptyset$. Pick $r \in (0, 1/8)$ and $u_1 \in L_M \backslash E_M$ such that $\|u_1\| < r$. Pick $u_2 \in E_M$ such that $\|u_2\| = 8$. Then it is easy to see that $u_1 + u_2 \in L_M \backslash E_M$, $\|u_1 + u_2\| > 8 - r$ and

$$\operatorname{dist}(u_1 + u_2, E_M) \leq \|u_1 + u_2 - u_2\| = \|u_1\| < r. \tag{99}$$

Let

$$u' = \frac{u_1 + u_2}{\|u_1 + u_2\|}. \tag{100}$$

Then, by Theorem 1.44 of [12], there exists $a \in (0, 1)$ such that

$$\begin{aligned} \theta(u') &= \inf\left\{\lambda > 0 : \rho_M\left(\frac{u'}{\lambda}\right) < \infty\right\} \\ &= \operatorname{dist}(u', E_M) = \frac{\operatorname{dist}(u_1 + u_2, E_M)}{\|u_1 + u_2\|} < \frac{r}{8 - r} \\ &= 8a < 1 - 4a < 1. \end{aligned} \tag{101}$$

Moreover, by the Bishop-Phelp theorem, there exist $u \in S(L_M^0)$ and $v \in S(L_N)$ such that $(u, v) = \int_G (u(t), v(t))dt = 1$ and $\|u - u'\| < a$. Since $\operatorname{dist}(u', E_M) = 8a$, we have $\operatorname{dist}(u, E_M) > 4a$. This implies that $u \in L_M \backslash E_M$. Therefore, by formula (101), we have

$$\begin{aligned} \theta(u) &= \inf\left\{\lambda > 0 : \rho_M\left(\frac{u}{\lambda}\right) < \infty\right\} = \operatorname{dist}(u, E_M) \\ &\leq 10a < 1. \end{aligned} \tag{102}$$

Since p is continuous, by formula (102) and Theorem 2.49 of [12], we obtain that u is a smooth point. Let

$$\begin{aligned} E_n &= \{t \in G : |u(t)| \leq n\} \\ \text{and } u_n &= u\chi_{E_n}. \end{aligned} \tag{103}$$

Then, by holder inequality, we have

$$\begin{aligned} \int_G \left(u\chi_{E_n}(t) - u(t), v(t)\right) dt &\leq \left\|u\chi_{E_n} - u\right\|^0 \left\|v\chi_{G\backslash E_n}\right\| \\ &\leq \left\|v\chi_{G\backslash E_n}\right\| \longrightarrow 0. \end{aligned} \tag{104}$$

This implies that $(u\chi_{E_n} - u, v) \longrightarrow 1$ as $n \longrightarrow \infty$. Moreover, by the Hahn-Banach theorem, there exists $\phi \in (L_M^0)^*$ such that $\phi(v) > 0$ and $\phi(E_M^0) = 0$. Hence

$$\phi\left(u\chi_{E_n} - u\right) = \phi\left(u\chi_{E_n}\right) - \phi(u) = -\phi(u) < 0. \tag{105}$$

Since u is a smooth point and $(u, v) = 1$, by formulas (104) and (105), we obtain that u is not a weakly exposed point of $B(L_M^0)$. Since q is continuous, we obtain that M is strictly convex. Therefore, by Theorem 2.4 of [12], we obtain that L_M^0 is strictly convex. This implies that u is a exposed point of $B(L_M^0)$.

Example 42. Let $X = E_N^0$ and $M \in \Delta_2$ and $N \notin \Delta_2$. Then $X^* = L_M$. Since $M \in \Delta_2$, we obtain that L_M is separable. Therefore, by Theorem 19 and $(E_N^0)^* = L_M$, we obtain that every bounded subset of E_N^0 is weak dentable. Moreover, by Theorem 40, we obtain that E_N^0 has not the Radon-Nikodym property. Hence there exists a bounded subset D of E_N^0 such that D is not dentable.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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