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Research Article

Hybrid Coupled Fixed Point Theorems in Metric Spaces with Applications

Muhammad Shoaib,[1] Muhammad Sarwar ,[1] and Thabet Abdeljawad [2]

[1] *Department of Mathematics, University of Malakand, Chakdara Dir(L), Pakistan*
[2] *Department Mathematics and General Sciences, Prince Sultan University, P. O. Box 66833, Riyadh 11586, Saudi Arabia*

Correspondence should be addressed to Thabet Abdeljawad; tabdeljawad@psu.edu.sa

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In this manuscript, using CLR property, coupled coincidence and common coupled fixed point results for two-hybrid pairs satisfying (F, φ)- contraction are demonstrated. Using the established results existence of solution to the coupled system of functional and nonlinear matrix equations is also discussed. We provide examples where the main theorem is applicable but most current relevant results in literature fail to have a common coupled fixed point.

1. Introduction and Preliminaries

The existence and uniqueness of solution of a nonlinear matrix equations and functional equations are very interesting research area. Metric fixed point theory provides beneficial and best techniques for the existence of the abovemention equations. In [1–6] the authors worked on matrix equations and demonstrated the existence and uniqueness in the form of their positive definite solutions. Matrix equations and functional equations often arise from various areas, such as ladder networks [7, 8], control theory [9, 10], and dynamic programming [11–14].

Banach [15] has sorted out successful and well-known result; such consequence was later on named as Banach contraction principle (BCP). The Banach principle has been generalized in various spaces. Nadler [16] in 1969 further modified and elaborated the Banach contraction principle (BCP) to set-valued mapping using the Hausdorff metric known as Nadler contraction principle (NCP). Later, thousands of articles appeared in literature to generalize the BCP. Very recently, some authors proved the contraction principle in metric in controlled metric type spaces where the triangle inequality possess control functions (see [17–19] and the references therein).

Aamri and Moutawakil [20] defined (E.A) property for self-mappings which contained the class of compatible and noncompatible mappings and proved common fixed point results under strict contractive conditions. Kamran [21] demonstrated the (E.A) property for hybrid pair and established fixed point and coincidence points results with hybrid strict contractions. Liu et al. [22] introduced common (E.A) property for hybrid pairs of single and multivalued mappings and presented new common fixed point theorems using hybrid contractive conditions. Sintunavarat and Kumam [23] brought together the idea of common limit range (CLR) property for single-valued mappings and displayed its superiority over the property (E.A). Imdad et al. [24] defined common limit range property for a hybrid pair of mappings and demonstrated fixed point results in the symmetric (semimetric) spaces. These concepts were converted by Abbas et al. [25] to multivalued mappings and formulated coupled coincidence point and common coupled fixed point theorems linking hybrid pair of mappings satisfying generalized contractive conditions. Deshpande and Handa [26, 27] defined (E.A) property and occasional w-compatibility for hybrid (pair) coupled maps and also presented common (E.A) property for two hybrid coupled mappings.

In 2012, Wardowski [28] introduced a new type of contraction called F-contraction. In this way Wardowski generalized the Banach contraction principle (BCP) in different manner from the known results of literature. Following this direction Sgroi and Vetro [14] studied multivalued F-contractions and discussed their application on certain functional and integral equations. Recently, Nashine et al.[29] introduced generalized (F,φ)-contractions and studied common fixed point results for a hybrid pair under common limit range property with applications to certain system of functional equations and Volterra integral inclusion.

Coupled fixed points for several type contraction mappings were studied by many authors in different type metric spaces [30–32]. For more details see [25, 33–37]. For other types of common fixed point results we refer to [38–40] and the references therein.

Motivated by the above results, we studied common coupled fixed point results by defining the concept of CLR property for two hybrid pairs of mapping via generalized (F,φ) type contraction. Using the established results we also studied the existence of solution for the coupled system of functional and coupled system of nonlinear matrix equations. All over the paper $\mathbb{R}^+$, $\mathbb{N}$, and $\mathbb{N}_0$ represent the set of all positive real numbers, the set of positive integers, and the set of nonnegative integers, respectively.

Definition 1. Suppose Θ is nonempty set and let $d:\Theta\times\Theta\longrightarrow\mathbb{R}^+$ be a function satisfying the conditions

(1) $d(\kappa,\xi)=0$ if and only if $\kappa=\xi$ for all $\kappa,\xi\in\Theta$;

(2) $d(\kappa,\xi)=d(\xi,\kappa)$, where $\kappa,\xi\in\Theta$;

(3) $d(\kappa,\xi)\leq d(\kappa,\widetilde{z})+d(\xi,\widetilde{z})$ for all $\kappa,\xi,\widetilde{z}\in\Theta$.

Then d is a metric on Θ and the pair (Θ,d) is called metric space.

Definition 2 (see [23]). Functions $f,g:\Theta\longrightarrow\Theta$ are said to satisfy the common limit range property of f w.r.t g (shortly, the (CLR_f)-property w.r.t g) if there exists a sequence $\{\xi_n\}$ in Θ such that, for some $u\in\Theta$, $\lim_{n\longrightarrow\infty}f\xi_n=\lim_{n\longrightarrow\infty}g\xi_n=fu$.

Definition 3 (see [41]). Suppose $f:\Theta\longrightarrow\Theta$, $S:\Theta\longrightarrow CB(\Theta)$ are defined on a metric space (Θ,d). Then f and S are said to satisfy the common limit range property of f w.r.t S (shortly, (CLR_f)-property w.r.t S) if there exists a sequence $\{\xi_n\}$ in Θ and $\Omega_1\in CB(\Theta)$ such that, for some $u\in\Theta$, $\lim_{n\longrightarrow\infty}f\xi_n=fu\in\Omega_1=\lim_{n\longrightarrow\infty}S\xi_n$.

Definition 4 (see [41]). Functions $f,g:\Theta\longrightarrow\Theta$ and $S,T:\Theta\longrightarrow CB(\Theta)$ defined on a metric space (Θ,d), are to satisfy the common limit in the range of f w.r.t S (shortly, (CLR_f)-property w.r.t to S) if there exist sequences $\{\xi_n\}$ and $\{\zeta_n\}$ in Θ and $\Omega_1,\Omega_2\in CB(\Theta)$ such that, for some $u\in\Theta$, we have $\lim_{n\longrightarrow\infty}S\xi_n=\Omega_1$, $\lim_{n\longrightarrow\infty}T\zeta_n=\Omega_2$, and $\lim_{n\longrightarrow\infty}f\xi_n=\lim_{n\longrightarrow\infty}g\zeta_n=fu\in\Omega_1\cap\Omega_2$.

Remark 5. Clearly, if $f=g$ and $S=T$ in Definition 4 then we reobtain Definition 3.

Definition 6 (see [25]). Let $f:\Theta\longrightarrow\Theta$ and $F:\Theta\times\Theta\longrightarrow CB(\Theta)$ be mappings.

(1) A point $(x,y)\in\Theta\times\Theta$ is called a coupled coincidence point of f and F if $f(x)\in F(x,y)$ and $f(y)\in F(y,x)$.

(2) A point $(x,y)\in\Theta\times\Theta$ is called a coupled common point of f and F if $x=f(x)\in F(x,y)$ and $y=f(y)\in F(y,x)$.

Definition 7 (see [25]). Let $f:\Theta\longrightarrow\Theta$ and $F:\Theta\times\Theta\longrightarrow CB(\Theta)$ be mappings. The mapping f is called $F-$weakly commuting at some point, point $(x,y)\in\Theta\times\Theta$ if $f^2(x)\in F(fx,fy)$ and $f^2(y)\in F(fy,fx)$.

Definition 8 (see [27]). Mappings $f:\Theta\longrightarrow\Theta$ and $S:\Theta\times\Theta\longrightarrow CB(\Theta)$ on metric space (Θ,d) are said to have the E.A property if there exist sequences $\{\xi_n\}$ and $\{\zeta_n\}$ in Θ and $\Omega_1,\Omega_2\in CB(\Theta)$ such that for some $u\in\Theta\lim_{n\longrightarrow\infty}f\xi_n=u\in\Omega_1=\lim_{n\longrightarrow\infty}S(\xi_n,\zeta_n)$, $\lim_{n\longrightarrow\infty}f\zeta_n=v\in\Omega_2=\lim_{n\longrightarrow\infty}S(\zeta_n,\xi_n)$.

Now, we recall some definitions for multivalued mappings defined in a metric space (Θ,d). Recall the Hausdorff metric $H:CB(\Theta)\times CB(\Theta)\longrightarrow\mathbb{R}^+$ for $\Omega_1,\Omega_2\in CB(\Theta)$ by

$$H(\Omega_1,\Omega_2)=\max\left\{\sup_{\varsigma\in\Omega_1}d(\varsigma,\Omega_2),\sup_{\zeta\in\Omega_2}d(\zeta,\Omega_1)\right\},\tag{1}$$

where

$$\begin{aligned}d(\xi,\Omega_1)&=\inf\{d(\xi,\zeta):\zeta\in\Omega_1\},\\ \delta(\Omega_1,\Omega_2)&=\sup\{d(\varsigma,\zeta):\varsigma\in\Omega_1,\zeta\in\Omega_2\}\end{aligned}\tag{2}$$

and

$$D(\Omega_1,\Omega_2)=\inf\{d(\varsigma,\zeta):\varsigma\in\Omega_1,\zeta\in\Omega_2\}.\tag{3}$$

Lemma 9 (see [42]). *Let (Θ,d) be a metric space. For any $\Omega_1,\Omega_2\in CB(\Theta)$. We have $d(\xi,\Omega_2)\leq H(\Omega_1,\Omega_2)$, for all $\xi\in\Omega_1$.*

Lemma 10 (see [16]). *Assume (Θ,d) is a metric space and $\Omega_1,\Omega_2\in CB(\Theta)$. Then for every $\lambda>1$ and for each $\varsigma\in\Omega_1$ there exists $\zeta(\varsigma)\in\Omega_2$ such that $d(\varsigma,\zeta)\leq\lambda H(\Omega_1,\Omega_2)$.*

In [16] it was shown that the above lemma is also true for $\lambda\geq 1$. In fact we have the following.

Lemma 11. *Assume (Θ,d) is a metric space and $\Omega_1,\Omega_2\in CB(\Theta)$. Then for every $\lambda\geq 1$ and for each $\varsigma\in\Omega_1$ there exists $\zeta(\varsigma)\in\Omega_2$ such that $d(\varsigma,\zeta)\leq\lambda H(\Omega_1,\Omega_2)$.*

Definition 12 (see [28]). Let F_s represent the family of all functions $F:\mathbb{R}^+\longrightarrow\mathbb{R}$, with the following conditions

(1) F is continuous and strictly increasing;

(2) $\lim_{n\longrightarrow\infty}\alpha_n=0\Longleftrightarrow\lim_{n\longrightarrow\infty}F(\alpha_n)=-\infty$;

(3) for $\{\alpha_n\}\subset\mathbb{R}^+$, $\lim_{n\longrightarrow\infty}\alpha_n=0$, there exists $k\in(0,1)$ such that $\lim_{\alpha\longrightarrow 0^+}(\alpha_n)^kF(\alpha_n)=0$.

Theorem 13 (see [27]). *Let (Θ,d) be a metric space. Assume $f,g:\Theta\longrightarrow\Theta$ and $F,G:\Theta\times\Theta\longrightarrow CB(\Theta)$ to be a mapping satisfying the following.*

(a) (F, f) *and* (G, g) *satisfy the common (EA) property.*

(b) *For all* $x, y, u, v \in \Theta$, *there exist some* $\varphi \in \Phi$ *and some* $\psi \in \Psi$ *such that*

$$\varphi\left(\frac{H(F(x,y),G(u,v))+H(F(y,x),G(v,u))}{2}\right) \leq \varphi\left(\frac{d(fx,gu)+d(fy,gv)}{2}\right) - \psi\left(\frac{d(fx,gu)+d(fy,gv)}{2}\right). \tag{4}$$

(c) $f(\Theta)$ *and* $g(\Theta)$ *are closed subsets of* Θ. *Then*

(A_1) (g, G) *have coupled coincidence point.*

(A_2) (f, F) *have coupled coincidence point.*

(A_3) *If* g *is* $G-$ *weakly commuting at* (w_1, w_2) *and* $g^2w_1 = gw_1$, $g^2w_2 = gw_2$ *for* $(w_1, w_2) \in C(G, g)$, *then* G *and* g *have a common coupled fixed point.*

(A_4) *If* f *is* $F-$ *weakly commuting at* (z_1, z_2) *and* $f^2z_1 = fz_1$, $f^2z_2 = fz_2$ *for* $(z_1, z_2) \in C(F, f)$, *then* F *and* f *have a common coupled fixed point.*

(A_5) F, G, f, *and* g *have common coupled fixed point if* (A_3) *and* (A_4) *are true.*

Theorem 14 (see [27]). *Let* (Θ, d) *be a metric space. Assume* $f, g : \Theta \longrightarrow \Theta$ *and* $F, G : \Theta \times \Theta \longrightarrow CB(\Theta)$ *to be mappings satisfying* (*a*) *and* (*b*) *of Theorem 13 and*

(1) (F, f) *and* (G, g) *are w-compatible.*

(2) *Suppose that either* $g(\Theta)$ *is closed subset of* Θ *or* $G(\Theta \times \Theta) \subseteq f(\Theta)$ *or* $f(\Theta)$ *is closed subset of* Θ *and* $F(\Theta \times \Theta) \subseteq g(\Theta)$. *Then* F, G, f, g *have a common coupled fixed point.*

2. Main Results

We define the CLR property for the study of common coupled fixed point in the following way in metric space.

Definition 15. Mappings $f : \Theta \longrightarrow \Theta$ and $S : \Theta \times \Theta \longrightarrow CB(\Theta)$ on metric space (Θ, d) are to satisfy the common limit in the range of f with respect to S (shortly, the (CLR_f)-property with respect to S) if there exist sequences $\{\xi_n\}$ and $\{\zeta_n\}$ in Θ and $\Omega_1, \Omega_2 \in CB(\Theta)$ such that, for some $u \in \Theta$, we have $\lim_{n\longrightarrow\infty} f\xi_n = u = f\xi \in \Omega_1 = \lim_{n\longrightarrow\infty} S(\xi_n, \zeta_n)$, $\lim_{n\longrightarrow\infty} f\zeta_n = v = f\zeta \in \Omega_2 = \lim_{n\longrightarrow\infty} S(\zeta_n, \xi_n)$.

Definition 16. Let $f, g : \Theta \longrightarrow \Theta$ and $T, G : \Theta \times \Theta \longrightarrow CB(\Theta)$ be mappings on metric space (Θ, d). Then (T, f) and (G, g) have (CLR)-property, if there exist sequences $\{x_n\}$, $\{y_n\}\{u_n\}$ and $\{v_n\}$ and $C_1, C_2, D_1, D_2 \in CB(\Theta)$ such that

$$\begin{aligned}
\lim_{n\longrightarrow\infty} T(x_n, y_n) &= C_1, \\
\lim_{n\longrightarrow\infty} G(u_n, v_n) &= C_2, \\
\lim_{n\longrightarrow\infty} fx_n &= fz_1 \in C_1, \\
\lim_{n\longrightarrow\infty} gu_n &= gz_1 \in C_2, \\
\lim_{n\longrightarrow\infty} T(y_n, x_n) &= D_1, \\
\lim_{n\longrightarrow\infty} G(v_n, u_n) &= D_2, \\
\lim_{n\longrightarrow\infty} fy_n &= fz_2 \in D_1, \\
\lim_{n\longrightarrow\infty} gv_n &= gz_2 \in D_2,
\end{aligned} \tag{5}$$

for some $z_1, z_2 \in \Theta$.

Example 17. Let $\Theta = [0, \infty)$ with the usual metric. Define $f, g : \Theta \longrightarrow \Theta$ and $F, G : \Theta \times \Theta \longrightarrow CB(\Theta)$ by $f(x) = 1 + x$, $g(x) = x/2$, $F(x, y) = [1, 2 + 2x + y]$, $G(x, y) = [1, 2 + (3x + y)/4]$, $\forall x, y \in X$.

Consider the sequences $\{x_n\} = \{1 + 1/n\}$, $\{y_n\} = \{4 + 1/n\}$, $\{u_n\} = \{1 - 1/n\}$, $\{v_n\} = \{4 - 1/n\}$.

Clearly $\lim_{n\longrightarrow\infty} F(x_n, y_n) = [1, 8]$, $\lim_{n\longrightarrow\infty} f(x_n) = 2 = f(1) \in [1, 8]$, $\lim_{n\longrightarrow\infty} G(u_n, v_n) = [1, 15/4]$, $\lim_{n\longrightarrow\infty} g(u_n) = 1/2 = g(1) \in [1, 15/4]$. Further, $\lim_{n\longrightarrow\infty} F(y_n, x_n) = [1, 11]$, $\lim_{n\longrightarrow\infty} f(y_n) = 5 = f(4) \in [1, 11]$, $\lim_{n\longrightarrow\infty} G(v_n, u_n) = [1, 21/4]$, $\lim_{n\longrightarrow\infty} g(v_n) = 2 = g(4) \in [1, 21/4]$. Therefore (f, F) and (g, G) satisfy CLR property.

Example 18. Assume $\Theta = [0, 1)$ to be endowed through usual metric and $f, g : \Theta \longrightarrow \Theta$, $T, S : \Theta \times \Theta \longrightarrow CB(\Theta)$, define by

$$\begin{aligned}
fx &= \begin{cases} \dfrac{3}{4} & \text{if } 0 \leq x < \dfrac{1}{2} \\ \dfrac{79}{100} & \text{if } \dfrac{1}{2} \leq x < 1, \end{cases} \\
gx &= \begin{cases} \dfrac{3}{4} & \text{if } 0 \leq x < \dfrac{1}{2} \\ \dfrac{11}{20} & \text{if } \dfrac{1}{2} \leq x < 1, \end{cases} \\
T(x, y) &= \begin{cases} \left[\dfrac{1}{2}, \dfrac{3}{4}\right] & \text{if } 0 \leq x, y < \dfrac{1}{2} \\ \left[\dfrac{3}{4}, \dfrac{4}{5}\right] & \text{Otherwise}, \end{cases} \\
S(x, y) &= \begin{cases} \left[\dfrac{3}{5}, \dfrac{4}{5}\right] & \text{if } 0 \leq x, y < \dfrac{1}{2} \\ \left[\dfrac{1}{2}, \dfrac{3}{5}\right] & \text{Otherwise}. \end{cases}
\end{aligned} \tag{6}$$

Let $\{x_n\} = \{1 - 1/n\}$, where $n = 3, 4, 5, 6\ldots$, $\{y_n\} = \{1 - 1/n^3\}$, where $n = 4, 5, 6\ldots$ $\{u_n\} = \{1/9 - 1/2n\}$, where $n = 3, 4, 5, 6\ldots$, and $\{v_n\} = \{1/4 - 1/n\}$, where $n = 4, 5, 6\ldots$.

Clearly $\lim_{n\longrightarrow\infty}T(x_n, y_n) = [3/4, 4/5]$, $\lim_{n\longrightarrow\infty}f(x_n) = 79/100 = f(1/2) \in [3/4, 4/5]$, $\lim_{n\longrightarrow\infty}S(u_n, v_n) = [3/5, 4/5]$, $\lim_{n\longrightarrow\infty}g(u_n) = 11/20 = g(1/2) \in [3/5, 4/5]$. Further, $\lim_{n\longrightarrow\infty}T(y_n, x_n) = [3/4, 4/5]$, $\lim_{n\longrightarrow\infty}f(y_n) = 79/100 = f(4/5) \in [3/4, 4/5]$, $\lim_{n\longrightarrow\infty}S(v_n, u_n) = [3/5, 4/5]$, $\lim_{n\longrightarrow\infty}g(v_n) = 11/20 = g(4/5) \in [3/5, 4/5]$. Therefore (f, F) and (g, G) satisfy CLR property.

Throughout the paper $CB(\Theta)$ denote the set of all closed and bounded subsets of Θ and

$$\Phi = \left\{\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+, \textit{increasing, upper semicontineous such that } \lim_{s\longrightarrow t^+} \varphi(s) < \varphi(t), \varphi(t) < t, \textit{ for all } t > 0\right\}. \quad (7)$$

Theorem 19. *Let $f, g : \Theta \longrightarrow \Theta$ and $T, G : \Theta \times \Theta \longrightarrow CB(\Theta)$ be maps on metric space (Θ, d). Suppose that (T, f) and (G, g) have (CLR)-property and furthermore assume that*

$$\begin{aligned}\tau + F(H^p(T(x,y), G(u,v))) \\ \leq F(\varphi(\Theta(x,y,u,v))),\end{aligned} \quad (8)$$

where $H(T(x, y), G(u, v)) > 0$ and

$$\begin{aligned}\Theta(x,y,u,v) &= \alpha\left[d^p(fx, gu)\right] \\ &+ \beta\left[\frac{d^p(fy, T(y,x))\, d^p(gv, G(v,u))}{1 + d^p(fx, gu)}\right] \\ &+ \gamma\left[d^p(fx, T(x,y)) + d^p(gu, G(u,v))\right] \\ &+ \sigma\left[d^p(fy, T(y,x))\right] + \eta\left[d^p(gu, G(u,v))\right].\end{aligned} \quad (9)$$

Here, $\tau \in \mathbb{R}^+$, $\alpha + \beta + \gamma + \sigma + \eta \leq 1$, $p \geq 1$, $F \in F_s$ and $\varphi \in \Phi$. Then the following holds.

- (A_1) *(g, G) have coupled coincidence point.*
- (A_2) *(f, T) have coupled coincidence point.*
- (A_3) *If g is $G-$ weakly commuting at (w_1, w_2) and $g^2w_1 = gw_1$, $g^2w_2 = gw_2$ for $(w_1, w_2) \in C(G, g)$, then G and g have a common coupled fixed point.*
- (A_4) *If f is $T-$ weakly commuting at (z_1, z_2) and $f^2z_1 = fz_1$, $f^2z_2 = fz_2$ for $(z_1, z_2) \in C(T, f)$. Then T and f have a common coupled fixed point.*
- (A_5) *T, G, f, and g have common coupled fixed point if (A_3) and (A_4) are true.*

Proof. Since (T, f) and (G, g) have (CLR)-property, therefore there exist sequences $\{x_n\}$, $\{y_n\}$ $\{u_n\}$ and $\{v_n\}$ and $C_1, C_2, D_1, D_2 \in CB(\Theta)$ such that

$$\begin{aligned}&\lim_{n\longrightarrow\infty} T(x_n, y_n) = C_1, \\ &\lim_{n\longrightarrow\infty} G(u_n, v_n) = C_2, \\ &\lim_{n\longrightarrow\infty} fx_n = fz_1 \in C_1, \\ &\lim_{n\longrightarrow\infty} gu_n = gz_1 \in C_2, \\ &\lim_{n\longrightarrow\infty} T(y_n, x_n) = D_1, \\ &\lim_{n\longrightarrow\infty} G(v_n, u_n) = D_2, \\ &\lim_{n\longrightarrow\infty} fy_n = fz_2 \in D_1, \\ &\lim_{n\longrightarrow\infty} gv_n = gz_2 \in D_2.\end{aligned} \quad (10)$$

Putting $x = x_n$, $y = y_n$, $u = u_n$, $v = v_n$ in inequality (8), we have

$$\begin{aligned}\tau + F(H^p(T(x_n,y_n), G(u_n,v_n))) \\ \leq F(\varphi(\Theta(x_n,y_n,u_n,v_n))),\end{aligned} \quad (11)$$

where

$$\begin{aligned}\Theta(x_n,y_n,u_n,v_n) &= \alpha\left[d^p(fx_n, gu_n)\right] \\ &+ \beta\left[\frac{d^p(fy_n, T(y_n,x_n))\, d^p(gv_n, G(v_n,u_n))}{1 + d^p(fx_n, gu_n)}\right] \\ &+ \gamma\left[d^p(fx_n, T(x_n,y_n)) + d^p(gu_n, G(u_n,v_n))\right] \\ &+ \sigma\left[d^p(fy_n, T(y_n,x_n))\right] \\ &+ \eta\left[d^p(gu_n, G(u_n,v_n))\right].\end{aligned} \quad (12)$$

Applying limit to Θ, we have

$$\begin{aligned}
&\lim_{n\longrightarrow\infty}\Theta(x_n,y_n,u_n,v_n)\\
&\quad=\alpha\left[d^p(fz_1,gz_1)\right]\\
&\qquad+\beta\left[\frac{d^p(fz_2,D_1)\,d^p(gz_2,D_2)}{1+d^p(fz_1,gz_1)}\right]\\
&\qquad+\gamma\left[d^p(fz_1,C_1)+d^p(gz_1,C_2)\right]\\
&\qquad+\sigma\left[d^p(fz_2,D_1)\right]+\eta\left[d^p(gz_1,C_2)\right],
\end{aligned}\tag{13}$$

which implies that

$$\lim_{n\longrightarrow\infty}\Theta(x_n,y_n,u_n,v_n)=\alpha\left(d^p(fz_1,gz_1)\right).\tag{14}$$

Applying limit to (11) and using (14), we have

$$\tau+F\left(H^p(C_1,C_2)\right)\le F\left(\varphi\left(\alpha\left(d^p(fz_1,gz_1)\right)\right)\right),\tag{15}$$

which implies that

$$F\left(H^p(C_1,C_2)\right)\le F\left(\varphi\left(\alpha\left(d^p(fz_1,gz_1)\right)\right)\right).\tag{16}$$

Using definitions of F and φ, we have

$$H^p(C_1,C_2)\le\alpha\left(d^p(fz_1,gz_1)\right).\tag{17}$$

But $\alpha\le 1$ and using Lemma 11

$$d^p(fz_1,gz_1)\le H^p(C_1,C_2)<d^p(fz_1,gz_1),\tag{18}$$

which is contradiction. Hence, $d^p(fz_1,gz_1)=0$. Therefore

$$fz_1=gz_1.\tag{19}$$

Putting $x=y_n$, $y=x_n$, $u=v_n$, $v=u_n$ in inequality (8), we have

$$\begin{aligned}
&\tau+F\left(H^p\left(T(y_n,x_n),G(v_n,u_n)\right)\right)\\
&\quad\le F\left(\varphi\left(\Theta(y_n,x_n,v_n,u_n)\right)\right),
\end{aligned}\tag{20}$$

where

$$\begin{aligned}
&\Theta(y_n,x_n,v_n,u_n)\\
&\quad=\alpha\left[d^p(fy_n,gv_n)\right]\\
&\qquad+\beta\left[\frac{d^p(fx_n,T(x_n,y_n))\,d^p(gu_n,G(u_n,v_n))}{1+d^p(fy_n,gv_n)}\right]\\
&\qquad+\gamma\left[d^p(fy_n,T(y_n,x_n))+d^p(gv_n,G(v_n,u_n))\right]\\
&\qquad+\sigma\left[d^p(fx_n,T(x_n,y_n))\right]\\
&\qquad+\eta\left[d^p(gv_n,G(v_n,u_n))\right].
\end{aligned}\tag{21}$$

Applying limit to Θ, we have

$$\begin{aligned}
&\lim_{n\longrightarrow\infty}\Theta(y_n,x_n,v_n,u_n)\\
&\quad=\alpha\left[d^p(fz_2,gz_2)\right]\\
&\qquad+\beta\left[\frac{d^p(fz_1,C_1)\,d^p(gz_1,C_2)}{1+d^p(fz_2,gz_2)}\right]\\
&\qquad+\gamma\left[d^p(fz_2,D_1)+d^p(gz_2,D_2)\right]\\
&\qquad+\sigma\left[d^p(fz_1,C_1)\right]+\eta\left[d^p(gz_2,D_2)\right].
\end{aligned}\tag{22}$$

$$\lim_{n\longrightarrow\infty}\Theta(y_n,x_n,v_n,u_n)=\alpha\left(d^p(fz_2,gz_2)\right).\tag{23}$$

Applying limit to (20) and using (23), we have

$$fz_2=gz_2.\tag{24}$$

Putting $x=y_n$, $y=x_n$, $u=z_1$, $v=z_2$ in inequality (8), we have

$$\begin{aligned}
&\tau+F\left(H^p\left(T(x_n,y_n),G(z_1,z_2)\right)\right)\\
&\quad\le F\left(\varphi\left(\Theta(x_n,y_n,z_1,z_2)\right)\right),
\end{aligned}\tag{25}$$

where

$$\begin{aligned}
&\Theta(x_n,y_n,z_1,z_2)\\
&\quad=\alpha\left[d^p(fx_n,gz_1)\right]\\
&\qquad+\beta\left[\frac{d^p(fy_n,T(y_n,x_n))\,d^p(gz_2,G(z_2,z_1))}{1+d^p(fx_n,gz_1)}\right]\\
&\qquad+\gamma\left[d^p(fx_n,T(x_n,y_n))+d^p(gz_1,G(z_1,z_2))\right]\\
&\qquad+\sigma\left[d^p(fy_n,T(y_n,x_n))\right]\\
&\qquad+\eta\left[d^p(gz_1,G(z_1,z_2))\right].
\end{aligned}\tag{26}$$

Applying limit to Θ, we have

$$\begin{aligned}
&\lim_{n\longrightarrow\infty}\Theta(x_n,y_n,z_1,z_2)\\
&\quad=\alpha\left[d^p(fz_1,gz_1)\right]\\
&\qquad+\beta\left[\frac{d^p(fz_2,D_1)\,d^p(gz_2,G(z_2,z_1))}{1+d^p(fz_1,gz_1)}\right]\\
&\qquad+\gamma\left[d^p(fz_1,C_1)+d^p(gz_1,G(z_1,z_2))\right]\\
&\qquad+\sigma\left[d^p(fz_2,D_1)\right]+\eta\left[d^p(gz_1,G(z_1,z_2))\right],
\end{aligned}\tag{27}$$

which implies that

$$\begin{aligned}
&\lim_{n\longrightarrow\infty}\Theta(x_n,y_n,z_1,z_2)\\
&\quad=\gamma\left[d^p(gz_1,G(z_1,z_2))\right]\\
&\qquad+\eta\left[d^p(gz_1,G(z_1,z_2))\right],
\end{aligned}\tag{28}$$

and we get

$$\lim_{n\longrightarrow\infty}\Theta\left(x_n,y_n,z_1,z_2\right) = (\gamma+\eta)\left(d^p\left(gz_1,G\left(z_1,z_2\right)\right)\right). \quad (29)$$

Applying limit to (25) and using (29), we get

$$\tau+F\left(H^p\left(C_1,G\left(z_1,z_2\right)\right)\right) \leq F\left(\varphi\left((\gamma+\eta)\left(d^p\left(gz_1,G\left(z_1,z_2\right)\right)\right)\right)\right), \quad (30)$$

which implies that

$$F\left(H^p\left(C_1,G\left(z_1,z_2\right)\right)\right) \leq F\left(\varphi\left((\gamma+\eta)\left(d^p\left(gz_1,G\left(z_1,z_2\right)\right)\right)\right)\right). \quad (31)$$

Using definitions of F and φ and using Lemma 9, we have

$$H^p\left(C_1,G\left(z_1,z_2\right)\right) \leq (\gamma+\eta)\left(d^p\left(gz_1,G\left(z_1,z_2\right)\right)\right), \quad (32)$$

which implies that

$$gz_1 \in G\left(z_1,z_2\right). \quad (33)$$

Putting $x = y_n, y = x_n, u = z_2, v = z_1$ in inequality (8), we have

$$\tau+F\left(H^p\left(T\left(y_n,x_n\right),G\left(z_2,z_1\right)\right)\right) \leq F\left(\varphi\left(\Theta\left(y_n,x_n,z_2,z_1\right)\right)\right), \quad (34)$$

where

$$\begin{aligned}\Theta\left(y_n,x_n,z_2,z_1\right) &= \alpha\left[d^p\left(fy_n,gz_2\right)\right] \\ &+\beta\left[\frac{d^p\left(fx_n,T\left(x_n,y_n\right)\right)d^p\left(gz_1,G\left(z_1,z_2\right)\right)}{1+d^p\left(fy_n,gz_2\right)}\right] \\ &+\gamma\left[d^p\left(fy_n,T\left(y_n,x_n\right)\right)+d^p\left(gz_2,G\left(z_2,z_1\right)\right)\right] \\ &+\sigma\left[d^p\left(fx_n,T\left(x_n,y_n\right)\right)\right] \\ &+\eta\left[d^p\left(gz_2,G\left(z_2,z_1\right)\right)\right].\end{aligned} \quad (35)$$

Applying limit to Θ, we have

$$\begin{aligned}\lim_{n\longrightarrow\infty}\Theta\left(y_n,x_n,z_2,z_1\right) &= \alpha\left[d^p\left(fz_2,gz_2\right)\right] \\ &+\beta\left[\frac{d^p\left(fz_1,C_1\right)d^p\left(gz_1,G\left(z_1,z_2\right)\right)}{1+d^p\left(fz_1,gz_1\right)}\right] \\ &+\gamma\left[d^p\left(fz_1,C_1\right)+d^p\left(gz_2,G\left(z_2,z_1\right)\right)\right] \\ &+\sigma\left[d^p\left(fz_1,C_1\right)\right]+\eta\left[d^p\left(gz_2,G\left(z_2,z_1\right)\right)\right],\end{aligned} \quad (36)$$

which implies that

$$\lim_{n\longrightarrow\infty}\Theta\left(y_n,x_n,z_2,z_1\right) = \gamma\left[d^p\left(gz_2,G\left(z_2,z_1\right)\right)\right] + \eta\left[d^p\left(gz_2,G\left(z_2,z_1\right)\right)\right], \quad (37)$$

$$\lim_{n\longrightarrow\infty}\Theta\left(y_n,x_n,z_2,z_1\right) = (\gamma+\eta)\left(d^p\left(gz_2,G\left(z_2,z_1\right)\right)\right). \quad (38)$$

Applying limit to (25) and using (38), we get

$$\tau+F\left(H^p\left(C_1,G\left(z_2,z_1\right)\right)\right) \leq F\left(\varphi\left((\gamma+\eta)\left(d^p\left(gz_2,G\left(z_2,z_1\right)\right)\right)\right)\right), \quad (39)$$

which implies that

$$F\left(H^p\left(C_1,G\left(z_2,z_1\right)\right)\right) \leq F\left(\varphi\left((\gamma+\eta)\left(d^p\left(gz_2,G\left(z_2,z_1\right)\right)\right)\right)\right). \quad (40)$$

Using definitions of F and φ and using Lemma 9, we have

$$H^p\left(C_1,G\left(z_2,z_1\right)\right) \leq (\gamma+\eta)\left(d^p\left(gz_2,G\left(z_2,z_1\right)\right)\right), \quad (41)$$

and we obtained

$$gz_2 \in G\left(z_2,z_1\right). \quad (42)$$

Similarly by putting $x = z_1, y = z_2$ and $u = u_n, v = v_n$ and $x = z_2, y = z_1$ and $u = v_n, v = u_n$ we can obtained

$$fz_1 \in T\left(z_1,z_2\right) \quad (43)$$

and

$$fz_2 \in T\left(z_2,z_1\right). \quad (44)$$

Since g and G-weakly are commuting then $g^2z_1 \in G(fz_1,gz_2)$, $g^2z_2 \in G(fz_2,fz_1)$. Since $g^2z_1 = gz_1$, $g^2z_2 = gz_2$. Thus (gz_1,gz_2) is a common fixed point. A similar argument proves (A_4). Then using (A_3) and (A_4), (A_5) hold immediately. □

Theorem 20. *Let $f,g:\Theta\longrightarrow\Theta$ and $T,G:\Theta\times\Theta\longrightarrow CB(\Theta)$ are mapping on metric space (Θ,d). Furthermore assume that (T,f) and (G,g) have (CLR)-property and*

$$\tau+F\left(H^p\left(T\left(x,y\right),G\left(u,v\right)\right)\right) \leq F\left(\varphi\left(\Theta\left(x,y,u,v\right)\right)\right), \quad (45)$$

where $H(T(x,y),G(u,v)) > 0$ and

$$\Theta(x,y,u,v) = \max\left\{d^p(fx, T(x,y)),\right.$$
$$d^p(gu, G(u,v)), d^p(fx, gu),$$
$$\frac{d^p(fx, T(x,y)) + d^p(gu, G(u,v))}{2},$$
$$\frac{d^p(fy, T(y,x))\, d^p(gv, G(v,u))}{1 + d^p(fx, gu)}, \tag{46}$$
$$\frac{d^p(gv, G(v,u))\, d^p(fy, T(y,x))}{1 + d^p(fx, fu)},$$
$$\left.\frac{d^p(fx, T(x,y))\, d^p(gu, G(u,v))}{1 + D^p(T(x,y), G(u,v))}\right\}.$$

Here, $\tau \in \mathbb{R}^+$, $p \geq 1$, $F \in F_s$, and $\varphi \in \Phi$. Then the following holds.

- (A_1) *(g, G) have coupled coincidence point.*
- (A_2) *(f, T) have coupled coincidence point.*
- (A_3) *If g is G weakly commuting at (w_1, w_2) and $g^2 w_1 = g w_1$, $g^2 w_2 = g w_2$ for $(w_1, w_2) \in C(G, g)$, then G and g have a common coupled fixed point.*
- (A_4) *If f is T weakly commuting at (z_1, z_2) and $f^2 z_1 = f z_1$, $f^2 z_2 = f z_2$ for $(z_1, z_2) \in C(T, f)$, then T and f have a common coupled fixed point.*
- (A_5) *T, G, f, and g have common coupled fixed point if (A_3) and (A_4) are true.*

Proof. Since (T, f) and (G, g) have (CLR)-property, therefore there exist sequences $\{x_n\}$, $\{y_n\}$ $\{u_n\}$ and $\{v_n\}$ and $C_1, C_2, D_1, D_2 \in CB(\Theta)$ such that

$$\begin{gathered}
\lim_{n\to\infty} T(x_n, y_n) = C_1,\\
\lim_{n\to\infty} G(u_n, v_n) = C_2,\\
\lim_{n\to\infty} fx_n = fz_1 \in C_1,\\
\lim_{n\to\infty} gu_n = gz_1 \in C_2,\\
\lim_{n\to\infty} T(y_n, x_n) = D_1,\\
\lim_{n\to\infty} G(v_n, u_n) = D_2,\\
\lim_{n\to\infty} fy_n = fz_2 \in D_1,\\
\lim_{n\to\infty} gv_n = gz_2 \in D_2.
\end{gathered} \tag{47}$$

Putting $x = x_n, y = y_n, u = u_n, v = v_n$ in inequality (45), we get

$$\begin{aligned}
\tau + F(H^p(T(x_n, y_n), G(u_n, v_n))) \\
\leq F(\varphi(\Theta(x_n, y_n, u_n, v_n))),
\end{aligned} \tag{48}$$

where

$$\Theta(x_n, y_n, u_n, v_n) = \max\left\{d^p(fx_n, T(x_n, y_n)),\right.$$
$$d^p(gu_n, G(u_n, v_n)), d^p(fx_n, gu_n),$$
$$\frac{d^p(fx_n, T(x_n, y_n)) + d^p(gu_n, G(u_n, v_n))}{2},$$
$$\frac{d^p(fy_n, T(y_n, x_n))\, d^p(gv_n, G(v_n, u_n))}{1 + d^p(fx_n, gu_n)}, \tag{49}$$
$$\frac{d^p(gv_n, G(v_n, u_n))\, d^p(fy_n, T(y_n, x_n))}{1 + d^p(fx_n, fu_n)},$$
$$\left.\frac{d^p(fx_n, T(x_n, y_n))\, d^p(gu_n, G(u_n, v_n))}{1 + D^p(T(x_n, y_n), G(u_n, v_n))}\right\}.$$

Applying limit to Θ, we have

$$\lim_{n\to\infty}\Theta(x_n, y_n, u_n, v_n) = \max\left\{d^p(fz_1, C_1),\right.$$
$$d^p(gz_2, C_2), d^p(fz_1, gz_1),$$
$$\frac{d^p(fz_1, C_1) + d^p(gz_1, C_2)}{2},$$
$$\frac{d^p(fz_2, D_1)\, d^p(gz_2, D_2)}{1 + d^p(fz_1, gz_2)}, \tag{50}$$
$$\frac{d^p(gz_2, D_2)\, d^p(fz_2, D_1)}{1 + d^p(fz_1, gz_2)},$$
$$\left.\frac{d^p(fz_1, C_1)\, d^p(gz_2, C_2)}{1 + D^p(C_1, C_2)}\right\}.$$

$$\lim_{n\to\infty}\Theta(x_n, y_n, u_n, v_n) = d^p(fz_1, gz_1). \tag{51}$$

Applying limit to (48) and using (51) we can deduce that

$$\begin{aligned}
\tau + F(H^p(C_1, C_2)) \\
\leq F\left(\lim_{n\to\infty}\varphi(\Theta(x_n, y_n, u_n, v_n))\right),
\end{aligned} \tag{52}$$

which implies that

$$F(H^p(C_1, C_2)) \leq F(d^p(fz_1, gz_1)). \tag{53}$$

Using definitions of F and φ, we have

$$H^p(C_1, C_2) \leq \varphi(d^p(fz_1, gz_1)). \tag{54}$$

Using Lemma 11, we have

$$d^p(fz_1, gz_1) \leq H^p(C_1, C_2) \leq \varphi(d^p(fz_1, gz_1)). \tag{55}$$

Thus, we have

$$fz_1 = gz_1. \tag{56}$$

Similarly by taking $x = y_n, y = x_n, u = v_n, v = u_n$ in inequality (45) we get

$$\begin{aligned}\tau + F\left(H^p\left(T\left(y_n, x_n\right), G\left(v_n, u_n\right)\right)\right) \\ \leq F\left(\varphi\left(\Theta\left(y_n, x_n, v_n, u_n\right)\right)\right),\end{aligned} \tag{57}$$

where

$$\begin{aligned}\Theta\left(y_n, x_n, v_n, u_n\right) = \max\Bigg\{&d^p\left(fy_n, T\left(y_n, x_n\right)\right), \\ &d^p\left(gv_n, G\left(v_n, u_n\right)\right), d^p\left(fy_n, gv_n\right), \\ &\frac{d^p\left(fy_n, T\left(y_n, x_n\right)\right) + d^p\left(gv_n, G\left(v_n, u_n\right)\right)}{2}, \\ &\frac{d^p\left(fx_n, T\left(x_n, y_n\right)\right) d^p\left(gu_n, G\left(u_n, v_n\right)\right)}{1 + d^p\left(fy_n, gv_n\right)}, \\ &\frac{d^p\left(gu_n, G\left(u_n, v_n\right)\right) d^p\left(fx_n, T\left(x_n, y_n\right)\right)}{1 + d^p\left(fy_n, fv_n\right)}, \\ &\frac{d^p\left(fy_n, T\left(y_n, x_n\right)\right) d^p\left(gv_n, G\left(v_n, u_n\right)\right)}{1 + D^p\left(T\left(y_n, x_n\right), G\left(v_n, u_n\right)\right)}\Bigg\}.\end{aligned} \tag{58}$$

Applying limit to Θ we have

$$\begin{aligned}\lim_{n\longrightarrow\infty} \Theta\left(y_n, x_n, v_n, u_n\right) = \max\Bigg\{&d^p\left(fz_2, D_1\right), \\ &d^p\left(gz_2, D_2\right), d^p\left(fz_2, gz_2\right), \\ &\frac{d^p\left(fz_2, D_1\right) + d^p\left(gz_2, D_2\right)}{2}, \\ &\frac{d^p\left(fz_1, C_1\right) d^p\left(gz_2, C_2\right)}{1 + d^p\left(fz_2, gz_2\right)}, \\ &\frac{d^p\left(gz_2, C_2\right) d^p\left(fz_1, C_1\right)}{1 + d^p\left(fz_2, gz_2\right)}, \\ &\frac{d^p\left(fz_2, D_1\right) d^p\left(gz_2, D_2\right)}{1 + D^p\left(D_1, D_2\right)}\Bigg\},\end{aligned} \tag{59}$$

$$\lim_{n\longrightarrow\infty} \Theta\left(y_n, x_n, v_n, u_n\right) = d^p\left(fz_2, gz_2\right). \tag{60}$$

Applying limit to (57) and using (60) we can deduce that

$$fz_2 = gz_2. \tag{61}$$

By taking $x = x_n, y = y_n, u = z_1, v = z_2$ in inequality (45), we get

$$\begin{aligned}\tau + F\left(H^p\left(T\left(x_n, y_n\right), G\left(z_1, z_2\right)\right)\right) \\ \leq F\left(\varphi\left(\Theta\left(x_n, y_n, z_1, z_2\right)\right)\right),\end{aligned} \tag{62}$$

where

$$\begin{aligned}\Theta\left(x_n, y_n, z_1, z_2\right) = \max\Bigg\{&d^p\left(fx_n, T\left(x_n, y_n\right)\right), \\ &d^p\left(gz_1, G\left(z_1, z_2\right)\right), d^p\left(fx_n, gz_1\right), \\ &\frac{d^p\left(fx_n, T\left(x_n, y_n\right)\right) + d^p\left(gz_1, G\left(z_1, z_2\right)\right)}{2}, \\ &\frac{d^p\left(fy_n, T\left(y_n, x_n\right)\right) d^p\left(gz_2, G\left(z_2, z_1\right)\right)}{1 + d^p\left(fx_n, gz_1\right)}, \\ &\frac{d^p\left(gz_2, G\left(z_2, z_1\right)\right) d^p\left(fy_n, T\left(y_n, x_n\right)\right)}{1 + d^p\left(fx_n, fz_1\right)}, \\ &\frac{d^p\left(fx_n, T\left(x_n, y_n\right)\right) d^p\left(gz_1, G\left(z_1, z_2\right)\right)}{1 + D^p\left(T\left(x_n, y_n\right), G\left(z_1, z_2\right)\right)}\Bigg\}.\end{aligned} \tag{63}$$

Applying limit to Θ, we have

$$\begin{aligned}\lim_{n\longrightarrow\infty} \Theta\left(x_n, y_n, z_1, z_2\right) = \max\Bigg\{&d^p\left(fz_1, C_1\right), \\ &d^p\left(gz_1, G\left(z_1, z_2\right)\right), d^p\left(fz_1, gz_1\right), \\ &\frac{d^p\left(fx_n, T\left(x_n, y_n\right)\right) + d^p\left(gz_1, G\left(z_1, z_2\right)\right)}{2}, \\ &\frac{d^p\left(fy_n, T\left(y_n, x_n\right)\right) d^p\left(gz_2, G\left(z_2, z_1\right)\right)}{1 + d^p\left(fx_n, gz_1\right)}, \\ &\frac{d^p\left(gz_2, G\left(z_2, z_1\right)\right) d^p\left(fy_n, T\left(y_n, x_n\right)\right)}{1 + d^p\left(fx_n, fz_1\right)}, \\ &\frac{d^p\left(fx_n, T\left(x_n, y_n\right)\right) d^p\left(gz_1, G\left(z_1, z_2\right)\right)}{1 + D^p\left(T\left(x_n, y_n\right), G\left(z_1, z_2\right)\right)}\Bigg\},\end{aligned} \tag{64}$$

$$\lim_{n\longrightarrow\infty} \Theta\left(x_n, y_n, z_1, z_2\right) = d^p\left(gz_1, G\left(z_1, z_2\right)\right). \tag{65}$$

Applying limit to (62) using (65), we have

$$gz_1 \in G\left(z_1, z_2\right). \tag{66}$$

By taking $x = y_n, y = x_n, u = z_2, v = z_1$ in inequality (45) we get

$$\begin{aligned}\tau + F\left(H^p\left(T\left(y_n, x_n\right), G\left(z_2, z_1\right)\right)\right) \\ \leq F\left(\varphi\left(\Theta\left(y_n, x_n, z_2, z_1\right)\right)\right)\end{aligned} \tag{67}$$

where

$$\Theta\left(y_n, x_n, z_2, z_1\right) = \max\left\{d^p\left(fy_n, T\left(y_n, x_n\right)\right),\right.$$
$$d^p\left(gz_2, G\left(z_2, z_1\right)\right), d^p\left(fy_n, gz_2\right),$$
$$\frac{d^p\left(fy_n, T\left(y_n, x_n\right)\right) + d^p\left(gz_2, G\left(z_2, z_1\right)\right)}{2},$$
$$\frac{d^p\left(fx_n, T\left(x_n, y_n\right)\right) d^p\left(gz_1, G\left(z_1, z_2\right)\right)}{1 + d^p\left(fy_n, gz_2\right)}, \quad (68)$$
$$\frac{d^p\left(gz_1, G\left(z_1, z_2\right)\right) d^p\left(fx_n, T\left(x_n, y_n\right)\right)}{1 + d^p\left(fy_n, fz_2\right)},$$
$$\left.\frac{d^p\left(fy_n, T\left(y_n, x_n\right)\right) d^p\left(gz_2, G\left(z_2, z_1\right)\right)}{1 + D^p\left(T\left(y_n, x_n\right), G\left(z_2, z_1\right)\right)}\right\}.$$

Applying limit to Θ, we have

$$\lim_{n\to\infty}\Theta\left(y_n, x_n, z_2, z_2\right) = \max\left\{d^p\left(fz_2, C_2\right),\right.$$
$$d^p\left(gz_2, G\left(z_2, z_1\right)\right), d^p\left(fz_2, gz_2\right),$$
$$\frac{d^p\left(fz_2, C_2\right) + d^p\left(gz_2, G\left(z_2, z_1\right)\right)}{2},$$
$$\frac{d^p\left(fz_1, C_1\right) d^p\left(gz_1, G\left(z_1, z_2\right)\right)}{1 + d^p\left(fz_2, gz_2\right)}, \quad (69)$$
$$\frac{d^p\left(gz_1, G\left(z_1, z_2\right)\right) d^p\left(fz_1, C_1\right)}{1 + d^p\left(fz_2, fz_2\right)},$$
$$\left.\frac{d^p\left(fz_2, C_2\right) d^p\left(gz_2, G\left(z_2, z_1\right)\right)}{1 + D^p\left(C_2, G\left(z_1, z_2\right)\right)}\right\},$$
$$\lim_{n\to\infty}\Theta\left(x_n, y_n, z_1, z_2\right) = d^p\left(gz_2, G\left(z_2, z_1\right)\right). \quad (70)$$

Applying limit to (67) using (70), we have

$$gz_2 \in G\left(z_2, z_1\right). \quad (71)$$

Following the similar line of Theorem 19 we can obtain that T, G, f, and g have common coupled fixed point. □

Example 21. Let $\Theta = (-10, 10)$ with the usual metric. Define $T, G : \Theta \times \Theta \longrightarrow CB(\Theta)$, $f, g : \Theta \longrightarrow \Theta$, $\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ and $F : \mathbb{R}^+ \longrightarrow \mathbb{R}$ by $T(x, y) = [-5, 1 + \alpha x/4]$, $G(x, y) = [-5, 1 + (\alpha/6)x]$, $f(x) = x/2$, $g(x) = x/3$ $\forall x, y \in \Theta$, $\varphi(t) = \alpha t$, $0 < \alpha < 1$ and $F = \ln(x)$.

Consider the sequences $\{x_n\} = \{1 + 1/n\}$, $\{y_n\} = \{2 - 3/n^2\}$, $\{u_n\} = \{1 - 1/n\}$, $\{v_n\} = \{2 - 2/n\}$.

Now,

$$\lim_{n\to\infty} T\left(x_n, y_n\right) = \left[-5, 1 + \frac{\alpha}{4}\right],$$
$$\lim_{n\to\infty} f\left(x_n\right) = \frac{1}{2} = f(1) \in \left[-5, 1 + \frac{\alpha}{4}\right].$$
$$\lim_{n\to\infty} G\left(u_n, v_n\right) = \left[-5, 1 + \frac{\alpha}{6}\right],$$
$$\lim_{n\to\infty} g\left(u_n\right) = \frac{1}{3} = g(1) \in \left[-5, 1 + \frac{\alpha}{6}\right]. \quad (72)$$
$$\lim_{n\to\infty} T\left(y_n, x_n\right) = \left[-5, 1 + \frac{\alpha}{2}\right],$$
$$\lim_{n\to\infty} f\left(y_n\right) = 1 = f(2) \in \left[-5, 1 + \frac{\alpha}{2}\right],$$
$$\lim_{n\to\infty} G\left(v_n, u_n\right) = \left[-5, 1 + \frac{\alpha}{3}\right],$$
$$\lim_{n\to\infty} g\left(v_n\right) = \frac{2}{3} = g(2) \in \left[-5, 1 + \frac{\alpha}{3}\right].$$

Therefore (f, T) and (g, G) satisfy CLR property. Now,

$$H\left(T\left(x, y\right), G\left(u, v\right)\right) = H\left(\left[-5, 1 + \frac{\alpha x}{4}\right], \left[-5, 1 + \frac{\alpha u}{6}\right]\right)$$
$$= \max\left\{d\left(\left[-5, 1 + \frac{\alpha x}{4}\right], \left[-5, 1 + \frac{\alpha u}{6}\right]\right),\right.$$
$$\left.d\left(\left[-5, 1 + \frac{\alpha x}{6}\right], \left[-5, 1 + \frac{\alpha u}{4}\right]\right)\right\}, \quad (73)$$
$$= \max\left\{\left|\frac{\alpha x}{6} - \frac{\alpha u}{4}\right|, 0\right\}, = \frac{\alpha}{2} d\left(gx, fu\right) = \frac{1}{2}$$
$$\cdot \varphi\left(d\left(gx, fu\right)\right) \le \frac{1}{2}\varphi\left(\Theta\left(x, y, u, v\right)\right)$$
$$\le e^{-1/6}\varphi\left(\Theta\left(x, y, u, v\right)\right).$$

Taking logarithm on both sides and $p = 1$, we conclude that all the other conditions of Theorem 19 are satisfied. Therefore T, f and G, g have common coupled fixed point.

Example 22. Let $\Theta = (-1, \infty)$ with the usual metric. Define $T, G : \Theta \times \Theta \longrightarrow CB(\Theta)$, $f, g : \Theta \longrightarrow \Theta$, $\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, $F : \mathbb{R}^+ \longrightarrow \mathbb{R}$ by $T(x, y) = [-0.5, 12 + \alpha x^2]$, $G(x, y) = [-0.5, 12 + (3\alpha/2)x^2]$, $f(x) = 2x^2$, $g(x) = 3x^2$ $\forall x, y \in \Theta$, $\varphi(t) = \alpha t$, $0 < \alpha < 1$ and $F = \ln(x)$.

Consider the sequences $\{x_n\} = \{1 + 1/n\}$, $\{y_n\} = \{2 - 3/n^2\}$, $\{u_n\} = \{1 - 1/n\}$, $\{v_n\} = \{2 - 2/n\}$.

Now,

$$\begin{aligned}
&\lim_{n\to\infty} T(x_n, y_n) = [-0.5, 12+\alpha],\\
&\quad \lim_{n\to\infty} f(x_n) = 2 = f(1) \in [-0.5, 12+\alpha].\\
&\lim_{n\to\infty} G(u_n, v_n) = \left[-0.5, 12+\frac{3}{2}\alpha\right],\\
&\quad \lim_{n\to\infty} g(u_n) = 3 = g(1) \in \left[-0.5, 12+\frac{3}{2}\alpha\right].\\
&\lim_{n\to\infty} T(y_n, x_n) = [-0.5, 12+4\alpha],\\
&\quad \lim_{n\to\infty} f(y_n) = 8 = f(2) \in [-0.5, 12+4\alpha],\\
&\lim_{n\to\infty} G(v_n, u_n) = [-0.5, 12+6\alpha],\\
&\quad \lim_{n\to\infty} g(v_n) = 12 = g(2) \in [-0.5, 12+6\alpha].
\end{aligned} \tag{74}$$

Therefore (f, T) and (g, G) satisfy CLR property. Now,

$$\begin{aligned}
&H(T(x,y), G(u,v)) = H\Big(\left[-0.5, 12+\alpha x^2\right],\\
&\left[-0.5, 12+\frac{3\alpha}{2}u^2\right]\Big) = \max\Big\{d\Big(\left[-0.5, 12+\alpha x^2\right],\\
&\left[-0.5, 12+\frac{3\alpha}{2}u^2\right]\Big), d\Big(\left[-0.5, 12+\frac{3\alpha}{2}x^2\right],\\
&\left[-0.5, 12+\alpha u^2\right]\Big)\Big\}, = \max\left\{\left|\frac{3\alpha}{2}u^2 - \alpha x^2\right|, 0\right\},\\
&= \frac{\alpha}{2}d(gu, fx) = \frac{1}{2}\varphi(d(gu, fx)) \le \frac{1}{2}\varphi(\Theta(x, y,\\
&u, v)) \le e^{-1/6}\varphi(\Theta(x, y, u, v)).
\end{aligned} \tag{75}$$

Taking logarithm on both sides and $p = 1$, we conclude that all the other conditions of Theorem 19 are satisfied. Therefore, T, f and G, g have common coupled fixed point.

Remark 23. From the above examples the following is clear.

(i) Theorem 13 is not applicable to Example 21 because $f(\Theta) = (-5, 5)$ nor $g(\Theta) = (-10/3, 10/3)$ are closed.

(ii) Theorem 14 is not applicable to Example 22, because neither $T(\Theta \times \Theta) \subseteq f(\Theta)$ nor $G : (\Theta \times \Theta) \subseteq g(\Theta)$.

(iii) Similarly the main results of [43] Theorem 2.1 and Theorem 2.6 are not applicable to the above examples

Next, we explain Example 27 of [27] to which our Theorem 19 is also applicable.

Example 24. Let $\Theta = [0, 1]$, equipped with the metric $d : \Theta \times \Theta \longrightarrow [0, \infty)$ define by

$$d(x, y) = \begin{cases} \max\{x, y\}, & \text{for all } x \ne y \in \Theta \\ d(x, y) = 0, & \text{for all } x = y \in \Theta. \end{cases} \tag{76}$$

Define $T, G : \Theta \times \Theta \longrightarrow CB(\Theta)$, $f, g : \Theta \longrightarrow \Theta$, $\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ and $F : \mathbb{R}^+ \longrightarrow \mathbb{R}$ by

$$\begin{aligned}
T(x, y) &= \begin{cases} \{0\}, & \text{if } x, y = 1 \\ \left[0, \dfrac{x^2 + y^2}{4}\right] & \text{if } x, y \in [0, 1), \end{cases}\\
G(x, y) &= \begin{cases} \{0\}, & \text{if } x, y = 1 \\ \left[0, \dfrac{x + y}{8}\right] & \text{if } x, y \in [0, 1), \end{cases}\\
f(x) &= \begin{cases} x^2, & \text{if } x \ne 1 \\ \dfrac{3}{2}, & \text{if } x = 1, \end{cases}\\
g(x) &= \begin{cases} \dfrac{x}{2}, & \text{if } x \ne 1 \\ 1, & \text{if } x = 1, \end{cases}\\
\varphi(t) &= \frac{t}{2}.
\end{aligned} \tag{77}$$

And

$$F = \ln\left(\frac{x}{3}\right) \tag{78}$$

Consider the sequences $\{x_n\} = \{1/n\}$, $\{y_n\} = \{3/n^2\}$, $\{u_n\} = \{1/n^2\}$, $\{v_n\} = \{2/n\}$.

Now,

$$\begin{aligned}
&\lim_{n\to\infty} T(x_n, y_n) = \{0\},\\
&\quad \lim_{n\to\infty} f(x_n) = 0 = f(0) \in \{0\}.\\
&\lim_{n\to\infty} G(u_n, v_n) = \{0\},\\
&\quad \lim_{n\to\infty} g(u_n) = 0 = g(0) \in \{0\}.\\
&\lim_{n\to\infty} T(y_n, x_n) = \{0\},\\
&\quad \lim_{n\to\infty} f(y_n) = 0 = f(0) \in \{0\},\\
&\lim_{n\to\infty} G(v_n, u_n) = \{0\},\\
&\quad \lim_{n\to\infty} g(v_n) = 0 = g(0) \in \{0\}.
\end{aligned} \tag{79}$$

Therefore (f, T) and (g, G) satisfy CLR property.

Now for $x, y, u, v \in [0, 1)$, we discuss the following cases.

Case 1. If $(x^2 + y^2)/4 = (u + v)/8$, then

$$
\begin{aligned}
\frac{1}{3}H(T(x,y),G(u,v)) &= \frac{1}{3}\frac{u+v}{8} \\
&\leq \frac{1}{3}\frac{1}{4}\left[\max\left\{x^2,\frac{u}{2}\right\}+\max\left\{\frac{v}{2},\frac{u+v}{8}\right\}\right] \\
&= \frac{1}{3}\frac{((1/2)\,d(fx,gu)+(1/2)\,d(gv,G(u,v))}{2} \\
&= \frac{1}{3}\varphi\left(\frac{1}{2}\,d(fx,gu)+\frac{1}{2}d(gv,G(u,v))\right. \\
&= \frac{1}{3}\varphi(\Theta(x,y,u,v)) \\
&\approx e^{-0.000000001}\frac{1}{3}\varphi(\Theta(x,y,u,v)).
\end{aligned} \tag{80}
$$

Case 2. If $(x^2+y^2)/4 \neq (u+v)/8$ and $(x^2+y^2)/4 < (u+v)/8$, then

$$
\begin{aligned}
\frac{1}{3}H(T(x,y),G(u,v)) &= \frac{1}{3}\frac{u+v}{8} \\
&\leq \frac{1}{3}\frac{1}{4}\left[\max\left\{x^2,\frac{u}{2}\right\}+\max\left\{\frac{v}{2},\frac{u+v}{8}\right\}\right] \\
&= \frac{1}{3}\frac{((1/2)\,d(fx,gu)+(1/2)\,d(gv,G(u,v))}{2} \\
&= \frac{1}{3}\varphi\left(\frac{1}{2}\,d(fx,gu)+\frac{1}{2}d(gv,G(u,v))\right. \\
&= \frac{1}{3}\varphi(\Theta(x,y,u,v)) \\
&\approx e^{-0.000000001}\frac{1}{3}\varphi(\Theta(x,y,u,v)).
\end{aligned} \tag{81}
$$

Case 3. If $(x^2+y^2)/4 > (u+v)/8$, then

$$
\begin{aligned}
\frac{1}{3}H(T(x,y),G(u,v)) &= \frac{1}{3}\frac{x^2+y^2}{4} \\
&\leq \frac{1}{3}\frac{1}{4}\left[\max\left\{x^2,\frac{u}{2}\right\}+\max\left\{y^2,\frac{x^2+y^2}{4}\right\}\right] \\
&= \frac{1}{3}\frac{((1/2)\,d(fx,gu)+(1/2)\,d(fy,T(y,x))}{2} \\
&= \frac{1}{3}\varphi\left(\frac{1}{2}\,d(fx,gu)+\frac{1}{2}d(gv,G(u,v))\right. \\
&= \frac{1}{3}\varphi(\Theta(x,y,u,v)) \\
&\approx e^{-0.000000001}\frac{1}{3}\varphi(\Theta(x,y,u,v)),
\end{aligned} \tag{82}
$$

Similarly it is easy to show the same result for $x, y \in [0,1)$ and $u, v = 1$ and for $x, y, u, v = 1$. Taking logarithm on both sides and $p = 1$. we conclude that all conditions of our Theorem 19 are satisfied. Therefore T, f and G, g have common coupled fixed point.

Remark 25. (i) From Example 24 it is clear that all conditions of our Theorem 19 are satisfied for Example 27 of [27] and hence the corresponding conclusions hold.

3. Applications to System of Functional Equations

In this section, we discuss common solution for two coupled functional equations with the help of Theorem 19. Throughout this unit $\widehat{Z}$ and $\widehat{Y}$ stand for Banach spaces, the state space is $\widetilde{E} \subset \widehat{Z}$, the decision space is $\widetilde{F} \subset \widehat{Y}$, and the space of all bounded real-valued functions on $\widetilde{E}$ is $\Theta = B(\widetilde{E})$ which is Banach space.

Define $d : \Theta \times \Theta \longrightarrow \mathbb{R}^+$, by

$$
d(\acute{u}_1,\acute{u}_2) = \sup_{x\in\widetilde{E}}\left|\acute{u}_1(x)-\acute{u}_2(x)\right| = \left\|\acute{u}_1-\acute{u}_2\right\|. \tag{83}
$$

Here,

$$
\|\acute{u}\| = \sup\left\{|\acute{u}(x)| : \kappa_{11}\in\widetilde{E}\right\} \quad \forall \acute{u}\in B\left(\widetilde{E}\right). \tag{84}
$$

Consider the following system

$$
\begin{aligned}
g_1(\kappa_{11}) &= \sup\nolimits_{\kappa_{22}\in\widetilde{F}}\{\mu(\kappa_{11},\kappa_{22}) \\
&\quad + \Psi_1(\kappa_{11},\kappa_{22},g_1(b_1(\kappa_{11},\kappa_{22}))), \\
&\quad g_2(b_2(\kappa_{11},\kappa_{22})))\} \quad \forall\kappa_{11}\in\widetilde{E}, \\
g_2(\kappa_{11}) &= \sup\nolimits_{\kappa_{22}\in\widetilde{F}}\{\mu(\kappa_{11},\kappa_{22}) \\
&\quad + \Psi_2(\kappa_{11},\kappa_{22},g_2(b_2(\kappa_{11},\kappa_{22}))), \\
&\quad g_1(b_1(\kappa_{11},\kappa_{22})))\} \quad \forall\kappa_{11}\in\widetilde{E},
\end{aligned} \tag{85}
$$

where $\mu : \widetilde{E}\times\widetilde{F} \longrightarrow \mathbb{R}$, $b_i : \widetilde{E}\times\widetilde{F} \longrightarrow \widetilde{E}$, $\Psi_i : \widetilde{E}\times\widetilde{F}\times\mathbb{R} \longrightarrow \mathbb{R}$ for $i = 1,2$ and κ_{11}, κ_{22} denote the state vectors and decision vectors, respectively, b_1, b_2 signify the transformations of the process, and $g_1(\kappa_{11}), g_2(\kappa_{11})$ symbolized the sup return functions under the initial state κ_{11}.

Let $T, G : B(\widetilde{E})\times B(\widetilde{E}) \longrightarrow B(\widetilde{E})$, defined by

$$
\begin{aligned}
T(g_1,g_2)(\kappa_{11}) &= \sup\nolimits_{\kappa_{22}\in\widetilde{F}}\{\mu(\kappa_{11},\kappa_{22}) \\
&\quad + \Psi_1(\kappa_{11},\kappa_{22},g_1(b_1(\kappa_{11},\kappa_{22}))), \\
&\quad g_2(b_2(\kappa_{11},\kappa_{22})))\}, \\
G(g_1,g_2)(\kappa_{11}) &= \sup\nolimits_{\kappa_{22}\in\widetilde{F}}\{\mu(\kappa_{11},\kappa_{22}) \\
&\quad + \Psi_2(\kappa_{11},\kappa_{22},g_2(b_2(\kappa_{11},\kappa_{22}))), \\
&\quad g_1(b_1(\kappa_{11},\kappa_{22})))\}.
\end{aligned} \tag{86}
$$

Theorem 26. *Assume $T, G : B(\widetilde{E})\times B(\widetilde{E}) \longrightarrow B(\widetilde{E})$ to be maps given by (86) which holds the following conditions.*

(1) *μ and Ψ_i, for $i = 1, 2$, are bounded.*

(2) *For* $(\kappa_{11}, \kappa_{22}) \in \widetilde{E} \times \widetilde{F}$ *and* $g_1, g_1', g_2, g_2' \in B(\widetilde{E})$

$$\begin{aligned}&\Big|\Psi_1(\kappa_{11},\kappa_{22},g_1(b_1(\kappa_{11},\kappa_{22}))),g_2(b_2(\kappa_{11},\kappa_{22})))\\&\quad-\Psi_2\left(\kappa_{11},\kappa_{22},g_2'(b_2(\kappa_{11},\kappa_{22}))\right),\\&\quad g_1'(b_1(\kappa_{11},\kappa_{22})))\Big|\le e^{-\tau}\varphi(\Theta(x,y,u,v)).\end{aligned}\tag{87}$$

Here,

$$\begin{aligned}&\Theta(x,y,u,v)\\&\quad=\alpha\left[d^p(x,u)\right]\\&\qquad+\beta\left[\frac{d^p(y,T(y,x))\,d^p(v,G(v,u))}{1+d^p(x,u)}\right]\\&\qquad+\gamma\left[d^p(x,T(x,y))+d^p(u,G(u,v))\right]\\&\qquad+\sigma\left[d^p(x,T(x,y))\right]+\eta\left[d^p(u,G(u,v))\right].\end{aligned}\tag{88}$$

Then, system (85) has a common solution in $B(\widetilde{E})$.

Proof. Let λ be an arbitrary positive real number and there exist $g_1, g_1', g_2, g_2' \in B(\widetilde{E})$, for arbitrary $\kappa_{11} \in \widetilde{E}$, $\kappa_{22} \in \widetilde{F}$ such that

$$\begin{aligned}&T(g_1,g_2)(\kappa_{11})\le\mu(\kappa_{11},\kappa_{22})\\&\quad+\Psi_1(\kappa_{11},\kappa_{22},g_1(b_1(\kappa_{11},\kappa_{22}))),g_2(b_2(\kappa_{11},\kappa_{22})))\\&\quad+\lambda,\end{aligned}\tag{89}$$

$$\begin{aligned}&G\left(g_1',g_2'\right)(\kappa_{11})\le\mu(\kappa_{11},\kappa_{22})\\&\quad+\Psi_2\left(\kappa_{11},\kappa_{22},g_2'(b_2(\kappa_{11},\kappa_{22}))\right),g_1'(b_1(\kappa_{11},\kappa_{22})))\\&\quad+\lambda.\end{aligned}\tag{90}$$

From definition of T and G we have

$$\begin{aligned}&T(g_1,g_2)(\kappa_{11})>\mu(\kappa_{11},\kappa_{22})\\&\quad+\Psi_1(\kappa_{11},\kappa_{22},g_1(b_1(\kappa_{11},\kappa_{22}))),\\&\quad g_2(b_2(\kappa_{11},\kappa_{22}))),\end{aligned}\tag{91}$$

$$\begin{aligned}&G\left(g_1',g_2'\right)(\kappa_{11})>\mu(\kappa_{11},\kappa_{22})\\&\quad+\Psi_2\left(\kappa_{11},\kappa_{22},g_2'(b_2(\kappa_{11},\kappa_{22}))\right),\\&\quad g_1'(b_1(\kappa_{11},\kappa_{22}))).\end{aligned}\tag{92}$$

Next, from (89) and (92) we have

$$\begin{aligned}&T(g_1,g_2)(\kappa_{11})-G\left(g_1',g_2'\right)(\kappa_{11})\le\mu(\kappa_{11},\kappa_{22})\\&\quad+\Psi_1(\kappa_{11},\kappa_{22},g_1(b_1(\kappa_{11},\kappa_{22}))),g_2(b_2(\kappa_{11},\kappa_{22})))\\&\quad-\mu(\kappa_{11},\kappa_{22})-\Psi_2\left(\kappa_{11},\kappa_{22},g_2'(b_2(\kappa_{11},\kappa_{22}))\right),\\&\quad g_1'(b_1(\kappa_{11},\kappa_{22})))+\lambda\\&\quad\le\Big|\Psi_1(\kappa_{11},\kappa_{22},g_1(b_1(\kappa_{11},\kappa_{22}))),\\&\quad g_2(b_2(\kappa_{11},\kappa_{22})))\}\\&\quad-\Psi_2\left(\kappa_{11},\kappa_{22},g_2'(b_2(\kappa_{11},\kappa_{22}))\right),\\&\quad g_1'(b_1(\kappa_{11},\kappa_{22})))\Big|+\lambda\le e^{-\tau}\varphi(\Theta(x,y,u,v))+\lambda.\end{aligned}\tag{93}$$

Similarly from (90) and (91) we get

$$\begin{aligned}&G\left(g_1',g_2'\right)(\kappa_{11})-T(g_1,g_2)(\kappa_{11})\\&\quad\le e^{-\tau}\varphi(\Theta(x,y,u,v))+\lambda.\end{aligned}\tag{94}$$

Combining (93) and (94) we conclude that

$$\begin{aligned}&\left|G\left(g_1',g_2'\right)(\kappa_{11})-T(g_1,g_2)(\kappa_{11})\right|\\&\quad\le e^{-\tau}\varphi(\Theta(x,y,u,v))+\lambda.\end{aligned}\tag{95}$$

By taking $f(x) = g(x) = x$, $F = \ln(x)$ and $p = 1$ in Theorem 19. Then we deduce that the mappings T, G have a common coupled fixed point in $B(\widetilde{E})$; that is the system (85) has a solution. □

4. Applications to Matrix Equations

In this section, we study the nonlinear matrix equations with the help of Theorem 20.

$$\Delta_1 = Q + \sum_{i=1}^{m} A_i^* G_1(\Delta_1) A_i - \sum_{j=1}^{k} B_j^* K_1(\Delta_1) B_j, \tag{96}$$

$$\Delta_2 = Q + \sum_{i=1}^{m} A_i^* G_2(\Delta_1) A_i - \sum_{j=1}^{k} B_j^* K_2(\Delta_1) B_j. \tag{97}$$

Here Q is a positive definite matrix, A_i, B_j are arbitrary $n \times n$ matrices, and continuous order preserving maps are G_1, G_2, K_1, K_2 defined from $\mathbb{H}(n)$ into $\mathbb{P}(n)$ such that $G_1(0) = G_2(0) = K_1(0) = K_2(0) = 0$.

In this unit we will use the following notations:

$\mathbb{M}(n)$ symbolizes the set of all $n \times n$ complex matrices, $\mathbb{H}(n) \subset \mathbb{M}(n)$ the set of all $n \times n$ Hermitian matrices, and $\mathbb{P}(n) \subset \mathbb{H}(n)$ is the set of all $n \times n$ positive definite matrices. As a replacement for of $\Delta_1 \in \mathbb{P}(n)$ we will also write $\Delta_1 > 0$. Similarly, positive semidefinite matrix Δ_1 is denoted by $\Delta_1 \geq 0$. We also signify by $\|.\|$ the spectral norm, i.e.,

$$\|B\| = \sqrt{\lambda^+ (B^* B)}), \tag{98}$$

where the biggest eigenvalue of $B^* B$ is $\lambda^+(B^* B)$. We will use the metric induced by the trace norm $\|.\|_1$ defined by $\|B\|_1 =$

$\sum_{q=1}^{n} S_q(B)$, where $S_q(B), q = 1, \ldots, n$ are the singular values of B. The set $\mathbb{H}(n)$ is a complete metric space endowed with this norm.

The following lemma which is taken from [6] will be useful in the study of the matrix equations.

Lemma 27. *Let $A \geq 0$ and $B \geq 0$ be $n \times n$ matrices; then $0 \leq tr(AB) \leq \|A\| tr(B)$.*

In this section, we define the mapping $T, G : \mathbb{H}(n) \times \mathbb{H}(n) \longrightarrow \mathbb{H}(n)$ by

$$T(\Delta_1, \Delta_2) = Q + \sum_{i=1}^{m} A_i^* G_1(\Delta_1) A_i - \sum_{j=1}^{k} B_j^* K_1(\Delta_2) B_j, \quad (99)$$

$$G(\Delta_1, \Delta_2) = Q + \sum_{i=1}^{m} A_i^* G_2(\Delta_1) A_i - \sum_{j=1}^{k} B_j^* K_2(\Delta_2) B_j. \quad (100)$$

Here $Q \in \mathbb{P}(n)$, $A_i, B_j \in M(n)$, and G_1, G_2, K_1, K_2 are continuous order-preserving maps. In the following theorem we first discuss the existence of common coupled fixed point of T and G in $\mathbb{H}(n) \times \mathbb{H}(n)$.

Theorem 28. *Let $Q \in \mathbb{P}(n)$ such that*

(1) *for every $(\Delta_1, \Delta_2), (\Delta_3, \Delta_4) \in \mathbb{H}(n) \times \mathbb{H}(n)$, and*

$$|tr(G_1(\Delta_3) - G_2(\Delta_1))| \leq e^{-\tau} |tr(\Delta_3 - \Delta_1)|, \quad (101)$$

and

$$|tr(K_1(\Delta_2) - K_2(\Delta_4))| \leq e^{-\tau} |tr(\Delta_2 - \Delta_4)|, \quad (102)$$

(2) $\|\Delta_2 - \Delta_4\| \leq \|\Delta_3 - \Delta_1\|$,

(3) $\|\sum_{i=1}^{m} A_i A_i^*\| < 1/4$ *and* $\|\sum_{j=1}^{k} B_j B_j^*\| < 1/4$.

Then, there exist $\Delta_1^*, \Delta_2^* \in \mathbb{H}(n)$ such that $T(\Delta_1^*, \Delta_2^*) = \Delta_1^*$ and $T(\Delta_2^*, \Delta_1^*) = \Delta_2^*$. $G(\Delta_1^*, \Delta_2^*) = \Delta_1^*$ and $G(\Delta_2^*, \Delta_1^*) = \Delta_2^*$.

Proof. Let $(\Delta_1, \Delta_2), (\Delta_3, \Delta_4) \in \mathbb{H}(n) \times \mathbb{H}(n)$; then

$$\begin{aligned}
&\|T(\Delta_3, \Delta_4) - G(\Delta_1, \Delta_2)\|_1 \\
&= \Big(tr(T(\Delta_3, \Delta_4) - G(\Delta_1, \Delta_2)) \\
&= \sum_{i=1}^{m} tr(A_i^* (G_1(\Delta_3) - G_2(\Delta_1)) A_i) \\
&+ \sum_{j=1}^{k} tr(B_j^* (K_1(\Delta_2) - K_2(\Delta_4)) B_j) \\
&= \sum_{i=1}^{m} tr(A_i A_i^* (G_1(\Delta_3) - G_2(\Delta_1))) \\
&+ \sum_{j=1}^{k} tr(BjB_j^* (K_1(\Delta_2) - K_2(\Delta_4))) \\
&= tr\left(\left(\sum_{i=1}^{m} A_i A_i^*\right)(G_1(\Delta_3) - G_2(\Delta_1))\right) \\
&+ tr\left(\sum_{j=1}^{k} B_j B_j^*\right)(K_1(\Delta_2) - K_2(\Delta_4))) \\
&\leq \left\|\sum_{i=1}^{m} A_i A_i^*\right\| |tr(G_1(\Delta_3) - G_2(\Delta_1))| \\
&+ \left\|\sum_{j=1}^{k} B_j B_j^*\right| tr(K_1(\Delta_2) - K_2(\Delta_4))\Big| \\
&\leq \left\|\sum_{i=1}^{m} A_i A_i^*\right\| e^{-\tau} \|\Delta_3 - \Delta_1\|_1 + \left\|\sum_{j=1}^{k} B_j B_j^*\right\| e^{-\tau} \|\Delta_2 \\
&- \Delta_4\|_1 \leq \left\|\sum_{i=1}^{m} A_i A_i^*\right\| e^{-\tau} \|\Delta_3 - \Delta_1\|_1 + \left\|\sum_{j=1}^{k} B_j B_j^*\right\| \\
&\cdot e^{-\tau} \|\Delta_3 - \Delta_1\|_1 \leq \frac{1}{2} e^{-\tau} \|\Delta_3 - \Delta_1\|_1, \\
&\leq e^{-\tau} \varphi(\Theta(\Delta_1, \Delta_2, \Delta_3, \Delta_4)).
\end{aligned} \quad (103)$$

Thus, the contractive condition of Theorem 20 is satisfied for all $(\Delta_1, \Delta_2), (\Delta_3, \Delta_4) \in \mathbb{H}(n) \times \mathbb{H}(n)$. By taking $f(x) = g(x) = x$, $\varphi(t) = (1/2)t$, $F = \ln(x)$ and $p = 1$ in Theorem 20. From Theorem 20, T and G have a common coupled fixed point. □

Data Availability

Data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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