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*Research Article*

# Commutators of the Bilinear Hardy Operator on Herz Type Spaces with Variable Exponents

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In this paper, we obtain the boundedness of bilinear commutators generated by the bilinear Hardy operator and BMO functions on products of Herz spaces and Herz-Morrey spaces with variable exponents.

## 1. Introduction

Denote by $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ the set of all complex-valued locally integrable functions on $\mathbb{R}^n$. The Hardy operator was first considered in [1] as follows:

$$Hf(x) := \frac{1}{x}\int_0^x f(t)\,dt, \quad x \neq 0, \tag{1}$$

for $f \in L^1_{\mathrm{loc}}(\mathbb{R})$. In 1976, Faris [2] generalized it to the $n$ dimensional Euclidean space as

$$Hf(x) := \frac{1}{\Omega_n |x|^n}\int_{|y|<|x|} f(y)\,dy, \quad x \in \mathbb{R}^n \setminus \{0\}, \tag{2}$$

where $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\Omega_n = \pi^{n/2}/\Gamma(1+n/2)$ is the volume of the unit ball in $\mathbb{R}^n$. In [3], Christ and Grafakos proved that the Hardy operator is bounded on $L^p(\mathbb{R}^n)$.

For $m \in \mathbb{N}$, the $m$-linear Hardy operator was defined in [4] as

$$\begin{aligned} &H(f_1, \cdots f_m) \\ &\quad := \frac{1}{\Omega_{mn}|x|^{mn}}\int_{|(y_1,\cdots,y_m)|<|x|}\prod_{i=1}^m f_i(y_i)\,dy_1\cdots dy_m, \\ &\qquad\qquad x \in \mathbb{R}^n \setminus \{0\}, \end{aligned} \tag{3}$$

for $f_1, \ldots, f_m$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. The 2-linear operator will be referred to as bilinear operator.

A function $b$ belongs to $\mathrm{BMO}(\mathbb{R}^n)$ (bounded mean oscillation), if $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and

$$\|b\|_* := \sup_B \frac{1}{|B|}\int_B |b(x) - b_B|\,dx < \infty, \tag{4}$$

where the supremum is taken over all balls in $\mathbb{R}^n$, $b_B$ is the mean of $b$ on $B$, and what follows $|E|$ is the Lebesgue measure of measurable set $E$ in $\mathbb{R}^n$.

Let $b_i \in \mathrm{BMO}(\mathbb{R}^n)$ for $i = 1, \ldots, m$. Then the commutator generated by the $m$-linear Hardy operator and $b_1, \ldots, b_m$ is defined by

$$H_{\vec{b}}(f_1, \ldots, f_m)(x) = \sum_{i=1}^m H^i_{b_i}(f_1, \ldots, f_m)(x), \tag{5}$$

where

$$\begin{aligned} &H^i_{b_i}(f_1, \ldots, f_m)(x) \\ &\quad = b_i(x) H(f_1, \ldots, f_m)(x) \\ &\qquad - H(f_1, \ldots, f_{i-1}, f_i b_i, f_{i+1}, \ldots, f_m)(x). \end{aligned} \tag{6}$$

When $m = 1$, the operator is $H_b(f)(x) = b(x)Hf(x) - H(bf)(x)$.

Since the fundamental paper [5] by Kováčik and Rákosník appeared in 1991, the Lebesgue spaces with variable exponent have attracted a great attention. Many constant exponent

function spaces are generalized to variable exponent setting, such as Bessel potential spaces with a variable exponent [6], Besov and Trieble-Lizorkin spaces with variable exponents [7–12], Hardy spaces with variable exponents [13], Morrey spaces with variable exponents [14], Herz spaces with variable exponents [15], and Herz-Morrey spaces with variable exponents [16, 17]. These spaces have many applications, such as in the electrorheological fluid [18]; image restoration [19–23]; the Black-Scholes equation [24]; ordinary and partial differential equations [25–27] and references therein.

The boundedness of the Hardy operator was considered in many function spaces, such as in variable Lebesgue spaces [28, 29], variable exponent Sobolev spaces [30]. The boundedness of commutators of the Hardy operator was obtained in $\lambda$-central BMO spaces [31], Herz spaces $\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$ with variable exponent $p(\cdot)$ [32]. In [33], Wu proved the boundedness of multilinear commutators of fractional Hardy operators on Herz-Morrey spaces $M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)$. In [34], Wu considered the boundedness for fractional Hardy type operator on Herz-Morrey spaces $M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)$ with variable exponent $q(\cdot)$ but fixed $\alpha \in \mathbb{R}$ and $p \in (0,\infty)$. Wu and Zhang obtained the boundedness of multilinear Hardy type operators on the product of Herz-Morrey spaces $M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)$ with variable exponent $q(\cdot)$ and the boundedness of fractional Hardy type operators on Herz-Morrey spaces $M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)$ with variable exponent $q(\cdot)$ in [35, 36]. Wu and Zhang considered the boundedness of commutators of the fractional Hardy operators on Herz-Morrey spaces $M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)$ with variable exponent $q(\cdot)$ in [17]. Shu, Wang and Meng obtained the boundedness of commutators of Hardy type operators on Herz spaces $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}$ with variable exponents $\alpha(\cdot)$ and $p(\cdot)$ in [37]. Wu and Zhao proved the boundedness for variable fractional Hardy type operator on variable exponent Herz-Morrey spaces $M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n)$ in [38]. In [39], Xu and Yang introduced the Herz-Morrey-Hardy spaces $HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}$ with variable exponents and established their characterization in terms of atom. Moreover, applying the characterization, they obtained the boundedness of some singular integral operators on these spaces.

Motivated by the mentioned works, in this paper, we will consider the boundedness of $m$-linear commutators generated by the $m$-linear Hardy operator and BMO functions on Herz spaces and Herz-Morrey spaces with variable exponents.

## 2. Main Results

To state our results, let us first recall some definitions and notations. Let $\Omega$ be a positive measurable subset of $\mathbb{R}^n$, given a measurable function $p(\cdot) : \Omega \longrightarrow [1,\infty)$. The Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ is defined by

$$L^{p(\cdot)}(\Omega) := \left\{ f \text{ is measurable: } \int_{\mathbb{R}^n} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} dx < \infty \text{ for some } \lambda > 0 \right\}. \tag{7}$$

The Lebesgue space $L^{p(\cdot)}(\Omega)$ becomes a Banach function space equipped with the norm

$$\|f\|_{L^{p(\cdot)}} := \inf \left\{ \lambda > 0 : \int_{\Omega} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} dx \le 1 \right\}. \tag{8}$$

The space $L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)$ is the collection of all functions $f$ such that $f_{\chi_K} \in L^{p(\cdot)}(\mathbb{R}^n)$ for each compact subset $K \subset \mathbb{R}^n$. Here and what follows, $\chi_S$ denotes the characteristic function of a measurable set $S \subset \mathbb{R}^n$. Let $p(\cdot) : \mathbb{R}^n \longrightarrow (0,\infty)$; we denote $p_- := \operatorname{ess\,inf}_{x\in\mathbb{R}^n} p(x)$, $p_+ := \operatorname{ess\,sup}_{x\in\mathbb{R}^n} p(x)$. The set $\mathscr{P}(\mathbb{R}^n)$ consists of all $p(\cdot)$ such that $p_- > 1$ and $p_+ < \infty$; $\mathscr{P}^0(\mathbb{R}^n)$ consists of all $p(\cdot)$ such that $p_- > 0$ and $p_+ < \infty$. If $p(\cdot) \in \mathscr{P}^0(\mathbb{R}^n)$, then the space $L^{p(\cdot)}$ is similarly defined as above. $p'(\cdot)$ means that the conjugate exponent of $p(\cdot)$ that means $1/p(\cdot) + 1/p'(\cdot) = 1$.

The standard Hardy-Littlewood maximal operator $M$ is defined for $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ function $f$ by

$$Mf(x) = \sup_{B\ni x} \frac{1}{|B|} \int_B |f(y)|\, dy, \quad \forall x \in \mathbb{R}^n, \tag{9}$$

where $B$ is a ball. We denote by $\mathscr{B}(\mathbb{R}^n)$ the set of $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ such that $M$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. The common sufficient conditions for variable exponent to be in $\mathscr{B}(\mathbb{R}^n)$ are the following well known log-Hölder continuity, which introduced in [40–42].

*Definition 1.* Let $\alpha(\cdot)$ be a real-valued measurable function on $\mathbb{R}^n$.

(i) If there exists a constant $C_1$ such that

$$|\alpha(x) - \alpha(y)| \le \frac{C_1}{\log(e + 1/|x-y|)}, \quad x, y \in \mathbb{R}^n,\ |x-y| < \frac{1}{2}, \tag{10}$$

then the function $\alpha(\cdot)$ is called locally log-Hölder continuous.

(ii) If there exists a constant $C_2$ such that

$$|\alpha(x) - \alpha(0)| \le \frac{C_2}{\log(e + 1/|x|)}, \quad \forall x \in \mathbb{R}^n, \tag{11}$$

then the function $\alpha(\cdot)$ is called log-Hölder continuous at the origin and denoted by $\alpha(\cdot) \in \mathscr{P}^{\log}_0(\mathbb{R}^n)$;

(iii) If there exist $\alpha_\infty \in \mathbb{R}$ and a positive constant $C_3$ such that

$$|\alpha(x) - \alpha_\infty| \le \frac{C_3}{\log(e + |x|)}, \quad \forall x \in \mathbb{R}^n, \tag{12}$$

then the function $\alpha(\cdot)$ is called log-Hölder continuous at infinity and denoted by $\alpha(\cdot) \in \mathscr{P}^{\log}_\infty(\mathbb{R}^n)$.

(iv) If $\alpha(\cdot)$ are both locally log-Hölder continuous and log-Hölder continuous at the infinity, then the function $\alpha(\cdot)$ is called global log-Hölder continuous and denoted by $\alpha(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$.

To state the definitions of Herz space and Herz-Morrey space with variable exponents, we use the following notations. For each $k \in \mathbb{Z}$ we denote $B_k := \{x \in \mathbb{R}^n : |x| \le 2^k\}$, $D_k := B_k \setminus B_{k-1}$, $\chi_k := \chi_{D_k}$, $\widetilde{\chi}_m = \chi_m$, $m \ge 1$, $\widetilde{\chi}_0 = \chi_{B_0}$.

*Definition 2.* Let $0 < q \le \infty$, $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, and $\alpha(\cdot) : \mathbb{R}^n \longrightarrow \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$.

(i) The homogeneous Herz space $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)$ is defined by

$$\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n) := \left\{ f \in L_{\mathrm{loc}}^{p(\cdot)}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}} < \infty \right\}, \quad (13)$$

where

$$\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}} := \left\{ \sum_{k=-\infty}^{\infty} \left\| 2^{k\alpha(\cdot)} f\chi_k \right\|_{L^{p(\cdot)}}^q \right\}^{1/q} < \infty. \quad (14)$$

(ii) The inhomogeneous Herz space $K_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)$ is defined by

$$K_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n) := \left\{ f \in L_{\mathrm{loc}}^{p(\cdot)}(\mathbb{R}^n) : \|f\|_{K_{p(\cdot)}^{\alpha(\cdot),q}} < \infty \right\}, \quad (15)$$

where

$$\|f\|_{K_{p(\cdot)}^{\alpha(\cdot),q}} := \left\{ \sum_{m=0}^{\infty} \left\| 2^{m\alpha(\cdot)} f\widetilde{\chi}_m \right\|_{L^{p(\cdot)}}^q \right\}^{1/q} < \infty. \quad (16)$$

*Remark 3.* Obviously, if $0 < q_1 \le q_2 \le \infty$, then $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q_1}(\mathbb{R}^n) \subset \dot{K}_{p(\cdot)}^{\alpha(\cdot),q_2}(\mathbb{R}^n)$. If both $\alpha(\cdot)$ and $p(\cdot)$ are constants, then $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n) = \dot{K}_p^{\alpha,q}(\mathbb{R}^n)$ is the classical Herz spaces in [43].

To generalize the above spaces to variable exponent $q(\cdot)$, we need the notation of the variable mixed sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$, which is defined as follows. Given a sequence of functions $\{f_j\}_{j\in\mathbb{Z}}$, define the modular

$$\begin{aligned} &\rho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left(\left(f_j\right)_j\right) \\ &\quad := \sum_{j\in\mathbb{Z}} \inf\left\{\lambda_j > 0 : \rho_{p(\cdot)}\left(\lambda_j^{-1/q(\cdot)} f_j\right) \le 1\right\}, \end{aligned} \quad (17)$$

where $\lambda^{1/\infty} = 1$. If $q^+ < \infty$ or $q(\cdot) \le p(\cdot)$, the above can be written as

$$\rho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left(\left(f_j\right)_j\right) = \sum_{\nu\in\mathbb{Z}} \left\| \left|f_j(\cdot)\right|^{q(\cdot)} \right\|_{L^{p(\cdot)/q(\cdot)}}. \quad (18)$$

The norm is

$$\begin{aligned} &\left\|\left(f_j\right)_j\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \\ &\quad := \inf\left\{\mu > 0 : \rho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left(\left(\frac{f_j}{\mu}\right)_j\right) \le 1\right\}. \end{aligned} \quad (19)$$

Now, the space $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}$ is the collection of all functions $f \in L_{\mathrm{loc}}^{p(\cdot)}(\mathbb{R}^n \setminus \{0\})$ such that

$$\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)} := \left\| \left\{ 2^{\alpha(\cdot)k} f\chi_k \right\}_{k\in\mathbb{Z}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty, \quad (20)$$

and the space $K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}$ is the collection of all functions in $f \in L_{\mathrm{loc}}^{p(\cdot)}(\mathbb{R}^n)$ such that

$$\|f\|_{K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)} := \left\| \left\{ 2^{\alpha(\cdot)k} f\widetilde{\chi}_k \right\}_{k\in\mathbb{N}_0} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty. \quad (21)$$

In [44], Drihem and Seghiri proved the following result.

**Lemma 4** (see [44, Proposition 1]). *Let $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$, $p(\cdot)$, $q(\cdot) \in \mathscr{P}_0(\mathbb{R}^n)$. If $\alpha(\cdot)$ and $q(\cdot)$ are log-Hölder continuous at infinity, then*

$$K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n) = K_{p(\cdot)}^{\alpha_\infty,q_\infty}(\mathbb{R}^n). \quad (22)$$

*Additionally, if $\alpha(\cdot)$ and $q(\cdot)$ have a log decay at the origin, then*

$$\begin{aligned} \|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}} \approx &\left( \sum_{k\le 0} \left\| 2^{k\alpha(0)} f\chi_k \right\|_{L^{p(\cdot)}}^{q(0)} \right)^{1/q(0)} \\ &+ \left( \sum_{k>0} \left\| 2^{k\alpha_\infty} f\chi_k \right\|_{L^{p(\cdot)}}^{q_\infty} \right)^{1/q_\infty}. \end{aligned} \quad (23)$$

*Definition 5.* Let $0 < q \le \infty$, $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, $0 \le \lambda < \infty$, and $\alpha(\cdot) : \mathbb{R}^n \longrightarrow \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$.

(i) The homogeneous Herz-Morrey space $M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}(\mathbb{R}^n)$ is defined by

$$\begin{aligned} &M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}(\mathbb{R}^n) \\ &\quad := \left\{ f \in L_{\mathrm{loc}}^{p(\cdot)}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{MK_{p(\cdot),\lambda}^{\alpha(\cdot),q}} < \infty \right\}, \end{aligned} \quad (24)$$

where

$$\begin{aligned} \|f\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} &:= \sup_{L\in\mathbb{Z}} 2^{-L\lambda} \left\{ \sum_{k=-\infty}^{L} \left\| 2^{k\alpha(\cdot)} f\chi_k \right\|_{L^{p(\cdot)}}^q \right\}^{1/q} \\ &< \infty. \end{aligned} \quad (25)$$

(ii) The inhomogeneous Herz space $MK_{p(\cdot),\lambda}^{\alpha(\cdot),q}(\mathbb{R}^n)$ is defined by

$$\begin{aligned} &MK_{p(\cdot),\lambda}^{\alpha(\cdot),q}(\mathbb{R}^n) \\ &\quad := \left\{ f \in L_{\mathrm{loc}}^{p(\cdot)}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{MK_{p(\cdot),\lambda}^{\alpha(\cdot),q}} < \infty \right\}, \end{aligned} \quad (26)$$

where

$$\begin{aligned} \|f\|_{MK_{p(\cdot),\lambda}^{\alpha(\cdot),q}(\mathbb{R}^n)} &:= \sup_{L\in\mathbb{N}} 2^{-L\lambda} \left\{ \sum_{m=-\infty}^{L} \left\| 2^{k\alpha(\cdot)} f\widetilde{\chi}_m \right\|_{L^{p(\cdot)}}^q \right\}^{1/q} \\ &< \infty. \end{aligned} \quad (27)$$

*Remark 6.* If $\alpha(\cdot)$ is constant, then $M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(\mathbb{R}^n) = M\dot{K}^{\alpha,q}_{p(\cdot),\lambda}(\mathbb{R}^n)$ was defined in [45]. If $\lambda = 0$, then $M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(\mathbb{R}^n) = \dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)$. If both $\alpha(\cdot)$ and $p(\cdot)$ are constants and $\lambda = 0$, then $M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(\mathbb{R}^n) = \dot{K}^{\alpha,q}_{p}(\mathbb{R}^n)$ is the classical Herz space in [43].

*Definition 7.* Let $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, $q(\cdot) \in \mathscr{P}^{\log}_0(\mathbb{R}^n) \cap \mathscr{P}^{\log}_\infty(\mathbb{R}^n)$, $0 \le \lambda < \infty$. Let $\alpha(\cdot)$ be a bounded real-valued measurable function on $\mathbb{R}^n$. The homogeneous Herz-Morrey space $M\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot),\lambda}$ and nonhomogeneous Herz-Morrey space $MK^{\alpha(\cdot),q(\cdot)}_{p(\cdot),\lambda}$ are defined, respectively, by

$$M\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot),\lambda} := \left\{ f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus 0) : \|f\|_{M\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot),\lambda}} < \infty \right\}, \tag{28}$$

and

$$MK^{\alpha(\cdot),q(\cdot)}_{p(\cdot),\lambda} := \left\{ f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{MK^{\alpha(\cdot),q(\cdot)}_{p(\cdot),\lambda}} < \infty \right\}, \tag{29}$$

where

$$\|f\|_{M\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot),\lambda}} := \sup_{L\in\mathbb{Z}} 2^{-L\lambda} \left\| \left(2^{\alpha(\cdot)k} f\chi_k\right)_{k\le L} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}, \tag{30}$$

and

$$\|f\|_{MK^{\alpha(\cdot),q(\cdot)}_{p(\cdot),\lambda}} := \sup_{L\in\mathbb{N}_0} 2^{-L\lambda} \left\| \left(2^{\alpha(\cdot)k} f\widetilde{\chi}_k\right)_{k=0}^{L} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}. \tag{31}$$

By the previous definition, we obtain the following proposition.

**Proposition 8.** *Let $p(\cdot), q(\cdot) \in \mathscr{P}_0(\mathbb{R}^n)$, $\lambda \in [0,\infty)$, and $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$.*

*(i) If $\alpha(\cdot)$, $q(\cdot) \in \mathscr{P}^{log}_0(\mathbb{R}^n) \cap \mathscr{P}^{log}_\infty(\mathbb{R}^n)$, then for any $f \in L^{p(\cdot)}_{loc}(\mathbb{R}^n \setminus 0)$,*

$$\begin{aligned} \|f\|_{M\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot),\lambda}} &\approx \max\Bigg\{ \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\| \left(2^{k\alpha(0)} f\chi_k\right)_{k\le L} \right\|_{\ell^{q(0)}(L^{p(\cdot)})}, \\ &\quad \sup_{L>0,L\in\mathbb{Z}} \Bigg[ 2^{-L\lambda} \left\| \left(2^{k\alpha(0)} f\chi_k\right)_{k<0} \right\|_{\ell^{q(0)}(L^{p(\cdot)})} \\ &\quad + 2^{-L\lambda} \left\| \left(2^{k\alpha_\infty} f\chi_k\right)_{k=0}^{L} \right\|_{\ell^{q_\infty}(L^{p(\cdot)})} \Bigg] \Bigg\}, \end{aligned} \tag{32}$$

*(ii) If $\alpha(\cdot)$, $q(\cdot) \in \mathscr{P}^{log}_\infty(\mathbb{R}^n)$, then*

$$MK^{\alpha(\cdot),q(\cdot)}_{p(\cdot),\lambda} = MK^{\alpha_\infty,q_\infty}_{p(\cdot),\lambda}. \tag{33}$$

*Proof.* Obviously,

$$\begin{aligned} \|f\|_{M\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot),\lambda}} &= \max\Bigg\{ \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda} \\ &\quad \cdot \left\| \left(2^{k\alpha(\cdot)} f\chi_k\right)_{k\le L} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}, \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda} \\ &\quad \cdot \left\| \left(2^{k\alpha(\cdot)} f\chi_k\right)_{k\le L} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \Bigg\}. \end{aligned} \tag{34}$$

When $L \le 0$, from Lemma 4 we know that

$$\left\| \left(2^{k\alpha(\cdot)} f\chi_k\right)_{k\le L} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \approx \left\| \left(2^{k\alpha(0)} f\chi_k\right)_{k\le L} \right\|_{\ell^{q(0)}(L^{p(\cdot)})}. \tag{35}$$

When $L > 0$, from Lemma 4 again we also obtain that

$$\begin{aligned} &\left\| \left(2^{k\alpha(\cdot)} f\chi_k\right)_{k<L} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \\ &\quad \approx \left\| \left(2^{k\alpha(0)} f\chi_k\right)_{k<0} \right\|_{\ell^{q(0)}(L^{p(\cdot)})} \\ &\quad + \left\| \left(2^{k\alpha_\infty} f\chi_k\right)_{k=0}^{L} \right\|_{\ell^{q_\infty}(L^{p(\cdot)})}. \end{aligned} \tag{36}$$

Thus we proved (i). The proof of (ii) is similar. □

**Lemma 9** (see [46, Proposition 2]). *Let $0 < q \le \infty$, $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, and $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$. If $\alpha(\cdot)$ is log-Hölder continuous both at the origin and at infinity, then*

$$\begin{aligned} &\|f\|_{M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}} \\ &\quad \approx \max\Bigg\{ \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda} \Bigg( \sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \\ &\quad \cdot \|f\chi_k\|^q_{L^{p(\cdot)}} \Bigg)^{1/q} \sup_{L>0,L\in\mathbb{Z}} \Bigg[ 2^{-L\lambda} \Bigg( \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q} \\ &\quad \cdot \|f\chi_k\|^q_{L^{p(\cdot)}} \Bigg)^{1/q} + 2^{-L\lambda} \Bigg( \sum_{k=0}^{L} 2^{k\alpha_\infty q} \\ &\quad \cdot \|f\chi_k\|^q_{L^{p(\cdot)}} \Bigg)^{1/q} \Bigg] \Bigg\}. \end{aligned} \tag{37}$$

**Lemma 10** (see [45, Lemma 1 and (10)]). *Suppose $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$, then there exist constants $\delta_1$, $\delta_2 \in (0,1)$, and $C > 0$ such that for all balls $B$ in $\mathbb{R}^n$ and all measurable subsets $S \subset B$,*

$$\frac{\|\chi_S\|_{L^{p(\cdot)}}}{\|\chi_B\|_{L^{p(\cdot)}}} \le C \left( \frac{|S|}{|B|} \right)^{\delta_1}, \tag{38}$$

$$\frac{\|\chi_S\|_{L^{p'(\cdot)}}}{\|\chi_B\|_{L^{p'(\cdot)}}} \le C \left( \frac{|S|}{|B|} \right)^{\delta_2}. \tag{39}$$

The following is our main results. Although the results are also hold for all multilinear cases, for brevity, we only consider the bilinear case.

**Theorem 11.** *Let $H$ be a bilinear Hardy operator; $p_1(\cdot), p_2(\cdot) \in \mathscr{B}(\mathbb{R}^n)$ satisfying $1/p(\cdot) = 1/p_1(\cdot) + 1/p_2(\cdot)$, $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathscr{P}_0^{log}(\mathbb{R}^n) \cap \mathscr{P}_\infty^{log}(\mathbb{R}^n)$, $\alpha(0) = \alpha_1(0) + \alpha_2(0)$, $\alpha_\infty = \alpha_{1\infty} + \alpha_{2\infty}$, $q(\cdot) \in \mathscr{P}_0^{log}(\mathbb{R}^n) \cap \mathscr{P}_\infty^{log}(\mathbb{R}^n)$, $1/q(0) = 1/q_1(0) + 1/q_2(0)$, $1/q_\infty = 1/q_{1\infty} + 1/q_{2\infty}$, $\lambda = \lambda_1 + \lambda_2$, $0 \le \lambda_1, \lambda_2 < \infty$, $\delta_{j1}, \delta_{j2} \in (0,1)$ are the constants in Lemma 10 for exponents $p_j(\cdot)$. If for $j = 1, 2$ with $\lambda_j + n\delta_{j2} > \alpha_{j\infty} \ge \alpha_j(0)$, then $H_{\vec{b}}^1$ is bounded from $M\dot{K}_{p_1(\cdot),\lambda_1}^{\alpha_1(\cdot),q_1(\cdot)}(\mathbb{R}^n) \times M\dot{K}_{p_2(\cdot),\lambda_2}^{\alpha_2(\cdot),q_2(\cdot)}(\mathbb{R}^n)$ to $M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$, where $\vec{b} = (b_1, b_2), b_1, b_2 \in BMO$.*

**Corollary 12.** *Let $H$ be a bilinear Hardy operator; $p_1(\cdot)$, $p_2(\cdot) \in \mathscr{B}(\mathbb{R}^n)$ satisfying $1/p(\cdot) = 1/p_1(\cdot) + 1/p_2(\cdot)$, $0 < q \le \infty$, $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathscr{P}_0^{log}(\mathbb{R}^n) \cap \mathscr{P}_\infty^{log}(\mathbb{R}^n)$, $\alpha(0) = \alpha_1(0) + \alpha_2(0)$, $\alpha_\infty = \alpha_{1\infty} + \alpha_{2\infty}$, $1/q = 1/q_1 + 1/q_2$, $\delta_{j1}, \delta_{j2} \in (0,1)$ are the constants in Lemma 10 for exponents $p_j(\cdot)$. If for $j = 1, 2$ with $n\delta_{j2} > \alpha_{j\infty} \ge \alpha_j(0)$, then $H_{\vec{b}}^1$ is bounded from $\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot),q_1}(\mathbb{R}^n) \times \dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot),q_2}(\mathbb{R}^n)$ to $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)$, where $\vec{b} = (b_1, b_2), b_1, b_2 \in BMO$.*

**Corollary 13.** *Let $H$ be a bilinear Hardy operator; $p_1(\cdot)$, $p_2(\cdot) \in \mathscr{B}(\mathbb{R}^n)$ satisfying $1/p(\cdot) = 1/p_1(\cdot) + 1/p_2(\cdot)$, $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathscr{P}_0^{log}(\mathbb{R}^n) \cap \mathscr{P}_\infty^{log}(\mathbb{R}^n)$, $\alpha(0) = \alpha_1(0) + \alpha_2(0)$, $\alpha_\infty = \alpha_{1\infty} + \alpha_{2\infty}$, $q(\cdot) \in \mathscr{P}_0^{log}(\mathbb{R}^n) \cap \mathscr{P}_\infty^{log}(\mathbb{R}^n)$, $1/q(0) = 1/q_1(0) + 1/q_2(0)$, $1/q_\infty = 1/q_{1\infty} + 1/q_{2\infty}$, $\delta_{j1}, \delta_{j2} \in (0,1)$ are the constants in Lemma 10 for exponents $p_j(\cdot)$. If for $j = 1, 2$ with $n\delta_{j2} > \alpha_{j\infty} \ge \alpha_j(0)$, then $H_{\vec{b}}^1$ is bounded from $\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot),q_1(\cdot)}(\mathbb{R}^n) \times \dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot),q_2(\cdot)}(\mathbb{R}^n)$ to $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$, where $\vec{b} = (b_1, b_2), b_1, b_2 \in BMO$.*

**Corollary 14.** *Let $H$ be a bilinear Hardy operator; $p_1(\cdot)$, $p_2(\cdot) \in \mathscr{B}(\mathbb{R}^n)$ satisfying $1/p(\cdot) = 1/p_1(\cdot) + 1/p_2(\cdot)$, $0 < q \le \infty$, $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathscr{P}_0^{log}(\mathbb{R}^n) \cap \mathscr{P}_\infty^{log}(\mathbb{R}^n)$, $\alpha(0) = \alpha_1(0) + \alpha_2(0)$, $\alpha_\infty = \alpha_{1\infty} + \alpha_{2\infty}$, $1/q = 1/q_1 + 1/q_2$, $\lambda = \lambda_1 + \lambda_2$, $0 \le \lambda_1, \lambda_2 < \infty$, $\delta_{j1}, \delta_{j2} \in (0,1)$ are the constants in Lemma 10 for exponents $p_j(\cdot)$. If for $j = 1, 2$ with $\lambda_j + n\delta_{j2} > \alpha_{j\infty} \ge \alpha_j(0)$, then $H_{\vec{b}}^1$ is bounded from $M\dot{K}_{p_1(\cdot),\lambda_1}^{\alpha_1(\cdot),q_1}(\mathbb{R}^n) \times M\dot{K}_{p_2(\cdot),\lambda_2}^{\alpha_2(\cdot),q_2}(\mathbb{R}^n)$ to $M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}(\mathbb{R}^n)$, where $\vec{b} = (b_1, b_2), b_1, b_2 \in BMO$.*

## 3. Proof of Theorem 11

Since Corollaries 12–14 are special case of Theorem 11, we only need to prove Theorem 11.

To proceed, we need the following lemmas.

**Lemma 15** (see [45, Lemma 2]). *If $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$, then there exists a constant $C > 0$ such that, for all balls $B$ in $\mathbb{R}^n$,*

$$C^{-1} \le \frac{1}{|B|} \|\chi_B\|_{L^{p(\cdot)}} \|\chi_B\|_{L^{p'(\cdot)}} \le C. \tag{40}$$

**Lemma 16** (see [5, Theorem 2.1]). *Let $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$. Then for every $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and every $g \in L^{p'(\cdot)}(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} |f(x) g(x)| dx \le r_p \|f\|_{L^{p(\cdot)}} \|g\|_{L^{p'(\cdot)}}, \tag{41}$$

*where $r_p = 1 + 1/p_+ - 1/p_-$.*

**Lemma 17** (see [47, Theorem 2.3]). *Let $p, p_1, p_2 \in \mathscr{P}^0(\mathbb{R}^n)$ such that $1/p(x) = 1/p_1(x) + p_2(x)$. Then there exists a constant $C$ depending only on $p, p_1$ such that*

$$\|fg\|_{L^{p(\cdot)}} \le C \|f\|_{L^{p_1(\cdot)}} \|g\|_{L^{p_2(\cdot)}} \tag{42}$$

*holds for every $f \in L^{p_1(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p_2(\cdot)}(\mathbb{R}^n)$.*

**Lemma 18** (see [32, Theorem 3]). *Let $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$, $t \in \mathbb{N}$ and $k > i$ $(k, i \in \mathbb{N})$, then there exists a positive constant $C$ such that, for $b \in BMO(\mathbb{R}^n)$,*

$$\begin{aligned} C^{-1} \|b\|_*^t &\le \sup_{B \subset \mathbb{R}^n} \frac{1}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left\|(b - b_B)^t \chi_B\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\le C \|b\|_*^t, \\ \left\|\left(b - b_{B_i}\right)^t \chi_{B_k}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\le C (k-i)^t \|b\|_*^t \|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{43}$$

Indeed, if $t = 1$, then for all balls $B$ and $k, i \in \mathbb{Z}$ with $k > i$, we have

$$\begin{aligned} &\sup_{B \subset \mathbb{R}^n} \frac{1}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left\|(b - b_B) \chi_B\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C \|b\|_*, \\ &\left\|\left(b - b_{B_i}\right) \chi_{B_k}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C (k-i) \|b\|_* \|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{44}$$

From the definition of BMO, it is easy to know that

$$\left|b_{B_k} - b_{B_i}\right| \le (k-i) \|b\|_* \quad \text{for } k > i. \tag{45}$$

**Lemma 19** (see [48, Proposition 1.2]). *Let $0 < p \le \infty, \delta > 0$. Then we have*

$$\left(\sum_{j=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} 2^{-|k-j|\delta} a_k\right)^p\right)^{1/p} \le C \left(\sum_{j=-\infty}^{\infty} a_j^p\right)^{1/p} \tag{46}$$

*for nonnegative sequences $\{a_j\}_{j=-\infty}^{\infty}$. Here, when $p = \infty$, it is understood that (46) stands for*

$$\sup_{j \in \mathbb{Z}} \left(\sum_{k=-\infty}^{\infty} 2^{-|k-j|\delta} a_k\right) \le C \sup_{j \in \mathbb{Z}} a_j. \tag{47}$$

*Proof of Theorem 11.* Let $f_1 \in M\dot{K}_{p_1(\cdot),\lambda_1}^{\alpha_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)$, $f_2 \in M\dot{K}_{p_2(\cdot),\lambda_2}^{\alpha_2(\cdot),q_2(\cdot)}(\mathbb{R}^n)$. Let $(b_i)_{B_k}$ express the average of $b_i$ on the ball $B_k$ for $i = 1, 2$, and $k \in \mathbb{Z}$. If $x \in D_k$, then

$$\begin{aligned} \left|H_{b_1}^1 (f_1, f_2)(x)\right| &\lesssim 2^{-2kn} \int_{|(y_1,y_2)|<|x|} |f_1(y_1) f_2(y_2)| \\ &\quad \cdot |b_1(x) - b_1(y_1)| dy_1 dy_2 \\ &\lesssim 2^{-2kn} \int_{|(y_1,y_2)|<|x|} |f_1(y_1) f_2(y_2)| \\ &\quad \cdot \left|b_1(x) - (b_1)_{B_k}\right| dy_1 dy_2 \\ &\quad + 2^{-2kn} \int_{|(y_1,y_2)|<|x|} |f_1(y_1) f_2(y_2)| \\ &\quad \cdot \left|b_1(y_1) - (b_1)_{B_k}\right| dy_1 dy_2 =: \mathrm{I} + \mathrm{II}. \end{aligned} \tag{48}$$

Since $1/p = 1/p_1 + 1/p_2$, $1/p_1 + 1/p_1' = 1$ and $1/p_2 + 1/p_2' = 1$. Using Hölder's inequality, we have

$$\begin{aligned}
\mathrm{I} &\lesssim 2^{-2kn} \int_{B_k} \int_{B_k} |f_1(y_1) f_2(y_2)| \\
&\quad \cdot \left|b_1(x) - (b_1)_{B_k}\right| dy_1 dy_2 \lesssim 2^{-2kn} \left|b_1(x) \right. \\
&\quad \left. - (b_1)_{B_k}\right| \int_{B_k} \int_{B_k} |f_1(y_1) f_2(y_2)| dy_1 dy_2 \\
&= c 2^{-2kn} \left|b_1(x) - (b_1)_{B_k}\right| \sum_{i=-\infty}^{k} \int_{D_i} |f_1(y_1)| dy_1 \\
&\quad \cdot \sum_{j=-\infty}^{k} \int_{D_j} |f_2(y_2)| dy_2 \lesssim 2^{-2kn} \left|b_1(x) - (b_1)_{B_k}\right| \\
&\quad \cdot \sum_{i=-\infty}^{k} \|f_1 \chi_i\|_{L^1} \sum_{j=-\infty}^{k} \|f_2 \chi_j\|_{L^1} \lesssim 2^{-2kn} \left|b_1(x) \right. \\
&\quad \left. - (b_1)_{B_k}\right| \sum_{i=-\infty}^{k} \|f_1 \chi_i\|_{L^{p_1(\cdot)}} \|\chi_i\|_{L^{p_1'(\cdot)}} \\
&\quad \times \sum_{j=-\infty}^{k} \|f_2 \chi_j\|_{L^{p_2(\cdot)}} \|\chi_j\|_{L^{p_2'(\cdot)}}.
\end{aligned} \tag{49}$$

By Lemma 18, Hölder's inequality (Lemma 17), and Lemmas 10 and 15, we have

$$\begin{aligned}
\|\mathrm{I}\chi_k\|_{L^{p(\cdot)}} &\lesssim 2^{-2kn} \left\|\left(b_1 - (b_1)_{B_k}\right) \cdot \chi_k\right\|_{L^{p(\cdot)}} \\
&\quad \cdot \sum_{i=-\infty}^{k} \|f_1 \chi_i\|_{L^{p_1(\cdot)}} \|\chi_i\|_{L^{p_1'(\cdot)}} \\
&\quad \times \sum_{j=-\infty}^{k} \|f_2 \chi_j\|_{L^{p_2(\cdot)}} \|\chi_j\|_{L^{p_2'(\cdot)}} \lesssim 2^{-2kn} \|b_1\|_* \|\chi_{B_k}\|_{L^{p(\cdot)}} \\
&\quad \cdot \sum_{i=-\infty}^{k} \|f_1 \chi_i\|_{L^{p_1(\cdot)}} \|\chi_{B_i}\|_{L^{p_1'(\cdot)}} \\
&\quad \times \sum_{j=-\infty}^{k} \|f_2 \chi_j\|_{L^{p_2(\cdot)}} \|\chi_{B_j}\|_{L^{p_2'(\cdot)}} \lesssim 2^{-2kn} \|b_1\|_* \\
&\quad \cdot \sum_{i=-\infty}^{k} \|f_1 \chi_i\|_{L^{p_1(\cdot)}} \|\chi_{B_i}\|_{L^{p_1'(\cdot)}} \|\chi_{B_k}\|_{L^{p_1(\cdot)}} \\
&\quad \times \sum_{j=-\infty}^{k} \|f_2 \chi_j\|_{L^{p_2(\cdot)}} \|\chi_{B_j}\|_{L^{p_2'(\cdot)}} \|\chi_{B_k}\|_{L^{p_2(\cdot)}} \lesssim \|b_1\|_* \\
&\quad \cdot \sum_{i=-\infty}^{k} \|f_1 \chi_i\|_{L^{p_1(\cdot)}} \times \frac{\|\chi_{B_i}\|_{L^{p_1'(\cdot)}}}{\|\chi_{B_k}\|_{L^{p_1'(\cdot)}}} \times \sum_{i=-\infty}^{k} \|f_2 \chi_j\|_{L^{p_2(\cdot)}} \\
&\quad \times \frac{\|\chi_{B_j}\|_{L^{p_2'(\cdot)}}}{\|\chi_{B_k}\|_{L^{p_2'(\cdot)}}} \lesssim \|b_1\|_* \sum_{i=-\infty}^{k} 2^{-(k-i)n\delta_{12}} \|f_1 \chi_i\|_{L^{p_1(\cdot)}} \\
&\quad \times \sum_{j=-\infty}^{k} 2^{-(k-j)n\delta_{22}} \|f_2 \chi_j\|_{L^{p_2(\cdot)}}.
\end{aligned} \tag{50}$$

Then we consider the term II.

$$\begin{aligned}
\mathrm{II} &\lesssim 2^{-2kn} \sum_{i=-\infty}^{k} \int_{D_i} |f_1(y_1)| \left|b_1(y_1) - (b_1)_{B_i}\right| dy_1 \\
&\quad \times \sum_{j=-\infty}^{k} \int_{D_j} |f_2(y_2)| dy_2 \\
&\quad + 2^{-2kn} \sum_{i=-\infty}^{k} \int_{D_i} |f_1(y_1)| \left|(b_1)_{B_k} - (b_1)_{B_i}\right| dy_1 \\
&\quad \times \sum_{j=-\infty}^{k} \int_{D_j} |f_2(y_2)| dy_2 =: \mathrm{II}_1 + \mathrm{II}_2.
\end{aligned} \tag{51}$$

By Lemma 18 and Hölder's inequality (Lemma 16), we have

$$\begin{aligned}
\mathrm{II}_1 &\lesssim 2^{-2kn} \sum_{i=-\infty}^{k} \|f_1 \chi_i\|_{L^{p_1(\cdot)}} \left\|\left(b_1 - (b_1)_{B_i}\right) \chi_i\right\|_{L^{p_1'(\cdot)}} \\
&\quad \times \sum_{j=-\infty}^{k} \|f_2 \chi_j\|_{L^{p_2(\cdot)}} \|\chi_j\|_{L^{p_2'(\cdot)}} \\
&\lesssim 2^{-2kn} \|b_1\|_* \sum_{i=-\infty}^{k} \|f_1 \chi_i\|_{L^{p_1(\cdot)}} \|\chi_{B_i}\|_{L^{p_1'(\cdot)}} \\
&\quad \times \sum_{j=-\infty}^{k} \|f_2 \chi_j\|_{L^{p_2(\cdot)}} \|\chi_{B_j}\|_{L^{p_2'(\cdot)}}.
\end{aligned} \tag{52}$$

By Lemma 18 and Hölder's inequality (Lemma 16), we have

$$\begin{aligned}
\mathrm{II}_2 &\lesssim 2^{-2kn} \sum_{i=-\infty}^{k} \left|(b_1)_{B_k} - (b_1)_{B_i}\right| \|f_1 \chi_i\|_{L^1} \\
&\quad \times \sum_{j=-\infty}^{k} \|f_2 \chi_j\|_{L^1} \\
&\lesssim 2^{-2kn} \sum_{i=-\infty}^{k} \left|(b_1)_{B_k} - (b_1)_{B_i}\right| \|f_1 \chi_i\|_{L^{p_1}(\cdot)} \|\chi_i\|_{L^{p_1'}(\cdot)} \\
&\quad \times \sum_{j=-\infty}^{k} \|f_2 \chi_j\|_{L^{p_2}(\cdot)} \|\chi_j\|_{L^{p_2'}(\cdot)} \\
&\lesssim 2^{-2kn} \|b_1\|_* \sum_{i=-\infty}^{k} (k-i) \|f_1 \chi_i\|_{L^{p_1(\cdot)}} \|\chi_i\|_{L^{p_1'(\cdot)}} \\
&\quad \times \sum_{j=-\infty}^{k} \|f_2 \chi_j\|_{L^{p_2(\cdot)}} \|\chi_j\|_{L^{p_2'(\cdot)}}.
\end{aligned} \tag{53}$$

By Hölder's inequality (Lemma 16) and Lemmas 10 and 15, we have

$$\begin{aligned}
&\|\mathrm{II}_1\chi_k\|_{L^{p(\cdot)}}\\
&\quad\lesssim 2^{-2kn}\|b_1\|_*\|\chi_{B_k}\|_{L^{p_1(\cdot)}}\sum_{i=-\infty}^{k}\|f_1\chi_i\|_{L^{p_1(\cdot)}}\|\chi_{B_i}\|_{L^{p_1'(\cdot)}}\\
&\qquad\times\|\chi_{B_k}\|_{L^{p_2(\cdot)}}\sum_{j=-\infty}^{k}\|f_2\chi_j\|_{L^{p_2(\cdot)}}\|\chi_{B_j}\|_{L^{p_2'(\cdot)}}\\
&\quad\lesssim\|b_1\|_*\sum_{i=-\infty}^{k}\|f_1\chi_i\|_{L^{p_1(\cdot)}}\frac{\|\chi_{B_i}\|_{L^{p_1'(\cdot)}}}{\|\chi_{B_k}\|_{L^{p_1'(\cdot)}}}\\
&\qquad\times\sum_{j=-\infty}^{k}\|f_2\chi_j\|_{L^{p_2(\cdot)}}\frac{\|\chi_{B_j}\|_{L^{p_2'(\cdot)}}}{\|\chi_{B_k}\|_{L^{p_2'(\cdot)}}}\\
&\quad\lesssim\|b_1\|_*\sum_{i=-\infty}^{k}2^{-(k-i)n\delta_{12}}\|f_1\chi_i\|_{L^{p_1(\cdot)}}\\
&\qquad\times\sum_{j=-\infty}^{k}2^{-(k-j)n\delta_{22}}\|f_2\chi_j\|_{L^{p_2(\cdot)}}.
\end{aligned}\tag{54}$$

By Hölder's inequality (Lemma 16) and Lemmas 10 and 15, we get

$$\begin{aligned}
&\|\mathrm{II}_2\chi_k\|_{L^{p(\cdot)}}\lesssim 2^{-2kn}\|b_1\|_*\|\chi_{B_k}\|_{L^{p(\cdot)}}\\
&\quad\cdot\sum_{i=-\infty}^{k}(k-i)\|f_1\chi_i\|_{L^{p_1(\cdot)}}\|\chi_{B_i}\|_{L^{p_1'(\cdot)}}\\
&\quad\times\sum_{j=-\infty}^{k}\|f_2\chi_j\|_{L^{p_2(\cdot)}}\|\chi_{B_j}\|_{L^{p_2'(\cdot)}}\lesssim 2^{-2kn}\|b_1\|_*\\
&\quad\cdot\|\chi_{B_k}\|_{L^{p_1(\cdot)}}\sum_{i=-\infty}^{k}(k-i)\|f_1\chi_i\|_{L^{p_1(\cdot)}}\|\chi_{B_i}\|_{L^{p_1'(\cdot)}}\\
&\quad\times\|\chi_{B_k}\|_{L^{p_2(\cdot)}}\sum_{j=-\infty}^{k}\|f_2\chi_j\|_{L^{p_2(\cdot)}}\|\chi_{B_j}\|_{L^{p_2'(\cdot)}}\lesssim\|b_1\|_*\\
&\quad\cdot\sum_{i=-\infty}^{k}(k-i)\|f_1\chi_i\|_{L^{p_1(\cdot)}}\times\frac{\|\chi_{B_i}\|_{L^{p_1'(\cdot)}}}{\|\chi_{B_k}\|_{L^{p_1'(\cdot)}}}\\
&\quad\times\sum_{j=-\infty}^{k}\|f_2\chi_j\|_{L^{p_2(\cdot)}}\times\frac{\|\chi_{B_j}\|_{L^{p_2'(\cdot)}}}{\|\chi_{B_k}\|_{L^{p_2'(\cdot)}}}\lesssim\|b_1\|_*\\
&\quad\cdot\sum_{i=-\infty}^{k}(k-i)2^{-(k-i)n\delta_{12}}\|f_1\chi_i\|_{L^{p_1(\cdot)}}\\
&\quad\times\sum_{j=-\infty}^{k}2^{-(k-j)n\delta_{22}}\|f_2\chi_j\|_{L^{p_2(\cdot)}}.
\end{aligned}\tag{55}$$

Therefore, we have

$$\begin{aligned}
&\|H_{b_1}^1(f_1,f_2)\chi_k\|_{L^{p(\cdot)}}\\
&\quad\lesssim\|b_1\|_*\sum_{i=-\infty}^{k}(k-i)2^{-(k-i)n\delta_{12}}\|f_1\chi_i\|_{L^{p_1(\cdot)}}\\
&\qquad\times\sum_{j=-\infty}^{k}2^{-(k-j)n\delta_{22}}\|f_2\chi_j\|_{L^{p_2(\cdot)}}.
\end{aligned}\tag{56}$$

By Proposition 8, we obtain that

$$\begin{aligned}
&\|H_{b_1}^1(f_1,f_2)\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q(\cdot)}}\\
&\quad\approx\max\Big\{\sup_{L\le0,L\in\mathbb{Z}}2^{-L\lambda}\|(2^{k\alpha(0)}H_{b_1}^1(f_1,f_2)\chi_k)_{k\le L}\|_{\ell^{q(0)}(L^{p(\cdot)})},\\
&\quad\sup_{L>0,L\in\mathbb{Z}}\Big[2^{-L\lambda}\|(2^{k\alpha(0)}H_{b_1}^1(f_1,f_2)\chi_k)_{k<0}\|_{\ell^{q(0)}(L^{p(\cdot)})}\\
&\quad+2^{-L\lambda}\|(2^{k\alpha_\infty}H_{b_1}^1(f_1,f_2)\chi_k)_{k=0}^{L}\|_{\ell^{q_\infty}(L^{p(\cdot)})}\Big]\Big\}\\
&\quad:=\max\{E,F+G\},
\end{aligned}\tag{57}$$

where

$$\begin{aligned}
E&:=\sup_{L\le0,L\in\mathbb{Z}}2^{-L\lambda}\|(2^{k\alpha(0)}H_{b_1}^1(f_1,f_2)\chi_k)_{k\le L}\|_{\ell^{q(0)}(L^{p(\cdot)})},\\
F&:=\sup_{L>0,L\in\mathbb{Z}}2^{-L\lambda}\|(2^{k\alpha(0)}H_{b_1}^1(f_1,f_2)\chi_k)_{k<0}\|_{\ell^{q(0)}(L^{p(\cdot)})},\\
G&:=\sup_{L>0,L\in\mathbb{Z}}2^{-L\lambda}\|(2^{k\alpha_\infty}H_{b_1}^1(f_1,f_2)\chi_k)_{k=0}^{L}\|_{\ell^{q_\infty}(L^{p(\cdot)})}.
\end{aligned}\tag{58}$$

Since $\alpha(0)=\alpha_1(0)+\alpha_2(0)$, $1/q(0)=1/q_1(0)+1/q_2(0)$, and $\lambda=\lambda_1+\lambda_2$, by (56) and Hölder's inequality, we have

$$\begin{aligned}
&E\\
&\quad=\sup_{L\le0,L\in\mathbb{Z}}2^{-L\lambda}\Big(\sum_{k=-\infty}^{L}\|2^{k\alpha(0)}H_{b_1}^1(f_1,f_2)\\
&\qquad\cdot\chi_k\|_{L^{p(\cdot)}}^{q(0)}\Big)^{1/q(0)}\lesssim\|b_1\|_*\sup_{L\le0,L\in\mathbb{Z}}2^{-L\lambda_1}\\
&\qquad\times\Big\{\sum_{k=-\infty}^{L}\Big(\sum_{i=-\infty}^{k}2^{i\alpha_1(0)}\\
&\qquad\cdot\|f_1\chi_i\|_{L^{p_1(\cdot)}}(k-i)2^{(i-k)(n\delta_{12}-\alpha_1(0))}\Big)^{q_1(0)}\Big\}^{1/q_1(0)}\\
&\qquad\times2^{-L\lambda_2}\Big\{\sum_{k=-\infty}^{L}\Big(\sum_{j=-\infty}^{k}2^{j\alpha_2(0)}
\end{aligned}$$

$$\cdot \left\| f_2\chi_j \right\|_{L^{p_2(\cdot)}} 2^{(j-k)(n\delta_{22}-\alpha_2(0))} \Bigg)^{q_2(0)} \Bigg\}^{1/q_2(0)} =: \left\| b_1 \right\|_* \cdot E_1 \times E_2. \tag{59}$$

Since $n\delta_{12} - \alpha_1(0) > 0$, by Lemma 19 we obtain that

$$\begin{aligned} E_1 &\lesssim \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_1} \left( \sum_{i=-\infty}^{k} \left\| 2^{i\alpha_1(\cdot)} f_1\chi_i \right\|_{L^{p_1(\cdot)}}^{q_1(0)} \right)^{1/q_1(0)} \\ &\lesssim \left\| f_1 \right\|_{M\dot{K}_{p_1(\cdot),\lambda_1}^{\alpha_1(\cdot),q_1(\cdot)}}, \end{aligned} \tag{60}$$

where we wrote $(k-i)2^{-|k-i|(n\delta_{12}-\alpha_1(0))} \lesssim 2^{-|k-i|\varepsilon_1}$ for some $\varepsilon_1 \in (0, n\delta_{12} - \alpha_1(0))$.

Since $n\delta_{22} - \alpha_2(0) > 0$, again by Lemma 19 we obtain that

$$\begin{aligned} E_2 &\lesssim \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_2} \left( \sum_{j=-\infty}^{k} \left\| 2^{j\alpha_{22}(\cdot)} f_2\chi_j \right\|_{L^{p_2(\cdot)}}^{q_2(0)} \right)^{1/q_2(0)} \\ &\lesssim \left\| f_2 \right\|_{M\dot{K}_{p_2(\cdot),\lambda_2}^{\alpha_2(\cdot),q_2(\cdot)}}. \end{aligned} \tag{61}$$

Thus, we obtain

$$E \lesssim \left\| b_1 \right\|_* \left\| f_1 \right\|_{M\dot{K}_{p_1(\cdot),\lambda_1}^{\alpha_1(\cdot),q_1(\cdot)}} \left\| f_2 \right\|_{M\dot{K}_{p_2(\cdot),\lambda_2}^{\alpha_2(\cdot),q_2(\cdot)}}. \tag{62}$$

Since $\alpha(0) = \alpha_1(0) + \alpha_2(0)$, $1/q(0) = 1/q_1(0) + 1/q_2(0)$, and $\lambda = \lambda_1 + \lambda_2$, by (56) and Hölder's inequality, we have

$$\begin{aligned} F &= \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda} \Bigg( \sum_{k=-\infty}^{-1} \left\| 2^{k\alpha(0)} H_{b_1}^1 (f_1, f_2) \cdot \chi_k \right\|_{L^{p(\cdot)}}^{q(0)} \Bigg)^{1/q(0)} \le \left\| b_1 \right\|_* \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda_1} \\ &\quad \times \Bigg\{ \sum_{k=-\infty}^{-1} \Bigg( \sum_{i=-\infty}^{k} 2^{i\alpha_1(0)} \cdot \left\| f_1\chi_i \right\|_{L^{p_1(\cdot)}} (k-i) 2^{(i-k)(n\delta_{12}-\alpha_1(0))} \Bigg)^{q_1(0)} \Bigg\}^{1/q_1(0)} \\ &\quad \times 2^{-L\lambda_2} \Bigg\{ \sum_{k=-\infty}^{-1} \Bigg( \sum_{j=-\infty}^{k} 2^{j\alpha_2(0)} \left\| f_2\chi_j \right\|_{L^{p_2(\cdot)}} \cdot 2^{(j-k)(n\delta_{22}-\alpha_2(0))} \Bigg)^{q_2(0)} \Bigg\}^{1/q_2(0)} =: \left\| b_1 \right\|_* F_1 \times F_2. \end{aligned} \tag{63}$$

Here

$$\begin{aligned} F_1 &:= \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda_1} \times \Bigg\{ \sum_{k=-\infty}^{-1} \Bigg( \sum_{i=-\infty}^{k} 2^{i\alpha_1(0)} \left\| f_1\chi_i \right\|_{L^{p_1(\cdot)}} \cdot (k-i) 2^{(i-k)(n\delta_{12}-\alpha_1(0))} \Bigg)^{q_1(0)} \Bigg\}^{1/q_1(0)}, \\ F_2 &:= \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda_2} \times \Bigg\{ \sum_{k=-\infty}^{-1} \Bigg( \sum_{j=-\infty}^{k} 2^{j\alpha_2(0)} \left\| f_2\chi_j \right\|_{L^{p_2(\cdot)}} \cdot 2^{(j-k)(n\delta_{22}-\alpha_2(0))} \Bigg)^{q_2(0)} \Bigg\}^{1/q_2(0)}. \end{aligned} \tag{64}$$

Since $n\delta_{12} - \alpha_1(0) > 0$, by Lemma 19 we obtain that

$$\begin{aligned} F_1 &\lesssim \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda_1} \left( \sum_{i=-\infty}^{-1} \left\| 2^{i\alpha_1(\cdot)} f_1\chi_i \right\|_{L^{p_1(\cdot)}}^{q_1(0)} \right)^{1/q_1(0)} \\ &\lesssim \left\| f_1 \right\|_{M\dot{K}_{p_1(\cdot),\lambda_1}^{\alpha_1(\cdot),q_1(\cdot)}}, \end{aligned} \tag{65}$$

where we wrote $(k-i)2^{-|k-i|(n\delta_{12}-\alpha_1(0))} \lesssim 2^{-|k-i|\varepsilon_1}$ for some $\varepsilon_1 \in (0, n\delta_{12} - \alpha_1(0))$.

Since $n\delta_{22} - \alpha_2(0) > 0$, by Lemma 19 we obtain that

$$\begin{aligned} F_2 &\lesssim \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda_2} \left( \sum_{j=-\infty}^{-1} \left\| 2^{j\alpha_{22}(\cdot)} f_2\chi_j \right\|_{L^{p_2(\cdot)}}^{q_2(0)} \right)^{1/q_2(0)} \\ &\lesssim \left\| f_2 \right\|_{M\dot{K}_{p_2(\cdot),\lambda_2}^{\alpha_2(\cdot),q_2(\cdot)}}. \end{aligned} \tag{66}$$

Therefore, we get

$$F \lesssim \left\| b_1 \right\|_* \left\| f_1 \right\|_{M\dot{K}_{p_1(\cdot),\lambda_1}^{\alpha_1(\cdot),q_1(\cdot)}} \left\| f_2 \right\|_{M\dot{K}_{p_2(\cdot),\lambda_2}^{\alpha_2(\cdot),q_2(\cdot)}}. \tag{67}$$

Now we turn to estimate $G$. Since $q_\infty = 1/q_{1\infty} + 1/q_{2\infty}$, $\lambda = \lambda_1 + \lambda_2$, by (56) and Hölder's inequality, we have

$$\begin{aligned} G &= \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda} \left( \sum_{k=0}^{L} \left\| 2^{k\alpha_\infty} H_{b_1}^1 (f_1, f_2) \chi_k \right\|_{L^{p(\cdot)}}^{q_\infty} \right)^{1/q_\infty} \\ &\le \left\| b_1 \right\|_* \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda_1} \times \Bigg\{ \sum_{k=0}^{L} 2^{k\alpha_{1\infty}q_{1\infty}} \cdot \Bigg( \sum_{i=-\infty}^{k} \left\| f_1\chi_i \right\|_{L^{p_1(\cdot)}} (k-i) 2^{(i-k)n\delta_{12}} \Bigg)^{q_{1\infty}} \Bigg\}^{1/q_{1\infty}} \\ &\quad \times 2^{-L\lambda_2} \Bigg\{ \sum_{k=0}^{L} 2^{k\alpha_{2\infty}q_{2\infty}} \cdot \Bigg( \sum_{j=-\infty}^{k} \left\| f_2\chi_j \right\|_{L^{p_2(\cdot)}} 2^{(j-k)n\delta_{22}} \Bigg)^{q_{2\infty}} \Bigg\}^{1/q_{2\infty}} =: \left\| b_1 \right\|_* \cdot G_1 \times G_2, \end{aligned} \tag{68}$$

where

$$G_1 := \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda_1}\left\{\sum_{k=0}^{L} 2^{k\alpha_{1\infty}q_{1\infty}} \cdot\left(\sum_{i=-\infty}^{k}\|f_1\chi_i\|_{L^{p_1(\cdot)}}(k-i)\,2^{(i-k)n\delta_{12}}\right)^{q_{1\infty}}\right\}^{1/q_{1\infty}},$$
$$G_2 := \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda_2}\left\{\sum_{k=0}^{L} 2^{k\alpha_{2\infty}q_{2\infty}} \cdot\left(\sum_{j=-\infty}^{k}\|f_2\chi_j\|_{L^{p_2(\cdot)}}\,2^{(j-k)n\delta_{22}}\right)^{q_{2\infty}}\right\}^{1/q_{2\infty}}. \tag{69}$$

To go on, we need further preparation.

If $i<0$, by Proposition 8, we have

$$\begin{aligned}\|f_1\chi_i\|_{L^{p_1(\cdot)}} &= 2^{-i\alpha_1(0)}\left(2^{i\alpha_1(0)q_1(0)}\|f_1\chi_i\|_{L^{p_1(\cdot)}}^{q_1(0)}\right)^{1/q_1(0)}\\ &\lesssim 2^{-i\alpha_1(0)}\left(\sum_{l=-\infty}^{i}2^{l\alpha_1(0)q_1(0)}\|f_1\chi_l\|_{L^{p_1(\cdot)}}^{q_1(0)}\right)^{1/q_1(0)}\\ &\lesssim 2^{i(\lambda_1-\alpha_1(0))}2^{-i\lambda_1}\left(\sum_{l=-\infty}^{i}\left\|2^{l\alpha(0)}f_1\chi_l\right\|_{L^{p_1(\cdot)}}^{q_1(0)}\right)^{1/q_1(0)}\\ &\lesssim 2^{i(\lambda_1-\alpha_1(0))}\|f_1\|_{M\dot{K}_{p_1(\cdot),\lambda_1}^{\alpha_1(\cdot),q_1(\cdot)}}.\end{aligned} \tag{70}$$

Similarly, if $j<0$, we have

$$\|f_2\chi_j\|_{L^{p_2(\cdot)}} \lesssim 2^{j(\lambda_2-\alpha_2(0))}\|f_2\|_{M\dot{K}_{p_2(\cdot),\lambda_2}^{\alpha_2(\cdot),q_2(\cdot)}}. \tag{71}$$

If $i\geq 0$, we have

$$\begin{aligned}\|f_1\chi_i\|_{L^{p_1(\cdot)}} &= 2^{-i\alpha_{1\infty}}\left(2^{i\alpha_{1\infty}q_{1\infty}}\|f_1\chi_i\|_{L^{p_1(\cdot)}}^{q_{1\infty}}\right)^{1/q_{1\infty}}\\ &\lesssim 2^{-i\alpha_{1\infty}}\left(\sum_{l=-\infty}^{i}2^{l\alpha_{1\infty}q_{1\infty}}\|f_1\chi_l\|_{L^{p_1(\cdot)}}^{q_{1\infty}}\right)^{1/q_{1\infty}}\\ &\lesssim 2^{i(\lambda_1-\alpha_{1\infty})}2^{-i\lambda_1}\left(\sum_{l=-\infty}^{i}\left\|2^{l\alpha_{1\infty}}f_1\chi_l\right\|_{L^{p_1(\cdot)}}^{q_{1\infty}}\right)^{1/q_{1\infty}}\\ &\lesssim 2^{i(\lambda_1-\alpha_{1\infty})}\|f_1\|_{M\dot{K}_{p_1(\cdot),\lambda_1}^{\alpha_1(\cdot),q_1(\cdot)}}.\end{aligned} \tag{72}$$

Similarly, if $j\geq 0$, we have

$$\|f_2\chi_j\|_{L^{p_2(\cdot)}} \lesssim 2^{j(\lambda_2-\alpha_{2\infty})}\|f_2\|_{M\dot{K}_{p_2(\cdot),\lambda_2}^{\alpha_2(\cdot),q_1(\cdot)}}. \tag{73}$$

Since $\lambda_1+n\delta_{12}>\alpha_{1\infty}\geq\alpha_1(0)$ and by (70) and (72), we obtain that

$$\begin{aligned}G_1 &\lesssim \|f_1\|_{M\dot{K}_{p_1(\cdot),\lambda_1}^{\alpha_1(\cdot),q_1(\cdot)}}\sup_{L>0,L\in\mathbb{Z}}2^{-L\lambda_1}\times\left\{\sum_{k=0}^{L}2^{k\alpha_{1\infty}q_{1\infty}}\cdot\left(\sum_{i=-\infty}^{-1}(k-i)\,2^{(i-k)n\delta_{12}}2^{i(\lambda_1-\alpha_1(0))}+\sum_{i=0}^{k}(k-i)\cdot 2^{(i-k)n\delta_{12}}2^{i(\lambda_1-\alpha_{1\infty})}\right)^{q_{1\infty}}\right\}^{1/q_{1\infty}}\\ &\lesssim \|f_1\|_{M\dot{K}_{p_1(\cdot),\lambda_1}^{\alpha_1(\cdot),q_1(\cdot)}}\cdot\sup_{L>0,L\in\mathbb{Z}}2^{-L\lambda_1}\left\{\sum_{k=0}^{L}2^{k\alpha_{1\infty}q_{1\infty}}\times\left(\sum_{i=-\infty}^{-1}(k-i)\cdot 2^{(i-k)(n\delta_{12}+\lambda_1-\alpha_{1\infty})}2^{i(\alpha_{1\infty}-\alpha_1(0))}2^{k(\lambda_1-\alpha_{1\infty})}+\sum_{i=0}^{k}(k-i)\,2^{(i-k)(n\delta_{12}+\lambda_1-\alpha_{1\infty})}2^{k(\lambda_1-\alpha_{1\infty})}\right)^{q_{1\infty}}\right\}^{1/q_{1\infty}}\\ &\lesssim \|f_1\|_{M\dot{K}_{p_1(\cdot),\lambda_1}^{\alpha_1(\cdot),q_1(\cdot)}}\sup_{L>0,L\in\mathbb{Z}}2^{-L\lambda_1}\times\left\{\sum_{k=0}^{L}2^{k\lambda_1q_{1\infty}}\cdot\left(\sum_{i=-\infty}^{k}(k-i)\,2^{(i-k)(n\delta_{12}-\alpha_{1\infty}+\lambda_1)}\right)^{q_{1\infty}}\right\}^{1/q_{1\infty}}\\ &\lesssim \|f_1\|_{M\dot{K}_{p_1(\cdot),\lambda_1}^{\alpha_1(\cdot),q_1(\cdot)}}\sup_{L>0,L\in\mathbb{Z}}2^{-L\lambda_1}\left(\sum_{k=0}^{L}2^{k\lambda_1q_{1\infty}}\right)^{1/q_{1\infty}}\\ &\lesssim \|f_1\|_{M\dot{K}_{p_1(\cdot),\lambda_1}^{\alpha_1(\cdot),q_1(\cdot)}}.\end{aligned} \tag{74}$$

Similarly, since $\lambda_2+n\delta_{22}>\alpha_{2\infty}\geq\alpha_2(0)$, by (71) and (73), we obtain that

$$\begin{aligned}G_2 &\lesssim \|f_2\|_{M\dot{K}_{p_2(\cdot),\lambda_2}^{\alpha_2(\cdot),q_2(\cdot)}}\sup_{L>0,L\in\mathbb{Z}}2^{-L\lambda_2}\times\left\{\sum_{k=0}^{L}2^{k\lambda_2q_{2\infty}}\left(\sum_{j=-\infty}^{k}2^{(j-k)(n\delta_{22}-\alpha_{2\infty}+\lambda_2)}\right)^{q_{2\infty}}\right\}^{1/q_{2\infty}}\\ &\lesssim \|f_2\|_{M\dot{K}_{p_2(\cdot),\lambda_2}^{\alpha_2(\cdot),q_2(\cdot)}}\sup_{L>0,L\in\mathbb{Z}}2^{-L\lambda_2}\left(\sum_{k=0}^{L}2^{k\lambda_2q_{2\infty}}\right)^{1/q_{2\infty}}\\ &\lesssim \|f_2\|_{M\dot{K}_{p_2(\cdot),\lambda_2}^{\alpha_2(\cdot),q_2(\cdot)}}.\end{aligned} \tag{75}$$

Thus, we obtain

$$G\lesssim\|b_1\|_*\|f_1\|_{M\dot{K}_{p_1(\cdot),\lambda_1}^{\alpha_1(\cdot),q_1(\cdot)}}\|f_2\|_{M\dot{K}_{p_2(\cdot),\lambda_2}^{\alpha_2(\cdot),q_2(\cdot)}}. \tag{76}$$

Combining all the estimates for $E,F,G$ together, we obtain

$$\begin{aligned}&\left\|H_{b_1}^1(f_1,f_2)\right\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q(\cdot)}}\\ &\quad\lesssim\|b_1\|_*\|f_1\|_{M\dot{K}_{p_1(\cdot),\lambda_1}^{\alpha_1(\cdot),q_1(\cdot)}}\|f_2\|_{M\dot{K}_{p_2(\cdot),\lambda_2}^{\alpha_2(\cdot),q_2(\cdot)}}.\end{aligned} \tag{77}$$

Similarly, we have

$$\begin{aligned}&\left\|H_{b_2}^1(f_1,f_2)\right\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q(\cdot)}}\\&\quad\lesssim\|b_2\|_*\|f_1\|_{M\dot{K}_{p_1(\cdot),\lambda_1}^{\alpha_1(\cdot),q_1(\cdot)}}\|f_2\|_{M\dot{K}_{p_2(\cdot),\lambda_2}^{\alpha_2(\cdot),q_2(\cdot)}}.\end{aligned}\tag{78}$$

Therefore, the proof of Theorem 11 is completed. □

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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