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Research Article

Bilinear Multipliers on Banach Function Spaces

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Let X_1, X_2, X_3 be Banach spaces of measurable functions in $L^0(\mathbb{R})$ and let $m(\xi,\eta)$ be a locally integrable function in $\mathbb{R}^2$. We say that $m \in \mathscr{BM}_{(X_1,X_2,X_3)}(\mathbb{R})$ if $B_m(f,g)(x) = \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{f}(\xi)\hat{g}(\eta)m(\xi,\eta)e^{2\pi i<\xi+\eta,x>}d\xi d\eta$, defined for f and g with compactly supported Fourier transform, extends to a bounded bilinear operator from $X_1 \times X_2$ to X_3. In this paper we investigate some properties of the class $\mathscr{BM}_{(X_1,X_2,X_3)}(\mathbb{R})$ for general spaces which are invariant under translation, modulation, and dilation, analyzing also the particular case of r.i. Banach function spaces. We shall give some examples in this class and some procedures to generate new bilinear multipliers. We shall focus on the case $m(\xi,\eta) = M(\xi-\eta)$ and find conditions for these classes to contain nonzero multipliers in terms of the Boyd indices for the spaces.

1. Introduction

Throughout the paper $L^0(\mathbb{R}^n)$ stands for the space of complex valued measurable functions defined on $\mathbb{R}^n$, $C_c(\mathbb{R}^n)$ and $C_0(\mathbb{R}^n)$ for the spaces of continuous function with compact support and vanishing at infinity, respectively, $\mathcal{S}(\mathbb{R}^n)$ for the Schwartz class on $\mathbb{R}^n$, and $\mathscr{P}(\mathbb{R}^n)$ for the set of functions in $\mathcal{S}(\mathbb{R}^n)$ such that supp $\hat{f}$ is compact. The Fourier transform of $f \in L^1(\mathbb{R}^n)$ is defined by $\mathscr{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i\langle x,\xi\rangle}dx$. For $x \in \mathbb{R}^n$ and $\lambda > 0$, we denote τ_x, M_x, and D_λ are the translation, modulation, and dilation operators given by $\tau_x f(y) = f(y-x)$, by $M_x f(y) = e^{2\pi i\langle x,y\rangle} f(y)$, and by $D_\lambda f(y) = f(\lambda y)$ for $y \in \mathbb{R}^n$. We also recall the notation $f_t = t^{-n}D_{1/t}f$.

Throughout the paper we shall be considering $X \subset L^0(\mathbb{R}^n)$ such that $(X, \|\cdot\|_X)$ is a Banach space and satisfies

$$\|\tau_x f\|_X = \|f\|_X, \quad f \in X,\ x \in \mathbb{R}^n, \tag{1}$$

$$\|M_x f\|_X = \|f\|_X, \quad f \in X,\ x \in \mathbb{R}^n, \tag{2}$$

$$D_\lambda f \in X \quad \forall f \in X,\ \lambda > 0. \tag{3}$$

We denote by $\mathscr{B}_0$ the class of Banach spaces X satisfying (1), (2), and (3).

We say that $X \in \mathscr{B}_0$ is *homogeneous*, to be denoted $X \in \mathscr{B}_h$, whenever $L^1(\mathbb{R}^n) \cap X$ is dense in X and, for any $f \in X$, the maps $x \longrightarrow \tau_x f$ and $x \longrightarrow M_x f$ are continuous from $\mathbb{R}^n$ into X.

If $X \in \mathscr{B}_h$ then $\mathscr{P}(\mathbb{R}^n)$ is dense in X. Indeed, using Minkowski's inequality, for $\phi \in L^1(\mathbb{R}^n)$ and $f \in X$ one has

$$\phi * f = \int_{\mathbb{R}^n} \phi(y)\tau_y f\, dy \in X. \tag{4}$$

Hence given $f \in X$ we first approximate by $g \in X \cap L^1(\mathbb{R}^n)$ and then, by a standard argument, we approximate by $h \in \mathscr{P}(\mathbb{R}^n) \cap X$ using the continuity of the map $x \longrightarrow \tau_x g \in X$.

Let $X \in \mathscr{B}_0$ and $\lambda > 0$; we write

$$D_X(\lambda) = \sup\{\|D_\lambda(f)\|_X : \|f\|_X \le 1\}. \tag{5}$$

For instance, in the case $X = L^p(\mathbb{R})$ one has $D_X(\lambda) = \lambda^{1/p}$ and for $X = L^\Phi(\mathbb{R})$, where Φ is a submultiplicative Young functions with $\Phi(1) = 1$, one has $D_X(\lambda) = 1/\Phi^{-1}(\lambda)$ (see [1, Remark 2.6]).

If X_1 and X_2 are Banach spaces in $L^0(\mathbb{R}^n)$, we denote by $M(X_1, X_2)$ the space of "pointwise" multipliers; that is,

$$M(X_1, X_2) = \left\{f \in L^0(\mathbb{R}^n) : f \cdot g \in X_2,\ \forall g \in X_1\right\}. \tag{6}$$

This becomes a Banach space under the norm

$$\|f\|_{M(X_1,X_2)} = \sup\left\{\|f \cdot g\|_{X_2} : \|g\|_{X_1} = 1\right\}. \tag{7}$$

For $X_2 = L^1(\mathbb{R})$ one obtains the Köethe dual $X_1' = M(X_1, L^1(\mathbb{R}))$. Also notice that Hölder's inequality gives $M(L^{p_1}(\mathbb{R}), L^{p_2}(\mathbb{R})) = L^{p_3}(\mathbb{R})$ for $1 \leq p_1, p_2 < \infty$ and $1/p_3 = 1/p_1 + 1/p_2$. Also for Orlicz spaces (see [2], [3, page 64]) if Φ_i, $i = 1, 2, 3$ is Young functions satisfying

$$\Phi_1^{-1}(x)\,\Phi_2^{-1}(x) \leq \Phi_3^{-1}(x), \quad x \geq 0 \tag{8}$$

then $M(L^{\Phi_1}(\mathbb{R}), L^{\Phi_2}(\mathbb{R})) = L^{\Phi_3}(\mathbb{R})$.

It is straightforward to see that if $X_1, X_2 \in \mathscr{B}_0$ then $M(X_1, X_2) \in \mathscr{B}_0$ and that

$$D_{M(X_1,X_2)}(\lambda) \leq D_{X_1}\left(\lambda^{-1}\right) D_{X_2}(\lambda). \tag{9}$$

Given a couple $X_1 \in \mathscr{B}_h, X_2 \in \mathscr{B}_0$ we shall use the notation $\mathscr{M}_{X_1,X_2}(\mathbb{R}^n)$ for the space of locally integrable functions M defined on $\mathbb{R}^n$ such that

$$T_M(f)(x) = \int_{\mathbb{R}^n} \hat{f}(\xi)\,M(\xi)\,e^{2\pi i\langle x,\xi\rangle}\,d\xi, \tag{10}$$

well defined for $f \in \mathscr{P}(\mathbb{R}^n)$, satisfying that

$$\|T_M(f)\|_{X_2} \leq C\,\|f\|_{X_1}, \quad \forall f \in \mathscr{P}(\mathbb{R}^n) \cap X_1. \tag{11}$$

We endow the space with the norm $\|M\|_{X_1,X_2} = \|T_M\|$.

The reader should be aware that sometimes $\mathscr{M}_{X_1,X_2}(\mathbb{R}^n)$ is defined as the space of distributions $u \in \mathscr{S}'(\mathbb{R}^n)$ such that $u * f \in X_2$ for all $f \in X_1$. We are only restricting to those distributions such that $\hat{u} \in L^1_{loc}(\mathbb{R}^n)$.

For $X_1, X_2 \in \mathscr{B}_0$ we shall write (X_1, X_2) for the space of "convolution" multipliers; that is,

$$(X_1, X_2) = \left\{g \in L^0(\mathbb{R}^n) : g * f \in X_2,\ \forall f \in X_1\right\}. \tag{12}$$

This becomes a Banach space under the norm

$$\|g\|_{(X_1,X_2)} = \sup\left\{\|g * f\|_{X_2} : \|f\|_{X_1} = 1\right\}. \tag{13}$$

Of course $\mathscr{F}((X_1, X_2) \cap L^1(\mathbb{R}^n)) \subseteq \mathscr{M}_{X_1,X_2}(\mathbb{R}^n)$, i.e., $\hat{g} \in \mathscr{M}_{X_1,X_2}(\mathbb{R}^n)$ for any $g \in (X_1, X_2) \cap L^1(\mathbb{R}^n)$ and $\|\hat{g}\|_{X_1,X_2} \leq \|g\|_{(X_1,X_2)}$.

Using that $(\tau_x f) * g = \tau_x(f * g)$, $(M_x f) * g = M_x(f * M_{-x} g)$, and $(D_\lambda f) * g = (1/\lambda) D_\lambda(f * D_{1/\lambda} g)$ one obtains that $\mathscr{B}_0$ is stable under convolution; that is, $(X_1, X_2) \in \mathscr{B}_0$ whenever $X_1, X_2 \in \mathscr{B}_0$. Moreover,

$$D_{(X_1,X_2)}(\lambda) \leq \frac{1}{\lambda} D_{X_2}(\lambda)\,D_{X_1}\left(\frac{1}{\lambda}\right). \tag{14}$$

On the other hand Young's inequality gives $L^{p_3}(\mathbb{R}^n) \subseteq (L^{p_1}(\mathbb{R}^n), L^{p_2}(\mathbb{R}^n))$ for $1 \leq p_1, p_2 < \infty$ with $1/p_1 + 1/p_2 \geq 1$ and $1/p_3 + 1 = 1/p_1 + 1/p_2$. Also for Orlicz spaces (see [2], [3, page 64]) we have that if Φ_i, $i = 1, 2, 3$ are Young functions satisfying

$$\Phi_1^{-1}(x)\,\Phi_2^{-1}(x) \leq x\Phi_3^{-1}(x), \quad x \geq 0 \tag{15}$$

then $L^{\Phi_3}(\mathbb{R}^n) \subseteq (L^{\Phi_1}(\mathbb{R}^n), L^{\Phi_2}(\mathbb{R}^n))$.

From (4) we see that $L^1(\mathbb{R}^n) \subseteq (X, X)$ for any $X \in \mathscr{B}_h$. Actually, using approximations of the identity, one has $(L^1(\mathbb{R}^n), X) = X$ whenever $X \in \mathscr{B}_h$.

With the notation $\mathscr{M}_{p,q}(\mathbb{R}^n)$ for $X_1 = L^p(\mathbb{R}^n)$ and $X_2 = L^q(\mathbb{R}^n)$ and $1 \leq p, q \leq \infty$, we recall some well-known properties of the space of linear multipliers (see [4, 5]): $\mathscr{M}_{p,q}(\mathbb{R}^n) = \{0\}$ whenever $q < p$, $\mathscr{M}_{p,q}(\mathbb{R}^n) = \mathscr{M}_{q',p'}(\mathbb{R}^n)$ for $1 < p \leq q < \infty$ and for $1 \leq p \leq 2$,

$$\begin{gathered}\mathscr{M}_{1,1}(\mathbb{R}^n) \subset \mathscr{M}_{p,p}(\mathbb{R}^n) \subset \mathscr{M}_{2,2}(\mathbb{R}^n),\\ \mathscr{M}_{2,2}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n),\\ \left(L^1(\mathbb{R}^n), L^q(\mathbb{R}^n)\right) = L^q(\mathbb{R}^n), \quad 1 \leq q < \infty.\\ \mathscr{M}_{1,1}(\mathbb{R}^n) = \{\hat{\mu} : \mu \in M(\mathbb{R}^n)\}.\end{gathered} \tag{16}$$

In this paper we shall be concerned with the bilinear analogues and extensions of the above formulas for general function spaces. We shall extend several results shown by the author in the setting of Lebesgue and Orlicz spaces ([1, 6]). We present now the definition of a bilinear multiplier we shall be dealing with.

Definition 1. Let $m(\xi, \eta)$ be a locally integrable function on $\mathbb{R}^n \times \mathbb{R}^n$. Define

$$\begin{aligned}&B_m(f, g)(x)\\ &\quad = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\,\hat{g}(\eta)\,m(\xi, \eta)\,e^{2\pi i\langle\xi+\eta,x\rangle}\,d\xi\,d\eta\end{aligned} \tag{17}$$

for $f, g \in \mathscr{P}(\mathbb{R}^n)$.

Let $X_1, X_2 \in \mathscr{B}_h$ and $X_3 \in \mathscr{B}_0$. A locally integrable function m is said to be a bilinear multiplier on $\mathbb{R}^n$ of type (X_1, X_2, X_3) if there exists $C > 0$ such that

$$\|B_m(f, g)\|_{X_3} \leq C\,\|f\|_{X_1}\,\|g\|_{X_2} \tag{18}$$

for any $f \in \mathscr{P}(\mathbb{R}^n) \cap X_1$ and $g \in \mathscr{P}(\mathbb{R}^n) \cap X_2$.

We write $\mathscr{BM}_{(X_1,X_2,X_3)}(\mathbb{R}^n)$ for such a space and $\|m\|_{X_1,X_2,X_3} = \|B_m\|$ where $\|B_m\|$ stands for the norm of the bounded bilinear map $B_m : X_1 \times X_2 \longrightarrow X_3$.

The theory of multilinear multipliers acting on Lebesgue spaces for "nice" symbols was originated in the work by R. Coiffman and C. Meyer [7] in the eighties and continued by L. Grafakos and R. Torres [8] and many others (see [9, 10]). The theory was retaken and pushed in the nineties after the celebrated result by M. Lacey and C. Thiele, solving the old standing conjecture of Calderón on the boundedness of the bilinear Hilbert transform (see [11, 12]). The bilinear versions of several classical linear operators appearing in Harmonic Analysis, such as the Hilbert transform or the fractional integral, are the motivation for the class of bilinear multipliers that we shall analyze in the paper. Recall that the *bilinear Hilbert transform* and the *bilinear fractional integral* are defined by

$$H(f, g)(x) = \lim_{\varepsilon \longrightarrow 0} \frac{1}{\pi} \int_{|y|>\varepsilon} \frac{f(x - y)\,g(x + y)}{y}\,dy, \tag{19}$$

and

$$I_\alpha(f,g)(x) = \int_{\mathbb{R}} \frac{f(x-y)\,g(x+y)}{|y|^{1-\alpha}}\,dy, \qquad 0<\alpha<1 \tag{20}$$

where $f, g \in \mathcal{S}(\mathbb{R})$.

It is easy to see that (19) and (20) correspond to the bilinear multipliers given by the symbols $m(\xi,\eta) = \text{sign}(\xi-\eta)$ and $m(\xi,\eta) = |2\pi(\xi-\eta)|^{-\alpha}$, respectively; i.e.,

$$H(f,g)(x) = \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{f}(\xi)\,\hat{g}(\eta)\,\text{sign}(\xi-\eta)\,e^{2\pi i(\xi+\eta)x}\,d\xi\,d\eta. \tag{21}$$

$$I_\alpha(f,g)(x) = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\hat{f}(\xi)\,\hat{g}(\eta)}{|2\pi(\xi-\eta)|^{\alpha}}\,e^{2\pi i(\xi+\eta)x}\,d\xi\,d\eta. \tag{22}$$

This motivates the following particular class of bilinear multipliers.

Definition 2. We denote by $\mathcal{M}_{(X_1,X_2,X_3)}(\mathbb{R}^n)$ the space of measurable functions $M:\mathbb{R}^n \longrightarrow \mathbb{C}$ such that $m(\xi,\eta) = M(\xi-\eta) \in \mathcal{BM}_{(X_1,X_2,X_3)}(\mathbb{R}^n)$. We keep the notation $\|M\|_{X_1,X_2,X_3} = \|B_M\|$.

The boundedness results on L^p-spaces for the bilinear H and I_α took long time to be achieved. In particular it was shown that $M(\xi) = \text{sign}(\xi) \in \mathcal{M}_{(L^{p_1},L^{p_3},L^{p_3})}(\mathbb{R}^n)$ for $1 < p_1, p_2 < \infty$, $1/p_3 = 1/p_1 + 1/p_2$, and $2/3 < p_3 < \infty$; i.e., there exists $C>0$ such that

$$\|H(f,g)\|_{p_3} \le C\,\|f\|_{p_1}\,\|g\|_{p_2} \tag{23}$$

(Lacey-Thiele, [11–13]) and that $M(\xi) = 1/|\xi|^\alpha \in \mathcal{M}_{(L^{p_1},L^{p_2},L^{q})}(\mathbb{R}^n)$ for $1<p_1,p_2<\infty$, $0<\alpha<1/p_1+1/p_2$, and $1/q = 1/p_1+1/p_2-\alpha$; i.e., there exists $C>0$ such that

$$\|I_\alpha(f,g)\|_{q} \le C\,\|f\|_{p_1}\,\|g\|_{p_2} \tag{24}$$

(Kenig-Stein [10], Grafakos-Kalton [9]).

The case of more general nonsmooth symbols was also analyzed by J. Gilbert and A. Namod (see [14, 15]).

The study of bilinear multipliers acting on other function spaces has been addressed in the literature. Lorentz spaces have been studied mainly by O. Blasco and F. Villarroya (see [16, 17]), weighted Lebesgue spaces or Lebesgue spaces with variable exponent by T. Gürkanli and O. Kulak [9], rearrangement invariant quasi-Banach spaces by S. Rodriguez-López [18], and more recently Orlicz spaces by O. Blasco and A. Osancliol [1].

Our objective is to study the basic properties of the classes $\mathcal{BM}_{(X_1,X_2,X_3)}(\mathbb{R})$ and $\mathcal{M}_{X_1,X_2,X_3}(\mathbb{R})$, to find examples of bilinear multipliers in these classes, and to get methods to produce new ones. We shall restrict ourselves to rearrangement Banach function spaces to recover some known results under some conditions on the Boyd indices. The results presented in what follows could be formulated for any $n\in\mathbb{N}$, but we shall write our results only for $n=1$ for simplicity.

2. Bilinear Multipliers: The Basics

Throughout this section $X_1, X_2 \in \mathcal{B}_h$ and $X_3 \in \mathcal{B}_0$. Let us start with some elementary properties of the bilinear multipliers when composing with translations, modulations, and dilations. Next result, already established in [6] for Lebesgue spaces and in [1] for Orlicz spaces, follows easily from the basic formulas

$$\begin{aligned}\widehat{(\tau_y f)}(\xi) &= M_{-y}\hat{f}(\xi),\\ \widehat{(M_x f)}(\xi) &= \tau_x\hat{f}(\xi),\\ \widehat{(D_\lambda f)}(\xi) &= \hat{f}_\lambda(\xi).\end{aligned} \tag{25}$$

Proposition 3. *Let $m \in \mathcal{BM}_{(X_1,X_2,X_3)}(\mathbb{R})$.*

(a) *$\tau_{(\xi_0,\eta_0)}m \in \mathcal{BM}_{(X_1,X_2,X_3)}(\mathbb{R})$ for each $(\xi_0,\eta_0)\in\mathbb{R}^2$ and*

$$\left\|\tau_{(\xi_0,\eta_0)}m\right\|_{X_1,X_2,X_3} = \|m\|_{X_1,X_2,X_3}. \tag{26}$$

(b) *$M_{(\xi_0,\eta_0)}m \in \mathcal{BM}_{(X_1,X_2,X_3)}(\mathbb{R})$ for each $(\xi_0,\eta_0)\in\mathbb{R}^2$ and*

$$\left\|M_{(\xi_0,\eta_0)}m\right\|_{X_1,X_2,X_3} = \|m\|_{X_1,X_2,X_3}. \tag{27}$$

(c) *$D_t m \in \mathcal{BM}_{(X_1,X_2,X_3)}(\mathbb{R})$ for each $t>0$ and*

$$\|D_t m\|_{(X_1,X_2,X_3)} \le D_{X_3}\left(\frac{1}{t}\right) D_{X_1}(t)\,D_{X_2}(t)\,\|m\|_{(X_1,X_2,X_3)}. \tag{28}$$

Proof. (a) Let $(\xi_0,\eta_0)\in\mathbb{R}^2$. It is easily seen that

$$B_{\tau_{(\xi_0,\eta_0)}m}(f,g) = M_{\xi_0+\eta_0}B_m\left(M_{-\xi_0}f, M_{-\eta_0}g\right). \tag{29}$$

Hence $\tau_{(\xi_0,\eta_0)}m \in \mathcal{BM}_{(X_1,X_2,X_3)}(\mathbb{R})$ and

$$\left\|\tau_{(\xi_0,\eta_0)}m\right\|_{X_1,X_2,X_3} = \|m\|_{X_1,X_2,X_3}. \tag{30}$$

(b) If $(\xi_0,\eta_0)\in\mathbb{R}^2$ then one has

$$B_{M_{(\xi_0,\eta_0)}m}(f,g) = B_m\left(\tau_{-\xi_0}f, \tau_{-\eta_0}g\right). \tag{31}$$

Therefore, $M_{(\xi_0,\eta_0)}m \in \mathcal{BM}_{(X_1,X_2,X_3)}(\mathbb{R})$ and

$$\left\|M_{(\xi_0,\eta_0)}m\right\|_{X_1,X_2,X_3} = \|m\|_{X_1,X_2,X_3}. \tag{32}$$

(c) Let $t>0$. We first observe that

$$B_{D_t m}(f,g) = D_{1/t}B_m(D_t f, D_t g) \tag{33}$$

for each $f,g\in\mathcal{P}(\mathbb{R})$. Indeed,

$$\begin{aligned} B_m(D_t f, D_t g)(x) &= \int_{\mathbb{R}^2} \frac{1}{t}\hat{f}\left(\frac{\xi}{t}\right)\frac{1}{t}\hat{g}\left(\frac{\eta}{t}\right) m(\xi,\eta)\,e^{2\pi i(\xi+\eta)x}\,d\xi d\eta \\ &= \int_{\mathbb{R}^2} \hat{f}(\xi)\,\hat{g}(\eta)\,m(t\xi,t\eta)\,e^{2\pi i(\xi+\eta)tx}\,d\xi d\eta \\ &= D_t B_{D_t m}(f,g)(x). \end{aligned} \tag{34}$$

This gives

$$\begin{aligned}\left\|B_{D_t m}(f,g)\right\|_{X_3} &\le D_{X_3}\left(\frac{1}{t}\right)\left\|(B_m(D_t f, D_t g)\right\|_{X_3}\\ &\le D_{X_3}\left(\frac{1}{t}\right)\|m\|_{(X_1,X_2,X_3)}\left\|D_t f\right\|_{X_1}\left\|D_t g\right\|_{X_2}\\ &\le D_{X_3}\left(\frac{1}{t}\right)\|m\|_{(X_1,X_2,X_3)} D_{X_1}(t)\left\|f\right\|_{X_1} D_{X_2}(t)\\ &\quad\cdot\left\|g\right\|_{X_2},\end{aligned} \tag{35}$$

which shows that $D_t m \in \mathcal{BM}_{(X_1,X_2,X_3)}(\mathbb{R})$ and the desired estimate for the norm. □

We start presenting an elementary example of bilinear multipliers. Recall that if μ is a Borel regular measure in $\mathbb{R}$ then $\widehat{\mu}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x\xi} d\mu(x)$ is a bounded measurable function in $\mathbb{R}$.

Proposition 4. *Let $\mu \in M(\mathbb{R})$ be a Borel regular measure in $\mathbb{R}$, $(\alpha,\beta) \in \mathbb{R}^2$, and set $m(\xi,\eta) = \widehat{\mu}(\alpha\xi + \beta\eta)$. If $X_1 \subseteq M(X_2, X_3)$ with norm A then $m \in \mathcal{BM}_{(X_1,X_2,X_3)}(\mathbb{R})$. Moreover, $\|m\|_{X_1,X_2,X_3} \le A\|\mu\|_1$.*

Proof. Let us first rewrite the value $B_m(f,g)$ as follows:

$$\begin{aligned}B_m(f,g)(x) &= \int_{\mathbb{R}^2}\widehat{f}(\xi)\,\widehat{g}(\eta)\,\widehat{\mu}(\alpha\xi+\beta\eta)\\ &\quad\cdot e^{2\pi i(\xi+\eta)x}d\xi\,d\eta = \int_{\mathbb{R}^2}\widehat{f}(\xi)\,\widehat{g}(\eta)\\ &\quad\cdot\left(\int_{\mathbb{R}} e^{-2\pi i(\alpha\xi+\beta\eta)t}d\mu(t)\right)e^{2\pi i(\xi+\eta)x}d\xi\,d\eta\\ &= \int_{\mathbb{R}}\left(\int_{\mathbb{R}^2}\widehat{f}(\xi)\,\widehat{g}(\eta)\right.\\ &\quad\left.\cdot e^{2\pi i(x-\alpha t)\xi}e^{2\pi i(x-\beta t)\eta}d\xi\,d\eta\right)d\mu(t)\\ &= \int_{\mathbb{R}} f(x-\alpha t)\,g(x-\beta t)\,d\mu(t).\end{aligned} \tag{36}$$

Hence, using Minkowski's inequality, one has

$$\begin{aligned}\left\|B_m(f,g)\right\|_{X_3} &\le \int_{\mathbb{R}}\left\|f(\cdot-\alpha t)\,g(\cdot-\beta t)\right\|_{X_3} d|\mu|(t)\\ &\le \int_{\mathbb{R}}\left\|f(\cdot-\alpha t)\right\|_{M(X_2,X_3)}\left\|g(\cdot-\beta t)\right\|_{X_2} d|\mu|(t)\\ &\le \int_{\mathbb{R}} A\left\|f(\cdot-\alpha t)\right\|_{X_1}\left\|g(\cdot-\beta t)\right\|_{X_2} d|\mu|(t)\\ &= A\left\|f\right\|_{X_1}\left\|g\right\|_{X_2}\int_{\mathbb{R}} d|\mu|(t)\\ &= \|\mu\|_1 A\left\|f\right\|_{X_1}\left\|g\right\|_{X_2}.\end{aligned} \tag{37}$$

This completes the proof. □

Remark 5. Selecting $\alpha = \beta = 0$ in Proposition 4 one obtains $B_m(f,g) = \widehat{\mu}(0)f\cdot g$, selecting $\alpha = 1, \beta = 0$ one obtains $B_m(f,g) = (\mu * f)\cdot g$, and selecting $\alpha = 1, \beta = -1$ one has that $M = \widehat{\mu} \in \mathcal{M}_{(X_1,X_2,X_3)}(\mathbb{R})$.

Proposition 6. *Let $m \in \mathcal{BM}_{(X_1,X_2,X_3)}(\mathbb{R})$ and $M_1 \in \mathcal{M}_{Y_1,X_1}(\mathbb{R})$ and $M_2 \in \mathcal{M}_{Y_2,X_2}(\mathbb{R})$, where $Y_1, Y_2 \in \mathcal{B}_h$.*

(a) *If $\widetilde{m}(\xi,\eta) = M_1(\xi)m(\xi,\eta)M_2(\eta)$ then $\widetilde{m} \in \mathcal{BM}_{(Y_1,Y_2,X_3)}(\mathbb{R})$.*
Moreover $\|\widetilde{m}\|_{Y_1,Y_2,X_3} \le \|M_1\|_{Y_1,X_1}\|m\|_{X_1,X_2,X_3}\|M_2\|_{Y_2,X_2}$.

(b) *If $\Phi \in L^1(\mathbb{R}^2)$ then $\Phi * m \in \mathcal{BM}_{(X_1,X_2,X_3)}(\mathbb{R})$.*
Moreover $\|\Phi * m\|_{X_1,X_2,X_3} \le \|\Phi\|_1\|m\|_{X_1,X_2,X_3}$.

(c) *If $\nu \in M(\mathbb{R}^2)$ is a Borel regular measure on $\mathbb{R}^2$ then $\widehat{\nu}\cdot m \in \mathcal{BM}_{(X_1,X_2,X_3)}(\mathbb{R})$.*
Moreover $\|\widehat{\nu}\cdot m\|_{X_1,X_2,X_3} \le \|\nu\|_1\|m\|_{X_1,X_2,X_3}$.

Proof. (a) It follows trivially from

$$B_{\widetilde{m}}(f,g) = B_m\left(T_{M_1}f, T_{M_2}g\right). \tag{38}$$

(b) It was shown ([6, Proposition 2.5, (b)]) that

$$B_{\Phi * m}(f,g)(x) = \int_{\mathbb{R}^2} B_{\tau_{(u,v)}m}(f,g)(x)\,\Phi(u,v)\,du\,dv. \tag{39}$$

From the vector-valued Minkowski inequality and part (a) in Proposition 3, we have

$$\begin{aligned}&\|B_{\Phi * m}(f,g)\|_{X_3}\\ &\quad\le \int_{\mathbb{R}^2}\left\|B_{\tau_{(u,v)}m}(f,g)\right\|_{X_3}|\Phi(u,v)|\,du\,dv\\ &\quad\le \|m\|_{X_1,X_2,X_3}\left\|f\right\|_{X_1}\left\|g\right\|_{X_2}\|\Phi\|_1.\end{aligned} \tag{40}$$

(c) Observe that

$$\begin{aligned}B_{\widehat{\nu}\cdot m}(f,g)(x) &= \int_{\mathbb{R}^2}\widehat{f}(\xi)\,\widehat{g}(\eta)\\ &\quad\cdot\left(\int_{\mathbb{R}^2} M_{(-u,-v)}m(\xi,\eta)\,d\nu(u,v)\right)e^{2\pi i(\xi+\eta)x}d\xi\,d\eta\\ &= \int_{\mathbb{R}^2} B_{M_{(-u,-v)}m}(f,g)(x)\,d\nu(u,v).\end{aligned} \tag{41}$$

Argue as above, using now part (b) in Proposition 3, to conclude

$$\begin{aligned}\left\|B_{\widehat{\nu}\cdot m}(f,g)\right\|_{X_3} &\le \int_{\mathbb{R}^2}\left\|B_{M_{(-u,-v)}m}(f,g)\right\|_{X_3} d|\nu|(du,v)\\ &\le \|m\|_{X_1,X_2,X_3}\left\|f\right\|_{X_1}\left\|g\right\|_{X_2}\|\nu\|_1.\end{aligned} \tag{42}$$

□

With all these procedures we have several useful methods to produce examples of multipliers in $\mathcal{BM}_{(X_1,X_2,X_3)}(\mathbb{R})$, which extend those provided in particular cases in [1, 6].

Corollary 7. *Let $X_1 \subseteq M(X_2, X_3)$.*

(a) *If* $m_1 \in \mathcal{M}_{(X_1,X_1)}(\mathbb{R})$ *and* $m_2 \in \mathcal{M}_{(X_2,X_2)}(\mathbb{R})$ *then* $m(\xi,\eta) = m_1(\xi)m_2(\eta) \in \mathcal{BM}_{(X_1,X_2,X_3)}(\mathbb{R})$.

(b) *If* $m \in \mathcal{BM}_{(X_1,X_2,X_3)}(\mathbb{R})$ *and* Ω *is a bounded measurable set in* $\mathbb{R}^2$ *then*

$$m_1(\xi,\eta) = \int_{(\xi,\eta)+\Omega} m(u,v)\,du\,dv \in \mathcal{BM}_{(X_1,X_2,X_3)}(\mathbb{R}). \tag{43}$$

(c) *If* $\Phi \in L^1(\mathbb{R}^2)$ *then* $\widehat{\Phi} \in \mathcal{BM}_{(X_1,X_2,X_3)}(\mathbb{R})$.

(d) *Let* $W(t) = D^2_{X_3}(1/t)D^2_{X_2}(t)$ *and* $\psi \in L^1(\mathbb{R}^+,W)$ *and assume that* $t \longrightarrow m(t\xi,t\eta)\psi(t)$ *is integrable in* $\mathbb{R}^+$ *for each* $(\xi,\eta) \in \mathbb{R}^2$. *Define*

$$m_\psi(\xi,\eta) = \int_0^\infty m(t\xi,t\eta)\,\psi(t)\,dt. \tag{44}$$

Then $m_\psi \in \mathcal{BM}_{(X_1,X_2,X_3)}(\mathbb{R})$ *and* $\|m_\psi\|_{X_1,X_2,X_3} \le \|\psi\|_{L^1(\mathbb{R}^+,W)}\|m\|_{X_1,X_2,X_3}$.

Proof. (a), (b), and (c) follow trivially from Proposition 6.

(d) It is immediate to observe that

$$B_{m_\psi}(f,g)(x) = \int_0^\infty B_{D_t m}(f,g)(x)\,\psi(t)\,dt. \tag{45}$$

Hence Minkowski's inequality together with (1) and part (c) in Proposition 3 lead to the desired result and estimate. □

3. The Case $m(\xi,\eta)=M(\xi-\eta)$

As mentioned in the introduction a number of important bilinear multipliers, such as the bilinear fractional integral, the bilinear Hilbert transform, and other bilinear singular integrals, are defined for symbols $m(\xi,\eta) = M(\xi-\eta)$ for a given measurable function M defined in $\mathbb{R}$. Let us restrict ourselves to this family of multipliers. As in the previous section we always assume $X_1, X_2 \in \mathcal{B}_h$ and $X_3 \in \mathcal{B}_0$. We denote by $\mathcal{M}_{X_1,X_2,X_3}(\mathbb{R})$ the space of locally integrable functions $M : \mathbb{R} \longrightarrow \mathbb{C}$ such that $m(\xi,\eta) = M(\xi-\eta) \in \mathcal{BM}_{(X_1,X_2,X_3)}(\mathbb{R})$, that is to say,

$$B_M(f,g)(x) = \int_{\mathbb{R}^2} \hat{f}(\xi)\,\hat{g}(\eta)\,M(\xi-\eta)\,e^{2\pi i(\xi+\eta)x}d\xi\,d\eta; \tag{46}$$

defined for $\hat{f}$ and $\hat{g}$ compactly supported, satisfies the inequality

$$\|B_M(f,g)\|_{X_3} \le C\|f\|_{X_1}\|g\|_{X_2}, \quad \forall f \in \mathcal{P}(\mathbb{R}) \cap X_1,\ g \in \mathcal{P}(\mathbb{R}) \cap X_2. \tag{47}$$

We keep the notation $\|M\|_{X_1,X_2,X_3} = \|B_M\|$.

This class does have much richer properties than $\mathcal{BM}_{(X_1,X_2,X_3)}(\mathbb{R})$. Since the symbol is also defined on $\mathbb{R}$ we can establish the following behaviour of the bilinear map B_M under translations, modulations, and dilations:

$$\tau_y B_M(f,g) = B_M(\tau_y f, \tau_y g), \quad y \in \mathbb{R}, \tag{48}$$

$$M_{2y} B_M(f,g) = B_M(M_y f, M_y g), \quad y \in \mathbb{R}, \tag{49}$$

$$D_{1/\lambda}(B_M(f,g)) = B_{D_\lambda M}(D_{1/\lambda}f, D_{1/\lambda}g), \quad \lambda > 0. \tag{50}$$

Proposition 8. $\mathcal{M}_{(X_1,X_2,X_3)}(\mathbb{R}) \in \mathcal{B}_0$.

Proof. Let $M \in \mathcal{M}_{(X_1,X_2,X_3)}(\mathbb{R})$ and define $m(\xi,\eta) = M(\xi-\eta)$. We have that $\tau_{(y_1,y_2)}m = \tau_{y_1+y_2}M$, $M_{(y,-y))}m = M_y M$, and $D_\lambda m = D_\lambda M$ for any $y_1, y_2, y \in \mathbb{R}$ and $\lambda > 0$. Hence from the formulas for m we obtain the following ones for M:

$$B_{M_y M}(f,g) = B_M(\tau_{-y}f, \tau_y g), \quad y \in \mathbb{R}. \tag{51}$$

$$B_{\tau_y M}(f,g) = B_M(M_{y/2}f, M_{-y/2}g), \quad y \in \mathbb{R}. \tag{52}$$

$$B_{D_\lambda M}(f,g) = D_{1/\lambda}(B_M(D_\lambda f, D_\lambda g)), \quad \lambda > 0. \tag{53}$$

From them properties (1), (2), and (3) in $\mathcal{M}_{(X_1,X_2,X_3)}(\mathbb{R})$ are easily shown. □

For symbols $M = \hat{\mu}$ for a given $\mu \in M(\mathbb{R})$ we have the following expression.

Proposition 9. *Let* $M = \hat{\mu}$ *for a Borel regular measure* $\mu \in M(\mathbb{R})$. *Then*

$$B_M(f,g) = \int_{\mathbb{R}} f(x-t)\,g(x+t)\,d\mu(t), \quad \forall f,g \in \mathcal{P}(\mathbb{R}). \tag{54}$$

Proof. Given $f, g \in \mathcal{P}(\mathbb{R})$, we can write

$$\begin{aligned} B_M(f,g)(x) &= \int_{\mathbb{R}^2} \hat{f}(\xi)\,\hat{g}(\eta)\left(\int_{\mathbb{R}} e^{-2\pi i(\xi-\eta)t}d\mu(t)\right) \cdot e^{2\pi i(\xi+\eta)x}d\xi\,d\eta = \int_{\mathbb{R}}\left(\int_{\mathbb{R}^2} \hat{f}(\xi)\,\hat{g}(\eta) \cdot e^{2\pi i(x-t)\xi}e^{2\pi i(x+t)\eta}d\xi\,d\eta\right)d\mu(t) \\ &= \int_{\mathbb{R}} f(x-t)\,g(x+t)\,d\mu(t) \end{aligned} \tag{55}$$

and the proof is finished. □

Note that, selecting $\alpha = 1$ and $\beta = -1$ in Proposition 4, we obtain next example, but we would like to point out that it also follows from Proposition 9 even for spaces in $\mathcal{B}_0$.

Proposition 10. *Let* $X_1, X_2, X_3 \in \mathcal{B}_0$ *such that* $X_1 \subseteq M(X_2, X_3)$ *with norm* A *and let* $\mu \in M(\mathbb{R})$. *Then* $M = \hat{\mu} \in \mathcal{M}_{(X_1,X_2,X_3)}(\mathbb{R})$ *with* $\|M\|_{X_1,X_2,X_3} \le A\|\mu\|_1$.

We now produce a method to get multipliers in $\mathcal{BM}_{(X_1,X_2,X_3)}(\mathbb{R})$ from those in $\mathcal{M}_{(X_1,X_2,X_3)}(\mathbb{R})$.

Proposition 11. *Let* $X_1, X_2, X_3 \in \mathscr{B}_h$, $\mu \in M(\mathbb{R})$, $M \in \mathscr{M}_{(X_1,X_2,X_3)}(\mathbb{R})$, *and set* $m(\xi,\eta) = M(\xi-\eta)\widehat{\mu}(\xi+\eta)$. *Then* $m \in \mathscr{B}\mathscr{M}_{(X_1,X_2,X_3)}(\mathbb{R})$ *and* $\|m\|_{X_1,X_2,X_3} \le \|\mu\|_1 \|M\|_{X_1,X_2,X_3}$.

Proof. We use now the following formula for $f, g \in \mathscr{S}(\mathbb{R})$:

$$\begin{aligned} B_m(f,g)(x) &= \int_{\mathbb{R}^2} \widehat{f}(\xi)\widehat{g}(\eta) M(\xi-\eta) \\ &\cdot \left(\int_{\mathbb{R}} e^{-2\pi i(\xi+\eta)y} d\mu(y)\right) e^{2\pi i(\xi+\eta)x} d\xi\, d\eta \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}^2} \widehat{f}(\xi)\widehat{g}(\eta) M(\xi-\eta)\right. \\ &\cdot \left. e^{2\pi i(\xi+\eta)(x-y)} d\xi\, d\eta\right) d\mu(y) = \mu \\ &* B_M(f,g)(x). \end{aligned} \tag{56}$$

Now recall that $\mu * h(x) = \int_{\mathbb{R}} h(x-y)d\mu(y) = \int_{\mathbb{R}} \tau_y h(x) d\mu(y)$ we actually have that $\|\mu * h\|_{X_3} \le \|h\|_{X_3}\|\mu\|_1$ for any $h \in X_3$. Using that $h = B_M(f,g) \in X_3$ we conclude the result. □

As in the previous section we can generate new multipliers in $\mathscr{M}_{(X_1,X_2,X_3)}(\mathbb{R})$ extending [6, Proposition 3.5].

Proposition 12. *Let* $\phi \in L^1(\mathbb{R})$ *and* $M \in \mathscr{M}_{(X_1,X_2,X_3)}(\mathbb{R})$. *Then*

(a) $\phi * M \in \mathscr{M}_{(X_1,X_2,X_3)}(\mathbb{R})$ *and* $\|\phi * M\|_{X_1,X_2,X_3} \le \|\phi\|_1 \|M\|_{X_1,X_2,X_3}$.

(b) $\widehat{\phi} M \in \mathscr{M}_{(X_1,X_2,X_3)}(\mathbb{R})$ *and* $\|\widehat{\phi} M\|_{X_1,X_2,X_3} \le \|\phi\|_1 \|M\|_{X_1,X_2,X_3}$.

(c) *Let* $W(t) = D_{X_3}(1/t) D_{X_1}(t) D_{X_2}(t)$ *and* $\psi \in L^1(\mathbb{R}^+, W)$ *and assume that* $t \longrightarrow M(t\xi)\psi(t)$ *is integrable in* $\mathbb{R}^+$ *for each* $\xi \in \mathbb{R}$. *Define*

$$M_\psi(\xi) = \int_0^\infty M(t\xi)\psi(t)\, dt. \tag{57}$$

Then $M_\psi \in \mathscr{M}_{(X_1,X_2,X_3)}(\mathbb{R})$ *and* $\|M_\psi\|_{X_1,X_2,X_3} \le \|\psi\|_{L^1(\mathbb{R}^+,W)} \|M\|_{X_1,X_2,X_3}$.

Proof. (a) Apply Minkowski's inequality to the formula ([6, Proposition 3.5. (a)]:

$$B_{\phi * M}(f,g)(x) = \int_{\mathbb{R}} M_u B_M(M_{-u} f, g)(x)\phi(u)\, du. \tag{58}$$

(b) Use ([6, Proposition 3.5. (b)] which establishes that

$$B_{\widehat{\phi} m}(f,g)(x) = \int_{\mathbb{R}} B_{M_{-u}M}(f,g)(x)\phi(u)\, du \tag{59}$$

together with Minkowski's inequality and (51).

(c) Write making use of (50) the following formula:

$$\begin{aligned} B_{M_\psi}(f,g)(x) &= \int_{\mathbb{R}^2} \widehat{f}(\xi)\widehat{g}(\eta) \\ &\cdot \left(\int_0^\infty D_t M(\xi-\eta)\psi(t)\, dt\right) e^{2\pi i(\xi+\eta)x} d\xi\, d\eta \\ &= \int_0^\infty B_{D_t M}(f,g)(x)\psi(t)\, dt \\ &= \int_0^\infty D_{1/t} B_M(D_t f, D_t g)(x)\psi(t)\, dt. \end{aligned} \tag{60}$$

Therefore, from Minkowski's again one gets

$$\begin{aligned} \left\|B_{M_\psi}(f,g)(x)\right\|_{X_3} &\le \|M\|_{X_1,X_2,X_3} \|f\|_{X_1} \|g\|_{X_2} \\ &\cdot \int_0^\infty D_{X_3}\left(\frac{1}{t}\right) D_{X_1}(t) D_{X_2}(t) |\psi(t)|\, dt \end{aligned} \tag{61}$$

which finishes the proof. □

Let us show that the classes $\mathscr{M}_{(X_1,X_2,X_3)}(\mathbb{R})$ are reduced to $\{0\}$ for some values of the parameters. We follow the approach used first in [6] and later in [1].

Lemma 13. *Assume that* $W(x) = e^{-x^2} \in X_1 \cap X_2$ *and let* $M \in \mathscr{M}_{(X_1,X_2,X_3)}(\mathbb{R})$ *such that* $F_M(\lambda) = \int_{\mathbb{R}} e^{-\pi^2 v^2/2} M(\lambda v) dv < \infty$ *for all* $\lambda > 0$. *Then there exists a constant* $A > 0$ *such that*

$$F_M(\lambda) \le A\|M\|_{X_1,X_2,X_3} D_{X_1}(\lambda) D_{X_2}(\lambda) D_{X_3}\left(\frac{1}{\lambda}\right), \quad \lambda > 0. \tag{62}$$

Proof. It is known (see [6, Proposition 3.3]) that for $f, g \in \mathscr{S}(\mathbb{R})$ we can write

$$\begin{aligned} &B_M(f,g)(x) \\ &= \frac{1}{2}\int_{\mathbb{R}^2} \widehat{f}\left(\frac{u+v}{2}\right)\widehat{g}\left(\frac{u-v}{2}\right) M(v) e^{2\pi i u x} du\, dv. \end{aligned} \tag{63}$$

Let $G(x) = D_\pi W(x) = e^{-\pi^2 x^2}$. One has that $G \in X_1 \cap X_2$ and $\widehat{G} = \gamma G$ for certain constant γ. Making use of (50) and (63) we have that

$$\begin{aligned} B_M(D_\lambda G, D_\lambda G)(x) &= B_{D_\lambda M}(G,G)(\lambda x) = \frac{1}{2} \\ &\cdot \int_{\mathbb{R}}\int_{\mathbb{R}} \widehat{G}\left(\frac{u+v}{2}\right)\widehat{G}\left(\frac{u-v}{2}\right) M(\lambda v) e^{2\pi i u\lambda x} du\, dv \\ &= C\left(\int_{\mathbb{R}} e^{-\pi^2 u^2/2} e^{2\pi i u \lambda x} du\right) \\ &\cdot \left(\int_{\mathbb{R}} e^{-\pi^2 v^2/2} M(\lambda v)\, dv\right) = C' D_{\sqrt{2}\lambda} G(x) F_M(\lambda). \end{aligned} \tag{64}$$

Since $M \in \mathscr{M}_{(X_1,X_2,X_3)}(\mathbb{R})$ we have

$$\begin{aligned} &C' \left\|D_{\sqrt{2}\lambda} G\right\|_{X_3} F_M(\lambda) \\ &\quad \le \|M\|_{X_1,X_2,X_3} \left\|D_\lambda G\right\|_{X_1} \left\|D_\lambda G\right\|_{X_2}. \end{aligned} \tag{65}$$

Using that $G = D_{1/\sqrt{2}\lambda} D_{\sqrt{2}\lambda} G$ we have $\|G\|_{X_3} \leq D_{X_3}(1/\sqrt{2}) D_{X_3}(1/\lambda) \|D_{\sqrt{2}\lambda} G\|_{X_3}$.

Hence

$$\frac{\|G\|_{X_3}}{D_{X_3}(1/\lambda)} F_M(\lambda) \leq A' \|M\|_{X_1,X_2,X_3} D_{X_1}(\lambda) \|G\|_{X_1} D_{X_2}(\lambda) \|G\|_{X_2}. \quad (66)$$

The proof is then complete. □

Theorem 14. *Assume that* $W(x) = e^{-x^2} \in X_1 \cap X_2$ *and* $\mathscr{M}_{(X_1,X_2,X_3)}(\mathbb{R}) \neq \{0\}$. *Then*

$$\liminf_{\lambda \longrightarrow 0} D_{X_1}(\lambda) D_{X_2}(\lambda) D_{X_3}\left(\frac{1}{\lambda}\right) > 0 \quad (67)$$

and

$$\liminf_{\lambda \longrightarrow \infty} \lambda D_{X_1}(\lambda) D_{X_2}(\lambda) D_{X_3}\left(\frac{1}{\lambda}\right) > 0 \quad (68)$$

Proof. Using Proposition 12 we may assume that there exists a nonzero continuous and integrable function M belonging to $\mathscr{M}_{(X_1,X_2,X_3)}(\mathbb{R})$. Let $y \in \mathbb{R}$ such that $M(y) \neq 0$. By using Lemma 13 for the function $M(y - \cdot)$ we obtain

$$\left|\frac{1}{\lambda}\int_{\mathbb{R}} e^{-\pi^2\xi^2/2\lambda^2} M(y-\xi)\, d\xi\right| \leq C D_{X_1}(\lambda) D_{X_2}(\lambda) D_{X_3}\left(\frac{1}{\lambda}\right). \quad (69)$$

Since $M \in L^1(\mathbb{R})$ and continuous and in particular $M \in C_0(\mathbb{R})$, through the convolution with an approximation of the identity and taking limits as $\lambda \longrightarrow 0$ one obtains

$$\lim_{\lambda \longrightarrow 0} \frac{1}{\lambda} \left|\int_{\mathbb{R}} e^{-\pi^2\xi^2/2\lambda^2} M(y-\xi)\, d\xi\right| = C|M(y)| > 0. \quad (70)$$

This gives (67).

Since $\widehat{M} \neq 0$ there exists $y \in \mathbb{R}$ such that $\widehat{M}(y) \neq 0$. Using again Lemma 13, applied now to $M_{-y}M$, we obtain

$$\left|\int_{\mathbb{R}} e^{-\pi^2\xi^2/2\lambda^2} e^{-2\pi i \xi y} M(\xi)\, d\xi\right| \leq C\lambda D_{X_1}(\lambda) D_{X_2}(\lambda) D_{X_3}\left(\frac{1}{\lambda}\right). \quad (71)$$

Therefore, taking limits as $\lambda \longrightarrow \infty$ we get

$$\lim_{\lambda \longrightarrow \infty} \left|\int_{\mathbb{R}} e^{-\pi^2\xi^2/2\lambda^2} e^{-2\pi i \xi y} M(\xi)\, d\xi\right| = \left|\widehat{M}(y)\right| > 0. \quad (72)$$

Hence we get (68) and the proof is finished. □

Corollary 15. *Let* $X_1, X_2, X_3 \in \mathscr{B}_0$ *and let us write*

$$d_0(X_1,X_2,X_3) = \liminf_{\lambda \longrightarrow 0} D_{X_1}(\lambda) D_{X_2}(\lambda) D_{X_3}\left(\frac{1}{\lambda}\right)$$
$$d_\infty(X_1,X_2,X_3) = \liminf_{\lambda \longrightarrow \infty} \lambda D_{X_1}(\lambda) D_{X_2}(\lambda) D_{X_3}\left(\frac{1}{\lambda}\right). \quad (73)$$

If $d_0(X_1,X_2,X_3) d_\infty(X_1,X_2,X_3) = 0$ *then* $\mathscr{M}_{(X_1,X_2,X_3)}(\mathbb{R}) = \{0\}$.

In the cases $X_i = L^{p_i}(\mathbb{R})$ for $i = 1, 2, 3$ the constants $d_0(X_1,X_2,X_3)$ and $d_\infty(X_1,X_2,X_3)$ can be explicitly computed. Therefore one recovers the following result.

Corollary 16 (see [6, 17]). *Let* $1 \leq p_1, p_2, p_3 < \infty$ *and* $\mathscr{M}_{(L^{p_1}(\mathbb{R}),L^{p_2}(\mathbb{R}),L^{p_3}(\mathbb{R}))}(\mathbb{R}) \neq \{0\}$. *Then* $1/p_3 \leq 1/p_1 + 1/p_2 \leq 1/p_3 + 1$.

4. Bilinear Multipliers on Rearrangement Invariant Banach Function Spaces

In this section we shall restrict our study to Banach function spaces. A space $X \subset L^0(\mathbb{R})$ is called a "Banach function space" (see [19]), in short $X \in (BFS)$, if $(X, \|\cdot\|_X)$ is a Banach space which satisfies

(1) $f \in X$ and $|g| \leq |f|$ a.e. implies that $g \in X$ and $\|g\|_X \leq \|f\|_X$.

(2) If $0 \leq f_n \uparrow f$ a. e. then $\|f_n\|_X \uparrow \|f\|_X$.

(3) $\chi_E \in X$ whenever E is measurable and $|E| < \infty$.

(4) For each $|E| < \infty$ there exists $C_E > 0$ such that $\int_E |f| \leq C_E \|f\|_X$.

We shall denote $(BFS)_0 = (BFS) \cap \mathscr{B}_0$ and $(BFS)_h = (BFS) \cap \mathscr{B}_h$. It is clear that $C_c(\mathbb{R}^n) \subset X$ for any $X \in (BFS)$ and that $\mathscr{P}(\mathbb{R})$ is dense in X whenever $X \in (BFS)_h$.

Recall that $X \in (BFS)$ is said to have "absolutely continuous norm", in short $X \in (BFS)_a$, if $\|f\chi_{E_n}\|_X \longrightarrow 0$ for every $f \in X$ and every sequence of measurable sets E_n with $E_n \longrightarrow \emptyset$ a.e.

Proposition 17. *If* $X \in (BFS)_a$ *then* $X \in (BFS)_h$.

Proof. Let $X \in (BFS)_a$. The fact that $L^1(\mathbb{R}^n) \cap X$ is dense in X follows since bounded functions compactly supported are dense (see [19, Theorem 3.11]). To show that $x \longrightarrow \tau_x f$ and $x \longrightarrow M_x f$ are continuous for any $f \in X$ we shall make use of the Lebesgue dominated theorem (see [19, Proposition 3.6]) which holds because X has absolutely continuous norm. Now given $f \in X$ and a sequence $x_n \longrightarrow 0$ one has $M_{x_n} f - f \longrightarrow 0$ and $|M_{x_n} f - f| \leq 2|f|$ what gives that $\|M_{x_n} f - f\|_X \longrightarrow 0$. Therefore $x \longrightarrow M_x f$ is continuous at the origin and hence at any point. To study the translation we first assume that f is a bounded function supported on a finite set E and $|x_n| \leq 1$ with $x_n \longrightarrow 0$. In such a case $\tau_{x_n} f - f \longrightarrow 0$ and $|\tau_{x_n} f - f| \leq 2\|f\|_\infty \chi_{\widetilde{E}}$ with $\widetilde{E} = E \cup (E + [-1,1])$. This gives that $\|\tau_{x_n} f - f\|_X \longrightarrow 0$ and therefore $x \longrightarrow \tau_x f$ is continuous for any bounded function with finite support. Using the density of such functions in X one gets the result for any $f \in X$. □

Proposition 18. *Assume that* $X_1, X_2 \in \mathscr{B}_h$ *and* $X_3 \in (BFS)$. *If* $m_n(x,y) \longrightarrow m(x,y)$ *a.e. where* $m_n \in \mathscr{BM}_{(X_1,X_2,X_3)}(\mathbb{R})$ *with* $\sup_n \|m_n\|_{X_1,X_2,X_3} < \infty$ *then* $m \in \mathscr{BM}_{(X_1,X_2,X_3)}(\mathbb{R})$ *and* $\|m\|_{X_1,X_2,X_3} \leq \sup_n \|m_n\|_{X_1,X_2,X_3}$.

Proof. For each $f \in \mathscr{P}(\mathbb{R}) \cap X_1$ and $g \in \mathscr{P}(\mathbb{R}) \cap X_2$ one has that $B_{m_n}(f,g) \longrightarrow B_m(f,g)$ a.e. and $\liminf \|B_{m_n}(f,g)\|_{X_3} < \infty$. Hence using Fatou's lemma (see [19, Theorem 1.7]), one has

$$\begin{aligned}\|B_m(f,g)\|_{X_3} &\le \liminf \left\|B_{m_n}(f,g)\right\|_{X_3} \\ &\le \sup_n \|m_n\|_{X_1,X_2,X_3} \|f\|_{X_1} \|g\|_{X_2}.\end{aligned} \tag{74}$$

This gives the result. □

Recall that $X \in (BFS)$ is said to be invariant under rearrangement, in short $X \in (r.i.)$, whenever it satisfies the following.

(5) If $f \in X$ and g is equimeasurable to g then $g \in X$ and $\|f\|_X = \|g\|_X$.

Recall that if $X \in (r.i.)$ one defines

$$h_X(t) = \sup_{f\neq 0} \frac{\|D_{1/t} f^*\|_{\widetilde{X}}}{\|f^*\|_{\widetilde{X}}}, \quad t > 0 \tag{75}$$

where $f^*(t) = \inf\{s > 0 : |\{x : |f(x)| > s\}| \le t\}$ and $\widetilde{X}$ is the r.i. space defined on $(0,\infty)$ with the same distribution function. In particular $D_X(\lambda) = h_X(1/\lambda)$.

We observe that rearrangement invariant Banach function spaces preserve translations, modulations, and dilations; that is,

$$(r.i.) \subset (BFS)_0. \tag{76}$$

Indeed, it follows using that $\|M_x f\|_X = \||M_x f|\| = \|f\|_X$, $\tau_x f$ is equimeasurable to f for any x and $D_\lambda(f) < \infty$ for any $f \in X$ (see [19, Proposition 5.11]), since $(D_\lambda f)^* = D_\lambda f^*$. In particular if $X \in (r.i.) \cap (BFS)_a$ then $X \in (BFS)_h$. We shall write $(r.i.)_a$ the class of rearrangement invariant Banach function spaces with absolutely continuous norm.

Taking into account that $\widetilde{f}(x) = f(-x)$ is equimeasurable with f then

$$\mathscr{M}_{(X_1,X_2,X_3)}(\mathbb{R}) = \mathscr{M}_{(X_2,X_1,X_3)}(\mathbb{R}) \tag{77}$$

for any $X_1, X_2 \in (r.i.)_a$ and $X_3 \in (BFS)$.

In the setting of Banach function spaces we can always consider the associate space X', corresponding to the Köthe dual $M(X, L^1(\mathbb{R}))$. It is well-known that X' is isometrically embedded into the dual X^* (see [19, Lemma 2.8]) and that actually $X = X''$ (see [19, Theorem 2.7]). This allows us to give a characterization of bilinear multipliers in $\mathscr{BM}_{(X_1,X_2,X_3)}(\mathbb{R})$ in terms of the duality.

Proposition 19. *Let $X_i \in (r.i.)_a$ for $i = 1,2$ and $X_3 \in (r.i.)$ such that $X_3' \in (r.i.)_a$, and let m be a locally integrable function in $\mathbb{R}^2$. Then $m \in \mathscr{BM}_{(X_1,X_2,X_3)}(\mathbb{R})$ if and only if there exists $C > 0$ such that*

$$\begin{aligned}&\left|\iint_{\mathbb{R}^2} \widehat{f}(\xi)\,\widehat{g}(\eta)\,\widehat{h}(\xi+\eta)\,m(\xi,\eta)\,d\xi\,d\eta\right| \\ &\quad\le C\|f\|_{X_1}\|g\|_{X_2}\|h\|_{X_3'}\end{aligned} \tag{78}$$

for all $f \in \mathscr{P}(\mathbb{R}) \cap X_1$, $g \in \mathscr{P}(\mathbb{R}) \cap X_2$ and $h \in L^1(\mathbb{R}) \cap X_3'$.

Corollary 20. *Let $X_1, X_2, X_3', Y_3' \in (r.i.)_a$. Then $\mathscr{F}(Y_3' \cap L^1(\mathbb{R})) \subseteq \mathscr{M}_{(X_1,X_2,X_3)}(\mathbb{R})$ if and only if $\mathscr{F}(X_3' \cap L^1(\mathbb{R})) \subseteq \mathscr{M}_{(X_1,X_2,Y_3)}(\mathbb{R})$.*

Proof. Due to Proposition 19, $\widehat{k} \in \mathscr{M}_{(X_1,X_2,X_3)}(\mathbb{R})$ for some $k \in Y_3' \cap L^1(\mathbb{R})$ implies that

$$\begin{aligned}&\left|\iint_{\mathbb{R}^2} \widehat{f}(\xi)\,\widehat{g}(\eta)\,\widehat{h}(\xi+\eta)\,\widehat{k}(\xi-\eta)\,d\xi\,d\eta\right| \\ &\quad\le C\|f\|_{X_1}\|g\|_{X_2}\|h\|_{X_3'}\left\|\widehat{k}\right\|_{X_1,X_2,X_3} \\ &\quad\le C'\|f\|_{X_1}\|g\|_{X_2}\|h\|_{X_3'}\|k\|_{Y_3'}.\end{aligned} \tag{79}$$

Changing the variables $\xi = \xi'$ and $\eta = -\eta'$ implies that $\widehat{h} \in \mathscr{M}_{(X_1,X_2,Y_3)}(\mathbb{R})$. □

Let us give now a necessary condition for bilinear multipliers homogeneous of degree β in the setting of rearrangement invariant Banach function spaces. We need to recall the definition of Boyd indices (see [19, page 149]): these are given by

$$\begin{aligned}\underline{\alpha}_X &= \lim_{t\longrightarrow 0} \frac{\log h_X(t)}{\log t}, \\ \overline{\alpha}_X &= \lim_{t\longrightarrow\infty} \frac{\log h_X(t)}{\log t}.\end{aligned} \tag{80}$$

Proposition 21. *Let $X_1, X_2 \in (r.i.)_a$ and $X_3 \in (r.i.)$, $\beta \in \mathbb{R}$ and assume that $m \in \mathscr{BM}_{(X_1,X_2,X_3)}(\mathbb{R})$ be a nonzero multiplier such that $m(t\xi, t\eta) = t^\beta m(\xi,\eta)$ for any $t > 0$. Then*

$$\overline{\alpha}_{X_3} \ge \beta + \underline{\alpha}_{X_1} + \underline{\alpha}_{X_2} \tag{81}$$

and

$$\underline{\alpha}_{X_3} \le \beta + \overline{\alpha}_{X_1} + \overline{\alpha}_{X_2} \tag{82}$$

Proof. From assumption $D_t m = t^\beta m$ for $t > 0$. Using now Proposition 3 we can write

$$\begin{aligned}&t^\beta \|m\|_{(X_1,X_2,X_3)} \\ &\quad\le D_{X_3}\left(\frac{1}{t}\right) D_{X_1}(t)\, D_{X_2}(t)\, \|m\|_{(X_1,X_2,X_3)}, \quad t > 0.\end{aligned} \tag{83}$$

Since $D_X(t) = h_X(1/t)$ we have

$$h_{X_3}(t)\, h_{X_1}\left(\frac{1}{t}\right) h_{X_2}\left(\frac{1}{t}\right) \ge t^\beta, \quad t > 0. \tag{84}$$

Therefore,

$$\log h_{X_3}(t) + \log h_{X_1}\left(\frac{1}{t}\right) + \log h_{X_2}\left(\frac{1}{t}\right) \ge \beta, \quad t > 0 \tag{85}$$

This shows that

$$\frac{\log h_{X_3}(t)}{\log t} - \frac{\log h_{X_1}(1/t)}{\log(1/t)} - \frac{\log h_{X_2}(1/t)}{\log(1/t)} \geq \beta, \quad t \geq 1$$
$$\frac{\log h_{X_3}(t)}{\log t} - \frac{\log h_{X_1}(1/t)}{\log(1/t)} - \frac{\log h_{X_2}(1/t)}{\log(1/t)} \leq \beta, \quad 0 < t < 1. \tag{86}$$

Hence making limits as $t \longrightarrow \infty$ and $t \longrightarrow 0$ one obtains (81) and (82), respectively. □

Corollary 22. *Let $1 \leq p_1, p_2, p_3 < \infty$. If $m \in \mathscr{BM}_{(L^{p_1}(\mathbb{R}),L^{p_2}(\mathbb{R}),L^{p_3}(\mathbb{R}))}(\mathbb{R})$ is a nonzero and homogeneous of degree β then $1/p_1 + 1/p_2 + \beta = 1/p_3$.*

In particular the bilinear Hilbert transform H and the fractional integral I_α can only be bounded whenever $1/p_1 + 1/p_2 = 1/p_3$ and $1/p_1 + 1/p_2 - \alpha = 1/p_3$, respectively.

We use now our general approaches to get concrete examples of multipliers in $\mathscr{M}_{(X_1,X_2,X_3)}(\mathbb{R})$ and $\mathscr{BM}_{(X_1,X_2,X_3)}(\mathbb{R})$.

Proposition 23. *Let $X_1, X_2 \in (r.i.)_a$ and $X_3 \in (BFS)$ satisfy*

$$0 < \underline{\alpha}_{X_i} \leq \overline{\alpha}_{X_i} < 1, \quad i = 1, 2. \tag{87}$$

Let $Q = [a, b] \times [c, d]$ and $m \in \mathscr{BM}_{(X_1,X_2,X_3)}(\mathbb{R})$ then $m\chi_Q \in \mathscr{BM}_{(X_1,X_2,X_3)}(\mathbb{R})$ and $\|m\chi_Q\|_{X_1,X_2,X_3} \leq C\|m\|_{X_1,X_2,X_3}$.

Proof. We invoke firt Boyd's result (see [19, Theorem 5.18]) which establishes that $m(\xi) = \operatorname{sign}(\xi) \in \mathscr{M}_{(X,X)}(\mathbb{R})$ if and only if $0 < \underline{\alpha}_X \leq \overline{\alpha}_X < 1$. Since $\chi_{(-\infty,x]} = \tau_{-x}\chi_{(-\infty,0]}$ and $\chi_{(-\infty,0]} \in \mathscr{M}_{X_i,X_i}$ for $i = 1, 2$ one has that $\chi_{[a,b]} \in \mathscr{M}_{X_1,X_1}(\mathbb{R})$ and $\chi_{[c,d]} \in \mathscr{M}_{X_2,X_2}(\mathbb{R})$. Now applying part (a) in Proposition 6 one obtains the result. □

Proposition 24. *Let $X_2 \in (r.i.)_a$. Then*

$$\mathscr{F}\left(\left(X_2, L^2(\mathbb{R})\right) \cap L^1(\mathbb{R})\right) \subseteq \mathscr{M}_{(L^2(\mathbb{R}),X_2,L^\infty(\mathbb{R}))}(\mathbb{R}). \tag{88}$$

Proof. Let $h \in (X_2, L^2(\mathbb{R})) \cap L^1(\mathbb{R})$ and $M(\xi) = \check{h}(\xi) = \widehat{h}(-\xi)$. We shall see that $M \in \mathscr{M}_{(L^2(\mathbb{R}),X_2,L^\infty(\mathbb{R}))}(\mathbb{R})$ and

$$\|M\|_{L^2(\mathbb{R}),X_2,L^\infty(\mathbb{R})} \leq \|h\|_{(X_2,L^2(\mathbb{R}))}. \tag{89}$$

We use the formulation (see [6, Proposition 3.3]) given by

$$B_M(f, g)(x) = \int_{\mathbb{R}} \left(\widehat{\tau_{-x}g} * M\right)(\xi)\, \widehat{\tau_{-x}f}(\xi)\, d\xi. \tag{90}$$

Hence, since $\widehat{h} * g = h * \widehat{g}$ whenever it is well defined, for $f, g \in \mathscr{P}(\mathbb{R})$ we can write

$$\begin{aligned} \left|B_M(f, g)(x)\right| &\leq \int_{\mathbb{R}} \left|(\tau_{-x}g * h)(\xi)\right| \left|\widehat{\tau_{-x}f}(\xi)\right| d\xi \\ &\leq \left\|\tau_{-x}g * h\right\|_{L^2(\mathbb{R})} \left\|\widehat{\tau_{-x}f}\right\|_{L^2(\mathbb{R})} \\ &\leq \left\|\tau_{-x}g\right\|_{X_2} \|h\|_{(X_2,L^2(\mathbb{R}))} \left\|\widehat{\tau_{-x}f}\right\|_{L^2(\mathbb{R})} \\ &\leq \|g\|_{X_2} \|h\|_{(X_2,L^2(\mathbb{R}))} \|f\|_{L^2(\mathbb{R})}. \end{aligned} \tag{91}$$

This gives the result. □

We now shall combine our results with the method of interpolation for Banach lattices due to Calderón (see [4, 20]) to get some sufficient conditions on multipliers in $\mathscr{M}_{(X_1,X_2,X_3)}(\mathbb{R})$. Recall that for $0 < \theta < 1$ and $X_1, X_2 \in (BFS)$ we can define the Banach function space

$$X_1^\theta X_2^{1-\theta} = \left\{h \in L^0(\mathbb{R}) : |h| \leq \lambda |g|^\theta |h|^{1-\theta},\ \lambda > 0,\ \|f\|_{X_1} \leq 1,\ \|g\|_{X_2} \leq 1\right\}. \tag{92}$$

Proposition 25. *Let $X_1, X_2, X_3' \in (r.i.)_a$ satisfying that $X_1 \subseteq M(X_2, X_3)$ and let $0 \leq \theta \leq 1$. Set*

$$X = L^1(\mathbb{R})^\theta \left(X_3'\right)^{1-\theta}, \qquad \widetilde{X}_3 = X_3^\theta L^\infty(\mathbb{R})^{1-\theta} \tag{93}$$

and assume that $\mathscr{F}(X) \subset L^1_{loc}(\mathbb{R})$. Then $\mathscr{F}(X) \subseteq \mathscr{M}_{(X_1,X_2,\widetilde{X}_3)}(\mathbb{R})$.

Proof. Consider the trilinear form

$$T(K, f, g)(x) = \int_{\mathbb{R}} f(x - t)\, g(x + t)\, K(t)\, dt. \tag{94}$$

From Proposition 9, assuming that $K \in L^1(\mathbb{R})$, we have $B_M(f, g) = T(K, f, g)$ for $M = \widehat{K}$. Now from Proposition 10 we conclude that T is bounded from $L^1(\mathbb{R}) \times X_1 \times X_2$ into X_3 and it has norm bounded by 1.

On the other hand, if $K \in X_3'$, using Hölder's inequality,

$$\sup_x \left|\int_{\mathbb{R}} f(x - t)\, g(x + t)\, K(t)\, dt\right| \leq \|f\|_{X_1} \|g\|_{X_2} \|K\|_{X_3'}. \tag{95}$$

This shows that T is also bounded from $X_3' \times X_1 \times X_2$ into $L^\infty(\mathbb{R})$. Therefore, by interpolation, for each $0 < \theta < 1$ one obtains that T is bounded from $X \times X_1 \times X_2$ into $\widetilde{X}_3$. This shows that $\widehat{K} \in \mathscr{M}_{(X_1,X_2,\widetilde{X}_3)}(\mathbb{R})$ for any $K \in X$. □

Let us apply the previous proposition for $X_1 = L^{p_1}(\mathbb{R})$, $X_2 = L^{p_2}(\mathbb{R})$, and $X_3 = L^{p_3}(\mathbb{R})$ with $1/p_1 + 1/p_2 = 1/p_3$ with $1 \leq p_1, p_2 < \infty$ and $1 < p_3 \leq \infty$.

Corollary 26. *Let $1 \leq p_1, p_2 < \infty$ and $1 < p_3 \leq \infty$ with $1/p_1 + 1/p_2 = 1/p_3$. If $0 < q - p_3 \leq p_3 q/2$ then*

$$\mathscr{F}\left(L^p(\mathbb{R})\right) \subseteq \mathscr{M}_{(L^{p_1}(\mathbb{R}),L^{p_2}(\mathbb{R}),L^q(\mathbb{R}))}(\mathbb{R}) \tag{96}$$

where $1 \leq p \leq 2$ is such that $1/p_3 - 1/q = 1/p'$.

Proof. Since $p \leq 2$ invoking Hausdorff-Young, $\mathscr{F}(L^p(\mathbb{R})) \subseteq L^{p'}(\mathbb{R})$. Select $0 < \theta < 1$ such that $L^1(\mathbb{R})^\theta L^{p_3'}(\mathbb{R})^{1-\theta} = L^p(\mathbb{R})$ for $1/p = \theta + (1-\theta)/p_3' = \theta/p_3 + 1/p_3'$ and $L^{p_3}(\mathbb{R})^\theta L^\infty(\mathbb{R})^{1-\theta} = L^q(\mathbb{R})$ for $1/q = \theta/p_3$. Hence $1/p = 1/q + 1/p_3'$. The result now follows from Proposition 25. □

Proposition 27. *Let $X_1, X_2, X_3' \in (r.i.)_a$ satisfying that $X_1 \subseteq (X_2, X_3)$ and let $0 \le \theta \le 1$. Set*

$$X = L^{\infty}(\mathbb{R})^{\theta}\left(X_3'\right)^{1-\theta}, \qquad \widetilde{X}_3 = X_3^{\theta} L^1(\mathbb{R})^{1-\theta} \tag{97}$$

and assume that $\mathcal{F}(X) \subset L^1_{loc}(\mathbb{R})$. Then

$$\mathcal{F}(X) \subseteq \mathcal{M}_{(X_1, X_2, \widetilde{X}_3)}(\mathbb{R}). \tag{98}$$

Proof. Note that $\int_{\mathbb{R}} f(x-t)g(x+t)dt = f * g(2x)$. Hence each $\in L^{\infty}(\mathbb{R}) | \int_{\mathbb{R}} f(x-t)g(x+t)K(t)dt| \le \|K\|_\infty D_2(|f| * |g|)(x)$. Therefore,

$$\begin{aligned} &\left\| \int_{\mathbb{R}} f(x-t)\, g(x+t)\, K(t)\, dt \right\|_{X_3} \\ &\le \|K\|_\infty \left\| D_2(|f| * |g|) \right\|_{X_3} \\ &\le D_{X_3}(2) \|K\|_\infty \|f\|_{(X_2, X_3)} \|g\|_{X_2} \\ &\le C \|K\|_\infty \|f\|_{X_1} \|g\|_{X_2}. \end{aligned} \tag{99}$$

Hence, denoting as above $T(K, f, g)(x) = \int_{\mathbb{R}} f(x-t)g(x+t)K(t)dt$, one obtains that T is bounded from $L^{\infty}(\mathbb{R}) \times X_1 \times X_2$ into X_3.

Using duality, $\langle T(K, f, g), h\rangle = \langle T(h, \widetilde{f}, g), K\rangle$, where $\widetilde{f}(x) = f(-x)$, because

$$\begin{aligned} &\int_{\mathbb{R}} \int_{\mathbb{R}} f(x-t)\, g(x+t)\, K(t)\, dt) h(x)\, dx \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \widetilde{f}(t-x)\, g(x+t)\, h(x)\, dx \right) K(t)\, dt. \end{aligned} \tag{100}$$

Therefore T is also bounded from $X_3' \times X_1 \times X_2$ into $L^1(\mathbb{R})$.

Now the result follows again by interpolation. □

Let us apply the previous proposition for $X_1 = L^{p_1}(\mathbb{R})$, $X_2 = L^{p_2}(\mathbb{R})$, and $X_3 = L^{p_3}(\mathbb{R})$ with $1/p_1 + 1/p_2 - 1 = 1/p_3$ with $1 \le p_1, p_2 < \infty$ with $1 \le 1/p_1 + 1/p_2 < 3/2$ (in particular $p_3 > 2$).

Corollary 28. *Let $1 \le p_1, p_2 < \infty$ with $1 \le 1/p_1 + 1/p_2 < 3/2$ and $1/p_1 + 1/p_2 - 1 = 1/p_3$. If $p_3' < p \le 2$ then*

$$\mathcal{F}\left(L^p(\mathbb{R})\right) \subseteq \mathcal{M}_{(L^{p_1}(\mathbb{R}), L^{p_2}(\mathbb{R}), L^q(\mathbb{R}))}(\mathbb{R}) \tag{101}$$

where $1 \le q < \infty$ is such that $1/q = 1/p + 1/p_3$.

Proof. As above $\mathcal{F}(L^p(\mathbb{R})) \subseteq L^1_{\text{loc}}(\mathbb{R})$. Select $0 < \theta < 1$ such that $L^{\infty}(\mathbb{R})^{\theta} L^{p_3'}(\mathbb{R})^{1-\theta} = L^p(\mathbb{R})$ for $1/p = (1-\theta)/p_3'$ and $L^{p_3}(\mathbb{R})^{\theta} L^1(\mathbb{R})^{1-\theta} = L^q(\mathbb{R})$ for $1/q = \theta/p_3 + 1 - \theta = 1 - \theta/p_3' = 1/p_3 + (1-\theta)/p_3'$. Hence $1/q = 1/p + 1/p_3$ and the result follows from Proposition 27. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that they have no conflicts of interest.

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References

[1] O. Blasco and A. Osancliol, “Notes on bilinear multipliers on Orlicz spaces,” 2019, https://arxiv.org/abs/1902.01116.

[2] R. O’Neil, “Fractional integration in Orlicz spaces. I,” *Transactions of the American Mathematical Society*, vol. 115, pp. 300–328, 1965.

[3] M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, CRM Press, 1991.

[4] J. Bergh and J. Löfström, *Interpolation Spaces*, Springer Verlag, 1970.

[5] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, 1971.

[6] O. Blasco, “Notes on the spaces of bilinear multipliers,” *Revista de la Unión Matemática Argentina*, vol. 50, no. 2, pp. 20–34, 2009.

[7] R. R. Coifman and Y. Meyer, “Fourier analysis of multilinear convolutions, Calderón’s theorem, and analysis of Lipschitz curves,” in *Euclidean Harmonic Analysis (Proc. Sem. Univ. Maryland, College Univ., Md)*, vol. 779 of *Lecture Notes in Mathematics*, pp. 104–122, Springer, 1979.

[8] L. Grafakos and R. H. Torres, “Multilinear Calderón-Zygmund theory,” *Advances in Mathematics*, vol. 165, no. 1, pp. 124–164, 2002.

[9] L. Grafakos and N. Kalton, “Some remarks on multilinear maps and interpolation,” *Mathematische Annalen*, vol. 319, no. 1, pp. 151–180, 2001.

[10] C. E. Kenig and E. M. Stein, “Multilinear estimates and fractional integration,” *Mathematical Research Letters*, vol. 6, no. 1, pp. 1–15, 1999.

[11] M. Lacey and C. Thiele, “L^p estimates on the bilinear Hilbert transform for 2 < p < 8,” *Annals of Mathematics*, vol. 146, pp. 693–724, 1997.

[12] M. Lacey and C. Thiele, “Weak bounds for the bilinear Hilbert transform on L^p,” *Documenta Mathematica*, vol. ICM, pp. 1–1000, 1997.

[13] M. Lacey and C. Thiele, “On Calderón’s conjecture,” *Annals of Mathematics: Second Series*, vol. 149, no. 2, pp. 475–496, 1999.

[14] J. E. Gilbert and A. R. Nahmod, “Bilinear operators with non-smooth symbol, I,” *Journal of Fourier Analysis and Applications*, vol. 7, no. 5, pp. 435–467, 2001.

[15] J. E. Gilbert and A. R. Nahmod, “Boundedness of bilinear operators with nonsmooth symbols,” *Mathematical Research Letters*, vol. 7, no. 5-6, pp. 767–778, 2000.

[16] O. Blasco and F. Villarroya, “Transference of bilinear multipliers on Lorentz spaces,” *Illinois Journal of Mathematics*, vol. 47, no. 4, pp. 1327–1343, 2005.

[17] F. Villarroya, “Bilinear multipliers on Lorentz spaces,” *Czechoslovak Mathematical Journal*, vol. 58, no. 4, pp. 1045–1057, 2008.

[18] S. Rodríguez-López, “A homomorphism theorem for bilinear multipliers,” *Journal of the London Mathematical Society-Second Series*, vol. 88, no. 2, pp. 619–636, 2013.

[19] C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, London, UK, 1988.

[20] A.-P. Calderón, "Intermediate spaces and interpolation, the complex method," *Studia Mathematica*, vol. 24, pp. 113–190, 1964.