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Research Article

Endpoint Estimates for Oscillatory Singular Integrals with Hölder Class Kernels

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We prove the uniform $L^1 \longrightarrow L^{1,\infty}$ and $H^1_E \longrightarrow L^1$ boundedness of oscillatory singular integral operators whose kernels are the products of an oscillatory factor with bilinear phase and a Calderón-Zygmund kernel $K(x,y)$ satisfying a Hölder condition. This Hölder condition appreciably weakens the C^1 condition imposed in existing literature.

1. Introduction

Let $n \in \mathbb{N}$. We shall consider the following oscillatory singular integral operator:

$$T_{B,K} : f \longrightarrow \text{p.v.} \int_{\mathbb{R}^n} e^{iB(x,y)} K(x,y) f(y)\, dy \quad (1)$$

where $B(\cdot,\cdot)$ is a real-valued bilinear form. In past studies of this type of operators, $K(x,y)$ is typically assumed to be a Calderón-Zygmund kernel satisfying a C^1 condition away from the diagonal $\Delta = \{(x,x) : x \in \mathbb{R}^n\}$, i.e., there exists an $A > 0$ such that

(i) for all $(x,y) \in (\mathbb{R}^n \times \mathbb{R}^n) \setminus \Delta$,

$$|K(x,y)| \le \frac{A}{|x-y|^n}; \quad (2)$$

(ii) $K(x,y) \in C^1((\mathbb{R}^n \times \mathbb{R}^n) \setminus \Delta)$, and for $(x,y) \in (\mathbb{R}^n \times \mathbb{R}^n) \setminus \Delta$

$$|\nabla_x K(x,y)| + |\nabla_y K(x,y)| \le \frac{A}{|x-y|^{n+1}}; \quad (3)$$

(iii)

$$\|T_o\|_{L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)} \le A \quad (4)$$

where

$$T_o f(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x,y) f(y)\, dy. \quad (5)$$

Under conditions (i), (ii), and (iii), Phong and Stein proved the L^p boundedness of $T_{B,K}$ for $1 < p < \infty$ ([1]). The L^p result of Phong and Stein was then extended to operators with polynomial phases by Ricci and Stein ([2]), under the same conditions (i), (ii), and (iii) on $K(x,y)$, while the weak (1,1) boundedness of such operators was subsequently established by Chanillo and Christ in [3] (for all polynomial phase functions, bilinear or otherwise).

The C^1 property of K in condition (ii) was instrumental when van der Corput's lemma, a standard tool in the treatment of oscillatory integrals, was used in past studies, including the seminal papers cited above. There has been widespread interest in finding out what happens when the C^1 kernel $K(x,y)$ is replaced by a "rougher" kernel. Many interesting results have been obtained for kernels that are homogeneous and of convolutional type but lack smoothness (i.e., $K(x,y) = |x-y|^{-n}\Omega((x-y)/|x-y|)$). See, for example, [4–6].

In this paper we are interested in general kernels $K(x,y)$ for which condition (ii) is replaced by the following weaker condition of Hölder type:

(ii)$'$ there exists a $\delta > 0$ such that

$$\left|K(x,y) - K(x',y)\right| \le \frac{A\left|x-x'\right|^{\delta}}{\left(|x-y| + |x'-y|\right)^{n+\delta}} \quad (6)$$

whenever $|x-x'| < (1/2)\max\{|x-y|, |x'-y|\}$ and

$$\left|K(x,y) - K(x,y')\right| \le \frac{A\left|y-y'\right|^{\delta}}{\left(|x-y| + |x-y'|\right)^{n+\delta}} \quad (7)$$

whenever $|y-y'| < (1/2)\max\{|x-y|, |x-y'|\}$.

In a recent paper, we were able to prove the following uniform L^p boundedness of $T_{B,K}$ for $1 < p < \infty$:

Theorem 1 (see [7]). *Let $A, \delta > 0$ and $T_{B,K}$ be given as in (1). Suppose that $K(x,y)$ satisfies (i), (ii)$'$, and (iii). Then, for $1 < p < \infty$, there exists a positive C_p which may depend on p, n, δ, and A but is independent of the bilinear form $B(\cdot,\cdot)$, such that*

$$\|T_{B,K} f\|_{L^p(\mathbb{R}^n)} \le C_p \|f\|_{L^p(\mathbb{R}^n)} \quad (8)$$

for all $f \in L^p(\mathbb{R}^n)$.

In this paper we shall investigate the endpoint case $p = 1$ and obtain both the weak type (1,1) and Hardy space bounds. We begin with the weak (1,1) result.

Theorem 2. *Let $A, \delta > 0$ and $T_{B,K}$ be given as in (1). Suppose that $K(x,y)$ satisfies (i), (ii)$'$, and (iii). Then, $T_{B,K}$ is of weak type (1,1), i.e., there exists a positive C such that*

$$\left|\{x \in \mathbb{R}^n : |T_{B,K} f(x)| > \alpha\}\right| \le C\alpha^{-1} \|f\|_{L^1(\mathbb{R}^n)} \quad (9)$$

for all $f \in L^1(\mathbb{R}^n)$ and $\alpha > 0$. Moreover, while the constant C may depend on n, δ, and A, it is otherwise independent of $B(\cdot,\cdot)$ and $K(\cdot,\cdot)$.

In the statement above, we used $|S|$ to denote the Lebesgue measure of a measurable set S.

In order to describe our result on Hardy spaces, let $H^1_E(\mathbb{R}^n)$ be the Hardy space introduced by Phong and Stein in [1] as a variant of the standard Hardy space $H^1(\mathbb{R}^n)$ suitable for the study of oscillatory singular integrals. It was proved there that under conditions (i), (ii), and (iii) $T_{B,K}$ is a bounded operator from $H^1_E(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. As an improvement over their result, we have the following.

Theorem 3. *Under conditions (i), (ii)$'$, and (iii), $T_{B,K}$ is a bounded operator from $H^1_E(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. Moreover, the bound for the operator norm may depend on n, δ, and A but is otherwise independent of $B(\cdot,\cdot)$ and $K(\cdot,\cdot)$.*

It is already well-known that analogous results do not hold for H^p when $p < 1$.

The proof of the weak type (1,1) estimate will appear in Sections 2 and 3. It follows a $L^1 \longrightarrow L^2$ strategy pioneered by C. Fefferman in [8] (see also [3, 4, 9, 10]). The proof of the Hardy space estimate will be given in Section 4.

We now close this section by posing the following natural question which should be of interest to many working in this field: are the L^p and endpoint results for oscillatory singular integrals in Theorems 1–3 still true when the bilinear phase functions are replaced by general polynomials in x, y with real coefficients?

2. Basic Reductions for the $L^1 \longrightarrow L^{1,\infty}$ Bounds

We shall begin the proof of Theorem 2 with a few reductions. Since the one-dimensional case is relatively easier, we shall focus our attention on $n \ge 2$. For $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, $B(x,y)$ can be expressed as

$$B(x,y) = \sum_{j=1}^{n}\sum_{k=1}^{n} b_{jk} x_j y_k. \quad (10)$$

If $b_{jk} = 0$ for all $j,k \in \{1,\ldots,n\}$, then $T_{B,K} = T_o$. It is well-known that, under the conditions (2), (4), and (6)-(7), T_o is bounded from from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$ as well as from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Thus, from this point on, we may assume that $b_{jk} \ne 0$ holds for at least one pair (j,k). Let

$$b = \max\left\{\left|b_{jk}\right| : 1 \le j,k \le n\right\}. \quad (11)$$

If we let ρ denote the dilation operator

$$g(\cdot) \longrightarrow g\left(\frac{\cdot}{\sqrt{b}}\right), \quad (12)$$

then

$$\rho \circ T_{B,K} = T_{\rho B, b^{-n/2}\rho K} \circ \rho. \quad (13)$$

Since $b^{-n/2}\rho K$ satisfies (i), (ii)$'$, and (iii) with the same constants A and δ as K, it suffices to establish (9) under the additional assumption that, for some $k_0 \in \{1,\ldots,n\}$, $|b_{1k_0}| = b = 1$ (after reindexing the variables if necessary). Clearly, we may also assume that $\delta \le 1$.

For any cube Q in $\mathbb{R}^n$, let $l(Q)$ and x_Q denote its sidelength and center, respectively. For any $\beta > 0$, we let βQ denote the cube that has the same center as Q and sidelength $\beta l(Q)$. Also, let $j_Q = \log_2(l(Q))$.

Let $f \in L^1(\mathbb{R}^n)$ and $\alpha > 0$. Then there is a collection of dyadic cubes $\mathcal{F}$ with disjoint interiors such that the following are satisfied:

$$\|f\|_{L^\infty(\Omega^c)} \le \alpha \quad \text{where } \Omega = \bigcup_{Q\in\mathcal{F}} Q; \quad (14)$$

$$\frac{1}{|Q|}\int_Q |f| \le C\alpha \quad \forall Q \in \mathcal{F}; \quad (15)$$

$$\sum_{Q\in\mathcal{F}} |Q| \le C\alpha^{-1} \|f\|_{L^1(\mathbb{R}^n)}; \quad (16)$$

$$\left|j_Q - j_{Q'}\right| \le C \quad \text{whenever } Q, Q' \in \mathcal{F} \text{ and } \operatorname{dist}\left(Q,Q'\right) \le 2l(Q). \quad (17)$$

For $s \in \mathbb{N} \cup \{0\}$, let $K_s(x,y) = K(x,y)\chi_{[2^s,\infty)}(|x-y|)$. Let $\widetilde{K}(x,y) = K(x,y) - K_0(x,y)$,

$$\mathcal{G} = \{Q \in \mathcal{F} : l(Q) \ge 2\} \quad (18)$$

and

$$\mathcal{H} = \mathcal{F} \setminus \mathcal{G} = \{Q \in \mathcal{F} : l(Q) \le 1\}. \quad (19)$$

Then,

$$\{x \in \mathbb{R}^n : |T_{B,K} f(x)| > \alpha\} \subseteq \bigcup_{j=1}^{5} U_j \quad (20)$$

where

$$U_1 = \left\{x \in \mathbb{R}^n : \left|T_{B,K_0}(f\chi_{\Omega^c})(x)\right| > \frac{\alpha}{4}\right\},$$
$$U_2 = \bigcup_{Q\in\mathcal{F}} 4Q, \quad (21)$$

$$U_3 = \left\{x \in \left(\bigcup_{Q\in\mathcal{F}} 4Q\right)^c : \left|\sum_{Q\in\mathcal{G}} T_{B,K_0}(f\chi_Q)(x)\right| > \frac{\alpha}{4}\right\}, \quad (22)$$

$$U_4 = \left\{x \in \left(\bigcup_{Q\in\mathcal{F}} 4Q\right)^c : \left|\sum_{Q\in\mathcal{H}} T_{B,K_0}(f\chi_Q)(x)\right| > \frac{\alpha}{4}\right\}, \quad (23)$$

and

$$U_5 = \left\{x \in \mathbb{R}^n : \left|T_{B,\widetilde{K}} f(x)\right| > \frac{\alpha}{4}\right\}. \quad (24)$$

It follows from Theorem 1 and a standard argument that (8) remains valid when K is replaced by K_0 or $\widetilde{K}$. Thus, by (14)-(15),

$$|U_1| \le \left(\frac{\alpha}{4}\right)^{-2} \left\|T_{B,K_0}(f\chi_{\Omega^c})\right\|_{L^2(\mathbb{R}^n)} \le C\alpha^{-2}\int_{\Omega^c}|f|^2 \le C\alpha^{-1}\|f\|_{L^1(\mathbb{R}^n)}, \quad (25)$$

and

$$|U_2| = 4^n\left(\sum_{Q\in\mathcal{F}}|Q|\right) \le C\alpha^{-1}\|f\|_{L^1(\mathbb{R}^n)}. \quad (26)$$

The set U_5 can be treated by a finite overlapping argument. Let $Q_0 = (-1/2, 1/2]^n$. For each $h \in \mathbb{Z}^n$, let $h + Q_0 = \{h + y : y \in Q_0\}$ and

$$f_h(y) = e^{iB(y,y-h)}\chi_{h+Q_0}(y) f(y). \quad (27)$$

Clearly,

$$\operatorname{supp}\left(T_{B,\widetilde{K}}\left(\chi_{h+Q_0} f\right)\right) \subseteq h + 3Q_0. \quad (28)$$

By (2) and $b = 1$, for $x \in h + 3Q_0$,

$$\begin{aligned}
&\left|T_{B,\widetilde{K}}\left(\chi_{h+Q_0} f\right)(x) - e^{iB(x,h)}T_0(f_h)(x)\right| \\
&\le C\left(\int_{|x-y|\le 1}\frac{\left|\chi_{h+Q_0}(y) f(y)\right|}{|x-y|^{n-1}}dy + \int_{1\le|x-y|\le 2\sqrt{n}}\frac{\left|\chi_{h+Q_0}(y) f(y)\right|}{|x-y|^{n}}dy\right) \\
&\le C\left(g_0 * \left|\chi_{h+Q_0} f\right|\right)(x),
\end{aligned} \quad (29)$$

where $g_0(x) = |x|^{-n+1}\chi_{[0,2\sqrt{n}]}(|x|)$. Thus,

$$\begin{aligned}
|U_5| &\le \sum_{h\in\mathbb{Z}^n}\left|\left\{x\in\mathbb{R}^n : \left|T_{B,\widetilde{K}}\left(\chi_{h+Q_0} f\right)(x)\right| > \left(4\cdot 10^n\right)^{-1}\alpha\right\}\right| \\
&\le \sum_{h\in\mathbb{Z}^n}\left(\left|\left\{x\in\mathbb{R}^n : |T_0(f_h)(x)| > \left(8\cdot 10^n\right)^{-1}\alpha\right\}\right| + \left|\left\{x\in\mathbb{R}^n : \left|\left(g_0 * \left|\chi_{h+Q_0} f\right|\right)(x)\right| > \left(8C\cdot 10^n\right)^{-1}\alpha\right\}\right|\right) \\
&\le C\sum_{h\in\mathbb{Z}^n}\left(\alpha^{-1}\|f_h\|_{L^1(\mathbb{R}^n)} + \alpha^{-1}\|g_0\|_{L^1(\mathbb{R}^n)}\left\|\chi_{h+Q_0} f\right\|_{L^1(\mathbb{R}^n)}\right) \\
&\le C\alpha^{-1}\|f\|_{L^1(\mathbb{R}^n)}.
\end{aligned} \quad (30)$$

The proof of Theorem 2 has thus been reduced to the verification of the following:

$$\left|U_j\right| \le C\alpha^{-1}\|f\|_{L^1(\mathbb{R}^n)} \quad \text{for } j = 3, 4. \quad (31)$$

3. Estimates for $|U_3|$ and $|U_4|$

For two nonnegative integers s and m, let

$$L_{sm}(x, y) = \int_{\mathbb{R}^n} e^{iB(z,x-y)} K_s(z - x)\overline{K_m(z - y)}dz. \quad (32)$$

For $a \in \mathbb{R}^n$ and $r > 0$, let $\Gamma(a, r) = \{x \in \mathbb{R}^n : |x - a| \ge r\}$. Let

$$P(x) = \sum_{k=1}^{n} b_{1k}x_k. \quad (33)$$

Observe that

$$\left(\frac{1}{2}\right)\sum_{k=1}^{n}|x_k| \le |P(x)| + \sum_{k\ne k_0}|x_k| \le 2\sum_{k=1}^{n}|x_k|. \quad (34)$$

Lemma 4. *Let $s, m \in \mathbb{N}\cup\{0\}$ such that $s \ge m$. Suppose that K satisfies (2) and (6)-(7).*

(i) *For any $x, y \in \mathbb{R}^n$,*

$$\left|L_{sm}(x, y)\right| \le C2^{-sn}(1 + s - m) \quad (35)$$

(ii) *For any $x, y \in \mathbb{R}^n$,*

$$\left|L_{sm}(x, y)\right| \le C|x - y|^{-n}\ln\left(2 + 2^{-m}|x - y|\right) \quad (36)$$

(iii) *For any $x, y \in \mathbb{R}^n$ satisfying $|x - y| > 2^{s+1}$ and $|P(x - y)| > 4\pi$,*

$$\left|L_{sm}(x, y)\right| \le C|P(x - y)|^{-\delta}|x - y|^{-n}. \quad (37)$$

Proof. We shall omit the arguments for (i) and (ii) because they utilize (2) only and therefore can be found in [3] (see page 152).

Suppose that $x, y \in \mathbb{R}^n$, $|x-y| > 2^{s+1}$ and $|P(x-y)| > 4\pi$. For any $z = (z_1, z_2, \ldots, z_n) = (z_1, \widetilde{z})$, let $z^{\pm} = (z_1 \pm \theta, \widetilde{z})$ where $\theta = \pi/P(x - y)$. Thus,

$$\begin{aligned}\left|L_{s,m}(x,y)\right| &\le \int_{\mathbb{R}^{n-1}}\left|\int_{\mathbb{R}} e^{iz_1P(x-y)}K_s(z,x)\overline{K_m(z,y)}dz_1\right|d\tilde{z}\\ &= \frac{1}{2}\int_{\mathbb{R}^{n-1}}\left|\int_{\mathbb{R}} e^{iz_1P(x-y)}\left(K_s(z,x)\overline{K_m(z,y)}-K_s(z^+,x)\overline{K_m(z^+,y)}\right)dz_1\right|d\tilde{z}\\ &\le M_1(x,y)+M_2(x,y)+M_3(x,y)+M_4(x,y)\end{aligned} \tag{38}$$

where

$$\begin{aligned}M_1(x,y) &= \int_{\Gamma(x,2^s)\cap\Gamma(y,2^m)}\left|K(z,x)-K(z^+,x)\right|\cdot\left|K(z,y)\right|dz,\\ M_2(x,y) &= \int_{(\Gamma(x,2^s)\Delta\Gamma(x^-,2^s))\cap\Gamma(y,2^m)}\left|K(z^+,x)\right|\cdot\left|K(z,y)\right|dz,\\ M_3(x,y) &= \int_{\Gamma(x^-,2^s)\cap\Gamma(y,2^m)}\left|K(z,y)-K(z^+,y)\right|\cdot\left|K(z^+,x)\right|dz,\\ M_4(x,y) &= \int_{(\Gamma(y,2^m)\Delta\Gamma(y^-,2^m))\cap\Gamma(x^-,2^s)}\left|K(z^+,y)\right|\cdot\left|K(z^+,x)\right|dz.\end{aligned} \tag{39}$$

When $z\in\Gamma(x,2^s)$, we have

$$|z-x|\ge 2^s\ge 2|\theta| = 2\left|z-z^+\right|. \tag{40}$$

It follows from (2) and (6) that

$$\begin{aligned}M_1(x,y) &\le C|\theta|^\delta\int_{\Gamma(x,2^s)\cap\Gamma(y,2^m)}\frac{dz}{|z-x|^{n+\delta}|z-y|^n}\\ &\le C|\theta|^\delta\int_{\mathbb{R}^n}\frac{dz}{(1+|z-x|)^{n+\delta}(1+|z-y|)^n}\\ &\le C|\theta|^\delta\left(\int_{|z-y|\ge|x-y|/2}\frac{dz}{(1+|z-x|)^{n+\delta}|z-y|^n}\right.\\ &\quad\left.+\int_{|z-x|\ge|x-y|/2\ge|z-y|}\frac{dz}{|z-x|^n(1+|z-x|)^\delta(1+|z-y|)^n}\right)\\ &\le C|\theta|^\delta|x-y|^{-n}\int_{\mathbb{R}^n}\frac{dz}{(1+|z|)^{n+\delta}} = C\left|P(x-y)\right|^{-\delta}|x-y|^{-n}.\end{aligned} \tag{41}$$

When $z\in\Gamma(x^-,2^s)$,

$$|z-x|\le\left|z-x^-\right|+|\theta|\le 2\left|z-x^-\right| = 2\left|z^+-x\right|. \tag{42}$$

Thus,

$$M_3(x,y)\le C|\theta|^\delta\int_{\mathbb{R}^n}\frac{dz}{(1+|z-y|)^{n+\delta}(1+|z-x|)^n}. \tag{43}$$

It follows from the the arguments for $M_1(x,y)$ that

$$M_3(x,y)\le C\left|P(x-y)\right|^{-\delta}|x-y|^{-n}. \tag{44}$$

When $z\in\Gamma(x,2^s)\Delta\Gamma(x^-,2^s)$, we have

$$\left(\frac{3}{4}\right)|z-x|\le\left|z^+-x\right|\le\left(\frac{4}{3}\right)|z-x| \tag{45}$$

and

$$\begin{aligned}\left(\frac{3}{4}\right)2^s &\le |z-x|,\\ \left|z^+-x\right| &\le \left(\frac{4}{3}\right)2^s.\end{aligned} \tag{46}$$

Thus,

$$|z-x|\le\left(\frac{4}{3}\right)2^s\le\left(\frac{2}{3}\right)|x-y| \tag{47}$$

and

$$|z-y|\ge|x-y|-|z-x|\ge\left(\frac{1}{3}\right)|x-y|. \tag{48}$$

Therefore,

$$\begin{aligned}M_2(x,y) &\le C\int_{\Gamma(x,2^s)\Delta\Gamma(x^-,2^s)}\left|z^+-x\right|^{-n}|z-y|^{-n}dz\\ &\le C2^{-ns}|x-y|^{-n}\left|\Gamma(x,2^s)\Delta\Gamma(x^-,2^s)\right|\\ &\le C2^{-ns}|x-y|^{-n}2^{s(n-1)}|\theta|\\ &\le C\left|P(x-y)\right|^{-1}|x-y|^{-n}.\end{aligned} \tag{49}$$

Similarly,

$$\begin{aligned}M_4(x,y) &\le C2^{-nm}|x-y|^{-n}2^{m(n-1)}|\theta|\\ &\le C\left|P(x-y)\right|^{-1}|x-y|^{-n}.\end{aligned} \tag{50}$$

The proof of Lemma 4 is now complete. □

Let $d(S_1,S_2)$ denote the distance between two sets S_1 and S_2. For $Q\in\mathcal{G}$, let

$$\mathcal{G}(Q) = \left\{Q'\in\mathcal{G} : l(Q')\le l(Q)\right\}. \tag{51}$$

Lemma 5. *If $Q\in\mathcal{G}$ and $x\in Q$, then*

$$\sum_{Q'\in\mathcal{G}(Q)}\int_{Q'}\left|L_{j_Q,j_{Q'}}(x,y)f(y)\right|dy\le C\alpha. \tag{52}$$

Proof. Let $Q \in \mathcal{G}$ and

$$\mathcal{G}_1(Q) = \{Q' \in \mathcal{G}(Q) : d(Q', Q) \le 2l(Q)\}. \tag{53}$$

For each $x \in Q$, let

$$\mathcal{G}_2(Q,x) = \left\{Q' \in \mathcal{G} \setminus \mathcal{G}_1(Q) : \inf_{y\in Q'} |P(x-y)| \le (8n)\, l(Q')\right\}; \tag{54}$$

$$\mathcal{G}_3(Q,x) = \left\{Q' \in \mathcal{G} \setminus \mathcal{G}_1(Q) : \inf_{y\in Q'} |P(x-y)| > (8n)\, l(Q')\right\}. \tag{55}$$

It follows from (15), (17), and Lemma 4(i) that the cardinality of $\mathcal{G}_1(Q)$ is bounded by a constant and

$$\sum_{Q'\in\mathcal{G}_1(Q)} \int_{Q'} \left|L_{j_Q,j_{Q'}}(x,y) f(y)\right| dy \tag{56}$$

$$\le C \sum_{Q'\in\mathcal{G}_1(Q)} \left(1 + l(Q) - l(Q')\right) |Q|^{-1} \int_{Q'} |f(y)|\, dy \le C\alpha. \tag{57}$$

When $Q' \in \mathcal{G} \setminus \mathcal{G}_1(Q)$,

$$\sup_{y\in Q'} \left(\frac{\ln\left(2 + 2^{-j_{Q'}} |x-y|\right)}{|x-y|^n}\right) \le C \inf_{y\in Q'} \left(\frac{\ln\left(2 + 2^{-j_{Q'}} |x-y|\right)}{|x-y|^n}\right). \tag{58}$$

Also, for any $Q' \in \mathcal{G}_2(Q,x)$ and $y \in Q'$,

$$|P(x-y)| \le \inf_{v\in Q'} |P(x-v)| + nl(Q') \le 9nl(Q'). \tag{59}$$

Thus, by Lemma 4(ii), (15), and (58),

$$\begin{aligned}
&\sum_{Q'\in\mathcal{G}_2(Q,x)} \int_{Q'} \left|L_{j_Q,j_{Q'}}(x,y) f(y)\right| dy \\
&\le C\alpha \sum_{Q'\in\mathcal{G}_2(Q,x)} \int_{Q'} |x-y|^{-n} \ln\left(2 + 2^{-j_{Q'}} |x-y|\right) dy \\
&\le C\alpha \sum_{m=1}^{j_Q} \int_{\{y\in\mathbb{R}^n : |P(x-y)| \le (9n)2^m\}} \left(l(Q) + |x-y|\right)^{-n} \cdot \ln\left(2 + 2^{-m} |x-y|\right) dy \\
&= C\alpha \sum_{m=1}^{j_Q} \int_{\{u\in\mathbb{R}^n : |P(u)| \le 9n\}} \left(2^{j_Q-m} + |u|\right)^{-n} \ln(2 + |u|)\, du \\
&\le C\alpha \sum_{m=1}^{j_Q} \int_{\mathbb{R}^{n-1}} \int_{|v_1|\le 9n} \left(2^{j_Q-m} + |\tilde{v}| + |v_1|\right)^{-n} \cdot \ln\left(2 + |\tilde{v}| + |v_1|\right) dv_1\, d\tilde{v} \\
&\le C\alpha \sum_{m=1}^{j_Q} (1 + j_Q - m)\, 2^{-(j_Q-m)} \le C\alpha.
\end{aligned} \tag{60}$$

For any $Q' \in \mathcal{G}_3(Q,x)$,

$$\sup_{y\in Q'} |P(x-y)| \le \inf_{y\in Q'} |P(x-y)| + nl(Q') \le 2 \inf_{y\in Q'} |P(x-y)|. \tag{61}$$

It follows from Lemma 4(iii), (15), (58), and (61) that

$$\begin{aligned}
&\sum_{Q'\in\mathcal{G}_3(Q,x)} \int_{Q'} \left|L_{j_Q,j_{Q'}}(x,y) f(y)\right| dy \\
&\le C\alpha \sum_{Q'\in\mathcal{G}_3(Q,x)} \int_{Q'} |P(x-y)|^{-\delta} |x-y|^{-n}\, dy \\
&\le C\alpha \int_{\mathbb{R}^n} (1 + |P(u)|)^{-\delta} (1 + |u|)^{-n}\, du \\
&\le C\alpha \int_{\mathbb{R}^n} (1 + |v_1|)^{-\delta} (1 + |v|)^{-n}\, dv \le C\alpha.
\end{aligned} \tag{62}$$

Lemma 5 is proved. □

For $Q \in \mathcal{H}$, let $\mathcal{H}(Q) = \{Q' \in \mathcal{H} : l(Q') \le l(Q)\}$.

Lemma 6. *If $Q \in \mathcal{H}$ and $x \in Q$, then*

$$\sum_{Q'\in\mathcal{H}(Q)} \int_{Q'} |L_{0,0}(x,y) f(y)|\, dy \le C\alpha. \tag{63}$$

Proof. The argument is very similar to the proof of the previous lemma. We will point out the differences but omit most of the details.

Let $Q \in \mathcal{H}$ and $x \in Q$. Let

$$\begin{aligned}
&\mathcal{H}_1(Q) = \{Q' \in \mathcal{H}(Q) : d(Q', Q) \le 2\}, \\
&\mathcal{H}_2(Q,x) = \left\{Q' \in \mathcal{H} \setminus \mathcal{H}_1(Q) : \inf_{y\in Q'} |P(x-y)| \le 4\pi\right\}; \\
&\mathcal{H}_3(Q,x) = \left\{Q' \in \mathcal{H} \setminus \mathcal{H}_1(Q) : \inf_{y\in Q'} |P(x-y)| > 4\pi\right\}.
\end{aligned} \tag{64}$$

While there is no uniform bound on the cardinality of $\mathcal{H}_1(Q)$ (unlike $\mathcal{G}_1(Q)$), by using $|L_{0,0}(x,y)| \le C$, we still have

$$\sum_{Q' \in \mathcal{H}_1(Q)} \int_{Q'} |L_{0,0}(x,y) f(y)| \, dy \le C\alpha \sum_{Q' \in \mathcal{H}_1(Q)} |Q'| \le C\alpha. \tag{65}$$

By $|L_{0,0}(x,y)| \le C|x-y|^{-n} \ln(2+|x-y|)$,

$$\sum_{Q' \in \mathcal{H}_2(Q,x)} \int_{Q'} |L_{0,0}(x,y) f(y)| \, dy \le C\alpha \int_{\{u \in \mathbb{R}^n : |P(u)| \le 4\pi + n\}} (1+|u|)^{-n} \ln(2+|u|) \, du \le C\alpha. \tag{66}$$

Finally, $\mathcal{H}_3(Q,x)$ can be treated the same as $\mathcal{G}_3(Q,x)$, which finishes the proof of Lemma 6. □

We now employ a well-known $L^2 \longrightarrow L^1$ technique to obtain the desired estimates for $|U_3|$ and $|U_4|$ (see, for example, [3, 8–11]). By Lemma 5,

$$\begin{aligned} |U_3| &\le C\alpha^{-2} \left\| \sum_{Q \in \mathcal{G}} T_{B,K_0}(f\chi_Q) \right\|^2_{L^2((\bigcup_{Q \in \mathcal{F}} 4Q)^c)} \\ &\le C\alpha^{-2} \sum_{Q \in \mathcal{G}} \int_Q \left(\sum_{Q' \in \mathcal{G}(Q)} \left| L_{j_Q, j_{Q'}}(x,y) f(y) \right| dy \right) \cdot |f(x)| \, dx \\ &\le C\alpha^{-1} \sum_{Q \in \mathcal{G}} \int_Q |f(x)| \, dx \le C\alpha^{-1} \|f\|_{L^1(\mathbb{R}^n)}. \end{aligned} \tag{67}$$

By Lemma 6,

$$\begin{aligned} |U_4| &\le C\alpha^{-2} \sum_{Q \in \mathcal{H}} \int_Q \left(\sum_{Q' \in \mathcal{H}(Q)} |L_{0,0}(x,y) f(y)| \, dy \right) \cdot |f(x)| \, dx \\ &\le C\alpha^{-1} \sum_{Q \in \mathcal{H}} \int_Q |f(x)| \, dx \le C\alpha^{-1} \|f\|_{L^1(\mathbb{R}^n)}. \end{aligned} \tag{68}$$

The proof of Theorem 2 is now complete.

4. $H^1_E \longrightarrow L^1$ Boundedness

We shall begin by recalling the definition of the function space $H^1_E(\mathbb{R}^n)$.

Definition 7. A measurable function $a(\cdot)$ on $\mathbb{R}^n$ is called an atom if there exists a cube Q such that $\mathrm{supp}(a) \subseteq Q$, $\|a\|_\infty \le |Q|^{-1}$ and

$$\int_Q e^{iB(x_Q,y)} a(y) \, dy = 0. \tag{69}$$

A function f is in $H^1_E(\mathbb{R}^n)$ if there exist a sequence $\{\lambda_j\}$ in $\mathbb{C}$ and a sequence of atoms $\{a_j\}$ such that

$$f = \sum_j \lambda_j a_j. \tag{70}$$

The H^1_E norm of f is the infimum of $\sum_j |\lambda_j|$ over all possible expressions of f described in (70).

In order to prove Theorem 3, it suffices to establish that, for every atom $a(\cdot)$,

$$\|T_{B,K} a\|_{L^1(\mathbb{R}^n)} \le C. \tag{71}$$

By using a dilation (as in Section 2) as well as a translation, we can further reduce the task to the verification of (71) under the assumption that, for some $j_0 \in \{1, \dots, n\}$ and $h > 0$,

$$\begin{aligned} \max\left\{ |b_{jk}| : 1 \le j,k \le n \right\} &= |b_{j_0 1}| = 1, \\ \mathrm{supp}(a) &\subseteq [-h,h]^{-n}, \\ \|a\|_\infty &\le (2h)^{-n} \\ \text{and } \int_{[-h,h]^n} a(y) \, dy &= 0. \end{aligned} \tag{72}$$

Let $\eta = \max\{2nh, h^{-1}\}$. Then

$$\|T_{B,K} a\|_{L^1(\mathbb{R}^n)} \le I_1 + I_2 + I_3 + I_4 \tag{73}$$

where

$$\begin{aligned} I_1 &= \int_{|x| \le 2nh} |T_{B,K} a(x)| \, dx, \\ I_2 &= \int_{|x| \ge 2nh} \left| \int_{\mathbb{R}^n} e^{iB(x,y)} (K(x,y) - K(x,0)) \cdot a(y) \, dy \right| dx, \\ I_3 &= \int_{2nh \le |x| \le \eta} \left| K(x,0) \int_{\mathbb{R}^n} e^{iB(x,y)} a(y) \, dy \right| dx, \\ I_4 &= \int_{|x| \ge \eta} \left| K(x,0) \int_{\mathbb{R}^n} e^{iB(x,y)} a(y) \, dy \right| dx. \end{aligned} \tag{74}$$

By Theorem 1,

$$I_1 \le C h^{n/2} \|T_{B,K} a\|_{L^2(\mathbb{R}^n)} \le C h^{n/2} \|a\|_{L^2(\mathbb{R}^n)} \le C. \tag{75}$$

By (7),

$$\begin{aligned} I_2 &\le \int_{|x| \ge 2nh} \left(\int_{y \in [-h,h]^n} \frac{|y|^\delta |a(y)| \, dy}{|x|^{n+\delta}} \right) dx \\ &\le C h^\delta \|a\|_{L^1(\mathbb{R}^n)} \int_{|x| \ge 2nh} |x|^{-n-\delta} \, dx \le C. \end{aligned} \tag{76}$$

By the vanishing integral property of $a(\cdot)$,

$$I_3 = \int_{2nh \le |x| \le \eta} |K(x,0)| \cdot \left| \int_{y \in [-h,h]^n} \left(e^{iB(x,y)} - 1 \right) a(y) \, dy \right| dx$$

$$\leq C\int_{2nh\leq|x|\leq\eta}|x|^{-n+1}\left(\int_{y\in[-h,h]^n}|y||a(y)|\,dy\right)dx$$
$$\leq C\|a\|_{L^1(\mathbb{R}^n)}h(\eta-2nh)\leq C. \tag{77}$$

Let $\mathscr{F}_1$ denote the partial Fourier transform in the first variable. Then,

$$\begin{aligned}
I_4 &\leq C\int_{\mathbb{R}^n}(\eta+|x|)^{-n}\int_{\widetilde{y}\in\mathbb{R}^{n-1}}\Bigg|\mathscr{F}_1(a(\cdot,\widetilde{y}))\\
&\quad\cdot\left(-\sum_{j=1}^{n}b_{j1}x_j\right)\Bigg|\,d\widetilde{y}dx\\
&\leq C\int_{\widetilde{y}\in\mathbb{R}^{n-1}}\int_{u\in\mathbb{R}^n}(\eta+|u|)^{-n}\\
&\quad\cdot\left|\mathscr{F}_1(a(\cdot,\widetilde{y}))(u_1)\right|\,dud\widetilde{y}\\
&\leq C\int_{\widetilde{y}\in\mathbb{R}^{n-1}}\int_{\widetilde{u}\in\mathbb{R}^{n-1}}\left(\int_{\mathbb{R}}\frac{du_1}{(\eta+|u|)^{2n}}\right)^{1/2}\\
&\quad\cdot\left(\int_{\mathbb{R}}\left|\mathscr{F}_1(a(\cdot,\widetilde{y}))(u_1)\right|^2du_1\right)^{1/2}d\widetilde{u}d\widetilde{y}\\
&\leq C\left(\int_{\widetilde{u}\in\mathbb{R}^{n-1}}(\eta+|\widetilde{u}|)^{-n+1/2}d\widetilde{u}\right)\\
&\quad\cdot\int_{\widetilde{y}\in\mathbb{R}^{n-1}}\left(\int_{\mathbb{R}}|a(y)|^2dy_1\right)^{1/2}d\widetilde{y}\\
&\leq C(\eta h)^{-1/2}\leq C.
\end{aligned}\tag{78}$$

The proof of Theorem 3 is now complete.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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