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Research Article

Second-Order Linear Differential Equations with Solutions in Analytic Function Spaces

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This research is concerned with second-order linear differential equation $f'' + A(z)f = 0$, where $A(z)$ is an analytic function in the unit disc. On the one hand, some sufficient conditions for the solutions to be in α-Bloch (little α-Bloch) space are found by using exponential type weighted Bergman reproducing kernel formula. On the other hand, we find also some sufficient conditions for the solutions to be in analytic Morrey (little analytic Morrey) space by using the representation formula.

1. Introduction

We shall assume that the reader is familiar with the definitions of classical function spaces, for example, Hardy space, Bergman space, and Bloch space. One of main objectives in the research of complex linear differential equations with analytic coefficients in unit disc is to consider the relationship between the growth of coefficients and the growth of solutions. Many results concerning fast growing solutions have been obtained by Nevanlinna and Wiman-Valiron theories; for example, see [1–5] and reference therein. However, it is very difficult to study the slowly growing solutions of complex linear differential equations by using Nevanlinna and Wiman-Valiron theories; hence, the different approaches are employed, for example, Herold's comparison theorem [6], Gronwall's lemma [7], Picard's successive approximations [8], and some methods based on Carleson measures [9]. There are many results concerning the slowly growing solutions that have been obtained; for example, see [6, 7, 9–11] and reference therein.

In [9], Pommerenke considered the second-order equation:

$$f'' + A(z)f = 0, \tag{1}$$

where $A(z)$ is an analytic function in the unit disc $\mathbb{D} = \{z : |z| < 1\}$ and obtained a sharp sufficient conditions for $A(z)$ which guarantee all solutions of (1) are in the classical Hardy space $H^2(\mathbb{D})$. The coefficient condition was given in terms of Carleson measures by Green's formula and Carleson's theorem for Hardy space. The leading method of this approach has been extended to study, for example, solutions in the Hardy space and Bergman space [6] and Dirichlet type space [10].

Recently, Gröhn-Huusko-Rättyä [12] found sufficient conditions for the coefficient $A(z)$ guaranteeing all solutions of (1) are in Bloch (little Bloch) space by taking advantage of the reproducing formula in small weighted Bergman space [13]. Motivated from [12], now we use the reproducing formula in Bergman space with exponential type weights to get the conditions for the coefficient $A(z)$ such that all solutions of (1) are in α-Bloch (little α-Bloch) space. In the same paper, they obtained also some sufficient conditions on the coefficient $A(z)$ which guarantee all solutions of (1) belong to BMOA (VMOA) space, in which some properties of Bloch space were used. It follows from [14–16] that BMOA space is a special kind of analytic Morrey space, and the analytic Morrey space and Bloch space are different. Therefore, we obtain also some sufficient conditions on $A(z)$

which guarantee all solutions of (1) to be in analytic Morrey (little analytic Morrey) space by using different ways in [12]. The definition of BMOA space and analytic Morrey space are recalled in Section 2 below.

Note. If $A \lesssim B$, then there exists a positive constant C such that $A \leq CB$. Similarly, if $A \gtrsim B$, then there exists a positive constant C such that $A \geq CB$. If A and B satisfy $A \gtrsim B$ and $A \lesssim B$, then we say that A and B are comparable, by $A \approx B$. Note that $\sigma_a(z) = (a-z)/(1-\overline{a}z)$ is the Möbius transformation of $\mathbb{D}$ for $a, z \in \mathbb{D}$, and $dm(z)$ denotes the normalized Lebesgue area measure.

The paper is organized as follows: some definitions of related function spaces are recalled in Section 2. Some results in which all solutions of (1) are in α-Bloch spaces are discussed in Section 3. We obtain some results in which all solutions of (1) are in analytic Morrey spaces in Section 4.

2. Some Related Function Spaces and Notations

Let $\partial\mathbb{D}$ be the boundary of $\mathbb{D}$ and $\mathscr{A}(\mathbb{D})$ be the space of analytic functions in $\mathbb{D}$. For $1 \leq p < \infty$, Hardy space $H^p(\mathbb{D})$ consists of $f \in \mathscr{A}(\mathbb{D})$ with

$$\|f\|_{H^p}^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\theta}\right) \right|^p d\theta < \infty. \tag{2}$$

For a subarc $I \subset \partial\mathbb{D}$, the length of I is defined as

$$|I| = \frac{1}{2\pi} \int_I |d\zeta|, \tag{3}$$

and let

$$S(I) = \{r\zeta \in \mathbb{D} : 1 - |I| \leq r < 1, \zeta \in I\} \tag{4}$$

denote the Carleson square in $\mathbb{D}$.

Denote

$$f_I = \frac{1}{|I|} \int_I f(\zeta) \frac{|d\zeta|}{2\pi} \tag{5}$$

as the average of $f \in \mathscr{A}(\mathbb{D})$ over I. BMOA space consists of those functions $f \in H^2(\mathbb{D})$ such that

$$\|f\|_{BMOA} = \left(\sup_{I \subset \partial\mathbb{D}} \frac{1}{|I|} \int_I |f(\zeta) - f_I|^2 \frac{|d\zeta|}{2\pi} \right)^{1/2} < \infty. \tag{6}$$

VMOA space consists of those functions $f \in$ BMOA such that

$$\lim_{|I| \longrightarrow 0} \frac{1}{|I|} \int_I |f(\zeta) - f_I|^2 \frac{|d\zeta|}{2\pi} = 0. \tag{7}$$

Morrey spaces were introduced in the 1930s [17] in connection to partial differential equations and were subsequently studied as function spaces in harmonic analysis on Euclidean spaces. The analytic Morrey space $\mathscr{L}^{2,\lambda}(\mathbb{D})$ $(0 \leq \lambda \leq 1)$ was introduced recently by Wu and Xie [15], and then many researchers pay attention to the spaces; for example, see [16, 18, 19] and reference therein. The analytic Morrey space $\mathscr{L}^{2,\lambda}(\mathbb{D})$ consists of those functions $f \in H^2(\mathbb{D})$ such that

$$\|f\|_{\mathscr{L}^{2,\lambda}} = \sup_{I \subset \partial\mathbb{D}} \left(\frac{1}{|I|^\lambda} \int_I |f(\zeta) - f_I|^2 \frac{|d\zeta|}{2\pi} \right)^{1/2} < \infty. \tag{8}$$

Little analytic Morrey space $\mathscr{L}_0^{2,\lambda}(\mathbb{D})$ consists of those functions $f \in \mathscr{L}^{2,\lambda}(\mathbb{D})$ and

$$\lim_{|I| \longrightarrow 0} \frac{1}{|I|^\lambda} \int_I |f(\zeta) - f_I|^2 \frac{|d\zeta|}{2\pi} = 0. \tag{9}$$

Clearly, for $\lambda = 0$ or $\lambda = 1$, $\mathscr{L}^{2,\lambda}(\mathbb{D})$ reduces to $H^2(\mathbb{D})$ and BMOA, respectively; hence the case of $0 < \lambda < 1$ is marked, and then BMOA$\subset \mathscr{L}^{2,\lambda}(\mathbb{D}) \subset H^2(\mathbb{D})$.

The following lemma gives some equivalent conditions of $\mathscr{L}^{2,\lambda}(\mathbb{D})$ by [19, Theorem 3.1] or [20, Theorem 3.21].

Lemma 1. *Suppose that $0 < \lambda \leq 1$ and $f \in \mathscr{A}(\mathbb{D})$. Then the following statements are equivalent:*

(i) $f \in \mathscr{L}^{2,\lambda}(\mathbb{D})$.

(ii) $\sup_{a \in \mathbb{D}} (1-|a|^2)^{1-\lambda} \int_{\mathbb{D}} |f'(z)|^2 (1-|\sigma_a(z)|^2) dm(z) < \infty$.

(iii) $\sup_{I \subset \partial\mathbb{D}} (1/|I|^\lambda) \int_{S(I)} |f'(z)|^2 (1-|z|^2) dm(z) < \infty$.

(iv) $\sup_{a \in \mathbb{D}} (1-|a|^2)^{(1-\lambda)/2} \|f \circ \sigma_a - f(a)\|_{H^2} < \infty$.

From Lemma 1, we see the following.

Lemma 2. *Let f and λ be defined as Lemma 1. Then $f \in \mathscr{L}_0^{2,\lambda}(\mathbb{D})$ if and only if*

$$\lim_{|a| \longrightarrow 1^-} \left(1-|a|^2\right)^{1-\lambda} \int_{\mathbb{D}} \left|f'(z)\right|^2 \left(1-\left|\sigma_a(z)\right|^2\right) dm(z) = 0 \tag{10}$$

or

$$\lim_{|a| \longrightarrow 1^-} \left(1-|a|^2\right)^{(1-\lambda)/2} \|f \circ \sigma_a - f(a)\|_{H^2} = 0. \tag{11}$$

Here we use the reproducing formula in Bergman space with exponential type weights to study (1). To this end, let $0 < p < \infty$; the exponential type weighted Bergman space $A^P(\omega_{\gamma,\alpha})$ is the space of functions $f \in \mathscr{A}(\mathbb{D})$ such that

$$\|f\|_{A^P(\omega_{\gamma,\alpha})}^p := \int_{\mathbb{D}} |f(z)|^p \omega_{\gamma,\alpha}(|z|) dm(z) < \infty, \tag{12}$$

where $\omega_{\gamma,\alpha}$ is the exponential type weight as

$$\omega_{\gamma,\alpha}(\rho) = (1-\rho)^\gamma \exp\left(\frac{-c}{(1-\rho)^\alpha}\right), \quad \gamma \geq 0, \ c > 0, \ \alpha > 0. \tag{13}$$

In [21], it is proved that the point evaluation $L_\zeta(f(z)) = f(\zeta)$ is bounded linear function on $A^P(\omega_{\gamma,\alpha})$. Hence there exists a reproducing kernel $K_\zeta \in A^P(\omega_{\gamma,\alpha})$ with $\|K_\zeta\|_{A^2(\omega_{\gamma,\alpha})} = \|L_\zeta\|$ such that

$$f(\zeta) = L_\zeta(f(z)) := \int_{\mathbb{D}} f(z)\overline{K_\zeta(z)}\omega_{\gamma,\alpha}(|z|)\,dm(z), \quad f \in A^P(\omega_{\gamma,\alpha}), \tag{14}$$

where

$$K_\zeta(z) = \sum_{n=0}^{\infty} \frac{(z\bar{\zeta})^n}{2\int_0^1 \rho^{2n+1}\omega_{\gamma,\alpha}(\rho)\,d\rho}. \tag{15}$$

Finally, we recall the definition of α-Bloch space $\mathfrak{B}^\alpha$, which can be found in [22]. Let $\alpha \in (0,\infty)$; it is defined as

$$\mathfrak{B}^\alpha := \left\{h \in \mathscr{A}(\mathbb{D}) : \|h\|_{\mathfrak{B}^\alpha} := \sup_{z\in\mathbb{D}} |h'(z)|\left(1-|z|^2\right)^\alpha < \infty\right\}. \tag{16}$$

Little α-Bloch space $\mathfrak{B}_0^\alpha$ is the subspace of $\mathfrak{B}^\alpha$ consisting of functions h with

$$\lim_{|z|\to 1^-} |h'(z)|\left(1-|z|^2\right)^\alpha = 0. \tag{17}$$

Clearly, $\mathfrak{B}^\alpha$ ($\mathfrak{B}_0^\alpha$) is the Bloch (little Bloch) space for $\alpha = 1$.

3. Sufficient Conditions of Solutions in $\mathfrak{B}^\alpha$ ($\mathfrak{B}_0^\alpha$)

In the section, we consider the estimation of coefficient similarly to Section 7 in [12], in which the exponential type weighted Bergman reproducing kernel formula is used. Some sufficient conditions on $A(z)$ guaranteeing all solutions of (1) to be in $\mathfrak{B}^\alpha$ ($\mathfrak{B}_0^\alpha$) are obtained by these estimates as follows.

Theorem 3. *Let $\omega_{\gamma,\alpha}(\rho)$ be an exponential type weight. If*

$$\overline{\lim_{\rho\to 1^-}} \sup_{z\in\mathbb{D}} \left\{(1-|z|)^{2(1+\alpha)} \cdot \left|\int_{\mathbb{D}} \omega_{\gamma,\alpha}(|\eta|)\left(\int_0^z \overline{K'_\zeta(\eta)}A(\rho\zeta)\,d\zeta\right)dm(\eta)\right|\right\} \tag{18}$$

is sufficiently small, then all solutions of (1) belong to $\mathfrak{B}^{2(1+\alpha)}$.

Theorem 4. *Let $\omega_{\gamma,\alpha}(\rho)$ be an exponential type weight. Suppose that the condition of Theorem 3 is satisfied, and*

$$\overline{\lim_{|z|\to 1^-}}\ \overline{\lim_{\rho\to 1^-}} \left((1-|z|)^{2(1+\alpha)} \cdot \left|\int_{\mathbb{D}} \omega_{\gamma,\alpha}(|\eta|)\left(\int_0^z \overline{K'_\zeta(\eta)}A(\rho\zeta)\,d\zeta\right)dm(\eta)\right|\right) = 0. \tag{19}$$

Then all solutions of (1) belong to $\mathfrak{B}_0^{2(1+\alpha)}$.

In order to prove Theorems 3 and 4, we need the following lemma.

Lemma 5 (see [21]). *If $f, g \in A^2(\omega_{\gamma,\alpha})$, then*

$$\begin{aligned}\int_{\mathbb{D}} f(z)\overline{g(z)}\omega_{\gamma,\alpha}(|z|)\,dm(z) &= \int_{\mathbb{D}} f'(z)\overline{g'(z)}(1-|z|)^{2(1+\alpha)}\omega_{\gamma,\alpha}(|z|)\,dm(z) \\ &\quad + f(0)\overline{g(0)}.\end{aligned} \tag{20}$$

Proof of Theorem 3. Let f be any solution of (1), then f_ρ is analytic in $\overline{\mathbb{D}}$ and satisfies $f''_\rho(z) + \rho^2 A(\rho z) f_\rho(z) = 0$. Therefore,

$$f'_\rho(z) = -\int_0^z \rho^2 f_\rho(\zeta)A(\rho\zeta)\,d\zeta + f'_\rho(0), \quad z \in \mathbb{D}. \tag{21}$$

By the reproducing formula and Fubini's theorem,

$$\begin{aligned} f'_\rho(z) &= -\int_0^z\left(\int_{\mathbb{D}} f_\rho(\eta)\overline{K_\zeta(\eta)}\omega_{\gamma,\alpha}(|\eta|)\,dm(\eta)\right)\rho^2 A(\rho\zeta)\,d\zeta \\ &\quad + f'_\rho(0) \\ &= -\int_{\mathbb{D}} f_\rho(\eta)\omega_{\gamma,\alpha}(|\eta|)\left(\int_0^z \overline{K_\zeta(\eta)}\rho^2 A(\rho\zeta)\,d\zeta\right)dm(\eta) \\ &\quad + f'_\rho(0), \quad z \in \mathbb{D}.\end{aligned} \tag{22}$$

Applying Lemma 5,

$$\begin{aligned} f'_\rho(z) &= -\int_{\mathbb{D}} f'_\rho(\eta)(1-|\eta|)^{2(1+\alpha)}\omega_{\gamma,\alpha}(|\eta|) \\ &\quad \cdot\left(\int_0^z \overline{K'_\zeta(\eta)}\rho^2 A(\rho\zeta)\,d\zeta\right)dm(\eta) + f_\rho(0) \\ &\quad \cdot\left(\int_0^z \frac{\rho^2 A(\rho\zeta)}{2\int_0^1 \rho\omega_{\gamma,\alpha}(\rho)\,d\rho}d\zeta\right) + f'_\rho(0).\end{aligned} \tag{23}$$

Therefore,

$$\begin{aligned} \|f_\rho\|_{\mathfrak{B}^{2(1+\alpha)}} &\lesssim \|f_\rho\|_{\mathfrak{B}^{2(1+\alpha)}} \sup_{z\in\mathbb{D}}\left\{(1-|z|)^{2(1+\alpha)}\right. \\ &\quad \left.\cdot\left|\int_{\mathbb{D}} \omega_{\gamma,\alpha}(|\eta|)\left(\int_0^z \overline{K'_\zeta(\eta)}A(\rho\zeta)\,d\zeta\right)dm(\eta)\right|\right\} \\ &\quad + |f_\rho(0)| \sup_{z\in\mathbb{D}}\left\{(1-|z|)^{2(1+\alpha)}\left|\int_0^z \frac{A(\rho\zeta)}{2\int_0^1 \rho\omega_{\gamma,\alpha}(\rho)\,d\rho}d\zeta\right|\right\} \\ &\quad + |f'_\rho(0)|, \quad z \in \mathbb{D}.\end{aligned} \tag{24}$$

This implies that

$$\begin{aligned} &\|f_\rho\|_{\mathfrak{B}^{2(1+\alpha)}}\left(1 - C\sup_{z\in\mathbb{D}}\left\{(1-|z|)^{2(1+\alpha)}\right.\right. \\ &\quad \left.\left.\cdot\left|\int_{\mathbb{D}} \omega_{\gamma,\alpha}(|\eta|)\left(\int_0^z \overline{K'_\zeta(\eta)}A(\rho\zeta)\,d\zeta\right)dm(\eta)\right|\right\}\right) \\ &\lesssim |f_\rho(0)| \sup_{z\in\mathbb{D}}\left\{(1-|z|)^{2(1+\alpha)}\left|\int_0^z A(\rho\zeta)\,d\zeta\right|\right\} + |f'_\rho(0)|,\end{aligned} \tag{25}$$

where C is a positive constant and $0 < \rho < 1$. It follows from Fubini's theorem and the reproducing formula that

$$\begin{aligned}
&\sup_{z\in\mathbb{D}}\Bigg((1-|z|)^{2(1+\alpha)} \\
&\cdot\left|\int_{\mathbb{D}}\omega_{\gamma,\alpha}(|\eta|)\left(\int_0^z\overline{K'_\zeta(\eta)}A(\rho\zeta)\,d\zeta\right)dm(\eta)\right|\Bigg) \\
&=\sup_{z\in\mathbb{D}}\Bigg((1-|z|)^{2(1+\alpha)} \\
&\cdot\left|\int_0^z\left(\int_{\mathbb{D}}\omega_{\gamma,\alpha}(|\eta|)\overline{K'_\zeta(\eta)}dm(\eta)\right)A(\rho\zeta)\,d\zeta\right|\Bigg) \\
&\gtrsim\sup_{z\in\mathbb{D}}(1-|z|)^{2(1+\alpha)}\left|\int_0^z A(\rho\zeta)\,d\zeta\right|.
\end{aligned}\tag{26}$$

Therefore, we deduce $f \in \mathfrak{B}^{2(1+\alpha)}$ by the above formula and (25) and letting $\rho \longrightarrow 1^-$. □

Proof of Theorem 4. By Theorem 3, we know that $f \in \mathfrak{B}^{2(1+\alpha)}$. Fubini's theorem and the reproducing formula yield

$$\begin{aligned}
&\overline{\lim_{|z|\longrightarrow1^-}}\ \overline{\lim_{\rho\longrightarrow1^-}}\Bigg((1-|z|)^{2(1+\alpha)} \\
&\cdot\left|\int_{\mathbb{D}}\omega_{\gamma,\alpha}(|\eta|)\left(\int_0^z\overline{K'_\zeta(\eta)}A(\rho\zeta)\,d\zeta\right)dm(\eta)\right|\Bigg) \\
&\gtrsim\overline{\lim_{|z|\longrightarrow1^-}}\ \overline{\lim_{\rho\longrightarrow1^-}}(1-|z|)^{2(1+\alpha)}\left|\int_0^z A(\rho\zeta)\,d\zeta\right|.
\end{aligned}\tag{27}$$

The condition of Theorem 4 implies

$$\overline{\lim_{|z|\longrightarrow1^-}}\ \overline{\lim_{\rho\longrightarrow1^-}}(1-|z|)^{2(1+\alpha)}\left|\int_0^z A(\rho\zeta)\,d\zeta\right|=0.\tag{28}$$

By using the similar reason as in the proof of Theorem 3, we have

$$\begin{aligned}
&(1-|z|)^{2(1+\alpha)}\left|f'_\rho(z)\right|\lesssim\left\|f_\rho\right\|_{\mathfrak{B}^{2(1+\alpha)}}(1-|z|)^{2(1+\alpha)} \\
&\cdot\left|\int_{\mathbb{D}}\omega_{\gamma,\alpha}(|\eta|)\left(\int_0^z\overline{K'_\zeta(\eta)}A(\rho\zeta)\,d\zeta\right)dm(\eta)\right| \\
&+\left|f_\rho(0)\right|(1-|z|)^{2(1+\alpha)}\left|\int_0^z A(\rho\zeta)\,d\zeta\right|+(1-|z|)^{2(1+\alpha)} \\
&\cdot\left|f'_\rho(0)\right|,\quad z\in\mathbb{D}.
\end{aligned}\tag{29}$$

Then by the condition of Theorem 4,

$$\overline{\lim_{|z|\longrightarrow1^-}}\ \overline{\lim_{\rho\longrightarrow1^-}}(1-|z|)^{2(1+\alpha)}\left|f'_\rho(z)\right|=0.\tag{30}$$

Therefore, the assertion follows. □

4. Sufficient Conditions of Solutions in $\mathscr{L}^{2,\lambda}(\mathbb{D})$ $(\mathscr{L}_0^{2,\lambda}(\mathbb{D}))$

In the section, some sufficient conditions on $A(z)$ which guarantee all solutions of (1) belong to analytic Morrey space are obtained by using two ideas. On the one hand, a sufficient condition on $A(z)$ guaranteeing that all solutions of (1) belong to BMOA is obtained in [12, Theorem 3]; here we study the condition on $A(z)$ which guarantee that all solutions of (1) are in analytic Morrey space by using similar idea in [12, Theorem 3]. On the other hand, a sufficient condition on $A(z)$ guaranteeing that all solutions of (1) are in analytic Morrey space is shown by using representation formula.

It is also known that BMOA space is a subspace of Bloch space, and then the properties of Bloch space were used in the proof of [12, Theorem 3]. However, it was proved that analytic Morrey space and Bloch space are different in [16]. Therefore, we need different methods from Section 6 of [12] to deal with the case of analytic Morrey space; the following results are proved.

Theorem 6. *Let $A(z) \in \mathscr{A}(\mathbb{D})$ and $0 < \lambda < 1$. Suppose that*

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|A(z)|^2\left(1-|z|^2\right)^2\left(1-\left|\sigma_a(z)\right|^2\right)dm(z)\tag{31}$$

is sufficiently small and

$$\sup_{z\in\mathbb{D}}\left\{|A(z)|(1-|z|)^{(5-\lambda)/2}\int_0^{|z|}\frac{dr}{(1-r)^{(3-\lambda)/2}}\right\}<\frac{3-\lambda}{2}.\tag{32}$$

Then all solutions of (1) are in $\mathscr{L}^{2,\lambda}(\mathbb{D})$.

Theorem 7. *Let $A(z) \in \mathscr{A}(\mathbb{D})$ and $0 < \lambda < 1$. Suppose that the conditions of Theorem 6 are satisfied, and*

$$\lim_{|a|\longrightarrow1^-}\int_{\mathbb{D}}|A(z)|^2\left(1-|z|^2\right)^2\left(1-\left|\sigma_a(z)\right|^2\right)dm(z)=0\tag{33}$$

and

$$\lim_{|z|\longrightarrow1^-}\left(|A(z)|(1-|z|)^{(5-\lambda)/2}\int_0^{|z|}\frac{dr}{(1-r)^{(3-\lambda)/2}}\right)=0.\tag{34}$$

Then all solutions of (1) are in $\mathscr{L}_0^{2,\lambda}(\mathbb{D})$.

In order to prove Theorems 6 and 7, we recall the definition of Carleson measure. For a non-negative measure μ on $\mathbb{D}$, it is called a Carleson measure if

$$\|\mu\|=:\sup_{I\subset\partial\mathbb{D}}\left(\frac{\mu(S(I))}{|I|}\right)^{1/2}<\infty.\tag{35}$$

Moreover, if in addition

$$\lim_{|I|\longrightarrow0}\frac{\mu(S(I))}{|I|}=0,\tag{36}$$

then μ is called a compact Carleson measure.

For the proof of Theorems 6 and 7, the following lemmas are needed.

Lemma 8 (sse [14]). *Suppose that μ is a Carleson measure. Then*

$$\|\mu\|\approx\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\left|\sigma'_a(z)\right|d\mu(z).\tag{37}$$

Lemma 9 (see [23]). *Let $0 < p, q < \infty$. If $f \in \mathscr{A}(\mathbb{D})$, then*

$$\int_{\mathbb{D}} |f(z)|^p \left(1-|z|^2\right)^q dm(z) \lesssim \left(|f(0)|^p + \int_{\mathbb{D}} \left|f'(z)\right|^p \left(1-|z|^2\right)^{p+q} dm(z)\right). \quad (38)$$

Lemma 10 (see [24]). *Let $0 < p, q < \infty$. If $f \in \mathscr{A}(\mathbb{D})$, then*

$$\left|f^{(n)}(a)\right|^p \left(1-|a|^2\right)^{q+2} \lesssim \int_{\mathbb{D}} \left|f^{(n)}(z)\right|^p \left(1-|z|^2\right)^q \left(1-\left|\sigma_a(z)\right|^2\right) dm(z). \quad (39)$$

Lemma 11 (see [22]). *Suppose that $0 < p, q < \infty$. Then, for $f \in \mathscr{A}(\mathbb{D})$,*

$$\sup_{a\in\mathbb{D}} \left|f^{(n)}(a)\right|^p \left(1-|a|^2\right)^q \quad (40)$$

and

$$\sup_{a\in\mathbb{D}} \left|f^{(n+1)}(a)\right|^p \left(1-|a|^2\right)^{q+p} \quad (41)$$

are comparable.

Lemma 12. *Let $0 < \lambda < 1$ and $f \in \mathscr{L}^{2,\lambda}$. Then*

$$\|f\|^2_{\mathscr{L}^{2,\lambda}} \lesssim \left(\sup_{a\in\mathbb{D}} \left(\left|f'(a)\right|^2 \left(1-|a|^2\right)^{3-\lambda}\right) + \sup_{a\in\mathbb{D}} \left(\left(1-|a|^2\right)^{1-\lambda} \cdot \int_{\mathbb{D}} \left|f''(z)\right|^2 \left(1-|z|^2\right)^2 \left(1-\left|\sigma_a(z)^2\right|\right) dm(z)\right)\right). \quad (42)$$

Proof. An auxiliary function is constructed as follows:

$$g(z) = \frac{f'(\sigma_a(z))\left(1-|a|^2\right)}{(1-\bar{a}z)^2}. \quad (43)$$

By Lemmas 1 and 9,

$$\begin{aligned}\|f\|^2_{\mathscr{L}^{2,\lambda}} &\approx \left(1-|a|^2\right)^{1-\lambda} \int_{\mathbb{D}} \left|f'(z)\right|^2 \left(1-\left|\sigma_a(z)\right|^2\right) dm(z) \\ &= \left(1-|a|^2\right)^{1-\lambda} \int_{\mathbb{D}} \left|f'(\sigma_a(z))\right|^2 \left(1-\left|\sigma_a\circ\sigma_a(z)\right|^2\right) \cdot \left|\sigma_a'(z)\right|^2 dm(z) = \left(1-|a|^2\right)^{1-\lambda} \\ &\cdot \int_{\mathbb{D}} \left|f'(\sigma_a(z))\right|^2 \left(1-\left|\sigma_a(z)\right|^2\right)^2 \frac{dm(z)}{1-|z|^2} = \left(1 - |a|^2\right)^{1-\lambda} \int_{\mathbb{D}} |g(z)|^2 \left(1-|z|^2\right) dm(z) \lesssim \left(|g(0)|^2 \right. \\ &\cdot \left(1-|a|^2\right)^{1-\lambda} + \left(1-|a|^2\right)^{1-\lambda} \\ &\left. \cdot \int_{\mathbb{D}} \left|g'(z)\right|^2 \left(1-|z|^2\right)^3 dm(z)\right), \end{aligned} \quad (44)$$

where $|g(0)|^2 = |f'(a)|^2(1-|a|^2)^2$.

Note that

$$\left(1-|a|^2\right)^{1-\lambda} \int_{\mathbb{D}} \left|g'(z)\right|^2 \left(1-|z|^2\right)^3 dm(z) \le I_1 + I_2, \quad (45)$$

where

$$I_1 = \left(1-|a|^2\right)^{1-\lambda} \cdot \int_{\mathbb{D}} \frac{\left|\left(f'(\sigma_a(z))\right)'\right|^2}{|1-\bar{a}z|^4} \left(1-|a|^2\right)^2 \left(1-|z|^2\right)^3 dm(z) \quad (46)$$

and

$$I_2 = \left(1-|a|^2\right)^{1-\lambda} \cdot \int_{\mathbb{D}} \frac{\left|f'(\sigma_a(z))\right|^2 \left(1-|a|^2\right)^2}{|1-\bar{a}z|^6} \left(1-|z|^2\right)^3 dm(z). \quad (47)$$

Clearly,

$$\begin{aligned} I_1 &\lesssim \left(1-|a|^2\right)^{1-\lambda} \cdot \int_{\mathbb{D}} \left|f''(\sigma_a(z))\right|^2 \frac{\left(1-\left|\sigma_a(z)\right|^2\right)^4}{1-|z|^2} dm(z) \\ &= \left(1-|a|^2\right)^{1-\lambda} \cdot \int_{\mathbb{D}} \left|f''(z)\right|^2 \left(1-|z|^2\right)^2 \left(1-\left|\sigma_a(z)\right|^2\right) dm(z). \end{aligned} \quad (48)$$

It follows from Lemmas 10 and 11 that

$$\begin{aligned} I_2 &= \left(1-|a|^2\right)^{1-\lambda} \cdot \int_{\mathbb{D}} \frac{\left|f'(\sigma_a(z))\right|^2 \left(1-\left|\sigma_a(z)\right|^2\right)^2}{|1-\bar{a}z|^2} \left(1-|z|^2\right) dm(z) \\ &\le \left(1-|a|^2\right)^{1-\lambda} \sup_{a\in\mathbb{D}} \left\{\left|f'(a)\right|^2 \left(1-|a|^2\right)^2\right\} \cdot \int_{\mathbb{D}} \frac{1-|z|^2}{|1-\bar{a}z|^2} dm(z) \lesssim \left(1-|a|^2\right)^{1-\lambda} \\ &\cdot \sup_{a\in\mathbb{D}} \left\{\left|f''(a)\right|^2 \left(1-|a|^2\right)^4\right\} \int_{\mathbb{D}} \frac{1-|z|^2}{|1-\bar{a}z|^2} dm(z) \\ &\lesssim \left(1-|a|^2\right)^{1-\lambda} \cdot \int_{\mathbb{D}} \left|f''(z)\right|^2 \left(1-|z|^2\right)^2 \left(1-\left|\sigma_a(z)\right|^2\right) dm(z) \cdot \int_{\mathbb{D}} \frac{1-|z|^2}{|1-\bar{a}z|^2} dm(z). \end{aligned} \quad (49)$$

It is easy to get that $\sup_{a\in\mathbb{D}} \int_{\mathbb{D}}((1-|z|^2)/|1-\bar{a}z|^2) dm(z)$ is finite. Then the lemma is completely proved. ☐

Lemma 13 (see [25]). *Suppose that μ is a positive Borel measure and $0 < p < \infty$. Then μ is a Carleson measure if and only if $H^p(\mathbb{D}) \subset L^p(d\mu)$. Namely, if there exists a continuous including mapping $i : H^p(\mathbb{D}) \longrightarrow L^p(d\mu)$, then*

$$\|f\|_{L^p(d\mu)} \lesssim \|\mu\| \|f\|_{H^p(\mathbb{D})}. \tag{50}$$

Lemma 14 (see [18]). *Let $0 < \lambda < 1$. If $f \in \mathscr{L}^{2,\lambda}$, then*

$$|f(z)| \lesssim \frac{\|f\|_{\mathscr{L}^{2,\lambda}}}{\left(1 - |z|^2\right)^{(1-\lambda)/2}}, \quad z \in \mathbb{D}. \tag{51}$$

Proof of Theorem 6. We can obtain that all solutions f of (1) belong to $\mathfrak{B}^{(3-\lambda)/2}$ by [11, Corollary 4] and (32). It follows from Lemma 12 that

$$\begin{aligned}
\|f_\rho\|^2_{\mathscr{L}^{2,\lambda}} &\lesssim \sup_{a\in\mathbb{D}} \Big(\left(1 - |a|^2\right)^{3-\lambda} \left|f'(\rho z)\right|^2 \rho^2 \\
&\quad + \left(1 - |a|^2\right)^{1-\lambda} \\
&\quad \cdot \int_{\mathbb{D}} \rho^4 \left|f''(\rho z)\right|^2 \left(1 - |z|^2\right)^2 \left(1 - |\sigma_a(z)|^2\right) dm(z) \Big) \\
&\lesssim \|f_\rho\|^2_{\mathfrak{B}^{(3-\lambda)/2}} + \sup_{a\in\mathbb{D}} \left(1 - |a|^2\right)^{1-\lambda} \cdot \int_{\mathbb{D}} \left|f_\rho(z) - f_\rho(a)\right|^2 \\
&\quad \cdot |A(\rho z)|^2 \left(1 - |z|^2\right)^2 \left(1 - |\sigma_a(z)|^2\right) dm(z) \\
&\quad + \sup_{a\in\mathbb{D}} \left(1 - |a|^2\right)^{1-\lambda} \left|f_\rho(a)\right|^2 \int_{\mathbb{D}} |A(\rho z)|^2 \left(1 - |z|^2\right)^2 \Big(1 \\
&\quad - |\sigma_a(z)|^2\Big) dm(z) \lesssim \|f_\rho\|^2_{\mathfrak{B}^{(3-\lambda)/2}} + J_1 \\
&\quad + J_2,
\end{aligned} \tag{52}$$

where $f_\rho(z) = f(\rho z)$. By Lemmas 1, 8, and 13,

$$\begin{aligned}
J_1 &\lesssim \sup_{a\in\mathbb{D}} \left(1 - |a|^2\right)^{1-\lambda} \int_{\mathbb{D}} \left|f_\rho(\sigma(z)) - f_\rho(a)\right|^2 \\
&\quad \cdot |A(\rho\sigma_a(z))|^2 \cdot \left(1 - |\sigma_a(z)|^2\right)^3 \left|\sigma_a'(z)\right| dm(z) \\
&\lesssim \sup_{a\in\mathbb{D}} \left(1 - |a|^2\right)^{1-\lambda} \Big(\left\|f_\rho(\sigma(z)) - f_\rho(a)\right\|^2_{H^2} \\
&\quad \cdot \sup_{b\in\mathbb{D}} \int_{\mathbb{D}} |A(\rho\sigma_a(z))|^2 \left(1 - |\sigma_a(z)|^2\right)^3 \left|\sigma_a'(z)\right| \\
&\quad \cdot \left|\sigma_b'(z)\right| dm(z) \Big) \lesssim \|f_\rho\|^2_{\mathscr{L}^{2,\lambda}} \\
&\quad \cdot \sup_{a\in\mathbb{D}} \Big(\sup_{b\in\mathbb{D}} \int_{\mathbb{D}} |A(\rho\sigma_a(z))|^2 \left(1 - |\sigma_a(z)|^2\right)^3 \left|\sigma_a'(z)\right| \\
&\quad \cdot \left|\sigma_b'(z)\right| dm(z) \Big) \lesssim \|f_\rho\|^2_{\mathscr{L}^{2,\lambda}} \\
&\quad \cdot \sup_{a\in\mathbb{D}} \Big(\sup_{b\in\mathbb{D}} \int_{\mathbb{D}} |A(\rho z)|^2 \left(1 - |z|^2\right)^2 \Big(1 - \Big|\sigma_b \\
&\quad \circ \sigma_a^{-1}(z)\Big|^2\Big) dm(z) \Big) \lesssim \|f_\rho\|^2_{\mathscr{L}^{2,\lambda}} \\
&\quad \cdot \sup_{c\in\mathbb{D}} \int_{\mathbb{D}} |A(\rho z)|^2 \left(1 - |z|^2\right)^2 \Big(1 \\
&\quad - |\sigma_c(z)|^2\Big) dm(z),
\end{aligned} \tag{53}$$

where $d\mu(z) = |A(\rho\sigma_a(z))|^2(1 - |\sigma_a(z)|^2)^3|\sigma_a'(z)|dm(z)$ is a Carleson measure.

It follows from Lemma 14 that

$$\begin{aligned}
J_2 &\lesssim \sup_{a\in\mathbb{D}} \left(1 - |a|^2\right)^{1-\lambda} \frac{\|f_\rho\|^2_{\mathscr{L}^{2,\lambda}}}{\left(1 - |a|^2\right)^{1-\lambda}} \\
&\quad \cdot \int_{\mathbb{D}} |A(\rho z)|^2 \left(1 - |z|^2\right)^2 \left(1 - |\sigma_a(z)|^2\right) dm(z) \\
&= \|f_\rho\|^2_{\mathscr{L}^{2,\lambda}} \\
&\quad \cdot \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |A(\rho z)|^2 \left(1 - |z|^2\right)^2 \left(1 - |\sigma_a(z)|^2\right) dm(z) \\
&\lesssim \|f_\rho\|^2_{\mathscr{L}^{2,\lambda}} \\
&\quad \cdot \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |A(\rho z)|^2 \left(1 - |z|^2\right)^2 \left(1 - |\sigma_a(z)|^2\right) dm(z).
\end{aligned} \tag{54}$$

Denote

$$J(a,\rho) = \int_{\mathbb{D}} |A(\rho z)|^2 \left(1 - |z|^2\right)^2 \left(1 - |\sigma_a(z)|^2\right) dm(z), \quad 0 < \rho < 1. \tag{55}$$

For $|a| \le 1/2$, the estimate is trivial. Let $1/2 \le |a| \le 1/(2-\rho)$. Since $|1 - \bar{a}z| \le 2|1 - \bar{a}z/\rho|$ for $|z| \le \rho$,

$$\begin{aligned}
&J(a,\rho) \\
&\quad = \int_{\mathbb{D}(0,\rho)} |A(z)|^2 \left(1 - \left|\frac{z}{\rho}\right|^2\right)^2 \frac{1 - |a|^2}{|1 - \bar{a}(z/\rho)|} \frac{dm(z)}{\rho^2} \\
&\quad \le \frac{4}{\rho^2} \int_{\mathbb{D}} |A(z)|^2 \left(1 - |z|^2\right)^2 \left(1 - |\sigma_a(z)|^2\right) dm(z) \\
&\quad \le 16 \int_{\mathbb{D}} |A(z)|^2 \left(1 - |z|^2\right)^2 \left(1 - |\sigma_a(z)|^2\right) dm(z)
\end{aligned} \tag{56}$$

for any $1/2 < \rho < 1$. Let $1/(2-\rho) \le |a| \le 1$. Now

$$\begin{aligned}
J(a,\rho) &\le \sup_{a\in\mathbb{D}} |A(z)|^2 \left(1 - |z|^2\right)^4 \\
&\quad \cdot \int_{\mathbb{D}} \frac{\left(1 - |z|^2\right)^2 \left(1 - |\sigma_a(z)|^2\right)}{\left(1 - |\rho z|^2\right)^4} dm(z).
\end{aligned} \tag{57}$$

By Lemma 10, we get

$$J(a,\rho) \lesssim \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |A(z)|^2 \left(1-|z|^2\right)^2 \left(1-|\sigma_a(z)|^2\right) dm(z) \cdot \int_{\mathbb{D}} \frac{\left(1-|z|^2\right)^2 \left(1-|\sigma_a(z)|^2\right)}{\left(1-|\rho z|^2\right)^4} dm(z). \tag{58}$$

By [12, Theorem 3],

$$\begin{aligned} J(a,\rho) &\lesssim \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |A(z)|^2 \left(1-|z|^2\right)^2 \left(1-|\sigma_a(z)|^2\right) dm(z) \cdot \int_0^1 \frac{(1-s)^3(1-|a|)}{(1-\rho s)^4(1-|a|s)} ds \\ &\lesssim \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |A(z)|^2 \left(1-|z|^2\right)^2 \left(1-|\sigma_a(z)|^2\right) dm(z) \cdot \int_0^1 \frac{(1-|a|)}{(1-\rho s)(1-|a|s)} ds \\ &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |A(z)|^2 \left(1-|z|^2\right)^2 \left(1-|\sigma_a(z)|^2\right) dm(z) \cdot \frac{1-|a|}{|a|-\rho} \log\left(1+\frac{|a|-\rho}{1-|a|}\right). \end{aligned} \tag{59}$$

Since $\rho \le 2-1/|a|$, $((1-|a|)/(|a|-\rho))\log(1+(|a|-\rho)/(1-|a|))$ is bounded. Therefore,

$$J(a,\rho) \lesssim \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |A(z)|^2 \left(1-|z|^2\right)^2 \left(1-|\sigma_a(z)|^2\right) dm(z). \tag{60}$$

If (31) is sufficiently small, then the norm $\|f_\rho\|^2_{\mathscr{L}^{2,\lambda}}$ is uniformly bounded for $1/2 < \rho < 1$. By letting $\rho \longrightarrow 1^-$, we reduce $f \in \mathscr{L}^{2,\lambda}$. □

Proof of Theorem 7. We can get that all solutions of (1) belong to $\mathfrak{B}_0^{(3-\lambda)/2}$ by the similar reason as in the proof of [11, Theorem 1]; here it is omitted. The next proof is also similar to the proof of Theorem 6; here it is also omitted. □

Next we consider the estimate of coefficient similarly to Section 8 in [12], in which the well-known representation formula

$$h(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{h\left(e^{is}\right)}{1-ze^{-is}} ds, \quad z \in \mathbb{D}, \tag{61}$$

is used, where $h \in H^1(\mathbb{D})$; it is found in [26, Theorem 3.6]. Here, we prove the following results.

Theorem 15. *Let $A \in \mathscr{A}(\mathbb{D})$ and $0 < \lambda \le 1$. Suppose that*

$$\overline{\lim_{\rho\longrightarrow 1^-}} \sup_{a\in\mathbb{D}} \left\{\left(1-|a|^2\right)^{1-\lambda} \int_{\mathbb{D}} \left(1-|\sigma_a(z)|^2\right) \cdot \left|\int_0^z A(\rho\zeta)\, d\zeta\right|^2 dm(z)\right\} \tag{62}$$

is bounded and

$$\overline{\lim_{\rho\longrightarrow 1^-}} \sup_{a\in\mathbb{D}} \left\{\int_{\mathbb{D}} \left(1-|\sigma_a(z)|^2\right) \left(\int_{\mathbb{D}} \left|h'_{\rho,z}(s)\right|^2 \cdot \frac{(\log|s|)^2}{\left(1-|\sigma_a(s)|^2\right)} dm(s)\right) dm(z)\right\} \tag{63}$$

is sufficiently small, where $h_{\rho,z}(s) = \overline{\int_0^z (A(\rho\varsigma)/(1-\bar{s}\varsigma))d\varsigma}$. Then all solutions of (1) are in $\mathscr{L}^{2,\lambda}(\mathbb{D})$.

Theorem 16. *Let $A \in \mathscr{A}(\mathbb{D})$ and $0 < \lambda \le 1$. Suppose that the conditions of Theorem 15 are satisfied, and*

$$\begin{gathered} \overline{\lim_{|a|\longrightarrow 1^-}}\ \overline{\lim_{\rho\longrightarrow 1^-}} \left\{\left(1-|a|^2\right)^{1-\lambda} \cdot \int_{\mathbb{D}} \left(1-|\sigma_a(z)|^2\right) \left|\int_0^z A(\rho\zeta)\, d\zeta\right|^2 dm(z)\right\} = 0, \\ \overline{\lim_{|a|\longrightarrow 1^-}}\ \overline{\lim_{\rho\longrightarrow 1^-}} \int_{\mathbb{D}} \left(1-|\sigma_a(z)|^2\right) \cdot \left(\int_{\mathbb{D}} \left|h'_{\rho,z}(s)\right|^2 \frac{(\log|s|)^2}{\left(1-|\sigma_a(s)|^2\right)} dm(s)\right) dm(z) = 0. \end{gathered} \tag{64}$$

Then all solutions of (1) are in $\mathscr{L}_0^{2,\lambda}(\mathbb{D})$.

In order to prove Theorems 15 and 16, the following lemma is needed.

Lemma 17 (see [12]). *If $f, g \in H^2(\mathbb{D})$, then*

$$\frac{1}{2\pi}\int_0^{2\pi} f\left(e^{it}\right)\overline{g\left(e^{it}\right)} dt = 2\int_{\mathbb{D}} f'(s)\,\overline{g'(s)} \log\frac{1}{|s|} dm(s) + f(0)\,\overline{g(0)}. \tag{65}$$

Proof of Theorem 15. Let f be any solution of (1). Then f_ρ is analytic in $\overline{\mathbb{D}}$ and satisfies $f''_\rho(z) + \rho^2 A(\rho z) f_\rho(z) = 0$. Therefore,

$$f'_\rho(z) = -\int_0^z \rho^2 f_\rho(\varsigma)\, A(\rho\varsigma)\, d\varsigma + f'_\rho(0), \quad z \in \mathbb{D}. \tag{66}$$

By the representation formula and Fubini's theorem,

$$\begin{aligned} f'_\rho(z) &= -\frac{1}{2\pi}\int_0^{2\pi} f_\rho\left(e^{is}\right)\int_0^z \frac{\rho^2 A(\rho\varsigma)}{1-e^{-is}\varsigma}d\varsigma ds + f'_\rho(0) \\ &= -\frac{\rho^2}{2\pi}\int_0^{2\pi} f_\rho\left(e^{is}\right)\overline{h_{\rho,z}\left(e^{is}\right)}ds + f'_\rho(0), \quad z\in\mathbb{D}, \end{aligned} \tag{67}$$

where

$$h_{\rho,z}(s) = \overline{\int_0^z \frac{A(\rho\varsigma)}{1-\bar{s}\varsigma}d\varsigma}. \tag{68}$$

Since $f_\rho, h_{\rho,z} \in H^2(\mathbb{D})$, Lemma 17 implies

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi} f_\rho\left(e^{is}\right)\overline{h_{\rho,z}\left(e^{is}\right)}dt \\ &\quad = 2\int_{\mathbb{D}} f'_\rho(s)\,\overline{h'_{\rho,z}(s)}\log\frac{1}{|s|}dm(s) + f_\rho(0)\,\overline{h_{\rho,z}(0)}. \end{aligned} \tag{69}$$

So,

$$\begin{aligned} \left|f'_\rho(z)\right|^2 &\le 8\left|\int_{\mathbb{D}} f'_\rho(s)\,\overline{h'_{\rho,z}(s)}\log\frac{1}{|s|}dm(s)\right|^2 \\ &\quad + 2\left|f_\rho(0)\,\overline{h_{\rho,z}(0)} - f'_\rho(0)\right|^2 \\ &\lesssim \int_{\mathbb{D}}\left|f'_\rho(s)\right|^2\left(1-\left|\sigma_a(s)\right|^2\right)dm(s) \\ &\quad\cdot\int_{\mathbb{D}}\left|h'_{\rho,z}(s)\right|^2\frac{(\log|s|)^2}{\left(1-\left|\sigma_a(s)\right|^2\right)}dm(s) \\ &\quad + 4\left|f_\rho(0)\,\overline{h_{\rho,z}(0)}\right|^2 + 4\left|f'_\rho(0)\right|^2. \end{aligned} \tag{70}$$

Therefore, by Lemma 1,

$$\begin{aligned} \left\|f_\rho\right\|^2_{\mathscr{L}^{2,\lambda}} &\approx \sup_{a\in\mathbb{D}}\Bigg(\left(1-|a|^2\right)^{1-\lambda} \\ &\quad\cdot\int_{\mathbb{D}}\left|f'_\rho(z)\right|^2\left(1-\left|\sigma_a(z)\right|^2\right)dm(z)\Bigg) \lesssim \left\|f_\rho\right\|^2_{\mathscr{L}^{2,\lambda}} \\ &\quad\cdot\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\left(1-\left|\sigma_a(z)\right|^2\right) \\ &\quad\cdot\left(\int_{\mathbb{D}}\left|h'_{\rho,z}(s)\right|^2\frac{(\log|s|)^2}{\left(1-\left|\sigma_a(s)\right|^2\right)}dm(s)\right)dm(z) \\ &\quad + 4\left|f_\rho(0)\right|^2\sup_{a\in\mathbb{D}}\left\{\left(1-|a|^2\right)^{1-\lambda}\int_{\mathbb{D}}\left(1-\left|\sigma_a(z)\right|^2\right)\right. \\ &\quad\left.\cdot\left|h_{\rho,z}(0)\right|^2dm(z)\right\} + 4\left|f'_\rho(0)\right|^2 \\ &\quad\cdot\sup_{a\in\mathbb{D}}\left\{\left(1-|a|^2\right)^{1-\lambda}\int_{\mathbb{D}}\left(1-\left|\sigma_a(z)\right|^2\right)dm(z)\right\}. \end{aligned} \tag{71}$$

Then

$$\begin{aligned} &\left\|f_\rho\right\|^2_{\mathscr{L}^{2,\lambda}}\Bigg(1 - C\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\left(1-\left|\sigma_a(z)\right|^2\right)\Bigg(\int_{\mathbb{D}}\left|h'_{\rho,z}(s)\right|^2 \\ &\quad\cdot\frac{(\log|s|)^2}{\left(1-\left|\sigma_a(s)\right|^2\right)}dm(s)\Bigg)dm(z)\Bigg) \\ &\quad\lesssim 4\left|f_\rho(0)\right|^2\sup_{a\in\mathbb{D}}\left\{\left(1-|a|^2\right)^{1-\lambda}\int_{\mathbb{D}}\left(1-\left|\sigma_a(z)\right|^2\right)\right. \\ &\quad\left.\cdot\left|\int_0^z A(\rho\zeta)\,d\zeta\right|^2dm(z)\right\} + 4\left|f'_\rho(0)\right|^2, \end{aligned} \tag{72}$$

where C is a positive constant. By letting $\rho \longrightarrow 1^-$, (62) and (63), the assertion follows. □

Proof of Theorem 16. By the assumption and Theorem 15, we get $f \in \mathscr{L}^{2,\lambda}$. By using the similar reason as the proof of Theorem 15 and the condition of Theorem 16, we have

$$\begin{aligned} &\varlimsup_{|a|\longrightarrow 1^-}\varlimsup_{\rho\longrightarrow 1^-}\Bigg(\left(1-|a|^2\right)^{1-\lambda} \\ &\quad\cdot\int_{\mathbb{D}}\left|f'_\rho(z)\right|^2\left(1-\left|\sigma_a(z)\right|^2\right)dm(z)\Bigg) \lesssim \left\|f_\rho\right\|^2_{\mathscr{L}^{2,\lambda}} \\ &\quad\cdot\varlimsup_{|a|\longrightarrow 1^-}\varlimsup_{\rho\longrightarrow 1^-}\int_{\mathbb{D}}\left(1-\left|\sigma_a(z)\right|^2\right) \\ &\quad\cdot\left(\int_{\mathbb{D}}\left|h'_{\rho,z}(s)\right|^2\frac{(\log|s|)^2}{\left(1-\left|\sigma_a(s)\right|^2\right)}dm(s)\right)dm(z) \\ &\quad + 4\left|f_\rho(0)\right|^2\varlimsup_{|a|\longrightarrow 1^-}\varlimsup_{\rho\longrightarrow 1^-}\left\{\left(1-|a|^2\right)^{1-\lambda}\right. \\ &\quad\left.\cdot\int_{\mathbb{D}}\left(1-\left|\sigma_a(z)\right|^2\right)\left|\int_0^z A(\rho\zeta)\,d\zeta\right|^2dm(z)\right\} \\ &\quad + 4\left|f'_\rho(0)\right|^2\varlimsup_{|a|\longrightarrow 1^-}\left\{\left(1-|a|^2\right)^{1-\lambda}\right. \\ &\quad\left.\cdot\int_{\mathbb{D}}\left(1-\left|\sigma_a(z)\right|^2\right)dm(z)\right\} = 0. \end{aligned} \tag{73}$$

The assertion follows by Lemmas 1 and 2. □

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors have contributed equally to this manuscript. All authors read and approved the final manuscript.

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