Research Article

The Modulus of Nearly Uniform Smoothness in Orlicz Sequence Spaces

Shaoyong Zhang , Meiling Zhang, and Yujia Zhan

Department of Mathematics, Harbin University of Science and Technology, Harbin 150080, China

Correspondence should be addressed to Shaoyong Zhang; 13904603565@139.com

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It is well known that the modulus of nearly uniform smoothness related with the fixed point property is important in Banach spaces. In this paper, we prove that the modulus of nearly uniform smoothness in Köthe sequence spaces without absolutely continuous norm is $\Gamma_X(t) = t$. Meanwhile, the formula of the modulus of nearly uniform smoothness in Orlicz sequence spaces equipped with the Luxemburg norm is given. As a corollary, we get a criterion for nearly uniform smoothness of Orlicz sequence spaces equipped with the Luxemburg norm. Finally, the equivalent conditions of $R(a, l_{(\Phi)}) < 1 + a$ and $RW(a, l_{(\Phi)}) < 1 + a$ are given.

1. Introduction

In last century, the fixed point property has been studied by many scholars. A Banach space X is said to have the fixed point property (FPP, for short) if every nonexpansive mapping $T : C \rightarrow C$

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C \quad (1)$$

acting on a nonempty bounded closed and convex subset C of X has a fixed point. A natural generalization of FPP is the weak fixed point property (WFPP, for short). A Banach space X is said to have the WFPP, whenever it satisfies the above condition from the definition of FPP with “weakly compact” in place of “bounded closed”. In 1965 Kirk [1] proved that any reflexive Banach space with normal structure has the FPP and he also asserts that a Banach space with weak normal structure has WFPP. In this connection, see also the paper [2, 3]. In 1989 Prus [4] introduced a property of a Banach space X , called nearly uniform smoothness and he got that a Banach space is nearly uniform convex if and only if its dual space X^* is nearly uniformly smooth. In paper [5], it is proved that weak nearly uniformly smooth Banach spaces have the FPP, which strengthens the result by S. Prus [6]. In [7], Domínguez-Benavides proved an important result on the existence of fixed points for nonexpansive mappings. In order

to do this, he defined, for a Banach space X and a nonnegative real number a , the parameter

$$R(a, X) := \sup \left\{ \liminf_{n \rightarrow \infty} \|x_n + x\| \right\}, \quad (2)$$

where the supremum is taken over all $x \in X$ with $\|x\| \leq a$ and all weakly null sequences (x_n) in $B(X)$ such that $D[(x_n)] := \limsup_{n \rightarrow \infty, m \rightarrow \infty} \|x_n - x_m\| \leq 1$. He also defined the coefficient

$$M(X) := \sup \left\{ \frac{1+a}{R(a, X)} : a \geq 0 \right\}. \quad (3)$$

The result that a Banach space X has the WFPP, whenever $M(X) > 1$ was obtained in [7] using an embedding into $l_\infty(X)/c_0(X)$.

In this paper we prove that $\Gamma_X(t) = t$ in Köthe sequence spaces without absolutely continuous norm. Furthermore, the formula of the modulus of nearly uniform smoothness in Orlicz sequence spaces equipped with the Luxemburg norm is given. We find the criteria for nearly uniform smoothness of Orlicz sequence spaces equipped with the Luxemburg. Finally, we prove that the condition of $R(a, l_{(\Phi)}) < 1 + a$ is equivalent to $\Phi \in \delta_2$. $RW(a, l_{(\Phi)}) = \Gamma_{l_{(\Phi)}}(a)$ and $RW(a, l_{(\Phi)}) < 1 + a$ if and only if exist $\delta > 0$ and $u_0 > 0$ such that $\Phi(u/(1+a)) \leq ((1-\delta)/(1+a))\Phi(u)$ or $\Phi(u/a) \geq ((1-\delta)/a)\Phi(u)$ when $|u| \leq u_0$.

2. Preliminaries

Throughout this paper, X is a Banach space which is assumed not to have the Schur property, in which X has a weakly convergent sequence that is not norm convergent. By $S(X)$ and $B(X)$ we denote, respectively, the unit sphere and the unit ball of X , and by l^0 we denote the set of all real sequences.

Inspiring on the underlying ideas in the definition of nearly uniformly smooth spaces given by Prus [4] as well as on the modulus of uniform smoothness, in [8] Domínguez-Benavides defined the modulus of nearly uniform smoothness.

Definition 1. The modulus of nearly uniform smoothness is defined by

$$\Gamma_X(t) := \sup \left\{ \inf_{n \geq 1} \left(\frac{\|x_1 + tx_n\| + \|x_1 - tx_n\|}{2} - 1 \right) \right\}, \quad (4)$$

where the supremum is taken over all basic sequences (x_n) in $B(X)$.

Definition 2. For each $0 < a \leq 1$, we define

$$\begin{aligned} RW(a, X) &:= \sup \left\{ \left(\liminf_{n \rightarrow \infty} \|x_n + x\| \right) \right. \\ &\quad \wedge \left(\liminf_{n \rightarrow \infty} \|x_n - x\| \right) : \{x_n\} \in B(X), x_n \xrightarrow{\omega} 0, \|x\| \\ &\quad \left. \leq a \right\}. \end{aligned} \quad (5)$$

In [9], they have the important result that if a Banach space X is reflexive and $RW(a, X) < 1 + a$ for some $a > 0$ then $\Gamma'_X(0) < 1$ that the Banach space X has the FPP.

They use the above result to give a direct proof that the uniformly nonseparable Banach space has the FPP.

In paper [8] it is shown that a Banach space X is nearly uniformly smooth if and only if X is reflexive and

$$\Gamma'_X(0) := \lim_{t \rightarrow 0^+} \frac{\Gamma_X(t)}{t} = 0 \quad (6)$$

and

$$\Gamma_X(t) := \sup \left\{ \inf_{n \geq 1} \left(\frac{\|x_1 + tx_n\| + \|x_1 - tx_n\|}{2} - 1 \right) \right\}, \quad (7)$$

where the supremum is taken over all weak null sequences (x_n) in $B(X)$.

The following theorem was proved in [10].

Theorem 3. If X is a reflexive Banach space and $\Gamma'_X(0) < 1/2$, X has the FPP.

Definition 4. A Banach space X is called a Köthe sequence space if for every $x \in l^0$ and $y \in X$ satisfying $|x(i)| \leq |y(i)|$

for all $i \in N$, we have $x \in X$ and $\|x\| \leq \|y\|$ and if there is a $x = (x(i)) \in X$ with $x(i) > 0$ for all $i \in N$ (see [11, 12]). Let

$$\begin{aligned} X_a &:= \left\{ x \right. \\ &\quad \left. \in X : \lim_{n \rightarrow \infty} \|(0, 0, \dots, 0, x(n+1), x(n+2), \dots)\| \right. \\ &\quad \left. = 0 \right\}. \end{aligned} \quad (8)$$

A Köthe sequence space X is said to be norm absolutely continuous if $X_a = X$.

Definition 5. A Köthe sequence space X is said to have the semi-Fatou property if for every sequence $\{x_n\} \subset X$ and $x \in X$ satisfying $|x_n(i)| \uparrow |x(i)|$ for all $i \in N$ there holds $\|x_n\| \rightarrow \|x\|$.

Definition 6. $\Phi : R \rightarrow R_+$ is called an Orlicz function if it has the following properties:

- (1) Φ is even, continuous, convex and $\Phi(0) = 0$;
- (2) $\Phi(u) > 0$ for all $u \neq 0$.

Definition 7. For every Orlicz function Φ we define the complementary function $\Psi : R \rightarrow [0, \infty]$ by the formula

$$\Psi(u) := \sup_{v > 0} \{u|v| - \Phi(u)\} \quad (9)$$

for every $v \in R$.

Definition 8. We say that Φ satisfies the δ_2 -condition ($\Phi \in \delta_2$ for short) if there exist $k > 0, u_0 > 0$ such that the inequality

$$\Phi(2u) \leq k\Phi(u) \quad (10)$$

whenever $|u| \leq u_0$.

We say that Φ satisfies the $\overline{\delta}_2$ -condition ($\Phi \in \overline{\delta}_2$ for short) if its complementary function Ψ satisfies the δ_2 -condition.

The Orlicz sequence space $l_{(\Phi)}$ is defined to be the set $\{x \in l^0 : I_{\Phi}(\lambda x) = \sum_{i=1}^{\infty} \Phi(\lambda x(i)) < \infty \text{ for some } \lambda > 0\}$ equipped with the Luxemburg norm

$$\|x\|_{(\Phi)} := \inf \left\{ k > 0 : I_{\Phi} \left(\frac{x}{k} \right) \leq 1 \right\}. \quad (11)$$

To simplify notations, we put $l_{(\Phi)} = (l_{\Phi}, \|\cdot\|_{(\Phi)})$.

The basic informations on Orlicz spaces can be found in [13–15].

3. Main Results

Theorem 9. Let X be a Köthe sequence space with the Semi-Fatou property. If the norm of X is not absolutely continuous, $\Gamma_X(t) = t$.

Proof. Suppose that the norm of X is not absolutely continuous. Then there exist $\varepsilon_0 > 0$ and $x_0 \in S(X)$ such that

$$\lim_{n \rightarrow \infty} \left\| \sum_{i=n+1}^{\infty} x_0(i) e_i \right\| = \varepsilon_0, \quad (12)$$

where $e_i = (0, 0, \dots, \overset{\text{ith}}{1}, 0, \dots)$.

Take any $\varepsilon > 0$. By $\lim_{n \rightarrow \infty} \left\| \sum_{i=n+1}^{\infty} x_0(i) e_i \right\| = \varepsilon_0$, there exists $n_1 \in \mathbb{N}$ such that

$$\left\| \sum_{i=n_1}^{\infty} x_0(i) e_i \right\| \leq (1 + \varepsilon) \varepsilon_0. \quad (13)$$

Notice that

$$\lim_{m \rightarrow \infty} \left\| \sum_{i=n_1+1}^m x_0(i) e_i \right\| = \left\| \sum_{i=n_1+1}^{\infty} x_0(i) e_i \right\|, \quad (14)$$

so there exists $n_2 > n_1$ such that

$$(1 - \varepsilon) \varepsilon_0 \leq \left\| \sum_{i=n_1+1}^{n_2} x_0(i) e_i \right\| \leq (1 + \varepsilon) \varepsilon_0. \quad (15)$$

In this way, we get a sequence $\{n_i\}$ of natural numbers such that

$$(1 - \varepsilon) \varepsilon_0 \leq \left\| \sum_{j=n_i+1}^{n_{i+1}} x_0(j) e_j \right\| \leq (1 + \varepsilon) \varepsilon_0, \quad i = 1, 2, \dots \quad (16)$$

Put $x_i = \sum_{j=n_i+1}^{n_{i+1}} x_0(j) e_j$ and $v_k = \sum_{j=n_k+1}^{\infty} x_0(j) e_j$. Then

(a) $\|x_i\| \rightarrow \varepsilon_0$ as $i \rightarrow \infty$;

(b) $x_i \xrightarrow{w} 0$ as $i \rightarrow \infty$. It is well known that, for any Köthe space X , we have

$$X^* = X' \oplus S, \quad (17)$$

where S is the space of all singular functionals over X . $X' = \{y \in l^0 : \text{the series } \sum_{i=1}^{\infty} x(i) y(i) \text{ converges for all } x \in X\}$. This means that every $f \in X^*$ is uniquely represented in the form

$$f = T_y + \varphi, \quad (18)$$

where $\varphi \in S$ and for $y \in X'$ the function T_y is defined by

$$T_y(x) = \sum_{i=1}^{\infty} x(i) y(i) \quad (19)$$

for all $x \in X$.

Consider that the series $\sum_{i=1}^{\infty} x(i) y(i)$ converges. We have

$$\lim_{n \rightarrow \infty} \sum_{j=1}^{\infty} x_n(j) y(j) = \lim_{i \rightarrow \infty} \sum_{j=n_i+1}^{n_{i+1}} x_i(j) y(j) = 0. \quad (20)$$

(c) Put $z_i = v_i / \|v_i\|$ and $w_k = (x_k - x_{k+1}) / \|x_k - x_{k+1}\|$ for all $i, k \in \mathbb{N}$. Then for $k > i$ the following holds:

$$\begin{aligned} & \frac{1}{2} (\|z_i + tw_n\| + \|z_i - tw_n\|) \\ &= \frac{1}{2} \left(\left\| \frac{v_i}{\|v_i\|} + t \frac{x_k - x_{k+1}}{\|x_k - x_{k+1}\|} \right\| \right. \\ & \quad \left. + \left\| \frac{v_i}{\|v_i\|} - t \frac{x_k - x_{k+1}}{\|x_k - x_{k+1}\|} \right\| \right) \\ &\geq \frac{1}{2} \left(\left\| \frac{x_k}{\|v_i\|} + t \frac{x_k}{\|x_k - x_{k+1}\|} \right\| \right. \\ & \quad \left. + \left\| \frac{x_{k+1}}{\|v_i\|} + t \frac{x_{k+1}}{\|x_k - x_{k+1}\|} \right\| \right) \geq (1 - \varepsilon) \varepsilon_0 \left(\frac{1}{\|v_i\|} \right. \\ & \quad \left. + t \frac{1}{\|x_k - x_{k+1}\|} \right) \geq \frac{1 - \varepsilon}{1 + \varepsilon} (1 + t). \end{aligned} \quad (21)$$

By the arbitrariness of ε , we get $\Gamma_X(t) \geq t$. It is clear that $\Gamma_X(t) \leq t$. Therefore $\Gamma_X(t) = t$. $\square$

Theorem 9 also can be derived from the result of Hudzik and Mastyło [16].

Corollary 10. If $\Phi \notin \delta_2$, $\Gamma_X(t) = t$ for any $t > 0$.

Proof. Since $\Phi \notin \delta_2$, we have that the norm of $l_{(\Phi)}$ is not absolutely continuous. Hence $\Gamma_X(t) = t$ for any $t > 0$. $\square$

Theorem 11. Let X be a reflexive Köthe sequence space and the norm of X be absolutely continuous. Then

$$\Gamma_X(t) := \sup \left\{ \inf_{n \geq 1} (\|x_1 + tx_n\| - 1) \right\}, \quad (22)$$

where the supremum is taken over all basic sequences (x_n) in $S(X)$.

Proof. It is clear that if $\{x_n\}$ is a basic sequence in $B(X)$, $(x_n / \|x_n\|)$ is also a basic sequence in $S(X)$. We only prove that $\liminf_{n \rightarrow \infty} (\|x_1 + tx_n\| - \|x_1 - tx_n\|) = 0$. For any $\varepsilon > 0$, by the norm of X which is absolutely continuous, there is a $i_0 \in \mathbb{N}$ such that

$$\left\| \sum_{i=i_0+1}^{\infty} x_1(i) e_i \right\| < \varepsilon, \quad (23)$$

and since $\{x_n\}$ is a basic sequence, we have that the sequence $\{x_n\}$ converge to 0 in coordinate, and there exists a $n_1 \in \mathbb{N}$ such that

$$\left\| \sum_{i=1}^{i_0} x_n(i) e_i \right\| < \varepsilon \quad (24)$$

whenever $n > n_1$. So

$$\begin{aligned} & \left\| \|x_1 + tx_n\| - \|x_1 - tx_n\| \right\| \\ & \leq \left\| \sum_{i=1}^{i_0} x_1(i) e_i + t \sum_{i=i_0+1}^{\infty} x_n(i) e_i \right\| \\ & - \left\| \sum_{i=1}^{i_0} x_1(i) e_i - t \sum_{i=i_0+1}^{\infty} x_n(i) e_i \right\| + 2\varepsilon(1+t). \end{aligned} \quad (25)$$

Noting that X is a Köthe space, we have

$$\begin{aligned} & \left\| \sum_{i=1}^{i_0} x_1(i) e_i + t \sum_{i=i_0+1}^{\infty} x_n(i) e_i \right\| \\ & = \left\| \sum_{i=1}^{i_0} x_1(i) e_i - t \sum_{i=i_0+1}^{\infty} x_n(i) e_i \right\|. \end{aligned} \quad (26)$$

Therefore

$$\left\| \|x_1 + tx_n\| - \|x_1 - tx_n\| \right\| \leq 2\varepsilon(1+t) \quad (27)$$

whenever $n > n_1$, that is, $\liminf_{n \rightarrow \infty} (\|x_1 + tx_n\| - \|x_1 - tx_n\|) = 0$. $\square$

In order to get the criterion that an Orlicz sequence space $l_{(\Phi)}$ is nearly uniformly smooth, we may assume that $\Phi \in \bar{\delta}_2$ and $\Psi \in \delta_2$ and that the space $l_{(\Phi)}$ is reflexive. Put $N(x) = \{i \in N : x(i) \neq 0\}$. For any $x \in S(l_{(\Phi)})$ with finite $N(x)$ and $t > 0$, we define $c_{x,n}(t)$ as follows:

$$\begin{aligned} c_{x,n}(t) &:= \sup \left\{ c_{x,y,t} > 0 : I_{\Phi} \left(\frac{x}{c_{x,y,t}} \right) + I_{\Phi} \left(\frac{ty}{c_{x,y,t}} \right) \right. \\ & \left. = 1 \text{ for } y \in S(l_{(\Phi)}) \text{ with } n \leq \min N(y) < \infty \right\}. \end{aligned} \quad (28)$$

It is clear that $\{c_{x,n}(t)\}_{n=1}^{\infty}$ is decreasing. Hence $c_x(t) = \lim_{n \rightarrow \infty} c_{x,n}(t)$.

Theorem 12. *Supposed that $\Phi \in \delta_2$ and $\Psi \in \delta_2$. For an Orlicz sequence space $l_{(\Phi)}$, there holds*

$$\begin{aligned} \Gamma_{l_{(\Phi)}}(t) &:= \sup \{c_x(t) : x \in S(l_{(\Phi)}) \text{ with finite } N(x)\} \\ &= 1. \end{aligned} \quad (29)$$

Proof. Let

$$d_{\Phi} := \sup \{c_x(t) : x \in S(l_{(\Phi)}) \text{ with finite } N(x)\}. \quad (30)$$

Then for any $\varepsilon \in (0, d_{\Phi})$, there exists $\|x\|_{(\Phi)} = 1$ with finite $N(x)$ such that $c_x > d_{\Phi} - \varepsilon$. By the definition of $c_x(t)$, there exists $n_1 \in N$ such that

$$\begin{aligned} & \sup \left\{ c_{x,y,t} > 0 : I_{\Phi} \left(\frac{x}{c_{x,y,t}} \right) + I_{\Phi} \left(\frac{ty}{c_{x,y,t}} \right) = 1 \text{ for } y \right. \\ & \left. \in S(l_{(\Phi)}) \text{ and } \min N(y) \geq n_1 \right\} > d_{\Phi} - \varepsilon \end{aligned} \quad (31)$$

whenever $n \geq n_1$. By the definition of $c_x(t)$, there exists $y_1 \in S(l_{(\Phi)})$ with $\min N(y_1) > n_1$ such that $c_{x,y_1,t} > d_{\Phi} - \varepsilon$, that is to say,

$$I_{\Phi} \left(\frac{x}{d_{\Phi} - \varepsilon} \right) + I_{\Phi} \left(\frac{ty_1}{d_{\Phi} - \varepsilon} \right) > 1. \quad (32)$$

Hence there exists $n_2 > n_1$ such that

$$I_{\Phi} \left(\frac{x}{d_{\Phi} - \varepsilon} \right) + \sum_{i=n_1+1}^{n_2} \Phi_i \left(\frac{ty_1(i)}{d_{\Phi} - \varepsilon} \right) > 1. \quad (33)$$

Since $n_2 > n_1$, we also have

$$\begin{aligned} & \sup \left\{ c_{x,y} > 0 : I_{\Phi} \left(\frac{x}{c_{x,y}} \right) + I_{\Phi} \left(\frac{ty}{c_{x,y}} \right) \right. \\ & \left. = 1 \text{ and } \min N(y) \geq n_2 \right\} > d_{\Phi} - \varepsilon, \end{aligned} \quad (34)$$

and there exists $y_2 \in l_{(\Phi)}$ with $\min N(y_2) > n_1$ such that $c_{x,y_2} > d_{\Phi} - \varepsilon$, that is to say,

$$I_{\Phi} \left(\frac{x}{d_{\Phi} - \varepsilon} \right) + I_{\Phi} \left(\frac{ty_2}{d_{\Phi} - \varepsilon} \right) > 1. \quad (35)$$

Hence there also exists $y_2 \in S(l_{(\Phi)})$ with $\min N(y_2) > \max N(y_1)$ such that

$$I_{\Phi} \left(\frac{x}{d_{\Phi} - \varepsilon} \right) + \sum_{i=n_2+1}^{n_3} \Phi_i \left(\frac{ty_2(i)}{d_{\Phi} - \varepsilon} \right) > 1. \quad (36)$$

In such a way, there exist a sequence $\{y_k\}_{k=1}^{\infty} \subset S(l_{(\Phi)})$ and a sequence of natural numbers $n_1 < n_2 < n_3 < \dots$ such that

$$I_{\Phi} \left(\frac{x}{d_{\Phi} - \varepsilon} \right) + \sum_{i=n_k+1}^{n_{k+1}} \Phi \left(\frac{ty_k(i)}{d_{\Phi} - \varepsilon} \right) > 1 \quad (37)$$

for all $k \in N$. It is clear that $y_k \xrightarrow{w} 0$.

For any $k > i_0$, we have

$$\begin{aligned} I_{\Phi} \left(\frac{x + ty_k}{d_{\Phi} - \varepsilon} \right) &= \sum_{i=n_k+1}^{n_{k+1}} \Phi_i \left(\frac{ty_k(i)}{d_{\Phi} - \varepsilon} \right) + I_{\Phi} \left(\frac{x(i)}{d_{\Phi} - \varepsilon} \right) \\ &> 1, \end{aligned} \quad (38)$$

that is to say,

$$\|x + ty_k\|_{(\Phi)} > d_{\Phi} - \varepsilon. \quad (39)$$

Therefore $\Gamma_{l_{(\Phi)}}(t) \geq d_{\Phi} - \varepsilon - 1$ and by the arbitrariness of ε , we have $\Gamma_{l_{(\Phi)}}(t) \geq d_{\Phi} - 1$.

Now we prove that $\Gamma_{l_{(\Phi)}}(t) \leq d_{\Phi} - 1$. By the definition of d_{Φ} , we always have

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \left\{ c_{x,y,t} > 0 : I_{\Phi} \left(\frac{x}{c_{x,y,t}} \right) + I_{\Phi} \left(\frac{ty}{c_{x,y,t}} \right) \right. \\ & \left. = 1 \text{ for } y \in S(l_{(\Phi)}) \text{ and } \min N(y) \geq n \right\} \leq d_{\Phi} \end{aligned} \quad (40)$$

for any $x \in S(l_{(\Phi)})$ with finite $N(x)$.

Taking any weak null sequence $\{x_n\} \subset S(l_{(\Phi)})$ and $\varepsilon > 0$, by $\Phi \in \delta_2$ there exists $i_0 > 0$ such that $\|\sum_{i=i_0+1}^{\infty} x_1(i)e_i\|_{(\Phi)} < \varepsilon$. Since $x_n \xrightarrow{w} 0$, there is a $n_0 \in N$ such that $\|\sum_{i=1}^{i_0} x_n(i)e_i\|_{(\Phi)} < \varepsilon$ when $n \geq n_0$. Hence

$$\begin{aligned} \|x_1 + tx_n\|_{(\Phi)} &\leq \left\| \sum_{i=1}^{i_0} (x_1(i) + tx_n(i)) e_i \right. \\ &\quad \left. + \sum_{i=i_0+1}^{\infty} (x_1(i) + tx_n(i)) e_i \right\|_{(\Phi)} \leq \left\| \sum_{i=1}^{i_0} x_1(i) e_i \right\|_{(\Phi)} \\ &\quad + \left\| \sum_{i=i_0+1}^{\infty} tx_n(i) e_i \right\|_{(\Phi)} + (1+t)\varepsilon. \end{aligned} \quad (41)$$

We next estimate the $\|\sum_{i=1}^{i_0} x_1(i)e_i + \sum_{i=i_0+1}^{\infty} tx_n(i)e_i\|_{(\Phi)}$, when $n > n_0$. Put $z_1 = \sum_{i=1}^{i_0} x_1(i)e_i / \|\sum_{i=1}^{i_0} x_1(i)e_i\|_{(\Phi)}$ and $z_n = \sum_{i=i_0+1}^{\infty} x_n(i)e_i / \|\sum_{i=i_0+1}^{\infty} x_n(i)e_i\|_{(\Phi)}$ for $n > n_0$. There exists $n_1 \geq n_0$ such that if $I_{\Phi}(z_1/c_{z_1, z_n, t}) + I_{\Phi}(tz_n/c_{z_1, z_n, t}) = 1$, then $c_{z_1, z_n, t} \leq d_{\Phi} + \varepsilon$ for $n \geq n_1$.

$$\begin{aligned} &I_{\Phi} \left(\frac{\sum_{i=1}^{i_0} z_1(i) e_i + \sum_{i=i_0+1}^{\infty} tz_n(i) e_i}{d_{\Phi} + \varepsilon} \right) \\ &= I_{\Phi} \left(\frac{z_1}{d_{\Phi} + \varepsilon} \right) + I_{\Phi} \left(\frac{tz_n}{d_{\Phi} + \varepsilon} \right) \\ &\leq I_{\Phi} \left(\frac{z_1}{c_{z_1, z_n, t}} \right) + I_{\Phi} \left(\frac{tz_n}{c_{z_1, z_n, t}} \right) = 1, \end{aligned} \quad (42)$$

that is,

$$\left\| \sum_{i=1}^{i_0} z_1(i) e_i + \sum_{i=i_0+1}^{\infty} tz_n(i) e_i \right\|_{(\Phi)} \leq d_{\Phi} + \varepsilon \quad (43)$$

when $n \geq n_0$.

When $n \geq n_1$. It follows that

$$\begin{aligned} &\left\| \sum_{i=1}^{i_0} x(i) e_i - \sum_{i=i_0+1}^{\infty} x_n(i) e_i \right\|_{(\Phi)} \\ &\leq \left\| \sum_{i=1}^{i_0} x_1(i) e_i \right\|_{(\Phi)} + \left\| \sum_{i=1}^{i_0} z_1(i) e_i \right\|_{(\Phi)} \\ &\quad + \left\| \sum_{i=i_0+1}^{\infty} x_n(i) e_i \right\|_{(\Phi)} + \left\| \sum_{i=i_0+1}^{\infty} tz_n(i) e_i \right\|_{(\Phi)} \leq \left\| \sum_{i=1}^{i_0} z_1(i) e_i \right\|_{(\Phi)} \\ &\quad + \left\| \sum_{i=i_0+1}^{\infty} tz_n(i) e_i \right\|_{(\Phi)} \leq d_{\Phi} + \varepsilon. \end{aligned} \quad (44)$$

By the arbitrariness of ε , we get the inequality $\Gamma_{l_{(\Phi)}}(t) \leq d_{\Phi} - 1$. Summing up, we obtain the equality $\Gamma_{l_{(\Phi)}}(t) = d_{\Phi} - 1$. $\square$

Lemma 13. Let $\Phi \in \bar{\delta}_2$. Then for any $\varepsilon > 0$ and $u_0 > 0$, there exists $\delta > 0$ such that

$$\Phi(tu) \leq t\varepsilon\Phi(u) \quad (45)$$

whenever $|u| \leq \Phi^{-1}(1)$ and $t \in (0, \delta)$.

Proof. By $\Phi \in \bar{\delta}_2$, for any $u_0 > 0$, there exists $\theta \in (0, 1)$ such that

$$\Phi\left(\frac{u}{2}\right) \leq \frac{\theta}{2}\Phi(u) \quad (46)$$

whenever $|u| \leq \Phi^{-1}(1)$.

For any $\varepsilon > 0$, there is a $n \in N$ such that $\theta^n < \varepsilon$. Put $\delta = 1/2^n$. Then

$$\Phi(\delta u) = \Phi\left(\frac{u}{2^n}\right) \leq \frac{\theta^n}{2^n}\Phi(u) \leq \delta\varepsilon\Phi(u) \quad (47)$$

whenever $|u| \leq \Phi^{-1}(1)$. Hence

$$\begin{aligned} \Phi(tu) &= \Phi\left(\frac{t}{\delta}\delta u\right) \leq \frac{t}{\delta}\Phi(\delta u) \leq \frac{t}{\delta}\delta\varepsilon\Phi(u) \\ &= t\varepsilon\Phi(u) \end{aligned} \quad (48)$$

whenever $t \in (0, \delta)$ and $|u| \leq \Phi^{-1}(1)$. $\square$

Corollary 14. An Orlicz sequence space $l_{(\Phi)}$ is nearly uniformly smooth if and only if $\Phi \in \delta_2$ and $\Phi \in \bar{\delta}_2$.

Proof. Since $\Phi \in \delta_2$ and $\Phi \in \bar{\delta}_2$ imply that the space $l_{(\Phi)}$ is reflexive, we only need to prove the sufficiency. For any $x \in S(l_{(\Phi)})$ with finite $N(x)$ that is $\max\{i : i \in N(x)\} < n$, and $y \in S(l_{(\Phi)})$ with $\min N(y) \geq n$. For any $\varepsilon > 0$, by Lemma 13, there exists $\delta > 0$ such that $\Phi(tu) \leq t\varepsilon\Phi(u)$ when $|u| \leq \Phi^{-1}(1)$ and $t \in (0, \delta)$. Hence if $c_{x, y, t} \geq 1$ and $t \in (0, \delta)$

$$1 = I_{\Phi} \left(\frac{x}{c_{x, y, t}} \right) + I_{\Phi} \left(\frac{ty}{c_{x, y, t}} \right) \quad (49)$$

$$\leq \frac{1}{c_{x, y, t}} I_{\Phi}(x) + \frac{t\varepsilon}{c_{x, y, t}} I_{\Phi}(y) = \frac{1}{c_{x, y, t}} (1 + t\varepsilon).$$

This shows that $c_{x, y, t} \leq 1 + t\varepsilon$, that is, $\Gamma_{l_{(\Phi)}} \leq t\varepsilon$ when $t \in (0, \delta)$. Therefore $\Gamma'_{l_{(\Phi)}}(0) = 0$. $\square$

Corollary 15. Let $l_p(1 < p < \infty)$ be a Lebesgue sequence space. Then $\Gamma_{l_p}(t) = (1 + t^p)^{1/p} - 1$.

Proof. It is well known that the Lebesgue sequence space $l_p(1 < p < \infty)$ is an Orlicz sequence space generated by $\Phi(u) = |u|^p$. For any $x, y \in S(l_p)$, we now consider the following:

$$I_{\Phi} \left(\frac{x}{c_{x, y, t}} \right) + I_{\Phi} \left(\frac{ty}{c_{x, y, t}} \right) = 1. \quad (50)$$

That is,

$$\sum_{i=1}^{\infty} \left| \frac{x(i)}{c_{x,y,t}} \right|^p + \sum_{i=1}^{\infty} \left| \frac{ty(i)}{c_{x,y,t}} \right|^p = 1. \quad (51)$$

Hence

$$\frac{1}{c_{x,y,t}^p} \sum_{i=1}^{\infty} |x(i)|^p + \frac{t^p}{c_{x,y,t}^p} \sum_{i=1}^{\infty} |y(i)|^p = \frac{1+t^p}{c_{x,y,t}^p} = 1, \quad (52)$$

that is to say, $c_{x,y,t} = (1+t^p)^{1/p}$. So $\Gamma_{l_{(\Phi)}}(t) = (1+t^p)^{1/p} - 1$ for $t > 0$. $\square$

It is also well known that if Φ is an Orlicz function, $\lim_{u \rightarrow 0} (\Phi(u)/u) = a$. If $a > 0$, $l_{(\Phi)}$ has Schur property [17].

Theorem 16. Suppose that $\Phi \in \delta_2$ and $a \in (0, 1)$. $RW(a, l_{(\Phi)}) < 1 + a$ if and only if exist $\delta > 0$ and $u_0 > 0$ such that $\Phi(u/(1+a)) \leq ((1-\delta)/(1+a))\Phi(u)$ or $\Phi(u/a) \geq ((1+\delta)/a)\Phi(u)$ when $|u| \leq u_0$.

Proof.

Necessity. If there exists a sequence $u_n \searrow 0$ such that

$$\Phi\left(\frac{u_n}{1+a}\right) \geq \frac{1-1/n}{1+a}\Phi(u_n) \quad (53)$$

and

$$\Phi\left(\frac{u_n}{a}\right) \geq \frac{1+1/n}{a}\Phi(u_n) \quad (54)$$

for each $n \in \mathbb{N}$, take $l_n \in \mathbb{N}$ such that

$$l_n \Phi\left(\frac{u_n}{1+a}\right) \leq 1 < (l_n + 1) \Phi\left(\frac{u_n}{1+a}\right). \quad (55)$$

Putting $x_n = (\overbrace{u_n, u_n, \dots, u_n}^{l_n}, 0, \dots)$, then $I_{\Phi}(x_n/(1+a)) \rightarrow 1$. So $\|x_n\|_{(\Phi)} \rightarrow 1+a$ as $n \rightarrow \infty$.

$$\begin{aligned} \|x_n\|_{(\Phi)} &\leq I_{\Phi}(x_n) = l_n \Phi(u_n) \\ &\leq l_n \left(\frac{1+a}{1-1/n} \Phi\left(\frac{u_n}{1+a}\right) \right) \leq \frac{1+a}{1-1/n}. \end{aligned} \quad (56)$$

Therefore

$$\lim_{n \rightarrow \infty} I_{\Phi}(x_n) = 1+a. \quad (57)$$

Take $k_n \in \mathbb{N}$ such that

$$k_n \Phi(u_n) \leq 1 < (k_n + 1) \Phi(u_n). \quad (58)$$

It is clear that $k_n < l_n$ for n large enough. Let $y_1^{(n)} = (\overbrace{u_n, u_n, \dots, u_n}^{l_n - k_n}, 0, \dots)$,

$$\begin{aligned} y_2^{(n)} &= \left(\overbrace{0, 0, \dots, 0}^{l_n}, \overbrace{u_n, u_n, \dots, u_n}^{k_n}, 0, \dots \right), \dots, \\ &\dots \end{aligned} \quad (59)$$

$$y_m^{(n)} = \left(\overbrace{0, 0, \dots, 0}^{l_n + (m-2)k_n}, \overbrace{u_n, u_n, \dots, u_n}^{k_n}, 0, \dots \right), \dots$$

Then $I_{\Phi}(y_m^{(n)}) \rightarrow 1$ and $I_{\Phi}(y_1^{(n)}) = I_{\Phi}(x_n) - I_{\Phi}(y_m^{(n)}) \rightarrow a$ as $n \rightarrow \infty$.

Since

$$\begin{aligned} I_{\Phi}\left(\frac{y_1^{(n)}}{a}\right) &= (l_n - k_n) \Phi\left(\frac{u_n}{a}\right) \\ &\geq \frac{1+1/n}{a} (l_n - k_n) \Phi(u_n) \\ &= \frac{1+1/n}{a} I_{\Phi}(y_1^{(n)}) \rightarrow 1, \end{aligned} \quad (60)$$

that is, $\|y_1^{(n)}\|_{(\Phi)} \rightarrow a$.

By

$$\begin{aligned} \limsup_{\lambda \rightarrow 0} \left\{ \frac{I_{\Phi}(\lambda y_m^{(n)})}{\lambda} : m = 1, 2, \dots \right\} &= \lim_{\lambda \rightarrow 0} \frac{I_{\Phi}(\lambda x_n)}{\lambda} \\ &= 0, \end{aligned} \quad (61)$$

we have $y_m^{(n)} \xrightarrow{w} 0$ as $m \rightarrow \infty$.

Therefore

$$\begin{aligned} RW(a, l_{(\Phi)}) &\geq \sup \left\{ \liminf_{m \rightarrow \infty} \|y_1^{(n)} + y_m^{(n)}\|_{(\Phi)} : n = 1, 2, \dots \right\} \\ &= 1+a, \end{aligned} \quad (62)$$

that is to say $RW(a, l_{(\Phi)}) = 1+a$.

Sufficiency. Suppose that there exist $\delta > 0$ and $u_0 > 0$ such that $\Phi(u/(1+a)) \leq ((1-\delta)/(1+a))\Phi(u)$ or $\Phi(u/a) \geq ((1+\delta)/a)\Phi(u)$ when $|u| \leq u_0$. That means there exist $\delta > 0$ and $u_0 > 0$ such that $\Phi(u) \leq ((1-\delta)/(1+a))\Phi((1+a)u)$ or $\Phi(u) \leq (a/(1+\delta))\Phi(u/a)$ when $|u| \leq u_0$. It is well known that if Φ is an Orlicz function, $\lim_{u \rightarrow 0} (\Phi(u)/u) = b$. If $b > 0$, $l_{(\Phi)}$ has Schur property. That is to say, the norm convergence of sequence in $l_{(\Phi)}$ coincides with the weak convergence. In order to discuss the coefficient $RW(a, l_{(\Phi)})$ we may assume $b = 0$. Put $f(u) = (1+a)\Phi(u)/\Phi((1+a)u)$. By $b = 0$, we have $f(u) < 1$. Since the function f is continuous in $[u_0, \Phi^{-1}(1)]$, there exists a $u_1 \in [u_0, \Phi^{-1}(1)]$ such that $f(u) \leq f(u_1)$ for each $u \in [u_0, \Phi^{-1}(1)]$. Suppose $f(u_1) = 1$. Then $(1+a)\Phi(u_1)/\Phi((1+a)u_1) = 1$ and $\Phi(u)$ is affine function in $[0, v_1]$, where $v_1 = (1+a)u_1$. This contradicts with $b = 0$. Therefore $(1+a)\Phi(u)/\Phi((1+a)u) = f(u_1)$. This implies $\Phi(u) \leq (f(u_1)/(1+a))\Phi((1+a)u)$ when $u \in [u_0, \Phi^{-1}(1)]$. It is shown that there exist $\delta > 0$ and $u_0 > 0$ such that $\Phi(u) \leq ((1-\delta)/(1+a))\Phi((1+a)u)$ or $\Phi(u) \leq (a/(1+\delta))\Phi(u/a)$ when $|u| \leq \Phi^{-1}(1)$. Using $\Phi \in \delta_2$ there exists a $\beta > 0$ such that $I_{\Phi}(x) \geq \beta$ when $\|x\|_{(\Phi)} = a$.

Put

$$B_a = \left\{ u \in [0, \Phi^{-1}(1)] : \Phi\left(\frac{u}{a}\right) \geq \frac{1+\delta}{a} \Phi(u) \right\} \quad (63)$$

and

$$B_{1+a} = \left\{ u \in [0, \Phi^{-1}(1)] : \Phi\left(\frac{u}{1+a}\right) \geq \frac{1-\delta}{1+a}\Phi(u) \right\}. \quad (64)$$

Let $N_a = \{i : x(i) \in B_a\}$ and $N_{1+a} = \{i : x(i) \in B_{1+a}\}$.

We consider the following two cases.

Case I ($\sum_{i \in N_a} \Phi(x(i)) \geq \beta/2$). In this case, from

$$\begin{aligned} 1 &\geq I_\Phi\left(\frac{x}{a}\right) = \sum_{i \in N_a} \Phi\left(\frac{x(i)}{a}\right) + \sum_{i \in N_{1+a}} \Phi\left(\frac{x(i)}{a}\right) \\ &\geq \frac{1+\delta}{a} \sum_{i \in N_a} \Phi(x(i)) + \frac{1}{a} \sum_{i \in N_{1+a}} \Phi(x(i)) \\ &= \frac{1}{a} I_\Phi(x) + \frac{\delta\beta}{2a}, \end{aligned} \quad (65)$$

we get

$$I_\Phi(x) \leq a - \frac{\delta\beta}{2a}. \quad (66)$$

For any $y \in B(l_\Phi)$ with $\text{supp}(y) = \{i \in N : y(i) \neq 0\} \cap \text{supp}(x) = \emptyset$, the following inequality holds:

$$I_\Phi(x+y) = I_\Phi(x) + I_\Phi(y) \leq a - \frac{\delta\beta}{2a} + 1. \quad (67)$$

Therefore $\|x+y\|_{(\Phi)} \leq I_\Phi(x+y) \leq a - \delta\beta/2a + 1$, that is to say $RW(a, l_\Phi) \leq a - \delta\beta/2a + 1 < 1+a$.

Case II ($\sum_{i \in N_{1+a}} \Phi(x(i)) \geq \beta/2$). In this case, for any $y \in B(l_\Phi)$ with $\text{supp}(y) = \{i \in N : y(i) \neq 0\} \cap \text{supp}(x) = \emptyset$, the following inequality holds:

$$\begin{aligned} I_\Phi\left(\frac{x+y}{1+a}\right) &= I_\Phi\left(\frac{x}{1+a}\right) + I_\Phi\left(\frac{y}{1+a}\right) \\ &\leq \sum_{i \in N_a} \Phi\left(\frac{x(i)}{1+a}\right) + \sum_{i \in N_{1+a}} \Phi\left(\frac{x(i)}{1+a}\right) \\ &\quad + \frac{1}{1+a} I_\Phi(y) \\ &\leq \frac{1}{1+a} \sum_{i \in N_a} \Phi(x(i)) \\ &\quad + \frac{1-\delta}{1+a} \sum_{i \in N_{1+a}} \Phi(x(i)) + \frac{1}{1+a} \\ &= \frac{1}{1+a} I_\Phi(x) + \frac{1}{1+a} - \frac{\delta\beta}{2(1+a)}. \end{aligned} \quad (68)$$

From $I_\Phi(x) \leq \|x\|_{(\Phi)} = a \leq 1$, we have

$$I_\Phi\left(\frac{x+y}{1+a}\right) \leq 1 - \frac{\delta\beta}{2(1+a)}, \quad (69)$$

that is to say

$$\|x+y\|_{(\Phi)} \leq \frac{2+a-\delta\beta}{2}. \quad (70)$$

Therefore $RW(a, l_\Phi) \leq (2+a-\delta\beta)/2 < 1+a$. $\square$

Theorem 17. For $a \in (0, 1)$, $R(a, l_\Phi) < 1+a$ if and only if $\Phi \in \delta_2$.

Proof. It is easy to get the following formula:

$$\begin{aligned} R(a, X) &= \sup \left\{ \liminf_{n \rightarrow \infty} \|x_n + x\|_{(\Phi)} : x_n \xrightarrow{w} 0, \{x_n, x\} \right. \\ &\quad \left. \subset B(X), \|x_n - x_m\|_{(\Phi)} \leq 1 \right\}. \end{aligned} \quad (71)$$

Take any weak null sequence $\{x_n\} \subset B(l_\Phi)$ with $\|x_n - x_m\|_{(\Phi)} \leq 1$. Hence $I_\Phi(x_n - x_m) \leq 1$. Using $x_n \xrightarrow{w} 0$, for any $\varepsilon \in (0, 1/2)$ and subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $I_\Phi(x_{n_i}) \leq 1/2 + \varepsilon$. By $\Phi \in \delta_2$, there exists $\delta \in (0, 1)$ such that $\|x_{n_i}\|_{(\Phi)} \leq \delta$. Therefore for any $x \in l_\Phi$, $\|x\|_{(\Phi)} = a$. It holds

$$\|x_{n_i} + x\|_{(\Phi)} \leq \|x_{n_i}\|_{(\Phi)} + \|x\|_{(\Phi)} \leq \delta + a < 1+a. \quad (72)$$

$\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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