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Research Article

On (p, q)-Analogue of Gamma Operators

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In this paper, a kind of new analogue of Gamma type operators based on (p, q)-integers is introduced. The Voronovskaja type asymptotic formula of these operators is investigated. And some other results of these operators are studied by means of modulus of continuity and Peetre K−functional. Finally, some direct theorems concerned with the rate of convergence and the weighted approximation for these operators are also obtained.

1. Introduction

In recent years, with the rapid development of q-calculus, the study of approximation theory with q-integer has been discussed widely. Afterwards, with the generalization from q-calculus to (p, q)-calculus, it has been used efficiently in many areas of sciences such as algebras [1, 2] and CAGD [3]. And, recently, approximation by sequences of linear positive operators has been transferred to operators with (p, q)-integer. Some useful notations and definitions about q-calculus and (p, q)-calculus in this paper are reviewed in [4–6].

Let $0 < q < p \leq 1$. For each nonnegative integer n, the (p, q)-integer $[n]_{p,q}$ and (p, q)-factorial $[n]_{p,q}!$ are defined as

$$[n]_{p,q} = \frac{p^n - q^n}{p - q} \quad n = 1, 2, \ldots \tag{1}$$

and

$$[n]_{p,q}! = \begin{cases} [1]_{p,q} [2]_{p,q} \cdots [n]_{p,q}, & n \geq 1 \\ 1, & n = 0 \end{cases} \tag{2}$$

Further, the (p, q)−power basis is defined as

$$\begin{aligned}(x \oplus y)^n_{p,q} &= (x + y)(px + qy)(p^2x + q^2y) \cdots (p^{n-1}x + q^{n-1}y).\end{aligned} \tag{3}$$

And

$$\begin{aligned}(x \ominus y)^n_{p,q} &= (x - y)(px - qy)(p^2x - q^2y) \cdots (p^{n-1}x - q^{n-1}y).\end{aligned} \tag{4}$$

Let n be a nonnegative integer; the (p, q)-Gamma function is defined as

$$\Gamma_{p,q}(n + 1) = \frac{(p \ominus q)^n_{p,q}}{(p - q)^n} = [n]_{p,q}!, \quad 0 < q < p \leq 1. \tag{5}$$

Aral and Gupta [7] proposed (p, q)-Beta function of second kind for $m, n \in \mathbb{N}$ as

$$B_{p,q}(m, n) = \int_0^\infty \frac{x^{m-1}}{(1 \oplus px)^{m+n}_{p,q}} d_{p,q}x. \tag{6}$$

And the relationship between (p, q)-analogues of Beta and Gamma functions is as follows:

$$B_{p,q}(m, n) = \frac{q\Gamma_{p,q}(m)\Gamma_{p,q}(n)}{(p^{m+1}q^{m-1})^{m/2}\Gamma_{p,q}(m + n)}. \tag{7}$$

Particularly, when $p = q = 1$, $B(m, n) = \Gamma(m)\Gamma(n)/\Gamma(m + n)$. It may be observed that, in (p, q)-setting, order is important, which is the reason why (p, q)-variant of Beta function does not satisfy commutativity property; that is, $B_{p,q}(m,n) \neq B_{p,q}(n,m)$.

In [8], Mazhar studied some approximation properties of the Gamma operators as follows:

$$\overline{F}_n(f;x) = \frac{(2n)!x^{n+1}}{n!(n-1)!}\int_0^\infty \frac{t^{n-1}}{(x+t)^{2n+1}} f(t)\,\mathrm{d}t, \quad n>1,\ x>0. \tag{8}$$

Recently, Mursaleen first applied (p,q)-calculus in approximation theory and introduced the (p,q)-analogue of Bernstein operators [9], (p,q)-Bernstein-Stancu operators [10], and (p,q)-Bernstein-Schurer operators [11] and investigated their approximation properties. And many well-known approximation operators with (p,q)-integer have been introduced, such as (p,q)-Bernstein-Stancu-Schurer-Kantorovich operators [12], (p,q)-Szász-Baskakov operators [13], and (p,q)-Baskakov-Beta operators [14]. As we know, many researchers have studied approximation properties of the Gamma operators and their modifications (see [15–21], etc.). All this achievement motivates us to construct the (p,q)-analogue of the Gamma operators (8). First, we introduce (p,q)-analogue of Gamma operators as follows.

Definition 1. For $n\in\mathbb{N}$, $n>1$, $x\in(0,\infty)$, and $0<q<p\le 1$, the (p,q)-Gamma operators can be defined as

$$F_n^{p,q}(f;x) = \frac{x^{n+1}p^{n^2}q^{n^2+n}}{B_{p,q}(n,n+1)}\int_0^\infty \frac{t^{n-1}}{(p^nq^nx\oplus t)_{p,q}^{2n+1}} f(t)\,\mathrm{d}_{p,q}t \tag{9}$$

The paper is organized as follows. In the first section, we give the basic notations and the definition of (p,q)-Gamma operators. In the second section, we present the moments of the operators. In the third section, we obtain Voronovskaja type asymptotic formula. In the fourth section, we present a direct result of (p,q)-Gamma operators in terms of first- and second-order modulus of continuity. In the last section, we study the rate of convergence and the weighted approximation of the (p,q)-Gamma operators.

2. Auxiliary Results

In order to obtain the approximation properties of the operators $F_n^{p,q}(f;x)$, we need the following lemma and remarks.

Lemma 2. *The following equalities hold:*

(1) $F_n^{p,q}(1;x) = 1$.

(2) $F_n^{p,q}(t;x) = x$, *for* $n>1$.

(3) $F_n^{p,q}(t^2;x) = [n+1]_{p,q}x^2/pq[n-1]_{p,q}$, *for* $n>2$.

(4) $F_n^{p,q}(t^3;x) = [n+1]_{p,q}[n+2]_{p,q}x^3/(pq)^3[n-1]_{p,q}[n-2]_{p,q}$, *for* $n>3$.

(5) $F_n^{p,q}(t^4;x) = [n+1]_{p,q}[n+2]_{p,q}[n+3]_{p,q}x^4/(pq)^6[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}$, *for* $n>4$.

Proof. According to the properties of (p,q)-Beta function and (p,q)-Gamma function, we have

$$\begin{aligned}
F_n(t^k;x) &= \frac{x^{n+1}p^{n^2}q^{n^2+n}}{B_{p,q}(n,n+1)}\int_0^\infty \frac{t^{n+k-1}}{(p^nq^nx\oplus t)_{p,q}^{2n+1}}\mathrm{d}_{p,q}t \\
&= \frac{x^{n+1}p^{n^2}q^{n^2+n}}{B_{p,q}(n,n+1)} \\
&\quad\cdot\int_0^\infty \frac{1}{x^{2n+1}p^{n(2n+1)}q^{n(2n+1)}}\frac{t^{n+k-1}}{(1\oplus pt/(xp^{n+1}q^n))_{p,q}^{2n+1}}\mathrm{d}_{p,q}t = \frac{x^{n+1}p^{n^2}q^{n^2+n}}{B_{p,q}(n,n+1)} \\
&\quad\cdot\int_0^\infty \frac{x^{n+k}p^{(n+1)(n+k)}q^{n(n+k)}}{x^{2n+1}p^{n(2n+1)}q^{n(2n+1)}}\frac{(t/(xp^{n+1}q^n))^{n+k-1}}{(1\oplus pt/(xp^{n+1}q^n))_{p,q}^{2n+1}}\mathrm{d}_{p,q}\left(\frac{t}{xp^{n+1}q^n}\right) \\
&= \frac{x^kp^{nk+k}q^{nk}B_{p,q}(n+k,n-k+1)}{B_{p,q}(n,n+1)} \\
&= \frac{x^kp^{nk+k}q^{nk}(q^{n-1}p^{n+1})^{n/2}}{(q^{n+k-1}p^{n+k+1})^{(n+k)/2}} \\
&\quad\cdot\frac{\Gamma_{p,q}(n+k)\Gamma_{p,q}(n-k+1)}{\Gamma_{p,q}(n)\Gamma_{p,q}(n+1)} \\
&= \frac{x^k(pq)^{-k(k-1)/2}[n+k-1]_{p,q}![n-k]_{p,q}!}{[n-1]_{p,q}![n]_{p,q}!}
\end{aligned} \tag{10}$$

This proves Lemma 2. □

Remark 3. Let $n>2$, and $x\in(0,\infty)$; then, for $0<q<p\le 1$, we have the central moments as follows:

(1) $F_n^{p,q}(t-x;x) = 0$.

(2) $A(x) := F_n^{p,q}((t-x)^2;x) = ((q/p-1)+p^{n-2}[2]_{p,q}/q[n-1]_{p,q})x^2$.

Remark 4. The sequences (p_n) and (q_n) satisfy $0<q_n<p_n<1$ such that $p_n\longrightarrow 1$, $q_n\longrightarrow 1$, and $p_n^n\longrightarrow a$, $q_n^n\longrightarrow b$, and $[n]_{p_n,q_n}\longrightarrow\infty$ as $n\longrightarrow\infty$, where $0\le a,b<1$, and $x\in(0,\infty)$; then

(1) $\lim_{n\longrightarrow\infty}[n-1]_{p_n,q_n}F_n^{p_n,q_n}((t-x)^2;x) = (a+b)x^2$.

(2) $\lim_{n\longrightarrow\infty}[n-3]_{p_n,q_n}F_n^{p_n,q_n}((t-x)^4;x) = 0$.

Proof. (1) Using Remark 3,

$$\begin{aligned}
&\lim_{n\longrightarrow\infty}[n-1]_{p_n,q_n}F_n^{p_n,q_n}((t-x)^2;x) \\
&\quad= \lim_{n\longrightarrow\infty}\left(\frac{p_n^{n-2}[2]_{p_n,q_n}}{q_n} - \frac{(p_n-q_n)[n-1]_{p_n,q_n}}{p_n}\right)x^2 \\
&\quad= \lim_{n\longrightarrow\infty}\left(\frac{p_n^{n-2}[2]_{p_n,q_n}}{q_n} - \frac{(p_n^{n-1}-q_n^{n-1})}{p_n}\right)x^2 \\
&\quad= (2a-(a-b))x^2 = (a+b)x^2
\end{aligned} \tag{11}$$

(2) Let $k = n - 3$; we have

$$[n+1]_{p_n,q_n}[n+2]_{p_n,q_n}[n+3]_{p_n,q_n} = \left(q_n^4[k]_{p_n,q_n} + p_n^k[4]_{p_n,q_n}\right)\left(q_n^5[k]_{p_n,q_n} + p_n^k[5]_{p_n,q_n}\right) \cdot \left(q_n^6[k]_{p_n,q_n} + p_n^k[6]_{p_n,q_n}\right) \sim q_n^{15}[k]_{p_n,q_n}^3 + p_n^k\left(q_n^9[6]_{p_n,q_n} + q_n^{10}[5]_{p_n,q_n} + q_n^{11}[4]_{p_n,q_n}\right)[k]_{p_n,q_n}^2. \quad (12)$$

Similarly, we have

$$[n+1]_{p_n,q_n}[n+2]_{p_n,q_n}[n-3]_{p_n,q_n} \sim q_n^9[k]_{p_n,q_n}^3 + p_n^k\left(q_n^5[4]_{p_n,q_n} + q_n^4[5]_{p_n,q_n}\right)[k]_{p_n,q_n}^2. \quad (13)$$

$$[n+1]_{p_n,q_n}[n-2]_{p_n,q_n}[n-3]_{p_n,q_n} \sim q_n^5[k]_{p_n,q_n}^3 + p_n^k\left(q_n[4]_{p_n,q_n} + q_n^4\right)[k]_{p_n,q_n}^2 \quad (14)$$

$$[n-1]_{p_n,q_n}[n-2]_{p_n,q_n}[n-3]_{p_n,q_n} \sim q_n^3[k]_{p_n,q_n}^3 + p_n^k\left(q_n[2]_{p_n,q_n} + q_n^2\right)[k]_{p_n,q_n}^2 \quad (15)$$

Using Lemma 2, we obtain

$$F_n^{p_n,q_n}\left((t-x)^4; x\right) \sim \left(A_n + \frac{1}{[k]_{p_n,q_n}}B_n\right)x^4 \quad (16)$$

where $A_n = q_n^{15} - 4p_n^3q_n^{12} + 6p_n^5q_n^{10} - 3p_n^6q_n^q$ and

$$B_n = p_n^k\left(\left(q_n^9[6]_{p_n,q_n} + q_n^{10}[5]_{p_n,q_n} + q_n^{11}[4]_{p_n,q_n}\right) - 4p_n^3q_n^3\left(q_n^5[4]_{p_n,q_n} + q_n^4[5]_{p_n,q_n}\right) + 6p_n^5q_n^5\left(q_n[4]_{p_n,q_n} + q_n^4\right) - 3p_n^6q_n^6\left(q_n[2]_{p_n,q_n} + q_n^2\right)\right). \quad (17)$$

Combining with

$$[k]_{p_n,q_n}A_n \sim [k]_{p_n,q_n}\left(q_n^6 - 4p_n^3q_n^3 + 6p_n^5q_n - 3p_n^6\right) = [k]_{p_n,q_n}\left(4p_n^3\left(p_n^3 - q_n^3\right) - 6p_n^5\left(p_n - q_n\right) - \left(p_n^6 - q_n^6\right)\right) = [k]_{p_n,q_n}\left(4p_n^3[3]_{p_n,q_n}\left(p_n - q_n\right) - 6p_n^5\left(p_n - q_n\right) - [6]_{p_n,q_n}\left(p_n - q_n\right)\right) = [k]_{p_n,q_n} \cdot \left(\frac{p_n^n - q_n^n}{[n]_{p_n,q_n}}\left(4p_n^3[3]_{p_n,q_n} - 6p_n^5 - [6]_{p_n,q_n}\right)\right) \sim \left(p_n^n - q_n^n\right)\left(4p_n^3[3]_{p_n,q_n} - 6p_n^5 - [6]_{p_n,q_n}\right) \sim (a-b)(12 - 6 - 6) = 0 \quad (18)$$

and

$$B_n \sim (4+5+6) - 4\times(4+5) + 6\times(4+1) - 3\times(2+1) = 0, \quad (19)$$

we can obtain $\lim_{n\to\infty}[n-3]_{p_n,q_n}F_n^{p_n,q_n}((t-x)^4; x) = 0$. □

3. Voronovskaja Type Theorem

We give a Voronovskaja type asymptotic formula for $F_n^{p,q}(f; x)$ by means of the second and fourth central moments.

Theorem 5. *Let f be bounded and integrable on the interval $x \in (0,\infty)$; second derivative of f exists at a fixed point $x \in (0,\infty)$; the sequences (p_n) and (q_n) satisfy $0 < q_n < p_n < 1$ such that $p_n \longrightarrow 1$, $q_n \longrightarrow 1$, and $p_n^n \longrightarrow a$, $q_n^n \longrightarrow b$, and $[n]_{p_n,q_n} \longrightarrow \infty$ as $n \longrightarrow \infty$, where $0 \le a, b < 1$; then*

$$\lim_{n\to\infty}[n-3]_{p_n,q_n}\left(F_n^{p_n,q_n}(f;x) - f(x)\right) = \frac{a+b}{2}x^2 f''(x). \quad (20)$$

Proof. Let $x \in (0,\infty)$ be fixed. In order to prove this identity, we use Taylor's expansion:

$$f(t) - f(x) = (t-x)f'(x) + (t-x)^2\left(\frac{f''(x)}{2} + \Phi_{p_n,q_n}(t,x)\right), \quad (21)$$

where $\Phi_{p_n,q_n}(x,t)$ is bounded and $\lim_{t\to x}\Phi_{p_n,q_n}(t,x) = 0$. By applying the operator $F_n^{p_n,q_n}(f;x)$ to the equality above, we obtain

$$F_n^{p_n,q_n}(f;x) - f(x) = f'(x)F_n^{p_n,q_n}((t-x);x) + \frac{1}{2}f''(x)F_n^{p_n,q_n}\left((t-x)^2;x\right) + F_n^{p_n,q_n}\left(\Phi_{p_n,q_n}(t,x)(t-x)^2;x\right) = \frac{1}{2}f''(x)F_n^{p_n,q_n}\left((t-x)^2;x\right) + F_n^{p_n,q_n}\left(\Phi_{p_n,q_n}(t,x)(t-x)^2;x\right). \quad (22)$$

Since $\lim_{t\to x}\Phi_{p_n,q_n}(t,x) = 0$, for all $\epsilon > 0$, there exists $\delta > 0$ such that $|t-x| < \delta$ and it will imply $|\Phi_{p_n,q_n}(t,x)| < \epsilon$ for all fixed $x \in (0,\infty)$ as n is sufficiently large. Meanwhile, if $|t-x| \ge \delta$, then $|\Phi_{p_n,q_n}(t,x)| \le (C/\delta^2)(t-x)^2$, where $C > 0$ is a constant. Using Remark 4, we have

$$\lim_{n\to\infty}[n-3]F_n^{p_n,q_n}\left((t-x)^2;x\right) = \lim_{n\to\infty}[n-1]F_n^{p_n,q_n}\left((t-x)^2;x\right) = (a+b)x^2 \quad (23)$$

and

$$[n-3]_{p_n,q_n}\left|F_n^{p_n,q_n}\left(\Phi_{p_n,q_n}(t,x)(t-x)^2;x\right)\right| \le \epsilon[n-3]_{p_n,q_n}F_n^{p_n,q_n}\left((t-x)^2;x\right) + \frac{C}{\delta^2}[n-3]_{p_n,q_n}F_n^{p_n,q_n}\left((t-x)^4;x\right) \longrightarrow 0 \quad (n \longrightarrow \infty). \quad (24)$$

The proof is completed. □

4. Local Approximation

We denote the space of all real valued continuous bounded functions f defined on the interval $[0,+\infty)$ by $C_B[0,+\infty)$. The norm $\|\cdot\|$ on the space $C_B[0,+\infty)$ is given by

$$\|f\| = \sup\left\{\left|f(x)\right| : x \in [0,+\infty)\right\}. \tag{25}$$

Let us consider the following K-functional:

$$K(f,\delta) = \inf_{g\in C_B^2[0,\infty)} \left\{\|f-g\| + \delta\left\|g''\right\|\right\}, \tag{26}$$

where $\delta > 0$ and $C_B^2[0,\infty) = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. By [22] (p. 177, Theorem 2.4), there exists an absolute constant $C > 0$ such that

$$K(f,\delta) \le C\omega_2\left(f, \sqrt{\delta}\right) \tag{27}$$

where

$$\omega_2(f,\delta) = \sup_{0<|h|\le\delta}\ \sup_{x\in[0,\infty)} \left|f(x+2h) - 2f(x+h) + f(x)\right| \tag{28}$$

is the second-order modulus of smoothness of f. By

$$\omega(f,\delta) = \sup_{0<|h|\le\delta}\ \sup_{x\in[0,\infty)} \left|f(x+h) - f(x)\right|, \tag{29}$$

we denote the usual modulus of continuity of $f \in C_B[0,\infty)$.

Our first result is a direct local approximation theorem for the operators $F_n^{p,q}(f;x)$.

Theorem 6. *Let $f \in C_B[0,+\infty)$; $0 < q < p \le 1$; then, for every $x \in (0,\infty)$ and $n > 2$, we have*

$$\left|F_n^{p,q}(f;x) - f(x)\right| \le C\omega_2\left(f, \sqrt{A(x)}\right), \tag{30}$$

where C is some positive constant.

Proof. For all $g \in C_B^2[0,\infty)$, using Taylor's expansion for $x \in (0,\infty)$, we have

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)\,du. \tag{31}$$

Applying the operators $F_n^{p,q}$ to both sides of the equality above and using Remark 3, we get

$$\begin{aligned}\left|F_n^{p,q}(g;x) - g(x)\right| &= \left|F_n^{p,q}\left(\int_x^t (t-u)g''(u)\,du; x\right)\right| \\ &\le F_n^{p,q}\left(\left|\int_x^t (t-u)g''(u)\,du\right|; x\right) \\ &\le F_n^{p,q}\left(\left\|g''\right\|(t-x)^2; x\right) \\ &\le A(x)\left\|g''\right\|.\end{aligned} \tag{32}$$

Using $|F_n^{p,q}(f;x)| \le \|f\|$, we have

$$\begin{aligned}\left|F_n^{p,q}(f;x) - f(x)\right| &\le \left|F_n^{p,q}(f-g;x) - (f-g)(x)\right| \\ &\quad + \left|F_n^{p,q}(g;x) - g(x)\right| \\ &\le 2\|f-g\| + A(x)\left\|g''\right\|\end{aligned} \tag{33}$$

Lastly, taking infimum on both sides of the inequality above over all $g \in C_B^2[0,\infty)$,

$$\left|F_n^{p,q}(f;x) - f(x)\right| \le 2K(f; A(x)) \tag{34}$$

for which we have the desired result by (27). □

Theorem 7. *Let $0 < \gamma \le 1$ and let E be any bounded subset of the interval $[0,\infty)$. If $f \in C_B[0,\infty)$ is locally in $Lip(\gamma)$, i.e., the condition*

$$\left|f(x) - f(t)\right| \le L|x-t|^\gamma, \quad t \in E \text{ and } x \in (0,\infty) \tag{35}$$

holds, then, for each $x \in (0,\infty)$, we have

$$\left|F_n^{p,q}(f;x) - f(x)\right| \le L\left\{(A(x))^{\gamma/2} + 2\left(\mathrm{d}(x;E)^\gamma\right)\right\}, \tag{36}$$

where L is a constant depending on γ and f; and $d(x;E)$ is the distance between x and E defined by

$$\mathrm{d}(x;E) = \inf\{|t-x| : t \in E\}. \tag{37}$$

Proof. From the properties of infimum, there is at least one point t_0 in the closure of E; that is, $t_0 \in \overline{E}$, such that

$$\mathrm{d}(x;E) = \left|t_0 - x\right|. \tag{38}$$

Using the triangle inequality, we have

$$\begin{aligned}\left|F_n^{p,q}(f;x) - f(x)\right| &\le F_n^{p,q}\left(\left|f(t) - f(x)\right|; x\right) \\ &\le F_n^{p,q}\left(\left|f(t) - f(t_0)\right|; x\right) + F_n^{p,q}\left(\left|f(t_0) - f(x)\right|; x\right) \\ &\le L\left(F_n^{p,q}\left(\left|t-t_0\right|^\gamma; x\right) + F_n^{p,q}\left(\left|t_0-x\right|^\gamma; x\right)\right) \\ &\le L\left(F_n^{p,q}\left(|t-x|^\gamma; x\right) + 2\left|t_0-x\right|^\gamma\right)\end{aligned} \tag{39}$$

Choosing $a_1 = 2/\gamma$ and $a_2 = 2/(2-\gamma)$ and using the well-known Hölder inequality,

$$\begin{aligned}&\left|F_n^{p,q}(f;x) - f(x)\right| \\ &\quad\le L\left\{\left(F_n^{p,q}\left(|t-x|^{\gamma a_1}; x\right)\right)^{1/a_1}\left(F_n^{p,q}\left(1^{a_2}; x\right)\right)^{1/a_2}\right. \\ &\quad\left. + 2\left|t_0-x\right|^\gamma\right\} \le L\left\{\left(F_n^{p,q}\left((t-x)^2; x\right)\right)^{\gamma/2}\right. \\ &\quad\left. + 2\left|t_0-x\right|^\gamma\right\} \le L\left\{A^{\gamma/2}(x) + 2\left(\mathrm{d}(x;E)\right)^\gamma\right\}\end{aligned} \tag{40}$$

This completes the proof. □

5. Rate of Convergence and Weighted Approximation

Let $C_\rho[0,\infty)$ be the set of all functions defined on $[0,\infty)$ satisfying the condition $|f(x)| \le C_f \rho(x)$, where $C_f > 0$ is a constant depending only on f and $\rho(x)$ is a weight function. Let $C_\rho[0,\infty)$ be the space of all continuous functions in $C_\rho[0,\infty)$ with the norm $\|f\|_\rho = \sup_{x\in[0,\infty)}(|f(x)|/\rho(x))$ and $C_\rho^0[0,\infty) = \{f \in C_\rho[0,\infty) : \lim_{x\longrightarrow\infty}(|f(x)|/\rho(x)) < \infty\}$. We consider $\rho(x) = 1 + x^2$ in the following two theorems. Meanwhile, we denote the modulus of continuity of f on the closed interval $[0,a]$, $a > 0$, by

$$\omega_a(f,\delta) = \sup_{|t-x|\le\delta}\ \sup_{x,t\in[0,a]} |f(t) - f(x)|. \tag{41}$$

Obviously, for the function $f \in C_\rho[0,\infty)$, the modulus of continuity $\omega_a(f,\delta)$ tends to zero. Then we establish the following theorem on the rate of convergence for the operators $F_n^{p,q}(f;x)$.

Theorem 8. *Let $f \in C_\rho[0,\infty)$, $0 < q < p \le 1$, and $\omega_{a+1}(f,\delta)$ be its modulus of continuity on the finite interval $[0,a+1] \subset [0,\infty)$, where $a > 0$. Then, for every $n > 2$, $x > 0$,*

$$\begin{aligned}&\left\|F_n^{p,q}(f;x) - f(x)\right\|_{C[0,a]}\\ &\quad\le 4C_f\left(1+a^2\right)A(x) + 2\omega_{a+1}\left(f,\sqrt{A(x)}\right)\end{aligned} \tag{42}$$

Proof. For all $x \in [0,a]$ and $t > a+1$, we easily have $(t-x)^2 \ge (t-a)^2 \ge 1$; therefore,

$$\begin{aligned}|f(t) - f(x)| &\le |f(t)| + |f(x)| \le C_f\left(2 + x^2 + t^2\right)\\ &= C_f\left(2 + x^2 + (x - t - x)^2\right)\\ &\le C_f\left(2 + 3x^2 + 2(x-t)^2\right)\\ &\le C_f\left(4 + 3x^2\right)(t-x)^2\\ &\le 4C_f\left(1+a^2\right)(t-x)^2.\end{aligned} \tag{43}$$

And, for all $x \in [0,a]$, $t \in [0,a+1]$, and $\delta > 0$, we have

$$\begin{aligned}|f(t) - f(x)| &\le \omega_{a+1}(f,|t-x|)\\ &\le \left(1 + \frac{|t-x|}{\delta}\right)\omega_{a+1}(f,\delta)\end{aligned} \tag{44}$$

From (43) and (44), we get

$$\begin{aligned}|f(t) - f(x)| &\le 4C_f\left(1+a^2\right)(t-x)^2\\ &\quad + \left(1 + \frac{|t-x|}{\delta}\right)\omega_{a+1}(f,\delta).\end{aligned} \tag{45}$$

By Schwarz's inequality and Remark 3, we have

$$\begin{aligned}&\left|F_n^{p,q}(f;x) - f(x)\right| \le F_n^{p,q}\left(|f(t) - f(x)|;x\right)\\ &\quad\le 4C_f\left(1+a^2\right)F_n^{p,q}\left((t-x)^2;x\right)\\ &\qquad + F_n^{p,q}\left(\left(1 + \frac{|t-x|}{\delta}\right);x\right)\omega_{a+1}(f,\delta)\\ &\quad\le 4C_f\left(1+a^2\right)F_n^{p,q}\left((t-x)^2;x\right)\\ &\qquad + \omega_{a+1}(f,\delta)\left(1 + \frac{1}{\delta}\sqrt{F_n^{p,q}\left((t-x)^2;x\right)}\right)\\ &\quad\le 4C_f\left(1+a^2\right)A(x) + \omega_{a+1}(f,\delta)\left(1 + \frac{1}{\delta}\sqrt{A(x)}\right)\end{aligned} \tag{46}$$

By taking $\delta = \sqrt{A(x)}$, we get the proof of Theorem 8. ☐

The following is a direct estimate in weighted approximation.

Theorem 9. *Let (p_n) and (q_n) satisfy $0 < q_n < p_n \le 1$ such that $p_n \longrightarrow 1$, $q_n \longrightarrow 1$, $p_n^n \longrightarrow a$, $q_n^n \longrightarrow b$, and $[n]_{p_n,q_n} \longrightarrow \infty$. Then, for $f \in C_\rho^0[0,\infty)$, we have*

$$\lim_{n\longrightarrow\infty}\left\|F_n^{p_n,q_n}(f;x) - f(x)\right\|_\rho = 0. \tag{47}$$

Proof. Using the Korovkin theorem in [23], we see that it is sufficient to verify the following three conditions:

$$\lim_{n\longrightarrow\infty}\left\|F_n^{p_n,q_n}\left(t^k;x\right) - x^k\right\|_\rho = 0, \quad k = 0,1,2. \tag{48}$$

Since $F_n^{p_n,q_n}(1;x) = 1$ and $F_n^{p_n,q_n}(t;x) = x$, (48) holds true for $k = 0, 1$. By Remark 3, for $n > 2$, we have

$$\begin{aligned}&\lim_{n\longrightarrow\infty}\left\|F_n^{p_n,q_n}(f;x) - f(x)\right\|_\rho\\ &\quad = \sup_{x\in[0,\infty)}\left|\left(\frac{q_n}{p_n} - 1\right) + \frac{p_n^{n-2}[2]_{p_n,q_n}}{q_n[n-1]_{p_n,q_n}}\right|\frac{x^2}{1+x^2}\\ &\quad\le \left|\frac{q_n}{p_n} - 1\right| + \left|\frac{p_n^{n-2}[2]_{p_n,q_n}}{q_n[n-1]_{p_n,q_n}}\right| \longrightarrow 0 \quad (n \longrightarrow \infty).\end{aligned} \tag{49}$$

Thus the proof is completed. ☐

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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