Research Article

Type II Half Logistic Kumaraswamy Distribution with Applications

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In this paper, a new distribution with a unit interval named type II half logistic Kumaraswamy (TIIHLKw) distribution is proposed. Its density and distribution functions are presented using alternate expressions. This distribution is obtained by adding an extra parameter in the existing model to rise its ability fitting complex data sets. Some important statistical properties of TIIHLKw distribution are derived. The estimation of the parameters is obtained by numerous well-recognized approaches and simulation study confirmed the efficiencies of estimates such obtained. We apply the related model to practical datasets, and it is concluded that the proposed model is the best by model selection criteria than other competitive models.

1. Introduction

Over the last few years, inspired by the increasing demand of probability distributions in many fields, many generalized distributions have been studied. Most of them are proposed by the addition of more parameters to well-known probability distributions that exist in literature to make them flexible. For instance, Haq et al. [1] proposed and studied the generalized odd Burr III (GOBIII) family of distributions along with its important characterizations. Ahmed [2] proposed a new model derived by the transformation of the baseline model. Different shapes of failure function can be formed such as increasing and bathtub. Some mathematical properties are obtained. So many research works have been done for proposing more flexible generalized probability distributions such as Alzaatreh et al. [3], Haq et al. [4], Hashmi et al. [5], Elgarhy et al. [6], and ZeinEldin et al.

[7]. Alshenawy [8] suggested a new one-parameter distribution. Several statistical properties and characteristics of the proposed distribution are derived along with estimation under Type II censoring. The research is concluded on the basis of a simulation study and real data analysis.

The Kumaraswamy (Kw) distribution was introduced and studied by Kumaraswamy [9] with unit interval, denoted by $Kw(a, b)$, with cumulative distribution function (cdf) is

$$G(x) = 1 - (1 - x^a)^b, \quad (1)$$

and its related probability density function (pdf) is

$$g(x) = abx^{a-1}(1 - x^a)^{b-1}, \quad x, a, b > 0. \quad (2)$$

The Kw density has one of these shapes depending upon parameter values, unimodal ($a, b > 1$), bathtub ($a, b < 1$), increasing ($a > 1$ and $b \leq 1$), decreasing (D) ($a \leq 1$ and $b > 1$), or steady (both a and b equal to 1).

The behavior of Kw distribution is analogous to the beta distribution but simpler due to closed-form of both its pdf and cdf. Boundary behavior and the major special models are also the same in both Beta and Kw distributions. This distribution could be a good substitute in situations wherein reality the bounds are finite i.e., $(0, 1)$.

The Kw is originally developed as a lifetime distribution. Many authors studied and developed the generalizations of Kw distribution such as exponentiated Kw distribution studied by Lemonte et al. [10], El-Sherpieny, and Ahmed [11] proposed Kw Kw distribution, transmuted Kw distribution studied by Khan et al. [12], Sharma and Chakrabarty [13] studied size biased Kw distribution, George and Thobias [14] introduced Marshall-Olkin Kw distribution, exponentiated generalized Kw distribution studied by Elgarhy et al. [6], type II Topp Leone inverted Kw by ZeinEldin et al. [15], type I half logistic inverted Kw by ZeinEldin et al. [16], truncated inverted Kw by Bantan et al. [17], and Ghosh [18] introduced bivariate and multivariate weighted Kw distributions.

Hassan et al. [19] proposed Type II half logistic-G (TIIHL-G). The TIIHL-G distribution is expressed by its cdf given by

$$F(x; \lambda, \zeta) = 1 - \int_0^{-\log G(x; \zeta)} \frac{2\lambda e^{-\lambda t}}{(1 + e^{-\lambda t})^2} dt \quad (3)$$

$$= \frac{2[G(x; \zeta)]^\lambda}{1 + [G(x; \zeta)]^\lambda}, x > 0, \lambda > 0,$$

where $G(x; \zeta)$ is cdf of baseline model with parameter vector ζ and $F(x; \lambda, \zeta)$ is cdf derived by the T-X generator proposed by Alzaatreh [3]. The pdf of the TIIHL-G family is given as

$$f(x; \lambda, \zeta) = \frac{2\lambda g(x; \zeta)[G(x; \zeta)]^{\lambda-1}}{[1 + [G(x; \zeta)]^\lambda]^2}, x > 0, \lambda > 0. \quad (4)$$

This article is dedicated to both its mathematical and application features. A significant portion is kept for the estimation of the parameters through various methods including maximum likelihood estimation (MLE), least-square estimation (LSE), weighted least square estimation (WLSE), percentiles estimation (PCE), and Cramer-von Mises estimation (CVE).

The core purpose of this research is to suggest a simpler and more flexible model called Type II Half Logistic Kw (TIIHLKw) distribution. This article is organized in the following manner: Section 2 is dedicated for the proposition of type II half logistic- Kw (TIIHLKw) distribution. Section 3 deals with the leading statistical properties of this model. Important binomial expansions of density and distribution functions are presented which involve binomial expansions. In Section 4, an extensive study of five different methods of

estimation is carried out, with all derivations and detailed discussions. A comprehensive simulation study is conducted to compute the biases and efficiency for parameters and compare the performances of five estimation approaches stated above in the next section. Section 6 is devoted to the real data application of TIIHLKwD to show the importance of TIIHLKw distribution. Lastly, the conclusion is given in Section 7.

2. The TIIHLKw Distribution

In this section, we examine the usefulness, and flexibility of a new associate of type II half-logistic-G family having $(0, 1)$, using Kw distribution as the baseline. The pdf and cdf of Kw distribution (2 shape parameters: $a, b > 0$) are given as follows

$$g(x; a, b) = abx^{a-1}(1 - x^a)^{b-1}, 0 < x < 1, \quad (5)$$

$$G(x; a, b) = 1 - (1 - x^a)^b, 0 < x < 1. \quad (6)$$

The random variable (r.v.) X follows TIIHLKw distribution if its cdf is obtained by inserting (6) in (3)

$$F(x; \lambda, a, b) = \frac{2[1 - (1 - x^a)^b]^\lambda}{1 + [1 - (1 - x^a)^b]^\lambda}, a, b, \lambda > 0, 0 < x < 1. \quad (7)$$

The pdf of TIIHLKw distribution is as follows

$$f(x; \lambda, a, b) = \frac{2\lambda abx^{a-1}(1 - x^a)^{b-1}(1 - (1 - x^a)^b)^{\lambda-1}}{(1 + (1 - (1 - x^a)^b)^\lambda)^2}, a, b, \lambda > 0, 0 < x < 1. \quad (8)$$

The survival function of TIIHLKw distribution is

$$\bar{F}(x; \lambda, a, b) = \frac{1 - [1 - (1 - x^a)^b]^\lambda}{1 + [1 - (1 - x^a)^b]^\lambda} = \frac{2}{1 + (1 - (1 - x^a)^b)^\lambda} - 1. \quad (9)$$

The failure rate function of TIIHLKwD is

$$h(x; a, b) = \frac{2\lambda abx^{a-1}(1 - x^a)^{b-1}(1 - (1 - x^a)^b)^{\lambda-1}}{1 - (1 - (1 - x^a)^b)^{2\lambda}}. \quad (10)$$

The flexibility of TIIHLKw distribution can be illustrated in Figures 1, 2, and 3. The pdf plots for the TIIHLKw

FIGURE 1: Pdf plot of TIHLKwD.

FIGURE 2: Plots of some pdfs of TIHLKwD.

distribution are given in Figures 1 and 2, and plots of hrf are given in Figure 3.

3. Some Mathematical Properties

The properties of TIHLKwD are derived here. After this, we consider a r.v. X follows the pdf (8) and cdf (7).

3.1. Quantile Function. The quantile function of X is denoted by $Q(u)$, defined as

$$Q(u) = G^{-1} \left[\frac{u}{2-u} \right]^{1/\lambda}, \quad u \in (0, 1), \quad (11)$$

the inverse function of $G(\cdot)$ is $G^{-1}(\cdot)$, given in (6).

FIGURE 3: Plots of some hrf of TIIHLKwD.

Also,

$$x_u = \left(1 - \left(1 - \left(\frac{u}{u-2} \right)^{1/\lambda} \right)^{1/b} \right)^{1/a}. \quad (12)$$

Simulated values from TIIHLKwD can be utilized in the simulation study. The r. v. u follows rectangular distribution on the interval 0 to 1, i.e., $x_u = Q(u)$ follows TIIHLKw distribution. In particular, the first three quartiles are obtained by putting $u = 1/4, 1/2$, and $3/4$, respectively, in (11) and (12).

On differentiation, we can have the density of quantile function

$$q(u) = \frac{1}{ab\lambda} \left(\frac{u}{2-u} \right)^{-1+1/\lambda} \left[\frac{1}{2-u} + \frac{u}{(u-2)^2} \right] \cdot \left[1 - \left(\frac{u}{2-u} \right)^{1/\lambda} \right]^{-1+1/b} \left[1 - \left\{ 1 - \left(\frac{u}{2-u} \right)^{1/\lambda} \right\}^{1/b} \right]^{-1+1/a}. \quad (13)$$

3.2. Alternate Representation. Here, we present the expansion of the pdf and cdf for TIIHLKw distribution for further mathematical manipulation.

The binomial theorem, for $\beta > 0$ and $|z| < 1$, can be expressed as

$$(1-z)^{\beta-1} = \sum_{i=0}^{\infty} (-1)^i \binom{\beta-1}{i} z^i, \quad (14)$$

Then, by applying (13) in (8), the pdf of TIIHLKwD becomes

$$f(x; \lambda, a, b) = 2\lambda ab x^{a-1} \left(1 + \left(1 - (1-x^a)^b \right)^\lambda \right)^{-2} \cdot \sum_{i=0}^{\infty} (-1)^i \binom{\lambda-1}{i} (1-x^a)^{b(i+1)-1}, \quad (15)$$

for $a, \lambda, b > 0, 0 < x < 1$.

Another form of pdf (8) can be obtained by means of the following expansion

$$(1+z)^{-\beta} = \sum_{i=0}^{\infty} (-1)^i \binom{\beta+i-1}{i} z^i \quad (16)$$

for $|z| < 1$ and $\beta > 0$. By applying (16) in pdf (8), we get

$$f(x) = \sum_{i=0}^{\infty} \eta_i x^{a-1} (1-x^a)^{b-1} \left[1 - (1-x^a)^b \right]^{\lambda(i+1)-1}, \quad (17)$$

where $\eta_i = 2ab\lambda(-1)^i(i+1)$. Now considering (13), we have

$$\left[1 - (1-x^a)^b \right]^{\lambda(i+1)-1} = \sum_{j=0}^{\infty} (-1)^j \binom{\lambda(i+1)-1}{j} (1-x^a)^{bj}. \quad (18)$$

Inserting this expansion in (17), we have pdf of the following form

$$f(x) = \sum_{i,j=0}^{\infty} \eta_{i,j} x^{a-1} (1-x^a)^{b(j+1)-1}, \quad (19)$$

where $\eta_{i,j} = (-1)^j \binom{\lambda(i+1)-1}{j} \eta_i$.

Another formula can be formed from pdf (19), which is given in (20) using an infinite linear combination

$$f(x) = \sum_{i,j,k=0}^{\infty} \eta_{i,j,k} (-1)^k \binom{b(j+1)-1}{k} x^{a(k+1)-1}, \quad (20)$$

$$f(x) = \sum_{i,j,k=0}^{\infty} \eta_{i,j,k} x^{a(k+1)-1}, \quad (21)$$

where $\eta_{i,j,k} = (-1)^k \binom{b(j+1)-1}{k} \eta_{i,j}$.

Proposition 1. Let “ h ” be a positive integer. The expansion of cdf can be written in following form:

$$[F(x)]^h = \sum_{l=0}^h \sum_{m=0}^{\infty} \eta_{l,m} (1-x^a)^{bm}, \quad (22)$$

where $\eta_{l,m} = (2)^h (-1)^{l+m} \binom{h+l-1}{l} \binom{\lambda(l+h)}{m}$.

Proof. The cdf $[F(x)]^h$ is obtained, for ‘ h ’ an integer by using (16).

$$\begin{aligned} [F(x)]^h &= \frac{2^h [1 - (1-x^a)^b]^{\lambda h}}{\left(1 + [1 - (1-x^a)^b]^{\lambda}\right)^h}, \\ [F(x)]^h &= \sum_{l=0}^h (2)^h (-1)^l \binom{h+l-1}{l} \left(1 - (1-x^a)^b\right)^{\lambda(l+h)}. \end{aligned} \quad (23)$$

Once more binomial expansion is applied to $[F(x)]^h$ and it can be written as

$$[F(x)]^h = \sum_{l=0}^h \sum_{m=0}^{\infty} (2)^h (-1)^{l+m} \binom{h+l-1}{l} \binom{\lambda(l+h)}{m} (1-x^a)^{bm}. \quad (24)$$

The required result is obtained by combing some expression together, completing the proof.

3.3. Probability Weighted Moments: The probability-weighted moments (PWMs) are used to study some more

characteristics of the probability distribution. Under the specified setting discussed above, PWMs are denoted by $\tau_{r,s}$, can be defined as

$$\tau_{r,s} = E[X^r F(x)^s] = \int_{-\infty}^{\infty} x^r (F(x))^s f(x) dx. \quad (25)$$

The PWMs of TIIHLKw are obtained by substituting (21) and (22) into (25), as follows

$$\tau_{r,s} = \sum_{i,j,k=0}^{\infty} \eta_{i,j,k} \sum_{l=0}^s \sum_{m=0}^{\infty} \eta_{l,m} \int_0^1 x^{a(k+1)+r-1} (1-x^a)^{bm} dx. \quad (26)$$

Then,

$$\begin{aligned} \tau_{r,s} &= \sum_{i,j,k=0}^{\infty} \eta_{i,j,k} \sum_{l=0}^s \sum_{m=0}^{\infty} \eta_{l,m} \frac{\Gamma(1+bm)\Gamma(1+k+(r/a))}{a \Gamma(2+k+bm+(r/a))}, \\ \tau_{r,s} &= \sum_{i,j,k=0}^{\infty} \eta_{i,j,k} \sum_{l=0}^s \sum_{m=0}^{\infty} \eta_{l,m} \beta\left(1+bm, 1+k+\frac{r}{a}\right), \end{aligned} \quad (27)$$

where $\Gamma(s) = \int_0^{+\infty} t^s e^{-t} dt$, $s > 0$. Also, $\beta(a, b)$ is beta function.

3.4. Moments. The moments have a vital role in the study of the distribution and real data applications. Now, we get the r^{th} moment for the TIIHLKwD. Under the specified assumptions, the r^{th} moment is obtained as

$$\mu'_r = \int_{-\infty}^{\infty} x^r f(x) dx = \int_0^1 \sum_{i,j=0}^{\infty} \eta_{i,j} x^{r+a-1} (1-x^a)^{b(j+1)-1} dx, \quad (28)$$

$$\mu'_r = \sum_{i,j=0}^{\infty} \eta_{i,j} \beta(r+a, b(j+1)); r = 1, 2, 3, \dots \dots, \quad (29)$$

where $\eta_{i,j} = (-1)^j \binom{\lambda(i+1)-1}{j} \eta_i$.

The mean (μ'_1) and variance (var) of TIIHLKw distribution can be derived as

$$\begin{aligned} \mu'_1 &= \sum_{i,j=0}^{\infty} \eta_{i,j} \beta(1+a, b(j+1)), \\ \text{var}(X) &= \sum_{i,j=0}^{\infty} \eta_{i,j} \beta(2+a, b(j+1)) - \left(\sum_{i,j=0}^{\infty} \eta_{i,j} \beta(1+a, b(j+1)) \right)^2. \end{aligned} \quad (30)$$

Also, the coefficients of skewness and kurtosis of TIIHLKw distribution are given by

$$S_k = \frac{\mu'_3}{\mu'_2^{3/2}}, \quad K = \frac{\mu'_4}{\mu'_2^2}. \quad (31)$$

The summary measures, mean, variance, skewness (S_k) and kurtosis (K) values are presented in Table 1. The plots of mean and var for the TIIHLKwD are given in Figure 4 and graphs of skewness and kurtosis are presented in Figure 5 for different parameter ranges.

We see a monotonic variation in these measures caused by variation in parameters a , b , and λ .

3.5. Moment-Generating Function. The moment-generating function of X , using moments about the origin (29), is obtained as

$$M_X(t) = \sum \frac{t^r}{r!} \mu'_r = \sum_{i,j,r=0}^{\infty} \frac{t^r}{r!} \eta_{i,j} \beta(r+a, b(j+1)). \quad (32)$$

3.6. Incomplete Moments. The r^{th} incomplete moment of TIIHLKwD can be obtained by using (19) is

$$E_r(t) = \int_0^t x^r f(x) dx = \int_0^t \sum_{i,j=0}^{\infty} \eta_{i,j} x^{r+a-1} (1-x^a)^{b(j+1)-1} dx, \\ E_r(t) = \sum_{i,j=0}^{\infty} \eta_{i,j} \beta_t(r+a, b(j+1)), \quad r = 1, 2, 3, \dots, \quad (33)$$

where $\beta_t(\cdot, \cdot)$ is incomplete beta function.

3.7. The Mean Deviation. Following are expressions used to get mean deviation about mean and median, respectively

$$M_1(X) = 2\mu F(\mu) - 2T(\mu) \text{ and } M_2(X) = \mu - 2T(M_d), \quad (34)$$

where M_d = Median of X and $T(q) = \int_{-\infty}^q x f(x) dx$ is the initial incomplete moment. Now

$$T(\mu) = \int_0^{\mu} x f(x) dx = \sum_{i,j=0}^{\infty} \eta_{i,j} \beta_{\mu}(1+a, b(j+1)). \\ T(M_d) = \int_0^{M_d} x f(x) dx = \sum_{i,j=0}^{\infty} \eta_{i,j} \beta_{M_d}(1+a, b(j+1)). \quad (35)$$

3.8. Bonferroni and Lorenz Curves. Bonferroni and Lorenz curves are important applications of the first incomplete moment. These curves are mostly used in different fields of life such as economics, reliability analysis, demographic studies, life testing, life insurance, and medical technology. The

TABLE 1: Mean, variance, skewness (S_k), and kurtosis (K) values ($a = 3$).

λ	b	Mean	Var	S_k	K
1	2.0	0.56556	0.03908	-0.09702	2.30905
	3.0	0.50543	0.03430	0.04856	2.38086
	4.0	0.46478	0.03053	0.13105	2.45360
	5.0	0.43469	0.02758	0.18440	2.51360
	6.0	0.41114	0.02522	0.22179	2.56192
	7.0	0.39198	0.02330	0.24948	2.60112
1.5	2.0	0.64566	0.02830	-0.26723	2.53251
	3.0	0.57974	0.02601	-0.08852	2.52495
	4.0	0.53444	0.02373	0.01141	2.56666
	5.0	0.50062	0.02176	0.07559	2.61164
	6.0	0.47399	0.02011	0.12039	2.65173
	7.0	0.45225	0.01871	0.15347	2.68603
2.0	2.0	0.69540	0.02170	-0.35350	2.69440
	3.0	0.62680	0.02080	-0.15040	2.62680
	4.0	0.57910	0.01930	-0.03790	2.64750
	5.0	0.54310	0.01800	0.03400	2.68340
	6.0	0.51470	0.01680	0.08400	2.71910
	7.0	0.49140	0.01570	0.12090	2.75120
2.5	2.0	0.72980	0.01740	-0.40310	2.80610
	3.0	0.66000	0.01720	-0.18150	2.69350
	4.0	0.61070	0.01640	-0.05970	2.70010
	5.0	0.57340	0.01540	0.01780	2.73070
	6.0	0.54380	0.01440	0.07160	2.76440
	7.0	0.51950	0.01360	0.11120	2.79570
3.0	2.0	0.81500	0.00830	-0.48390	3.03600
	3.0	0.74510	0.00940	-0.21230	2.81430
	4.0	0.69370	0.00950	-0.06610	2.79630
	5.0	0.65380	0.00930	0.02610	2.82280
	6.0	0.62160	0.00900	0.08980	2.85860
	7.0	0.59490	0.00860	0.13640	2.89410

Lorenz and Bonferroni curves are obtained, respectively, as follows

$$L_F(x) = \frac{1}{E(X)} \int_0^x t f(t) dt = \frac{\sum_{i,j=0}^{\infty} \eta_{i,j} \beta_x(1+a, b(j+1))}{\sum_{i,j=0}^{\infty} \eta_{i,j} \beta(1+a, b(j+1))}, \quad (36)$$

and,

$$B_F(x) = \frac{L_F(x)}{F(x)} = \frac{1 + [1 - (1 - x^a)^b]^{\lambda} \left(\sum_{i,j=0}^{\infty} \eta_{i,j} \beta_x(1+a, b(j+1)) / \sum_{i,j=0}^{\infty} \eta_{i,j} \beta(1+a, b(j+1)) \right)}{2 [1 - (1 - x^a)^b]^{\lambda}}. \quad (37)$$

FIGURE 4: Graphs of mean and variance of TIHLKw distribution.

FIGURE 5: Graphs of skewness and kurtosis of TIHLKw distribution.

3.9. Order Statistics. In statistical theory, order statistics is widely applied and practiced. Let $X_1, X_2, \dots, X_n$ be r.v.s. with their corresponding cdfs $F(x)$. Let $X_{1:n}, X_{2:n}, \dots, X_{n:n}$ be the related ordered r. sample of size n , then the density of r^{th} order statistic is given as

$$f_{r:n}(x) = K \sum_{v=0}^{n-r} (-1)^v \binom{n-r}{v} f(x) F(x)^{v+r-1}, \quad (38)$$

where $K = 1/\beta(r, n-r+1)$ and $\beta(\cdot, \cdot)$ is the beta function. The pdf of the r^{th} order statistic of TIHLKwD is obtained by putting (21) and (22) in (38), changing h with $v+k-1$,

$$f_{r:n}(x) = K \sum_{v=0}^{n-r} \sum_{k,m=0}^{\infty} \eta_v^* x^{a(k+1)-1} (1-x^a)^{bm}, \quad (39)$$

where

$$\eta_v^* = \sum_{l=0}^{v+r-1} \sum_{i,j=0}^{\infty} (-1)^{k+l+m} (2)^{v+r-1} \binom{\lambda(i+1)-1}{j} \binom{b(j+1)-1}{k} \cdot \binom{v+r+l-2}{l} \binom{\lambda(l+v+r-1)}{m} \binom{n-r}{v} \eta_i. \quad (40)$$

More, the s^{th} moment of r^{th} order statistics for TIHLKwD is given by

$$E(X_{r:n}^s) = \int_{-\infty}^{\infty} x^s f_{r:n}(x) dx. \quad (41)$$

By substituting (39) in (41), we have

$$E(X_{r:n}^s) = \frac{1}{\beta(r, n-r+1)} \sum_{v=0}^{n-r} \sum_{k,m=0}^{\infty} \eta_v^* \int_0^1 x^{s+a(k+1)-1} (1-x^a)^{bm} dx. \quad (42)$$

Then,

$$E(X_{r:n}^s) = \frac{1}{B(r, n-r+1)} \sum_{v=0}^{n-r} \sum_{k,m=0}^{\infty} \eta_v^* \beta(s+a(k+1), bm+1). \quad (43)$$

3.10. Rényi Entropy. Rényi (1961) proposed and used this measure. It can be obtained by

$$I_{\delta}(X) = \frac{1}{1-\delta} \log \int_{-\infty}^{\infty} f(x)^{\delta} dx, \delta > 0 \text{ and } \delta \neq 1. \quad (44)$$

TABLE 2: Estimates and MSEs of parameters of TIIHLKw Distribution for the Set1 ($a = 2, \lambda = 2, b = 2$).

n	MLEs		LSEs		WLSEs		CVEs		PCEs	
	Estimates	MSEs	Estimates	MSEs	Estimates	MSEs	Estimates	MSEs	Estimates	MSE
50	2.285	0.821	1.733	0.263	1.833	0.442	1.751	0.293	1.727	0.317
	1.97	0.467	2.566	0.927	2.586	1.327	2.776	1.325	2.747	1.31
	2.092	0.077	1.961	0.258	2.035	0.23	2.099	0.268	2.078	0.372
100	2.226	0.393	1.741	0.239	1.755	0.392	1.703	0.235	1.616	0.304
	1.925	0.329	2.554	0.84	2.651	1.238	2.713	1.128	2.810	1.221
	2.056	0.031	1.961	0.123	1.952	0.093	2.014	0.099	1.929	0.078
200	1.838	0.187	1.759	0.20	1.975	0.327	1.710	0.217	1.700	0.236
	2.287	0.247	2.566	0.83	2.295	0.869	2.611	0.917	2.634	0.905
	1.95	0.018	1.996	0.065	2.001	0.08	1.957	0.06	1.967	0.069
500	1.901	0.079	1.709	0.205	1.872	0.26	1.764	0.173	1.706	0.207
	2.175	0.129	2.572	0.804	2.417	0.853	2.503	0.712	2.564	0.783
	1.973	3.377*	1.93	0.032	1.991	0.031	1.963	0.029	1.921	0.029

* 10^{-3} .TABLE 3: Estimates and MSEs of parameters of TIIHLKwD for the Set2 ($a = 2, \lambda = 3, b = 2$).

n	MLEs		LSEs		WLSEs		CVEs		PCEs	
	Estimates	MSEs	Estimates	MSEs	Estimates	MSEs	Estimates	MSEs	Estimates	MSEs
50	2.209	0.417	1.941	0.291	1.95	0.369	2.069	0.365	2.12	0.327
	2.833	0.492	3.461	1.365	3.499	1.871	3.454	1.49	3.19	1.119
	2.024	0.044	2.02	0.164	2.033	0.25	2.17	0.289	2.171	0.264
100	2.258	0.377	1.973	0.218	1.978	0.329	1.976	0.245	1.974	0.24
	2.689	0.449	3.335	1.12	3.504	1.752	3.35	1.127	3.3	1.01
	2.02	0.022	2.034	0.095	2.036	0.128	2.015	0.093	2.005	0.113
200	2.207	0.132	1.986	0.198	1.938	0.329	1.946	0.212	2.031	0.218
	2.728	0.2	3.285	1.006	3.559	1.708	3.491	1.213	3.277	0.99
	2.052	7.539*	2.002	0.046	2.017	0.055	2.053	0.062	2.063	0.074
500	2.14	0.03	1.95	0.182	1.961	0.323	1.979	0.189	1.958	0.186
	2.78	0.07	3.317	0.925	3.454	1.707	3.251	0.896	3.294	0.902
	2.039	3.826*	1.985	0.022	1.991	0.028	1.985	0.029	1.985	0.025

Applying binomial expansion (16) in (8) then $f(x)^\delta$ can be written as

$$f(x)^\delta = \sum_{i=0}^{\infty} \eta_i x^{\delta(a-1)} (1-x^a)^{\delta(b-1)} \left[1 - (1-x^a)^b\right]^{\lambda(i+\delta)-\delta}, \quad (45)$$

So, the Rényi entropy of TIIHLKw distribution is as follows

$$\begin{aligned} I_\delta(X) &= \frac{1}{1-\delta} \log \sum_{i=0}^{\infty} \eta_i \int_0^1 x^{\delta(a-1)} (1-x^a)^{\delta(b-1)} \\ &\quad \cdot \left[1 - (1-x^a)^b\right]^{\lambda(i+\delta)-\delta} dx. \\ &= \frac{1}{1-\delta} \log \sum_{i,j=0}^{\infty} \eta_{i,j}^* \int_0^1 x^{\delta(a-1)} (1-x^a)^{b(j+\delta)-\delta} dx, \end{aligned} \quad (46)$$

$$\text{where } \eta_{i,j}^* = (-1)^j \binom{\lambda(i+\delta)-\delta}{j} \eta_i.$$

$$I_\delta(X) = \frac{1}{1-\delta} \log \sum_{i,j=0}^{\infty} \eta_{i,j}^* \beta\left(\frac{\delta(a-1)+1}{a}, bj + (b-1)\delta + 1\right). \quad (47)$$

3.11. Stress-Strength Reliability. This subsection deals with the stress-strength parameter of TIIHLKw distribution. Let X_1 be the strength of a structure with a stress X_2 , and if X_1 follows TIIHLKw(λ_1, a_1, b_1) and X_2 follows TIIHLKw(λ_2, a_2, b_2), provided X_1 and X_2 are statistically independent r.v.s.,

TABLE 4: Estimates and MSEs of parameters of TIIHLKwD for the Set3 ($a = 2, \lambda = 2, b = 3$).

n	MLEs		LSEs		WLSEs		CVEs		PCEs	
	Estimates	MSEs	Estimates	MSEs	Estimates	MSEs	Estimates	MSEs	Estimates	MSEs
50	0.262	1.909	1.746	0.285	1.942	0.689	1.828	0.373	1.728	0.309
	0.85	2.376	2.635	1.26	2.569	1.538	2.702	1.461	2.844	1.888
	0.111	2.924	2.931	0.863	3.063	0.941	3.114	0.64	3.055	1.001
100	0.197	2.131	1.671	0.261	1.927	0.489	1.686	0.26	1.704	0.215
	0.231	1.983	2.717	1.11	2.554	1.44	2.731	1.195	2.765	1.32
	0.094	3.134	2.782	0.258	3.003	0.317	2.866	0.286	2.839	0.221
200	0.119	2.007	1.729	0.199	1.945	0.483	1.727	0.211	1.823	0.162
	0.16	2.063	2.598	0.962	2.478	1.167	2.643	0.968	2.509	0.904
	0.046	2.968	2.808	0.156	3.034	0.27	2.886	0.194	2.919	0.127
500	0.077	2.145	1.787	0.158	2.003	0.468	1.764	0.166	1.789	0.152
	0.099	1.876	2.464	0.637	2.391	0.983	2.497	0.727	2.531	0.793
	0.034	3.11	2.863	0.111	3.072	0.254	2.836	0.11	2.928	0.085

* 10^{-3} .

Proposition 2. Under the assumption discussed above, we have

$$R = \eta^* \beta \left(\frac{a_1(m+1)}{a_2}, lb_2 + 1 \right). \quad (48)$$

Proof. The reliability is defined by

$$R = P(X_2 < X_1) = \int_0^\infty f_1(x; \lambda_1, a_1, b_1) F_2(x; \lambda_2, a_2, b_2) dx, \quad (49)$$

Then, we can write

$$\begin{aligned} R &= 2 \sum_{i,j,k=0}^{\infty} (-1)^k \eta_{i,j} \int_0^1 x^{a_1-1} (1-x^{a_1})^{b_1(j+1)-1} \\ &\quad \cdot \left(1 - (1-x^{a_2})^{b_2} \right)^{\lambda_2(k+1)} dx. \\ &= \eta^* \int_0^1 x^{a_1(m+1)-1} (1-x^{a_2})^{lb_2} dx, \end{aligned} \quad (50)$$

where $\eta^* = 2 \sum_{i,j,k,l,m=0}^{\infty} (-1)^{k+l+m} \binom{\lambda_2(k+1)+1}{l} \binom{b_1(j+1)-1}{m} \eta_{i,j}$, which completes the proof.

4. Inference

This section is dedicated to estimation aspects of TIIHLKw distribution, assuming that population parameters (a, b, λ) are unknown and can be estimated using different methods of estimation including ML, LS, WLS, PC, and CV.

4.1. Maximum Likelihood Estimation. For a random sample of $x_1, x_2, x_3, \dots, x_n$ from the TIIHLKw(λ, a, b) distribution, the log-likelihood function for $\Phi = (\lambda, a, b)$ is

$$\begin{aligned} \log L(\Phi) &= n \log(2ab\lambda) + (a-1) \sum_{i=1}^n \log[x_i] \\ &\quad + (\lambda-1) \sum_{i=1}^n \log[1 - (1-x_i^a)^b] + (b-1) \sum_{i=1}^n \log[1 - x_i^a] \\ &\quad + \sum_{i=1}^n \log \left[\left(1 + \left(1 - (1-x_i^a)^b \right)^\lambda \right)^2 \right]. \end{aligned} \quad (51)$$

The members of $U(\Phi) = (U_\lambda, U_a, U_b)$ are given below

$$\begin{aligned} U_\lambda &= \frac{n}{\lambda} + \sum_{i=1}^n \text{Log} \left[1 - (1-x_i^a)^b \right] \\ &\quad - \sum_{i=1}^n \frac{2 \text{Log} \left[1 - (1-x_i^a)^b \right] \left(1 - (1-x_i^a)^b \right)^\lambda}{1 + \left(1 - (1-x_i^a)^b \right)^\lambda}, \end{aligned} \quad (52)$$

$$\begin{aligned} U_a &= \frac{n}{a} + \sum_{i=1}^n \text{Log}[x_i] + (\lambda-1) \sum_{i=1}^n \frac{b \text{Log}[x_i] x_i^a (1-x_i^a)^{-1+b}}{1 - (1-x_i^a)^b} \\ &\quad - (b-1) \sum_{i=1}^n \frac{\text{Log}[x_i] x_i^a}{1 - x_i^a} \\ &\quad - \sum_{i=1}^n \frac{2b\lambda \text{Log}[x_i] x_i^a (1-x_i^a)^{-1+b} \left(1 - (1-x_i^a)^b \right)^{-1+\lambda}}{1 + \left(1 - (1-x_i^a)^b \right)^\lambda}, \end{aligned} \quad (53)$$

TABLE 5: Estimates and MSEs of parameters of TIHLKW Distribution for the Set4 ($a = 1, \lambda = 2, b = 2$).

N	MLEs		LSEs		WLSEs		CVEs		PCEs	
	Estimates	MSEs	Estimates	MSEs	Estimates	MSEs	Estimates	MSEs	Estimates	MSEs
50	1.251	0.2	1.321	0.614	1.353	0.817	1.331	0.645	1.471	1.469
	1.885	0.408	1.822	0.74	1.883	1.103	1.838	0.792	1.865	0.874
	2.104	0.099	2.246	0.571	2.236	0.742	2.379	1.046	2.5	1.88
100	1.232	0.157	1.182	0.195	1.229	0.5	1.226	0.333	1.341	0.501
	1.674	0.239	1.828	0.463	1.97	0.793	1.954	0.701	1.75	0.751
	2.131	0.069	2.145	0.294	2.116	0.181	2.247	0.506	2.307	0.403
200	1.178	0.047	1.164	0.11	1.156	0.229	1.23	0.132	1.181	0.1
	1.66	0.157	1.79	0.34	1.962	0.486	1.713	0.315	1.81	0.319
	2.09	0.011	2.062	0.095	2.117	0.119	2.169	0.123	2.165	0.125
500	1.127	0.022	1.152	0.077	1.194	0.13	1.141	0.066	1.112	0.058
	1.753	0.089	1.746	0.222	1.746	0.261	1.781	0.201	1.849	0.163
	2.068	5.923*	2.092	0.038	2.098	0.056	2.066	0.035	2.065	0.034

* 10^{-3} .

$$U_b = \frac{n}{b} + \sum_{i=1}^n \text{Log}[1 - x_i^a] - (\lambda - 1) \sum_{i=1}^n \frac{\text{Log}1 - x_i^a^b}{1 - (1 - x_i^a)^b} + \sum_{i=1}^n \frac{2\lambda \text{Log}1 - x_i^a^b (1 - (1 - x_i^a)^b)^{-1+\lambda}}{1 + (1 - (1 - x_i^a)^b)^\lambda}. \quad (54)$$

equating (52), (53), and (54) to zero and solve them simultaneously give the ML estimates (MLEs) $\hat{\Phi} = (\lambda\wedge, a\wedge, b\wedge)^T$ of $\Phi = (\lambda, a, b)^T$. The iterative algorithm is used to obtain the numerical solution of these nonlinear equations such as the Newton-Raphson method.

4.2. Ordinary and Weighted LS Estimation (LSEs and WLSEs): The LSEs of λ, a , and b can be obtained by minimizing the sum of squares of errors with respect to parameters. Suppose $X_1, X_2, \dots, X_n$ is a random sample of size n from TIHLKWd and suppose $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ be the related ordered sample.

The sum is independent of the unknown parameters.

$$\sum_{i=1}^n \left[F_{(i)}(x) - \frac{i}{n+1} \right]^2, \quad (55)$$

Equivalently,

$$\sum_{i=1}^n \left[\frac{2 \left(1 - \left(1 - x_{(i)}^a \right)^b \right)^\lambda}{1 + \left(1 - \left(1 - x_{(i)}^a \right)^b \right)^\lambda} - \frac{i}{n+1} \right]^2, \quad (56)$$

TABLE 6: Ranks of all the methods of estimation for different parametric specifications.

Parameters	N	ML	LS	WLS	PC	CV
$a = 2$ $\lambda = 2$ $b = 2$	50	2	1	4.5	3	4.5
	100	1	3.5	5	2	3.5
	200	1	2	5	3	4
	500	1	4	5	2	3
$a = 2$ $\lambda = 3$ $b = 2$	50	2	1	4.5	4.5	3
	100	1.5	1.5	5	4	3
	200	1	2	5	3.5	3.5
	500	1	2	5	4	3
$a = 2$ $\lambda = 2$ $b = 3$	50	1	2	5	3	4
	100	1	3	5	4	2
	200	1	3	5	4	2
	500	1	3	5	4	2
$a = 1$ $\lambda = 2$ $b = 2$	50	1	2	4	3	5
	100	1	2	3.5	3.5	5
	200	1	2	5	3.5	3.5
	500	1	4	5	3	2
$\sum \text{Ranks}$		18.5	38	76.5	54	53
Overall rank		1	2	5	4	3

The following function is minimized with respect to λ, a and b to get WLS estimators of model parameters.

$$\text{WLS}(\lambda, a, b) = \sum_{i=1}^n \frac{(n+1)^2(n+2)}{i(n-i+1)} \cdot \left[\frac{2 \left(1 - \left(1 - x_{(i)}^a \right)^b \right)^\lambda}{1 + \left(1 - \left(1 - x_{(i)}^a \right)^b \right)^\lambda} - \frac{i}{n+1} \right]^2. \quad (57)$$

FIGURE 6: Total test time (TTT) for both data sets.

i.e., $\partial \text{WLS}(\lambda, a, b)/\partial \lambda = 0$, $\partial \text{WLS}(\lambda, a, b)/\partial a = 0$, and $\partial \text{WLS}(\lambda, a, b)/\partial b = 0$.

4.3. PC Estimators (PCEs). Under the specification defined above including order statistics having relationship $X_{(1)} < X_{(2)} < \dots < X_{(n)}$. In the PC method of estimation, the estimators of λ , a , and b are derived by minimizing the following expression

$$\sum_{i=1}^n \left[\ln \left(\frac{i}{n+1} \right) - \ln \left(\frac{2 \left(1 - \left(1 - x_{(i)}^a \right)^b \right)^\lambda}{1 + \left(1 - \left(1 - x_{(i)}^a \right)^b \right)^\lambda} \right) \right]^2. \quad (58)$$

with respect to λ , a , and b .

4.4. The Cramer-von Mises Minimum Distance Estimation. The CVE is another estimation method based on minimum distance. The CV estimators are obtained by minimizing, with respect to λ , a , and b .

$$CV = \frac{1}{12n} + \sum_{i=1}^n \left[\frac{2 \left(1 - \left(1 - x_{(i)}^a \right)^b \right)^\lambda}{1 + \left(1 - \left(1 - x_{(i)}^a \right)^b \right)^\lambda} - \frac{2i-1}{2n} \right]^2. \quad (59)$$

CV minimum distance estimators provide empirical evidence that the bias of this estimator is smaller than the other minimum distance estimators.

5. Simulation Study

In this section, we present a Monte Carlo (MC) simulation study in order to illustrate the behavior of different estimates. We consider four random sample sizes: $n = 50, 100, 200$, and 500 from the TIIHLKw distribution and the samples are drawn 10,000 times. Four specifications of the parameters

are used in this simulation study, given as, set1 = $(\lambda = 2, b = 0.5, a = 2)$, set2 = $(\lambda = 3, b = 2, a = 2)$, set3 = $(\lambda = 2, b = 3, a = 2)$, set4 = $(\lambda = 2, b = 2, a = 1)$. For sample generated, the MLE, LSE, WLSE, CVE, PCE, and MPSE of estimators are computed numerically. Then, the estimates of all methods and their mean square errors (MSEs) are documented in Tables 1–4.

Estimates of parameters and their corresponding MSEs are calculated using the same sample under different methods of estimation for four sets of parameters mentioned above for small sample size ($n = 50$) to sufficient large sample size, that is, $n = 500$.

The entries of Table 2 to Table 5 show that estimates are reliable and consistent. MSEs reduce as sample size (n) increases under each method of estimation. The summary of these four tables is presented in Table 5.

Using the entries of Table 6 for different parametric combinations, we can conclude that the MLE method outperforms than all other estimation methods (with an overall score of 18.5). Therefore, depending on the simulation study, the MLE method performs best for TIIHLKwD.

6. Applications

The TIIHLKw distribution aims at providing an alternative distribution to fit data on the unit interval to other distributions available in the literature. Here, we used the following probability distributions as competitor models;

(i) The Kumaraswamy (Kw) distribution

$$f(x; a, b) = abx^{a-1}(1-x^a)^{b-1}, 0 < x < 1. \quad (60)$$

(ii) Transmuted Kumaraswamy (TKw) distribution

$$f_{TKw}(x; a, b) = abx^{a-1}(1-x^a)^{b-1} \left(1 - \lambda + 2\lambda(1-x^a)^b \right). \quad (61)$$

TABLE 7: MLEs and goodness-of-fit measures for the first data set.

Model	$\hat{a}$	$\hat{b}$	$\hat{\lambda}$	L	AIC	BIC	A_*	W^*
TIHLKw	8.45118	28229.6	0.209229	57.6664	-109.333	-104.597	0.336229	0.014361
Kw	2.07740	33.1374	—	56.0687	-108.137	-104.313	0.688407	0.105252
Tkw	2.07740	16.5687	0.99999	56.0687	-106.137	-100.401	0.688407	0.105252
SBKw	1.44716	19.9669	—	55.2067	-106.413	-102.589	0.838017	0.127612
Beta	2.68257	13.8658	—	54.6067	-105.213	-97.4772	0.926850	0.155443

TABLE 8: MLEs and goodness-of-fit measures for the second data set.

Model	$\hat{a}$	$\hat{b}$	$\hat{\lambda}$	L	AIC	BIC	A^*	W^*
TIHLKw	0.32008	5.84284	222.766	58.5465	-111.093	-105.479	0.17531	0.025823
Kw	2.71874	44.6604	—	52.4915	-100.983	-97.2407	1.31057	0.208116
TKw	2.98542	43.9205	0.651548	53.7972	-101.594	-95.9808	1.02032	0.155492
SBKw	2.19509	32.9024	—	53.6088	-103.218	-99.4752	1.10420	0.174441
Beta	5.94177	21.2057	—	55.6002	-107.200	-99.5868	0.78992	0.131428

FIGURE 7: The fitted pdf, cdf, survival, and PP plots of the TIHLKw distribution for the first data.

FIGURE 8: The fitted pdf, cdf, survival, and PP plots of the TIIHLKw distribution for second data.

(iii) Size biased Kumaraswamy (SBKw) distribution

$$f_{SBKw}(x; a, b) = \frac{ax^a(1-x^a)^{b-1}}{\text{Beta}[1 + (1/a), b]}. \quad (62)$$

(iv) Beta distribution

$$f_{Beta}(x; a, b) = \frac{1}{\text{Beta}[a, b]} x^{a-1} (1-x)^{b-1}. \quad (63)$$

We use the following accuracy measures for model comparison: Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), log-likelihood, Anderson-Darling (A^*), and Cramer-von Mises (W^*). R-Language is used for numerical computations.

1st data set. The data set was taken from Dasgupta [20], and considered $n = 50$ on burr (in millimeters), with hole diameter and a sheet thickness of 12 mm and 3.15 mm, respectively.

2nd data set. This data set was taken from [21], and considered $n = 48$ measurements on petroleum rock samples from a petroleum reservoir.

The total test time (TTT) for both datasets are presented in Figure 6. We can observe that the shape of TTT plots is concave for both datasets, which demonstrates an increasing failure rate.

The MLEs for TIIHLKw distribution along with some adequacy measures are presented in Tables 7 and 8.

Hence, it is concluded that the new model provides the better fit. Figures 7 and 8 show the estimated densities, cdfs, estimated survival functions, and PP plots for the considered distributions of both data sets, respectively. We note that the proposed model is more appropriated to fit the data than the other competing models.

7. Conclusion

In this article, a new Type-II Half Logistic Kumaraswamy distribution is proposed. Some characteristics of the TIIHLKw distribution including linear combination expressions for the

density function, probability weighted moments, moments, incomplete moments, quantile function, mean deviation about mean and about median, Bonferroni and Lorenz curves, order statistics, stress-strength reliability, and Rényi entropy are derived. The ML method is used to estimate model parameters. An extensive simulation study is conducted to compare several well-known estimation methods, including the method of maximum likelihood estimation, methods of least squares and weighted least squares estimation, and method of Cramer-von Mises minimum distance estimation. The simulation study showed the reliability and efficiency of the estimates. Finally, by considering the method of maximum likelihood estimation, the new model is fitted to two practical data sets. The applications on real data sets validated the significance of the new distribution.

Appendix

A.1. Data I

0.04, 0.02, 0.06, 0.12, 0.14, 0.08, 0.22, 0.12, 0.08, 0.26, 0.24, 0.04, 0.14, 0.16, 0.08, 0.26, 0.32, 0.28, 0.14, 0.16, 0.24, 0.22, 0.12, 0.18, 0.24, 0.32, 0.16, 0.14, 0.08, 0.16, 0.24, 0.16, 0.32, 0.18, 0.24, 0.22, 0.16, 0.12, 0.24, 0.06, 0.02, 0.18, 0.22, 0.14, 0.06, 0.04, 0.14, 0.26, 0.18, 0.16.

A.2. Data II

0.0903296, 0.2036540, 0.2043140, 0.2808870, 0.1976530, 0.3286410, 0.1486220, 0.1623940, 0.2627270, 0.1794550, 0.3266350, 0.2300810, 0.1833120, 0.1509440, 0.2000710, 0.1918020, 0.1541920, 0.4641250, 0.1170630, 0.1481410, 0.1448100, 0.1330830, 0.2760160, 0.4204770, 0.1224170, 0.2285950, 0.1138520, 0.2252140, 0.1769690, 0.2007440, 0.1670450, 0.2316230, 0.2910290, 0.3412730, 0.4387120, 0.2626510, 0.1896510, 0.1725670, 0.2400770, 0.3116460, 0.1635860, 0.1824530, 0.1641270, 0.1534810, 0.1618650, 0.2760160, 0.2538320, 0.2004470.

Data Availability

Data is present in appendix.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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