Hindawi
Journal of Function Spaces
Volume 2020, Article ID 1540383, 13 pages
https://doi.org/10.1155/2020/1540383

Research Article

Existence and Global Asymptotic Behavior of S-Asymptotically ω-Periodic Solutions for Evolution Equation with Delay

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Received 2 July 2020; Revised 12 October 2020; Accepted 25 October 2020; Published 19 November 2020

Academic Editor: Mark A. McKibben

This paper is concerned with the abstract evolution equation with delay. Firstly, we establish some sufficient conditions to ensure the existence results for the S-asymptotically periodic solutions by means of the compact semigroup. Secondly, we consider the global asymptotic behavior of the delayed evolution equation by using the Gronwall-Bellman integral inequality involving delay. These results improve and generalize the recent conclusions on this topic. Finally, we give an example to exhibit the practicability of our abstract results.

1. Introduction

Let X be a Banach space with norm $\|\cdot\|$ and $r > 0$ be a constant. Let $\mathscr{B} := C([-r, 0], X)$ be the Banach space of continuous functions from $[-r, 0]$ into X provided with the uniform norm $\|\phi\|_{\mathscr{B}} = \sup_{s\in[-r,0]} \|\phi(s)\|$. If $u : [0,\infty) \longrightarrow X$ is a continuous bounded function, then $u_t \in \mathscr{B}$ for each $t \geq 0$, where u_t defined by $u_t(s) := u(t + s)$ for $s \in [-r, 0]$.

In this article, we discuss the following delayed evolution equation (DEE)

$$u'(t) + Au(t) = F(t, u(t), u_t),\ t \geq 0 \tag{1}$$

with initial value condition $u(t) = \varphi(t)$ for $t \in [-r, 0]$, where $A : D(A) \subset X \longrightarrow X$ be a closed linear operator, and $-A$ generate a C_0-semigroup $T(t)(t \geq 0)$ in X; $F : \mathbb{R}^+ \times X \times \mathscr{B} \longrightarrow X$ is a given function which will be specified later, $\varphi \in \mathscr{B}$.

Delayed partial differential equations play a major role in evolution equations. Due to its extensive background in physics, chemistry, realistic mathematical model, and other aspects, delayed partial differential equations have attracted attentions of many scholars in recent years, see [1, 2] and the references therein. On the other hand, periodic oscillations occur frequently in many fields, which are natural and significant phenomena. However, the real concrete systems are usually represented by internal variations and external perturbations, which are approximately periodic. Therefore, Henriquez and Pierri [3] first proposed the concept of S-asymptotically ω-periodicity and found that S-asymptotically ω-periodicity is a generalization for the classical asymptotically. Compared to asymptotically periodic systems, from an application perspective, S-asymptotically periodic systems can reflect the actual world more really and more exactly. Thus, it is necessary to study S-asymptotically ω-periodic solutions for the delayed evolution equations.

Some scholars have discussed the existence results about S-asymptotically ω-periodic solutions for differential equations (one can see [3–15]). In these works, under the assumption that the nonlinear terms satisfy the Lipschitz type conditions, the existence and uniqueness results about S-asymptotically ω-periodic solutions are explored by using the principle of contractive mapping. However, based on the fact that the nonlinear functions represent the source of population or material in many complicated reaction-diffusion equations, the nonlinear functions depend on time in diversified ways. Therefore, we expect to obtain more general growth conditions instead of Lipschitz type conditions for most cases.

In addition, the global asymptotic behavior is one of the major problem encountered in applications and has attracted considerable attentions. Some scholars study the global exponential stability of differential equations by constructing Lyapunov functions or applying matrix theory (one can see [16–21] and the references therein). However, it is hard to establish Lyapunov functions or apply the matrix theory to study the global exponential stability for delayed partial differential equations. On the other hand, in view of the asymptotical periodic phenomena in many applied disciplines, it has a profound application prospect to discuss the global asymptotical periodicity of differential equations. In particular, in [22, 23], significant results have been obtained on the global asymptotic periodicity of neural networks. However, as far as we know, no similar results have been published for abstract evolution equations.

Motivated by the above discussions, we consider S-asymptotically ω-periodic solutions about the delayed evolution equation. Our aims are to explore the existence result for the S-asymptotically ω-periodic solutions and consider the global asymptotic behavior for DEE (1). Firstly, the existence of S-asymptotically ω-periodic mild solutions of DEE (1) under the nonlinear function F satisfying some growth conditions is explored by applying the semigroup theory of operators and fixed point theorem. Secondly, by using the integral inequality of Gronwall-Bellman type involving delay, we consider not only the global exponential stability but also the global asymptotic periodicity for DEE (1), which fills the gap in this field. Compared with constructing Lyapunov functions or applying matrix theory, our avenue is simpler. Finally, an example is proposed to verify the applicability of abstract results. In the next section, some notions, definitions, and preliminary facts that we need are provided.

2. Preliminaries

Throughout this article, let $(X, \|\cdot\|)$ be a Banach space, and let $A : D(A) \subset X \longrightarrow X$ be a closed linear operator, and $-A$ generate a C_0-semigroup $T(t)(t \geq 0)$ in X.

Generally, for a C_0-semigroup $T(t)(t \geq 0)$, there exist $M \geq 1$ and $\nu \in \mathbb{R}$ such that

$$\|T(t)\| \leq Me^{\nu t}, t \geq 0. \tag{2}$$

The growth exponent of the C_0-semigroup $T(t)(t \geq 0)$ can be defined by

$$\nu_0 = \inf \left\{ \nu \in \mathbb{R} \mid \text{there exists } M \geq 1 \text{ such that } \|T(t)\| \leq Me^{\nu t},\ \forall t \geq 0 \right\}. \tag{3}$$

If the C_0-semigroup $T(t)(t \geq 0)$ is continuous in the uniform operator topology for every $t \geq 0$ in X, ν_0 can also be determined by $\sigma(A)$ (the spectrum of A),

$$\nu_0 = -\inf \{\mathrm{Re}\,\lambda \mid \lambda \in \sigma(A)\}. \tag{4}$$

As we all know, if $T(t)(t \geq 0)$ is a compact semigroup, then $T(t)(t \geq 0)$ is continuous in the uniform operator topology for $t \geq 0$. Furthermore, if $\nu_0 < 0$, then the C_0-semigroup $T(t)(t \geq 0)$ is said to be exponentially stable. For more detailed theory of semigroups of the linear operator, one can find in [24, 25].

Now, let $C_b(\mathbb{R}^+, X)$ denote the set of all bounded and continuous functions from $\mathbb{R}^+$ to X equipped with norm $\|u\|_C = \sup_{t\in\mathbb{R}^+} \|u(t)\|$; then, $C_b(\mathbb{R}^+, X)$ is a Banach space.

Let $h : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a continuous and nondecreasing function such that $h(t) \geq 1$ for all $t \in \mathbb{R}^+$ and $\lim_{t\to\infty} h(t) = \infty$. We consider the space

$$C_h(X) = \left\{ u \in C(\mathbb{R}^+, X) \colon \lim_{t\to\infty} \frac{\|u(t)\|}{h(t)} = 0 \right\} \tag{5}$$

endowed with the norm $\|u\|_h = \sup_{t\geq 0} (\|u(t)\|/h(t))$.

Lemma 1 ([26]). *A set $B \subset C_h(X)$ is relatively compact in $C_h(X)$ if and only if, (i) B is equicontinuous; (ii) $\lim_{t\to\infty} \|u(t)\|/h(t) = 0$, uniformly for $u \in B$; and (iii) the set $B(t) = \{u(t) \colon u \in B\}$ is relatively compact in X, for every $t \geq 0$.*

Define

$$\begin{aligned} \mathscr{B}C_h(X) = \{ & u \in C([-r, +\infty), X) \colon u(t) \\ & = \varphi(t), t \in [-r, 0], \varphi \in \mathscr{B};\ u|_{t\geq 0} \in C_h(X)\} \end{aligned} \tag{6}$$

endowed with the norm $\|u\|_{\mathscr{B},h} = \|\varphi\|_{\mathscr{B}} + \|u\|_h$.

We write

$$\begin{aligned} \mathscr{B}C_b(X) = \{ & u \in C([-r, +\infty), X) \colon u(t) \\ & = \varphi(t), t \in [-r, 0];\ \varphi \in \mathscr{B};\ u|_{t\geq 0} \in C_b(X)\} \end{aligned} \tag{7}$$

endowed with the norm $\|u\|_\infty = \|\varphi\|_{\mathscr{B}} + \|u\|_C$. It is not difficult to verify that $\mathscr{B}C_b(X)$ is a Banach space.

Next, we introduce a standard definition of the S-asymptotically ω-periodic function.

Definition 2 ([3]). A function $u \in C_b(\mathbb{R}^+, X)$ is said to be the S-asymptotically ω-periodic function, if there exists $\omega > 0$ such that $\lim_{t\to\infty} \|u(t + \omega) - u(t)\| = 0$. In this case, we say that ω is an asymptotic periodic of u. It is obvious that if ω is an asymptotic period for u, then every $k\omega$ is also an asymptotic period of u, $k = 1, 2$.

Let $SAP_\omega(X)$ represent the subspace of $C_b(\mathbb{R}^+, X)$ consisting of all the X value S-asymptotically ω-periodic functions equipped with the uniform convergence norm. Then, $SAP_\omega(X)$ is a Banach space (see [20, Proposition 3.5]). If $u \in SAP_\omega(X)$, then it is easy to verify that the function $t \longrightarrow u_t$ belongs to $SAP_\omega(\mathscr{B})$ (see [27, 28]).

In order to study the S-asymptotically ω-periodic mild solution, for any given $\varphi \in \mathscr{B}$, we define

$$\begin{aligned}\mathscr{B}SAP_{\omega}(X) = \{u \in C([-r,+\infty), X)\colon u(t) \\ = \varphi(t), t \in [-r,0]\,; u|_{t\geq 0} \in SAP_{\omega}(X)\}\end{aligned} \tag{8}$$

endowed with the norm $\|u\|_{\infty} = \|\varphi\|_{\mathscr{B}} + \|u\|_{C}$.

There are some basic definitions involved in this paper.

Definition 3. A function $u \in C([-r,\infty), X)$ is said to be called mild solution of DEE (1) if $u(t) = \varphi(t)$ for $t \in [-r,0]$,

$$u(t) = T(t)\varphi(0) + \int_0^t T(t-s)F(s,u(s),u_s)ds, t \geq 0. \tag{9}$$

Moreover, if $u \in \mathscr{B}SAP_{\omega}(X)$, then u is called an S-asymptotically ω-periodic mild solution of DEE (1.1).

Definition 4. Assume that u is a S-asymptotically ω-periodic mild solution of DEE (1) with the initial conditions $u(s) = \varphi(s)$ for $s \in [-r,0]$, if there exist positive constants N and α, such that $\|u(t) - v(t)\| \leq N\|\varphi - \phi\|_{\mathscr{B}} \cdot e^{-\alpha t}$ for all $t \geq 0$, then the S-asymptotically ω-periodic mild solution u is said to be globally exponentially stable, where $v(t)$ is a mild solution of DEE (1) corresponding to the initial conditions $v(s) = \phi(s)$,$s \in [-r,0]$.

Definition 5. DEE (1) is said to be globally asymptotically ω-periodic if there is an ω-periodic function $u^*(t)$, such that all solutions of DEE (1.1) convergent to $u^*(t)$.

In some proofs, the following inequality is also needed.

Lemma 6 ([29]). *Let $\psi \in C([-r,\infty), \mathbb{R}^+)$. If there are constants $l_1, l_2 > 0$ such that*

$$\psi(t) \leq \psi(0) + \int_0^t l_1\psi(s) + l_2 \sup_{\tau\in[-r,0]} \psi(s+\tau)ds, t \geq 0. \tag{10}$$

Then, $\psi(t) \leq \|\psi\|_{\mathscr{B}} \cdot e^{(l_1+l_2)t}$ for each $t \geq 0$.

3. Main Results

Theorem 7. *Let $-A$ generate a compact and exponentially stable C_0-semigroup $T(t)(t \geq 0)$ in X, whose growth exponent denotes ν_0. Let $F : \mathbb{R}^+ \times X \times \mathscr{B} \longrightarrow X$ be a continuous mapping. If the following conditions (H1) for all $x \in X$ and $\phi \in \mathscr{B}$, there is $\omega > 0$, such that*

$$\lim_{t\to\infty} \|F(t+\omega,x,\phi) - F(t,x,\phi)\| = 0\,; \tag{11}$$

(H2) for all $t \in \mathbb{R}^+$, $x \in X$, and $\phi \in \mathscr{B}$, there are integrable function $p_i : \mathbb{R}^+ \longrightarrow \mathbb{R}^+ (i = 1,2)$ and continuous nondecreasing function $\Phi_i : \mathbb{R}^+ \longrightarrow \mathbb{R}^+ (i = 1,2)$ and positive constant $\mathscr{K}$ such that

$$\begin{aligned}&\|F(t,h(t)x,h(t)\phi)\| \leq p_1(t)\Phi_1(\|x\|) + p_2(t)\Phi_2(\|\phi\|_{\mathscr{B}}) + \mathscr{K}, \\ &\liminf_{r\to\infty} \frac{\Phi_i(r)}{r} = \sigma_i < \infty \quad i = 1,2.\end{aligned} \tag{12}$$

(H3) $M(\rho_1\sigma_1 + \rho_2\sigma_2) < 1$, where $\rho_i = \sup_{t\geq 0}\int_0^t e^{\nu_0(t-s)}p_i(s)ds$, $(i = 1,2)$ hold, then DEE (1) has at least one S-asymptotically ω-periodic mild solution $u \in \mathscr{B}SAP_{\omega}(X)$.

Proof. Define an operator Γ on $\mathscr{B}C_h(X)$ by $\Gamma u(t) = \varphi(t)$ for any $t \in [-r,0]$,

$$\Gamma u(t) = T(t)\varphi(0) + \int_0^t T(t-s)F(s,u(s),u_s)ds,\ t \geq 0, \tag{13}$$

where $\varphi \in \mathscr{B}$. It is easy to test that $\Gamma : \mathscr{B}C_h(X) \longrightarrow \mathscr{B}C_h(X)$ is well defined. In fact, for any $u \in \mathscr{B}C_h(X)$, we have $\|u(t)\| \leq h(t)\|u\|_{\mathscr{B},h}$,

$$\begin{aligned}\|u_t\|_{\mathscr{B}} &= \sup_{s\in[-r,0]} \|u(t+s)\| \leq \sup_{t\in[-r,0]} \|u(t)\| + \sup_{t\in[0,\infty)} \|u(t)\| \\ &\leq \|\varphi\|_{\mathscr{B}} + h(t)\|u\|_h \leq h(t)\|\varphi\|_{\mathscr{B}} + h(t)\|u\|_h \\ &\leq h(t)\|u\|_{\mathscr{B},h}.\end{aligned} \tag{14}$$

By the condition (H2), we obtain

$$\begin{aligned}&\frac{\left\|\int_0^t T(t-s)F(s,u(s),u_s)ds\right\|}{h(t)} \\ &\leq \frac{1}{h(t)}\int_0^t \|T(t-s)\| \cdot \|F(s,u(s),u_s)\|ds \\ &\leq \frac{1}{h(t)}\int_0^t Me^{\nu_0(t-s)} \cdot \left(p_1(s)\Phi_1\left(\frac{\|u(s)\|}{h(s)}\right)\right. \\ &\quad \left. + p_2(s)\Phi_2\left(\frac{\|u_s\|_{\mathscr{B}}}{h(s)}\right) + \mathscr{K}\right)ds \\ &\leq \frac{1}{h(t)}\int_0^t Me^{\nu_0(t-s)} \cdot (p_1(s)\Phi_1(\|u\|_{\mathscr{B},h}) \\ &\quad + p_2(s)\Phi_2(\|u\|_{\mathscr{B},h}) + \mathscr{K})ds \\ &\leq \frac{M}{h(t)}\left(\frac{\mathscr{K}}{|\nu_0|} + \int_0^t e^{\nu_0(t-s)} \cdot (p_1(s)\Phi_1(\|u\|_{\mathscr{B},h})\right. \\ &\quad \left. + p_2(s)\Phi_2(\|u\|_{\mathscr{B},h}))ds\right) \\ &\leq \frac{M}{h(t)}\left(\frac{\mathscr{K}}{|\nu_0|} + \Phi_1(\|u\|_{\mathscr{B},h})\rho_1 + \Phi_2(\|u\|_{\mathscr{B},h})\rho_2\right),\end{aligned} \tag{15}$$

where $\rho_i = \sup_{t\geq 0}\int_0^t e^{\nu_0(t-s)}p_i(s)ds$, $(i = 1,2)$. Hence, $\Gamma : \mathscr{B}C_h(X) \longrightarrow \mathscr{B}C_h(X)$ is well defined. By (13) and Definition 3, we can assert $u \in \mathscr{B}C_h(X)$ is the mild solution for DEE (1) and is equal to u that is the fixed point for operator Γ.

To do this, we will carry the proof out in six steps.

Step 8. Γ is continuous on $\mathscr{B}C_h(X)$. In $\mathscr{B}C_h(X)$, there is a sequence $\{u^{(n)}\}$ such that $u^{(n)} \longrightarrow u$ as $n \longrightarrow \infty$; then, $u_t^{(n)} \longrightarrow u_t(n \longrightarrow \infty)$ for all $t \in [0,\infty)$. Combining this with the definition of Γ, for any $t \in [-r, 0]$, we know that

$$\frac{\|\Gamma u^{(n)}(t) - \Gamma u(t)\|}{h(t)} = \frac{\|\varphi(t) - \varphi(t)\|}{h(t)} = 0, \tag{16}$$

and we can conclude from the continuity of F that

$$F\left(t, u^{(n)}(t), u_t^{(n)}\right) \longrightarrow F(t, u(t), u_t) \text{ as } n \longrightarrow \infty \text{for any } t \in [0, +\infty). \tag{17}$$

Together with the Lebesgue dominated convergence theorem, we get

$$\begin{aligned} \frac{\|\Gamma u^{(n)}(t) - \Gamma u(t)\|}{h(t)} &= \frac{1}{h(t)} \left\| \int_0^t T(t-s) \cdot \left(F\left(s, u^{(n)}(s), u_s^{(n)}\right) \right.\right. \\ &\quad \left.\left. - F(s, u(s), u_s) \right) ds \right\| \\ &\leq \frac{1}{h(t)} \int_0^t \|T(t-s)\| \cdot \left\| F\left(s, u^{(n)}(s), u_s^{(n)}\right) \right. \\ &\quad \left. - F(s, u(s), u_s) \right\| ds \longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned} \tag{18}$$

Hence, we say that operator Γ is continuous from $\mathscr{B}C_h(X)$ to $\mathscr{B}C_h(X)$.

For any $R > 0$, let

$$\bar{\Omega}_R \coloneqq \left\{ u \in \mathscr{B}C_h(X) \mid \|u\|_{\mathscr{B},h} \leq R \right\}. \tag{19}$$

Obviously, $\bar{\Omega}_R$ is a closed ball in $\mathscr{B}C_h(X)$.

Step 9. There is a constant $R_0 > 0$ big enough such that $\Gamma(\bar{\Omega}_{R_0}) \subset \bar{\Omega}_{R_0}$.

If this is incorrect, there are $u \in \bar{\Omega}_R$ and $t \geq 0$ such that $\|\Gamma u(t)\| > R$ for any $R > 0$. Thus, by (H2), one can see that

$$\begin{aligned} R &< \frac{\|\Gamma u(t)\|}{h(t)} \leq \frac{1}{h(t)} \|T(t)\varphi(0)\| + \frac{1}{h(t)} \int_0^t \|T(t-s)\| \\ &\quad \cdot \|F(s, u(s), u_s)\| ds \\ &\leq \frac{M e^{\nu_0 t} \|\varphi\|_{\mathscr{B}}}{h(t)} + \frac{M}{h(t)} \int_0^t e^{\nu_0 (t-s)} \cdot (p_1(s) \Phi_1(R) \\ &\quad + p_2(s) \Phi_2(R) + \mathscr{K}) ds \\ &\leq \frac{M}{h(t)} \left(\|\varphi\|_{\mathscr{B}} + \frac{\mathscr{K}}{|\nu_0|} \right) + \frac{M}{h(t)} \int_0^t e^{\nu_0 (t-s)} \cdot (p_1(s) \Phi_1(R) \\ &\quad + p_2(s) \Phi_2(R)) ds \\ &\leq \frac{M}{h(t)} \left(\|\varphi\|_{\mathscr{B}} + \frac{\mathscr{K}}{|\nu_0|} \right) + \frac{M}{h(t)} (\rho_1 \Phi_1(R) + \rho_2 \Phi_2(R)) \\ &\leq M \left(\|\varphi\|_{\mathscr{B}} + \frac{\mathscr{K}}{|\nu_0|} \right) + M(\rho_1 \Phi_1(R) + \rho_2 \Phi_2(R)). \end{aligned} \tag{20}$$

Dividing both sides of (20) by R and taking the lower limit as $R \longrightarrow +\infty$, and comparing this with the condition (H3), it follows that

$$1 \leq M(\rho_1 \sigma_1 + \rho_2 \sigma_2) < 1, \tag{21}$$

which is a contradiction. Hence, the conclusion is valid.

Step 10. The set

$$\Lambda(t) \coloneqq \left\{ \Gamma u(t) \mid u \in \bar{\Omega}_{R_0}, t \in [-r, a] \right\} \tag{22}$$

is relatively compact on X for every $a \in (0,\infty)$. From $\Gamma u(t) = \varphi(t)$ for every $u \in \bar{\Omega}_{R_0}$ and $t \in [-r, 0]$, we can conclude that $\Lambda(t)$ is relatively compact on X for $t \in [-r, 0]$. For $t \in [0, a]$, a set $\{\Lambda_\varepsilon(t)\}$ is defined by

$$\Lambda_\varepsilon(t) \coloneqq \left\{ \Gamma_\varepsilon u(t) \mid u \in \bar{\Omega}_{R_0}, \varepsilon \in (0, t), t \in [0, a] \right\}, \tag{23}$$

with

$$\begin{aligned} \Gamma_\varepsilon u(t) &= T(t)\varphi(0) + \int_0^{t-\varepsilon} T(t-s) F(s, u(s), u_s) ds \\ &= T(t)\varphi(0) + T(\varepsilon) \int_0^{t-\varepsilon} T(t-\varepsilon-s) \cdot F(s, u(s), u_s) ds. \end{aligned} \tag{24}$$

According to the compactness of the semigroup $T(t)(t \geq 0)$, $\{\Lambda_\varepsilon(t)\}$ is relatively compact on X for $\varepsilon \in (0, t)$. Thus, for any $u \in \bar{\Omega}_{R_0}$, $t \in [0, a]$, from the condition (H2), we obtain

$$\begin{aligned} &\|\Gamma u(t) - \Gamma_\varepsilon u(t)\| \\ &= \left\| \int_{t-\varepsilon}^t T(t-s) \cdot F(s, u(s), u_s) ds \right\| \\ &\leq \int_{t-\varepsilon}^t \|T(t-s)\| \cdot \|F(s, u(s), u_s)\| ds \\ &\leq \int_{t-\varepsilon}^t \|T(t-s)\| \cdot \left(p_1(s) \Phi_1\left(\frac{\|u(s)\|}{h(s)} \right) \right. \\ &\quad \left. + p_2(s) \Phi_2\left(\frac{\|u_s\|_{\mathscr{B}}}{h(s)} \right) + \mathscr{K} \right) ds \\ &\leq \int_{t-\varepsilon}^t \|T(t-s)\| \cdot (p_1(s) \Phi_1(\|u\|_{\mathscr{B},h}) \\ &\quad + p_2(s) \Phi_2(\|u\|_{\mathscr{B},h}) + \mathscr{K}) ds \\ &\leq \int_{t-\varepsilon}^t \|T(t-s)\| \cdot (p_1(s) \Phi_1(R_0) + p_2(s) \Phi_2(R_0) + \mathscr{K}) ds \\ &\leq M \int_{t-\varepsilon}^t e^{\nu_0 (t-s)} \cdot (p_1(s) \Phi_1(R_0) + p_2(s) \Phi_2(R_0) + \mathscr{K}) ds \\ &\longrightarrow 0 \text{ as } \varepsilon \longrightarrow 0. \end{aligned} \tag{25}$$

Namely, there are relatively compact sets, which are arbitrarily close to the set $\Lambda(t)$. It means that for any $t \in [0, a]$, the set $\Lambda(t)$ is relatively compact in X.

Step 11. $\Gamma(\bar{\Omega}_{R_0})$ is equicontinuous. For any $u \in \bar{\Omega}_{R_0}$, in view of (13), we only need to verify it on $[0, \infty)$. In general, assume that $0 \le t_1 < t_2$, we know that

$$\begin{aligned}\Gamma u(t_2) - \Gamma u(t_1) &= T(t_2)\varphi(0) + \int_0^{t_2} T(t_2 - s) F(s, u(s), u_s) ds \\ &\quad - T(t_1)\varphi(0) - \int_0^{t_1} T(t_1 - s) \cdot F(s, u(s), u_s) ds \\ &= T(t_2)\varphi(0) - T(t_1)\varphi(0) + \int_0^{t_1} (T(t_2 - s) - T(t_1 - s)) \\ &\quad \cdot F(s, u(s), u_s) ds + \int_{t_1}^{t_2} T(t_2 - s) \cdot F(s, u(s), u_s) ds \\ &:= J_1 + J_2 + J_3.\end{aligned} \tag{26}$$

Obviously,

$$\|\Gamma u(t_2) - \Gamma u(t_1)\| \le \|J_1\| + \|J_2\| + \|J_3\|. \tag{27}$$

Moreover, since $t \longrightarrow \|T(t)\|$ is continuous for $t > 0$, then we have

$$\begin{aligned}\|J_1\| &= \|T(t_2)\varphi(0) - T(t_1)\varphi(0)\| \\ &\le \|T(t_2) - T(t_1)\| \|\varphi\|_{\mathscr{B}} \le \|T(t_2 - t_1) - I\| \cdot \|T(t_1)\| \|\varphi\|_{\mathscr{B}} \\ &\longrightarrow 0 \; as \; t_2 - t_1 \longrightarrow 0,\end{aligned} \tag{28}$$

and taking $\varepsilon > 0$ small enough which is independent of t_1 and t_2, by the condition (H2) and (19), we arrive at

$$\begin{aligned}\|J_2\| &= \left\| \int_0^{t_1} (T(t_2 - s) - T(t_1 - s)) \cdot F(s, u(s), u_s) ds \right\| \\ &\le \int_0^{t_1 - \varepsilon} \|T(t_2 - s) - T(t_1 - s)\| \cdot \|F(s, u(s), u_s)\| ds \\ &\quad + \int_{t_1 - \varepsilon}^{t_1} \|T(t_2 - s) - T(t_1 - s)\| \cdot \|F(s, u(s), u_s)\| ds \\ &\le \|T(t_2 - t_1 + \varepsilon) - T(\varepsilon)\| \cdot \int_0^{t_1 - \varepsilon} \|T(t_1 - s - \varepsilon)\| \\ &\quad \cdot (p_1(s)\Phi_1(R_0) + p_2(s)\Phi_2(R_0) + \mathscr{K}) ds \\ &\quad + \int_{t_1 - \varepsilon}^{t_1} (\|T(t_2 - s)\| + \|T(t_1 - s)\|) \cdot (p_1(s)\Phi_1(R_0) \\ &\quad + p_2(s)\Phi_2(R_0) + \mathscr{K}) ds \\ &\le \|T(t_2 - t_1 + \varepsilon) - T(\varepsilon)\| M \left(\Phi_1(R_0)\rho_1 + \Phi_2(R_0)\rho_2 + \frac{\mathscr{K}}{|\nu_0|} \right) \\ &\quad + 2M \int_{t_1 - \varepsilon}^{t_1} (p_1(s)\Phi_1(R_0) + p_2(s)\Phi_2(R_0) + \mathscr{K}) ds \\ &\longrightarrow 0 \; as \; t_2 - t_1 \longrightarrow 0.\end{aligned} \tag{29}$$

Due to the exponentially stable semigroup $T(t) (t \ge 0)$ that is uniformly bounded, one can see that

$$\begin{aligned}\|J_3\| &= \left\| \int_{t_1}^{t_2} T(t_2 - s) F(s, u(s), u_s) ds \right\| \\ &\le \int_{t_1}^{t_2} \|T(t_2 - s)\| \cdot \|F(s, u(s), u_s)\| ds \\ &\le \int_{t_1}^{t_2} \|T(t_2 - s)\| \cdot (p_1(s)\Phi_1(R_0) + p_2(s)\Phi_2(R_0) + \mathscr{K}) ds \\ &\le M \cdot \int_{t_1}^{t_2} (p_1(s)\Phi_1(R_0) + p_2(s)\Phi_2(R_0) + \mathscr{K}) ds \\ &\longrightarrow 0 \; as \; t_2 - t_1 \longrightarrow 0.\end{aligned} \tag{30}$$

Therefore, from the above discussion, we have $\|\Gamma u(t_2) - \Gamma u(t_1)\|$ tends to 0 independently of $u \in \bar{\Omega}_{R_0}$ as $t_2 - t_1 \longrightarrow 0$, and it implies that $\Gamma(\bar{\Omega}_{R_0})$ is equicontinuous.

Step 12. $\lim_{t \to \infty} \Gamma u(t)/h(t) = 0$, uniformly for $u \in \bar{\Omega}_{R_0}$.

For any $u \in \bar{\Omega}_{R_0}$, one can find that

$$\begin{aligned}\frac{\|\Gamma u(t)\|}{h(t)} &\le \frac{1}{h(t)} \|T(t)\varphi(0)\| + \frac{1}{h(t)} \int_0^t \|T(t - s)\| \\ &\quad \cdot \|F(s, u(s), u_s)\| ds \\ &\le \frac{M e^{\nu_0 t} \|\varphi\|_{\mathscr{B}}}{h(t)} + \frac{M}{h(t)} \int_0^t e^{\nu_0 (t - s)} \cdot (p_1(s)\Phi_1(R_0) \\ &\quad + p_2(s)\Phi_2(R_0) + \mathscr{K}) ds \\ &\le \frac{M}{h(t)} \left(\|\varphi\|_{\mathscr{B}} + \frac{\mathscr{K}}{|\nu_0|} \right) + \frac{M}{h(t)} \int_0^t e^{\nu_0 (t - s)} \\ &\quad \cdot (p_1(s)\Phi_1(R_0) + p_2(s)\Phi_2(R_0)) ds \\ &\le \frac{M}{h(t)} \left(\|\varphi\|_{\mathscr{B}} + \frac{\mathscr{K}}{|\nu_0|} \right) + \frac{M}{h(t)} (\rho_1 \Phi_1(R_0) + \rho_2 \Phi_2(R_0)) \\ &\le \frac{M}{h(t)} \left(\|\varphi\|_{\mathscr{B}} + \frac{\mathscr{K}}{|\nu_0|} \right) + \frac{M}{h(t)} (\rho_1 \Phi_1(R_0) + \rho_2 \Phi_2(R_0)).\end{aligned} \tag{31}$$

It implies that $\|\Gamma u(t)\|/h(t)$ tends to zero, as $t \longrightarrow \infty$, uniformly for $u \in \bar{\Omega}_{R_0}$.

Above all, we can conclude that $\Gamma(\bar{\Omega}_{R_0})$ is relatively compact in $\mathscr{B}C_h(X)$. Thus, Γ is completely continuous.

Step 13. One can prove that $\Gamma(\mathscr{B}SAP_\omega(X)) \subseteq \mathscr{B}SAP_\omega(X)$.

For any $u \in \mathscr{B}SAP_\omega(X)$, by the definition of Γ, one can find that for $t \in [-r, 0]$, $\Gamma u(t) \equiv \varphi(t)$, which implies that $(\Gamma u)|_{[-r,0]} \in \mathscr{B}$. Thus, we only show that $\Gamma u(t) \in SAP_\omega(X)$ for all $t \ge 0$ and $u|_{\mathbb{R}^+} \in SAP_\omega(X)$. It is noteworthy that $\|u(t)\| \le \|u\|_\infty$ and $\|u_t\| = \sup_{s \in [-r,0]} \|u(t + s)\| \le \|\varphi\|_{\mathscr{B}} + \|u\|_C \le \|u\|_\infty$. So, it is easy to find

$$\begin{aligned}
&(\Gamma u)(t+\omega)-(\Gamma u)(t)\\
&\quad = T(t+\omega)\varphi(0)+\int_0^{t+\omega}T(t+\omega-s)F(s,u(s),u_s)ds\\
&\qquad - T(t)\varphi(0)-\int_0^t T(t-s)F(s,u(s),u_s)ds\\
&\quad = T(t+\omega)\varphi(0)-T(t)\varphi(0)+\int_0^{\omega}T(t+\omega-s)F(s,u(s),u_s)ds\\
&\qquad + \int_0^t T(t-s)\cdot(F(s+\omega,u(s+\omega),u_{s+\omega})-F(s,u(s),u_s))ds\\
&\quad := I_1(t)+I_2(t)+I_3(t).
\end{aligned} \tag{32}$$

Next, we show that $\|I_i(t)\|$ tends 0 as $t\longrightarrow\infty$ $(i=1,2,3)$. In fact, by calculation, one can get that

$$\begin{aligned}
\|I_1(t)\| &\le \|T(t+\omega)\varphi(0)\|+\|T(t)\varphi(0)\|\\
&\le \left(Me^{\nu_0(t+\omega)}+Me^{\nu_0 t}\right)\cdot\|\varphi\|_{\mathscr{B}}\le 2Me^{\nu_0 t}\|\varphi\|_{\mathscr{B}},
\end{aligned} \tag{33}$$

and by the condition (H2), we can derive

$$\begin{aligned}
\|I_2(t)\| &\le \int_0^{\omega}\|T(t+\omega-s)\|\cdot\|F(s,u(s),u_s)\|ds\\
&\le M\int_0^{\omega}e^{\nu_0(t+\omega-s)}\cdot\left(p_1(s)\Phi_1\left(\|u\|_\infty\right)\right.\\
&\qquad \left.+p_2(s)\Phi_2\left(\|u\|_\infty\right)+\mathscr{K}\right)ds\\
&= M\Phi_1\left(\|u\|_\infty\right)e^{\nu_0 t}\cdot\int_0^{\omega}e^{\nu_0(\omega-s)}p_1(s)ds\\
&\qquad + M\Phi_2\left(\|u\|_\infty\right)e^{\nu_0 t}\cdot\int_0^{\omega}e^{\nu_0(\omega-s)}p_2(s)ds+\frac{M\mathscr{K}e^{\nu_0 t}}{|\nu_0|}\\
&\le Me^{\nu_0 t}\left(\rho_1\Phi_1\left(\|u\|_\infty\right)+\rho_2\Phi_2\left(\|u\|_\infty\right)+\frac{\mathscr{K}}{|\nu_0|}\right).
\end{aligned} \tag{34}$$

According to the fact that $T(t)(t\ge 0)$ is exponentially stable, we can derive immediately that $\|I_1(t)\|$, $\|I_2(t)\|$ tend to 0 as $t\longrightarrow\infty$.

In addition, it is easy to know that $u|_{\mathbb{R}^+}\in SAP_\omega(X)$ and $u_t\in SAP_\omega(\mathscr{B})$ for arbitrary $t\ge 0$; in other words, for any positive ε, there is constant $l_1>0$ such that $\|u(t+\omega)-u(t)\|\le\varepsilon$ and $\|u_{t+\omega}-u_t\|_{\mathscr{B}}\le\varepsilon$ for every $t\ge l_1$. Thus, according to the continuity of F, we can derive

$$\|F(t,u(t+\omega),u_{t+\omega})-F(t,u(t),u_t)\|\le\frac{|\nu_0|}{M}\varepsilon,\text{ for any } t\ge l_1. \tag{35}$$

Furthermore, by the condition (H1), it is not difficult to find that there is a positive constant l_2 large enough such that

$$\|F(t+\omega,u(t+\omega),u_{t+\omega})-F(t,u(t+\omega),u_{t+\omega})\|\le\frac{|\nu_0|}{M}\varepsilon,\text{ for } t\ge l_2. \tag{36}$$

Then, for $t>l:=\max\{l_1,l_2\}$, from (35), (36), and (H2), one can easily deduce

$$\begin{aligned}
\|I_3(t)\| &= \left\|\int_0^t T(t-s)\cdot(F(s+\omega,u(s+\omega),u_{s+\omega})-F(s,u(s),u_s))ds\right\|\\
&\le \int_0^l\|T(t-s)\|\cdot\|F(s+\omega,u(s+\omega),u_{s+\omega})-F(s,u(s),u_s)\|\\
&\qquad\cdot ds+\int_l^t\|T(t-s)\|\cdot\|F(s+\omega,u(s+\omega),u_{s+\omega})\\
&\qquad - F(s,u(s+\omega),u_{s+\omega})\|ds+\int_l^t\|T(t-s)\|\\
&\qquad\cdot\|F(s,u(s+\omega),u_{s+\omega})-F(s,u(s),u_s)\|ds\\
&\le \int_0^l\|T(t-s)\|\cdot(\|F(s+\omega,u(s+\omega),u_{s+\omega})\|\\
&\qquad + \|F(s,u(s),u_s)\|)ds+M\int_l^t e^{\nu_0(t-s)}ds\cdot\frac{|\nu_0|\,\varepsilon}{M}\\
&\qquad + M\int_l^t e^{\nu_0(t-s)}ds\cdot\frac{|\nu_0|\,\varepsilon}{M}\\
&\le M\Phi_1\left(\|u\|_\infty\right)\cdot\int_0^l e^{\nu_0(t-s)}(p_1(s+\omega)+p_1(s))ds\\
&\qquad + M\Phi_2\left(\|u\|_\infty\right)\cdot\int_0^l e^{\nu_0(t-s)}(p_2(s+\omega)+p_2(s))ds\\
&\qquad + 2M\mathscr{K}\int_0^l e^{\nu_0(t-s)}ds+2\int_l^t e^{\nu_0(t-s)}ds\cdot|\nu_0|\varepsilon\\
&\le M\Phi_1\left(\|u\|_\infty\right)e^{\nu_0(t-l)}\cdot\left(\int_0^{l+\omega}e^{\nu_0(l+\omega-s)}p_1(s)ds\right.\\
&\qquad \left.+\int_0^l e^{\nu_0(l-s)}p_1(s)ds\right)+M\Phi_2\left(\|u\|_\infty\right)e^{\nu_0(t-l)}\\
&\qquad \cdot\left(\int_0^{l+\omega}e^{\nu_0(l+\omega-s)}p_2(s)ds+\int_0^l e^{\nu_0(l-s)}p_2(s)ds\right)\\
&\qquad + \frac{2M\mathscr{K}e^{\nu_0(t-l)}}{|\nu_0|}+2\left(1-e^{\nu_0(t-l)}\right)\varepsilon\\
&\le 2Me^{\nu_0(t-l)}\left(\rho_1\Phi_1\left(\|u\|_\infty\right)+\rho_2\Phi_2\left(\|u\|_\infty\right)+\frac{\mathscr{K}}{|\nu_0|}\right)\\
&\qquad + 2\left(1-e^{\nu_0(t-l)}\right)\varepsilon.
\end{aligned} \tag{37}$$

This means that $\|I_3(t)\|$ tends to 0 as $t\longrightarrow\infty$.

We conclude from the above discussion that

$$\lim_{t\to\infty}\|\Gamma u(t+\omega)-\Gamma u(t)\|=0, \tag{38}$$

namely, $\Gamma u\in SAP_\omega(X)$. Therefore, $\Gamma(\mathscr{B}SAP_\omega(X))\subset\mathscr{B}SAP_\omega(X)$.

From the above results, one has that $\Gamma:\overline{\Omega}_{R_0}\cap\overline{\mathscr{B}SAP_\omega(X)}\longrightarrow\overline{\Omega}_{R_0}\cap\overline{\mathscr{B}SAP_\omega(X)}$ is a completely continuous operator. Meanwhile, by the Schauder fixed point theorem, the operator Γ has at least one fixed point u in $\overline{\Omega}_{R_0}\cap\overline{\mathscr{B}SAP_\omega(X)}$. Let $\{u^{(n)}\}$ be a sequence in $\overline{\Omega}_{R_0}\cap\overline{\mathscr{B}SAP_\omega(X)}$ that converges to u. One has that $\{\Gamma u^{(n)}\}$ converges to $\Gamma u=u$ uniformly in $[0,\infty)$. It implies that $u\in\mathscr{B}SAP_\omega(X)$.This completes the proof.

We further strengthen the condition (H2), namely, (H4) for all $t \in \mathbb{R}^+$, $x, y \in X$, and $\phi, \psi \in \mathscr{B}$, there are constants $C_1, C_2 > 0$ such that

$$\|F(t,x,\phi) - F(t,y,\psi)\| \le C_1\|x-y\| + C_2\|\phi-\psi\|_{\mathscr{B}}; \quad (39)$$

then, we can get the following results.

Theorem 14. *Let $-A$ generate a compact and exponentially stable C_0 − semigroup $T(t)(t \ge 0)$ in X. Let $F : \mathbb{R}^+ \times X \times \mathscr{B} \longrightarrow X$ be a continuous mapping and $\sup_{t\in\mathbb{R}^+}\|F(t,\theta,\theta)\| < \infty$. If the conditions (H1), (H4), and (H5) $M(C_1 + C_2) < |\nu_0|$ hold, there is a unique S-asymptotically ω-periodic mild solution for DEE (1). Moreover, if the condition (H5) replaced by $(H5')MC_1 + MC_2e^{-\nu_0 r} < |\nu_0|$, then the unique S-asymptotically ω-periodic mild solution of DEE (1) is globally exponentially stable.*

Proof. We consider the operator Γ be defined on $\mathscr{B}C_b(X)$ by (13). For any $u \in \mathscr{B}C_b(X)$, we have$\|u(t)\| \le \|u\|_\infty$,

$$\begin{aligned}\|u_t\| &= \sup_{s\in[-r,0]}\|u(t+s)\| \le \sup_{t\in[-r,0]}\|u(t)\| + \sup_{t\in[0,\infty)}\|u(t)\| \\ &\le \|\varphi\|_{\mathscr{B}} + \|u\|_C \le \|u\|_\infty.\end{aligned} \quad (40)$$

Hence, it is not difficult to find that

$$\|F(t,u(t),u_t)\| \le (C_1 + C_2)\|u\|_\infty + \|F(t,\theta,\theta)\| \coloneqq C. \quad (41)$$

By the definition of Γ, we have $\|\Gamma u(t)\| \equiv \|\varphi(t)\| \le \|\varphi\|_{\mathscr{B}}$ for $t \in [-r, 0]$. On the other hand, if $t \ge 0$, then by the condition (H4), we have

$$\left\|\int_0^t T(t-s)F(s,u(s),u_s)ds\right\| \le \frac{MC}{|\nu_0|}. \quad (42)$$

It means that $\Gamma : \mathscr{B}C_b(X) \to \mathscr{B}C_b(X)$ is well defined.

Next, we need to verify that $\Gamma(\mathscr{B}SAP_\omega(X)) \subset \mathscr{B}SAP_\omega(X)$. To do this, we just need to show that (32) tends 0 as $t \to \infty$. Similar to the proof of Theorem 7, From (35), (36), (41), and (H4), one has $\|I_1(t)\| \le 2Me^{\nu_0 t}\|\varphi\|_{\mathscr{B}}$

$$\begin{aligned}\|I_2(t)\| &\le \int_0^\omega \|T(t+\omega-s)\| \cdot \|F(s,u(s),u_s)\|ds \\ &\le \frac{MCe^{\nu_0 t}}{|\nu_0|}\|I_3(t)\| = \left\|\int_0^t T(t-s)\right. \\ &\quad \left.\cdot (F(s+\omega,u(s+\omega),u_{s+\omega}) - F(s,u(s),u_s))ds\right\| \\ &\le \int_0^l \|T(t-s)\| \cdot \|F(s+\omega,u(s+\omega),u_{s+\omega}) \\ &\quad - F(s,u(s),u_s)\|ds + \int_l^t \|T(t-s)\| \\ &\quad \cdot \|F(s+\omega,u(s+\omega),u_{s+\omega}) - F(s,u(s+\omega),u_{s+\omega})\| \\ &\quad \cdot ds + \int_l^t \|T(t-s)\| \cdot \|F(s,u(s+\omega),u_{s+\omega}) \\ &\quad - F(s,u(s),u_s)\|ds \\ &\le 2MC\int_0^l e^{\nu_0(t-s)}ds + M\int_l^t e^{\nu_0(t-s)}ds \cdot \frac{|\nu_0|\varepsilon}{M} \\ &\quad + M\int_l^t e^{\nu_0(t-s)}ds \cdot \frac{|\nu_0|\varepsilon}{M} \\ &\le \frac{2MCe^{\nu_0(t-l)}}{|\nu_0|} + 2\left(1 - e^{\nu_0(t-l)}\right)\varepsilon.\end{aligned} \quad (43)$$

According to the fact that $T(t)(t \ge 0)$ is exponentially stable, we infer that $\|I_i(t)\|$ tends to 0 as $t \longrightarrow \infty (i = 1, 2, 3)$. This means that $\Gamma(\mathscr{B}SAP_\omega(X)) \subset \mathscr{B}SAP_\omega(X)$.

Thus, for $u^{(1)}, u^{(2)} \in \mathscr{B}SAP_\omega(X)$, under the condition (H4), it is not difficult to derive that

$$\begin{aligned}&\left\|\Gamma u^{(1)}(t) - \Gamma u^{(2)}(t)\right\| \\ &= \left\|\int_0^t T(t-s) \cdot F\left(s,u^{(1)}(s),u_s^{(1)}\right)ds - \int_0^t T(t-s)\right. \\ &\quad \left.\cdot F\left(s,u^{(2)}(s),u_s^{(2)}\right)ds\right\| \\ &\le \int_0^t \|T(t-s)\| \cdot \left\|F\left(s,u^{(1)}(s),u_s^{(1)}\right) - F\left(s,u^{(2)}(s),u_s^{(2)}\right)\right\|ds \\ &\le M\int_0^t e^{\nu_0(t-s)}\left(C_1\left\|u^{(1)}(s) - u^{(2)}(s)\right\| + C_2\left\|u_s^{(1)} - u_s^{(2)}\right\|_{\mathscr{B}}\right)ds \\ &\le M(C_1 + C_2)\int_0^t e^{\nu_0(t-s)}ds \cdot \left\|u^{(1)} - u^{(2)}\right\|_\infty \\ &\le \frac{M(C_1 + C_2)}{|\nu_0|}\left\|u^{(1)} - u^{(2)}\right\|_\infty;\end{aligned} \quad (44)$$

by the condition (H5), we can conclude that Γ is a contraction mapping. Thus, there is a unique S-asymptotically ω -periodic mild solution for DEE (1).

Now, we verify the globally exponentially stability of the unique S-asymptotic ω-periodic mild solution. Let $u = u(t, \varphi) \in C([-r,\infty), X)$ be the unique S-asymptotic ω-periodic mild solution of DEE (1) with the initial value $\varphi \in \mathscr{B}$. From ([30], Theorem 3.2), it is easy to prove that for every $\phi \in \mathscr{B}$, the initial value problem corresponding to DEE (1) has a unique global mild solution $v = v(t, \phi) \in C([-r,\infty), X)$. By Definition 3, u and v satisfy the integral equation (2.4).

Since $T(t)(t \ge 0)$ is an exponentially stable C_0 − semigroup, whose growth exponent is $\nu_0 < 0$. Hence, by the condition $(H5')$, we can choose $\nu \in (MC_1 + MC_2e^{-\nu_0 r}, |\nu_0|)$, and it follows that $\|T(t)\| \le Me^{-\nu t}$ for $t \ge 0$. So, according to the condition (H4), for any $t \geq 0$, we can get

$$\begin{aligned}
&\|u(t)-v(t)\| \\
&\quad = \left\| T(t)\varphi(0) + \int_0^t T(t-s)F(s,u(s),u_s)ds \right. \\
&\qquad \left. - T(t)\phi(0) - \int_0^t T(t-s)F(s,v(s),v_s)ds \right\| \\
&\quad \leq \|T(t)\varphi(0) - T(t)\phi(0)\| + \int_0^t \|T(t-s)\| \\
&\qquad \cdot \|F(s,u(s),u_s) - F(s,v(s),v_s)\|ds \\
&\quad \leq Me^{-\nu t}\|\varphi(0)-\phi(0)\| + M\int_0^t e^{-\nu(t-s)} \\
&\qquad \cdot \left(C_1\|u(s)-v(s)\| + C_2\|u_s - v_s\|_{\mathscr{B}}\right)ds \\
&\quad \leq Me^{-\nu t}\|u(0)-v(0)\| + M\int_0^t e^{-\nu(t-s)} \\
&\qquad \times \left(C_1\|u(s)-v(s)\| + C_2 \sup_{\tau\in[-r,0]}\|u(s+\tau)-v(s+\tau)\|\right)ds.
\end{aligned} \tag{45}$$

For any $t \in [-r,\infty)$, let $\Psi(t) = e^{\nu t}\|u(t)-v(t)\|$, and one can find

$$\Psi(t) \leq M\Psi(0) + \int_0^t MC_1\Psi(s) + MC_2 e^{\nu r}\sup_{\tau\in[-r,0]}\Psi(s+\tau)ds. \tag{46}$$

Denote $l_1 = MC_1, l_2 = MC_2 e^{\nu r}$, by Lemma 6 and $\nu < |\nu_0|$, we can obtain

$$\begin{aligned}
e^{\nu t}\|u(t)-v(t)\| = \Psi(t) &\leq M\|\varphi-\phi\|_{\mathscr{B}} \cdot e^{(l_1+l_2)t} \\
&\leq M\|\varphi-\phi\|_{\mathscr{B}} \cdot e^{(MC_1+MC_2e^{-\nu_0 r})t}.
\end{aligned} \tag{47}$$

By $\alpha := \nu - (MC_1 + MC_2 e^{-\nu_0 r}) > 0$ and (47), one can obtain

$$\|u(t)-v(t)\| \leq M\|v\|_{\mathscr{B}} \cdot e^{-\alpha t} \tag{48}$$

for every $t \geq 0$, which implies that the S-asymptotically ω-periodic mild solution u of DEE (1) is globally exponentially stable. The proof is complete.

Theorem 15. *Let $-A$ generate an exponentially stable C_0-semigroup $T(t)(t \geq 0)$ in X. Let $F : \mathbb{R}^+ \times X \times \mathscr{B} \longrightarrow X$ be the continuous function and $\sup_{t\in\mathbb{R}^+}\|F(t,\theta,\theta)\| < \infty$. If the conditions (H1), (H4), (H5$'$) hold, then DEE (1) is globally asymptotically ω-periodic.*

Proof. We complete the proof by three steps.

Step 16. The solution of DEE (1) is bounded.

From ([30], Theorem 3.2), it follows that DEE (1) exists a unique global mild solution $u \in C([-r,\infty), X)$ for given $\varphi \in C([-r,0], X)$.

By Definition 3, for any $t \in [-r,0]$, $\|u(t)\| = \|\varphi(t)\| \leq \|\varphi\|_{\mathscr{B}}$, and if $t \geq 0$ and denote $C_0 := \sup_{t\in\mathbb{R}^+}\|F(t,\theta,\theta)\| < \infty$, then one can obtain that

$$\begin{aligned}
\|u(t)\| &= \left\| T(t)\varphi(0) + \int_0^t T(t-s)\cdot F(s,u(s),u_s)ds \right\| \\
&\leq \|T(t)\varphi(0)\| + \left\| \int_0^t T(t-s)\cdot F(s,u(s),u_s)ds \right\| \\
&\leq \|T(t)\|\cdot\|\varphi(0)\| + \int_0^t \|T(t-s)\|\cdot(\|F(s,u(s),u_s) \\
&\quad - F(s,\theta,\theta)\| + \|F(s,\theta,\theta)\|)ds \\
&\leq M\|\varphi\|_{\mathscr{B}} + M\int_0^t e^{\nu_0(t-s)} \\
&\quad \cdot \left(C_1\|u(s)\| + C_2\sup_{\tau\in[-r,0]}\|u(s+\tau)\| + C_0\right)ds \\
&\leq M\|\varphi\|_{\mathscr{B}} + \frac{MC_0}{|\nu_0|} + \frac{M(C_1+C_2)}{|\nu_0|}\|u\|_C.
\end{aligned} \tag{49}$$

Hence,

$$\|u\|_C \cdot \left(1 - \frac{M(C_1+C_2)}{|\nu_0|}\right) \leq M\|\varphi\|_{\mathscr{B}} + \frac{MC_0}{|\nu_0|}. \tag{50}$$

From $(H5')$, it follows that $1 - M(C_1+C_2)/|\nu_0| > 0$ holds, which implies the mild solution $u(t)$ of DEE (1) is bounded, namely, $u \in \mathscr{B}C_b(X)$.

Step 17. The mild solution $u \in \mathscr{B}C_b(X)$ of DEE (1) is S-asymptotically ω-periodic.

For this reason, we only need to verify $\lim_{t\to\infty}\|u(t+\omega) - u(t)\| = 0$. By Definition 3, we have

$$\begin{aligned}
&\|u(t+\omega)-u(t)\| \\
&\quad \leq \|T(t+\omega)\varphi(0) - T(t)\varphi(0)\| + \left\| \int_0^\omega T(t+\omega-s)F(s,u(s),u_s)ds \right\| \\
&\qquad + \left\| \int_0^t T(t-s)\cdot(F(s+\omega,u(s+\omega),u_{s+\omega}) - F(s,u(s),u_s))ds \right\| \\
&\quad := \|K_1(t)\| + \|K_2(t)\| + \|K_3(t)\|.
\end{aligned} \tag{51}$$

First of all, since $T(t)(t \geq 0)$ is an exponentially stable C_0 − semigroup, that is the growth exponent $\nu_0 < 0$. Hence, by the condition $(H5')$, we can choose $\nu \in (MC_1 + MC_2 e^{-\nu_0 r}, |\nu_0|)$, and it follows that $\|T(t)\| \leq Me^{-\nu t}$ for $t \geq 0$. Under the condition, we see that

$$\|K_1(t)\| \leq \|T(t+\omega)\varphi(0)\| + \|T(t)\varphi(0)\| \leq 2Me^{-\nu t}\|\varphi\|_{\mathscr{B}}. \tag{52}$$

Secondly, since the mild solution $u \in \mathscr{B}C_b(X)$, thus, there exists a positive constant R such that $\|u\|_\infty \leq R$. Combining this with the condition (H4), one can find that

$$\begin{aligned}\|F(t,u(t),u_t)\| &\le C_1\|u(t)\| + C_2\sup_{\tau\in[-r,0]}\|u(t+\tau)\| + C_0\\ &\le (C_1+C_2)R + C_0 := C\end{aligned} \tag{53}$$

for any $t \ge 0$. Therefore, one can see

$$\begin{aligned}\|K_2(t)\| &\le \int_0^{\omega}\|T(t+\omega-s)\|\cdot\|F(s,u(s),u_s)\|ds\\ &\le MC\int_0^{\omega}e^{-\nu(t+\omega-s)}ds = \frac{MC}{\nu}\left(e^{-\nu t}-e^{-\nu(t+\omega)}\right)\\ &\le \frac{MC}{\nu}e^{-\nu t}.\end{aligned} \tag{54}$$

Finally, by the condition (H1), for any $\varepsilon > 0$, there is a constant $l(l > 0)$ sufficiently large such that

$$\|F(t+\omega,u(t),u_t) - F(t,u(t),u_t)\| \le \varepsilon.\text{for } t > l. \tag{55}$$

Choosing $\varepsilon \le 2Ce^{-\nu t}/1 - e^{-\nu(t-l)}$, by the condition (H4), (53), and (55), one can deduce that

$$\begin{aligned}\|K_3(t)\| \le& \int_0^t\|T(t-s)\|\cdot\|F(s+\omega,u(s+\omega),u_{s+\omega})\\ &- F(s,u(s),u_s)\|ds\\ \le& \int_0^l\|T(t-s)\|\cdot\|F(s+\omega,u(s+\omega),u_{s+\omega})\\ &- F(s,u(s),u_s)\|ds + \int_l^t\|T(t-s)\|\\ &\cdot\|F(s+\omega,u(s+\omega),u_{s+\omega}) - F(s,u(s+\omega),u_{s+\omega})\|ds\\ &+ \int_l^t\|T(t-s)\|\cdot\|F(s,u(s+\omega),u_{s+\omega})\\ &- F(s,u(s),u_s)\|ds\\ \le& 2C\cdot\int_0^l Me^{-\nu(t-s)}ds + \varepsilon\int_l^t Me^{-\nu(t-s)}ds + \int_l^t Me^{-\nu(t-s)}\\ &\cdot\left(C_1\|u(s+\omega)-u(s)\| + C_2\|u_{s+\omega}-u_s\|_{\mathscr{B}}\right)ds\\ \le& \frac{2MC}{\nu}\cdot\left(e^{-\nu(t-l)}-e^{-\nu t}\right) + \frac{2Ce^{-\nu t}}{1-e^{-\nu(t-l)}}\\ &\cdot\frac{M}{\nu}\left(1-e^{-\nu(t-l)}\right) + \int_l^t Me^{-\nu(t-s)}(C_1\|u(s+\omega)\\ &- u(s)\| + C_2\sup_{\tau\in[-r,0]}\|u(s+\omega+\tau)-u(s+\tau)\|)ds\\ \le& \frac{2MC}{\nu}\cdot e^{-\nu(t-l)} + \int_0^t Me^{-\nu(t-s)}\Bigg(C_1\|u(s+\omega)-u(s)\|\\ &+ C_2\sup_{\tau\in[-r,0]}\|u(s+\omega+\tau)-u(s+\tau)\|\Bigg)ds.\end{aligned} \tag{56}$$

Therefore, based on the above results, one can find

$$\begin{aligned}\|u(t+\omega)-u(t)\| \le& 2Me^{-\nu t}\|\varphi\|_{\mathscr{B}} + \frac{MC}{\nu}e^{-\nu t} + \frac{2MC}{\nu}e^{-\nu(t-l)}\\ &+ \int_0^t Me^{-\nu(t-s)}\Bigg(C_1\|u(s+\omega)-u(s)\|\\ &+ C_2\sup_{\tau\in[-r,0]}\|u(s+\omega+\tau)-u(s+\tau)\|\Bigg)ds.\end{aligned} \tag{57}$$

For any $t \in [-r,\infty)$, let $\Psi(t) = e^{\nu t}\|u(t+\omega)-u(t)\|$; then,

$$\begin{aligned}\Psi(t) \le& 2M\|\varphi\|_{\mathscr{B}} + \frac{MC}{\nu}\cdot\left(1+2e^{\nu l}\right)\\ &+ \int_0^t MC_1\Psi(s) + MC_2e^{\nu r}\sup_{\tau\in[-r,0]}\Psi(s+\tau)ds.\end{aligned} \tag{58}$$

Let $l_1 = MC_1, l_2 = MC_2e^{\nu r}$, combined with Lemma 6 and $\nu < |\nu_0|$, and we can deduce

$$e^{\nu t}\|u(t+\omega)-u(t)\| = \Psi(t) \le \bar{M}\cdot e^{(l_1+l_2)t} \le \bar{M}\cdot e^{(MC_1+MC_2e^{-\nu_0 r})t}, \tag{59}$$

where $\bar{M} = 2M\|\varphi\|_{\mathscr{B}} + (MC/\nu)\cdot(1+2e^{\nu l})$. By $\alpha := \nu - (MC_1 + MC_2e^{-\nu_0 r}) > 0$ and (59), it is easy to know that

$$\|u(t+\omega)-u(t)\| \le \bar{M}\cdot e^{-\alpha t}. \tag{60}$$

Under this discussion, we have

$$\lim_{t\to\infty}\|u(t)-u(t+\omega)\| = 0. \tag{61}$$

That is to say, $u \in \mathscr{B}SAP_{\omega}(X)$.

Step 18. There is a nonconstant ω-periodic function, which makes the S-asymptotically ω-periodic mild solution $u \in \mathscr{B}SAP_{\omega}(X)$ asymptotically converges to the nonconstant ω-periodic function.

It is not difficult to verify that the sequence $\{u(t+k\omega)\}_{k\in\mathbb{N}}$ is of equicontinuity and of uniformly bound. We can choose a subsequence of $\{k\omega\}$ (for convenience, we still denote the subsequence as $\{k\omega\}$) such that sequence $\{u(t+k\omega)\}$ uniformly converges to a continuous function $u^*(t)$ on any compact set of $[0,\infty)$ by means of the Arzela-Ascoli theorem. Obviously, $u^*(t)$ is a ω-periodic function, i.e., $u^*(t+\omega) = u^*(t)$ for any $t \ge 0$ and $u^*(t) = \varphi(t)$ for $t \in [-r,0]$.

Now, for $t \ge 0$ and $k \in \mathbb{N}$, we consider

$$\begin{aligned}\|u(t)-u^*(t)\| \le& \|u(t)-u(t+\omega)\| + \|u(t+\omega)-u(t+k\omega)\|\\ &+ \|u(t+k\omega)-u^*(t)\|.\end{aligned} \tag{62}$$

Based on the S-asymptotically ω-periodicity of $u(t)$, one can obtain easily that

$$\lim_{t\to\infty} \|u(t) - u(t+\omega)\| = 0. \tag{63}$$

By globally exponentially stable of DEE (1),

$$\lim_{t\to\infty} \|u(t+\omega) - u(t+k\omega)\| = 0, \text{ for every } k \in \mathbb{N}. \tag{64}$$

According to the definition of $u^*(t)$, one has

$$\lim_{k\to\infty} \|u(t+k\omega) - u^*(t)\| = 0, \text{ for any } t \geq 0. \tag{65}$$

By (62), (63), (64), and (65), one can easily find

$$\lim_{t\to\infty} \|u(t) - u^*(t)\| = 0. \tag{66}$$

Thus, by Definition 5, DEE (1) is globally asymptotically periodic. This is the end of the proof.

4. Application

In this section, two examples are given to show the applicability and effectiveness of our main results.

Example 19. The functional partial differential equation is considered

$$\begin{cases} \dfrac{\partial}{\partial t} u(t,x) - \dfrac{\partial^2}{\partial x^2} u(t,x) = G(t,x,u(t,x),u_t(x)), \ t \in \mathbb{R}^+, x \in [0,\pi], \\ u(t,0) = u(t,\pi) = 0, \ t \in \mathbb{R}^+, \\ u(\tau,x) = \varphi(\tau,x), \ \tau \in [-r,0], \ x \in [0,\pi], \end{cases} \tag{67}$$

where $G : \mathbb{R}^+ \times [0,\pi] \times \mathbb{R} \times C([-r,0], L^2[0,\pi]) \longrightarrow \mathbb{R}$ is a continuous function, which is 1-asymptotic periodic with respect to t, $r > 0$ is a constant.

Let $X = L^2[0,\pi]$ with the norm $\|\cdot\|$. Operator $A : D(A) \subset X \longrightarrow X$ is defined by

$$\begin{aligned} D(A) &= \left\{u \in X : u' \in X, u'' \in X, u(0) = u(\pi) = 0\right\}, \\ Au(t,x) &= -\frac{\partial^2}{\partial x^2} u(t,x); \end{aligned} \tag{68}$$

then, $-A$ generates an exponentially stable compact analytic semigroup $\{T(t)\}(t \geq 0)$ in X. It means that A has a discrete spectrum with eigenvalues $n^2 (n \in \mathbb{N})$ and gives the corresponding normalized eigenfunctions by $e_n(x) = \sqrt{2/\pi} \cdot \sin(nx)$ for any $x \in [0,\pi]$. Consequently, for any $t \geq 0, u \in X$, the associated semigroup $\{T(t)\}(t \geq 0)$ is given by

$$T(t)u = \sum_{n=1}^{\infty} e^{-n^2 t} \langle u, e_n \rangle e_n. \tag{69}$$

Clearly, for all $t \geq 0$, $\|T(t)\| \leq e^{-t}$, namely, the growth exponent of the semigroup is -1.

Meanwhile, let $u(t)(x) = u(t,x)$ and $u_t(\tau)(x) = u(t+\tau,x)$ for any $t \in \mathbb{R}^+, x \in [0,\pi]$ and $\tau \in [-r,0]$; then, $u \in X$ and $u_t \in \mathscr{B} = C([-r,0],X)$. Thus, $F : \mathbb{R}^+ \times X \times \mathscr{B} \longrightarrow X$ is defined by

$$F(t,u(t),u_t)(x) = G(t,x,u(t,x),u_t(x)). \tag{70}$$

Therefore, equation (67) can be rewritten into DEE (1) in X.

Taking $h(t) = e^t$, let $G : \mathbb{R}^+ \times [0,\pi] \times \mathbb{R} \times C([-r,0], L^2[0,\pi]) \longrightarrow \mathbb{R}$ be a continuous function, which is 1-asymptotic periodic with respect to t.

Thus, the existence and uniqueness results are obtained from equation (67).

Theorem 20. *If the following condition holds: (P1) for any $t \in \mathbb{R}^+$, $x \in [0,\pi]$, $\eta \in \mathbb{R}$, $\zeta \in \mathscr{B}$,*

$$\left\|G\left(t,x,e^t\eta,e^t\zeta\right)\right\| \leq \frac{\pi \sin 2\pi t}{4e^t} \|\eta\| + \frac{\pi^2 \sin 2\pi t}{5e^t} \|\zeta\|_{\mathscr{B}} + 1, \tag{71}$$

then equation (67) has at least one S-asymptotically ω -periodic mild solution.

Proof. G is 1-asymptotic periodic with respect to t implying that the condition (H1) holds of Theorem 7, namely, $\omega = 1$. From the condition (P1), it is not difficult that the condition (H2) is satisfied. Indeed,

$$\begin{aligned} \left\|G\left(t,x,e^t\eta,e^t\zeta\right)\right\| &\leq \frac{\pi \sin 2\pi t}{4e^t} \|\eta\| + \frac{\pi^2 \sin 2\pi t}{5e^t} \|\zeta\|_{\mathscr{B}} + 1 \\ &:= p_1(t)\Phi_1(\|\eta\|) + p_2(t)\Phi_2\left(\|\zeta\|_{\mathscr{B}}\right) + \mathscr{K}, \end{aligned} \tag{72}$$

where $p_1(t) = \sin 2\pi t/4e^t$, $p_2 = \pi \sin 2\pi t/5e^t$, $\mathscr{K} = 1$, $\liminf_{r\to\infty} \Phi_i(r)/r = \pi < \infty i = 1, 2$. By $M = 1$, $\nu_0 = -1$,

$$\begin{aligned} &\sup_{t\geq 0} \int_0^t e^{-(t-s)} \frac{\pi \sin 2\pi s}{4e^s} ds + \sup_{t\geq 0} \int_0^t e^{-(t-s)} \frac{\pi^2 \sin 2\pi s}{5e^s} ds \\ &= \sup_{t\geq 0} e^{-t} \int_0^t \frac{\pi \sin 2\pi s}{4} ds + \sup_{t\geq 0} e^{-t} \int_0^t \frac{\pi^2 \sin 2\pi s}{5} ds \\ &= \sup_{t\geq 0} \frac{1-\cos 2\pi t}{8e^t} + \sup_{t\geq 0} \frac{\pi(1-\cos 2\pi t)}{10e^t} \leq \frac{1}{4} + \frac{\pi}{5} \\ &\approx 0.88 < 1, \end{aligned} \tag{73}$$

we can conclude that the condition (H3) of Theorem 7 is fulfilled. Thus, from Theorem 7, the functional partial differential equation (67) has at least one S-asymptotically 1-periodic mild solution.

Theorem 21. *If* $\sup_{t\in\mathbb{R}^+}\|G(t,x,\theta,\theta)\|<\infty$ *and the following condition: (P2) for any* $t\in\mathbb{R}^+$, $x\in[0,\pi]$, $\eta_1,\eta_2\in\mathbb{R}$, $\zeta_1,\zeta_2\in\mathscr{B}$,

$$\|G(t,x,\eta_1,\zeta_1)-G(t,x,\eta_2,\zeta_2)\|\le\frac{1}{4}\|\eta_1-\eta_2\|+\frac{\pi}{5}\|\zeta_1-\zeta_2\|_{\mathscr{B}} \tag{74}$$

hold, there is a unique S-asymptotically 1-periodic mild solution of equation (67). Moreover, if $0<r<\ln(15/4\pi)$ *holds, then the S-asymptotically 1-periodic mild solution of equation (67) is globally exponentially stable.*

Proof. Obviously, condition (H1) is true. Since $M=1$, $\nu_0=-1$,

$$\|G(t,x,\eta_1,\zeta_1)-G(t,x,\eta_2,\zeta_2)\|\le\frac{1}{4}\|\eta_1-\eta_2\|+\frac{\pi}{5}\|\zeta_1-\zeta_2\|_{\mathscr{B}}, \tag{75}$$

one has

$$M(C_1+C_2)=1\cdot\left(\frac{1}{4}+\frac{\pi}{5}\right)\approx 0.88<1=|\nu_0|, \tag{76}$$

which implies that condition (H5) holds with $C_1=1/4$, $C_2=\pi/5$. Hence, our conclusion follows from Theorem 14 that there is a unique S-asymptotically 1-periodic mild solution of equation (67).

From $0<r<\ln(15/4\pi)$, it is not difficult to know that condition $(H5')$ holds. In fact, $C_1=1/4$, $C_2=\pi/5$; then, $1/4+(\pi/5)<MC_1+MC_2e^r<3/4+1/4$, namely, $0.88<MC_1+MC_2e^r<1$. It suffices to apply Theorem 14, and one can find that the S-asymptotically 1-periodic mild solution of equation (67) is globally exponentially stable.

Theorem 22. *If the condition (P2) is satisfied and* $0<r<\ln(15/4\pi)$ *is valid, then equation (67) is globally asymptotically 1-periodic.*

Proof. Obviously, if conditions (H1), (H4), and $(H5')$ are true, our conclusion follows from Theorem 15. Hence, it suffices to apply Theorem 15, and we can obtain that equation (67) is globally asymptotically 1-periodic.

Example 23. Consider the integer-order neural networks with finite delay (INND)

$$\begin{cases} y_1'(t)+2y_1(t)=\dfrac{\sin t}{t+1}\left[\tanh\left(y_1(t+\theta)\right)+\dfrac{1}{5}\tanh\left(y_2(t+\theta)\right)+2\right], & t\ge 0,\\ y_2'(t)+2y_2(t)=\dfrac{\cos t}{t+1}\left[\dfrac{1}{10}\tanh\left(y_1(t+\theta)\right)+\tanh\left(y_2(t+\theta)\right)+3\right], & t\ge 0,\\ y_1(\theta)=y_2(\theta)=0.2, & \theta\in[-1,0]. \end{cases} \tag{77}$$

Let $X=\mathbb{R}^2$, the vector $y=(y_1,y_2)^T\in\mathbb{R}^2$ endowed with norm $\|y\|=\Sigma_{i=1}^2|y_i|$, define $\|A\|=\max_{1\le j\le 2}\Sigma_{i=1}^2|y_i|$ for the matrix $A=(a_{ij})_{2\times 2}$.

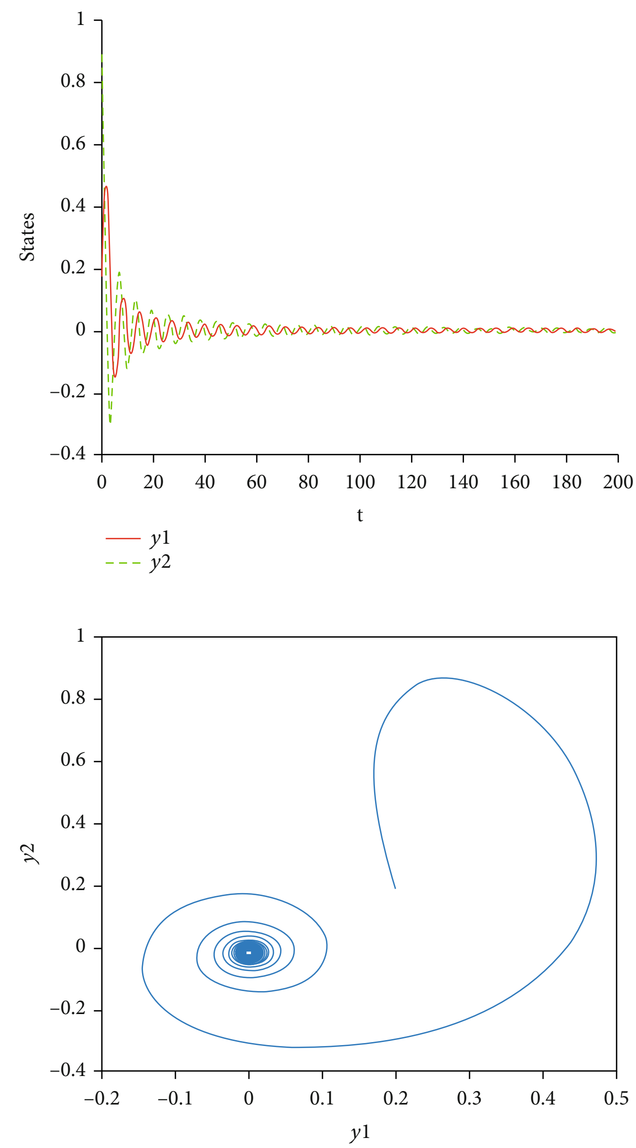

Figure 1: Numercial solution of Eq. (77).

In this way, equation (77) can be transformed into a vector form as follows:

$$\begin{gathered} y'(t)+By(t)=F(t,y(t),y_t)t\ge 0,\\ y(t)=0.2\ t\in[-1,0], \end{gathered} \tag{78}$$

where $y(t)=(y_1(t),y_2(t))^T$, $B=\operatorname{diag}(2,2)$, $F(t,y(t),y_t)=A(t)f(y_t)+C(t)$, $f(y_t)=(\tanh(y_{1,t}),\tanh(y_{2,t}))^T$,

$$A(t)=(a_{ij})_{2\times 2}=\begin{pmatrix}\dfrac{\sin t}{t+1} & \dfrac{\sin t}{5(t+1)}\\ \dfrac{\cos t}{10(t+1)} & \dfrac{\cos t}{t+1}\end{pmatrix}, \tag{79}$$

$C(t)=((2\sin t/t+1),(3\cos t/t+1))^T$. It is easy to see that B generates a bounded operator semigroup $T(t)=e^{-Bt}=\operatorname{diag}$

(e^{-2t}, e^{-2t}) and $\|T(t)\| \le e^{-2t}$, for $t \ge 0$, namely, $M = 1$, $\nu_0 = -2$, see [24]. For $x, y \in X$, $\varphi, \phi \in C([-1,0], X)$, one has

$$\begin{aligned} \|F(t,y,\varphi) - F(t,x,\phi)\| &\le \frac{6}{5}\|\varphi - \phi\|_{[-1,0]}, \\ \|F(t,0,0)\| &\le 5, \\ \|F(t+2\pi,y,\varphi) - F(t,y,\varphi)\| &\le \left(\frac{1}{t+1+2\pi} + \frac{1}{t+1}\right) \\ &\quad \cdot \left(\frac{6}{5}\|\varphi\|_{[-1,0]} + 5\right) \to 0\ t \to \infty. \end{aligned} \tag{80}$$

Then, all conditions in Theorem 14 hold; hence, INND (77) has a unique S-asymptotically 2π-periodic solution. Furthermore, the unique S-asymptotically 2π-periodic solution is globally exponentially stable and is global asymptotic 2π-periodic, see Figure 1.

Data Availability

Data and materials are not applicable.

Ethical Approval

H.Qiao, Q. Li and T.Yuan read and approved the final version of the manuscript.

Conflicts of Interest

H.Qiao, Q. Li and T.Yuan declare that they have no competing interests.

Authors' Contributions

H.Qiao, Q. Li, and T.Yuan contributed equally and significantly in writing this article. Authors read and approved the final manuscript. Hong Qiao, Qiang Li, and Tianjiao Yuan equally contributed this manuscript.

Acknowledgments

The author is most grateful to the editor professor and anonymous referees for the careful reading of the manuscript and valuable suggestions that helped in significantly improving an earlier version of this paper. This work was supported by the NNSF of China (11501342, 11261053), NSF of Shanxi, China (201901D211399), and STIP (2020L0243).

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