Research Article

A Characterization on Singular Value Inequalities of Matrices

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We obtain a characterization of pair matrices A and B of order n such that $s_j(A) \leq s_j(B)$, $j = 1, \dots, n$, where $s_j(X)$ denotes the j -th largest singular values of X . It can imply not only some well-known singular value inequalities for sums and direct sums of matrices but also Zhan's result related to singular values of differences of positive semidefinite matrices. In addition, some related and new inequalities are also obtained.

1. Introduction

Let M_n denote the vector space of all complex $n \times n$ matrices, and let H_n be the set of all Hermitian matrices of order n . We always denote the eigenvalues of $A \in H_n$ in decreasing order by $\lambda_1(A) \geq \lambda_2(A) \geq \dots \geq \lambda_n(A)$. For $A, B \in H_n$, we use the notation $A \leq B$ or $B \geq A$ to mean that $B - A$ is positive semidefinite. Clearly, " $\leq$ " and " $\geq$ " define two partial orders on H_n , each of which is called *Löwner partial order*. In particular, $B \geq O$ (res., $B > O$) means that B is positive semidefinite (res., B is positive definite). For $T \in M_n$, the singular values of T , denoted by $s_1(T), s_2(T), \dots, s_n(T)$, are the eigenvalues of the positive semidefinite matrix $|T| = (T^* T)^{1/2}$, enumerated as $s_1(T) \geq s_2(T) \geq \dots \geq s_n(T)$ and repeated according to multiplicity. It follows that the singular values of a normal matrix are just the moduli of its eigenvalues. In particular, if $T \in M_n$ is positive semidefinite, then singular values and eigenvalues of T are the same. For more information on this related topic, see [1–3]. Recall that a complex matrix $C \in M_n$ is called *contraction* if $C^* C \leq I$, or equivalently $\|C\| \leq 1$, where $\|\cdot\|$ denotes the spectral norm, the identity matrix of order n , respectively.

Here, we denote the block matrix $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ by $A \oplus B$.

This paper intends to give a characterization of pair matrices $A, B \in M_n$ such that $s_j(A) \leq s_j(B)$, $j = 1, \dots, n$. It can generalize some singular value inequalities for sums and

direct sums of matrices due to Hirzallah and Kittaneh [4]. Using this characterization, we give a new proof of Zhan's result related to singular values of differences of positive semidefinite matrices [5]. Several applications of this characterization are presented, and some related and new inequalities are also obtained.

2. Main Results

The following well-known results are due to Ky Fan.

Lemma 1. Let $A, B \in M_n$, $1 < i, j \leq n$, and $i + j - 1 \leq n$. Then

$$s_{i+j-1}(AB) \leq s_i(A)s_j(B). \quad (1)$$

In particular,

$$\begin{aligned} s_j(AB) &\leq A s_j(B), \\ s_j(AB) &\leq \|B\| s_j(A), \end{aligned} \quad (2)$$

$$j = 1, \dots, n,$$

where $\|\cdot\|$ denotes the spectral norm.

Next, we will prove the following useful result.

Theorem 1. Let $A, B \in M_n$. Then,

$$s_j(A) \leq s_j(B), \quad j = 1, 2, \dots, n, \quad (3)$$

if and only if contractive matrices W and V exist such that $A = WBV$. In particular, suppose A and B are positive semidefinite. Then,

$$s_j(A) \leq s_j(B), \quad j = 1, 2, \dots, n, \quad (4)$$

if and only if there exists a contractive matrix W such that $A = WBW^*$.

Proof. The sufficiency follows from Lemma 1. We prove the necessity.

Consider the singular value decompositions A and B . Then, unitary matrices U_1, U_2, V_1 , and V_2 exist such that

$$\begin{aligned} U_1 A V_1 &= \text{diag}(s_1(A), s_2(A), \dots, s_n(A)), \\ U_2 B V_2 &= \text{diag}(s_1(B), s_2(B), \dots, s_n(B)). \end{aligned} \quad (5)$$

Setting

$$\beta_i = \begin{cases} 1, & s_i(B) = 0, \\ \frac{s_i(A)}{s_i(B)}, & s_i(B) \neq 0. \end{cases} \quad (6)$$

Denote $W_0 = \text{diag}(\sqrt{\beta_1}, \sqrt{\beta_2}, \dots, \sqrt{\beta_n})$. It is clear that $0 \leq W_0 \leq I$. Then,

$$A = U_1^* W_0 U_2 B V_2 W_0 V_1^*. \quad (7)$$

Let $W = U_1^* W_0 U_2$ and $V = V_2 W_0 V_1^*$. It is trivial that both W and V are contractive. Then, $A = WBV$.

Let A and B be positive semidefinite. Note that for positive semidefinite matrices, singular values and eigenvalues are the same. Using the spectral decompositions of A and B , the unitary matrices U_1 and U_2 exist such that

$$\begin{aligned} U_1^* A U_1 &= \text{diag}(s_1(A), s_2(A), \dots, s_n(A)), \\ U_2^* B U_2 &= \text{diag}(s_1(B), s_2(B), \dots, s_n(B)). \end{aligned} \quad (8)$$

The left proof is similar to the above. This completes the proof.

It is clear that for $A \in M_n$,

$$s_j(A) = s_j(A \oplus O), \quad j = 1, 2, \dots, n, \quad (9)$$

where O is a zero square matrix of any order. An interesting consequence of Theorem 1 is the following. $\square$

Corollary 1 (see [4], Theorem 2.1). Let $A_i, B_i, X_i \in M_n$, $i = 1, \dots, m$. Then,

$$s_j \left(\sum_{i=1}^m A_i X_i B_i \right) \leq \left\| \sum_{i=1}^m |A_i^*|^2 \right\|^{1/2} \left\| \sum_{i=1}^m |B_i|^2 \right\|^{1/2} s_j \left(\bigoplus_{i=1}^m X_i \right), \quad (10)$$

for $j = 1, \dots, n$.

Proof. Denote

$$\begin{aligned} A &= \begin{pmatrix} A_1 & A_2 & \cdots & A_m \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \\ B &= \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ B_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_m & 0 & \cdots & 0 \end{pmatrix}, \\ X &= \bigoplus_{i=1}^m X_i. \end{aligned} \quad (11)$$

Then, $\|A\| = \|\sum_{i=1}^m |A_i^*|^2\|^{1/2}$ and $\|B\| = \|\sum_{i=1}^m |B_i|^2\|^{1/2}$. Then,

$$\frac{A}{\|A\|} X \frac{B}{\|B\|} = \frac{1}{\|A\| \|B\|} \left(\sum_{i=1}^m A_i X_i B_i \right) \oplus O, \quad (12)$$

where O is the zero matrix of order $n(m-1)$. It is clear that both $(1/\|A\|)A$ and $(1/\|B\|)B$ are contractive matrices. By Theorem 1, this completes the proof.

The following result generalizes the result due to Hirzallah and Kittaneh ([4], Corollary 2.1). $\square$

Theorem 2. Let $A_1, A_2, \dots, A_m \in M_n$, and let $\lambda_1, \lambda_2, \dots, \lambda_m$ be complex numbers such that $|\lambda_1| + |\lambda_2| + \dots + |\lambda_m| \leq 1$. Then, for $j = 1, \dots, n$,

$$s_j(\lambda_1 A_1 + \lambda_2 A_2 + \dots + \lambda_m A_m) \leq s_j(A_1 \oplus A_2 \oplus \dots \oplus A_m). \quad (13)$$

Proof. Note that each $\lambda_k = |\lambda_k| e^{i\theta_k}$, $k = 1, \dots, m$, for some $\theta_k \in [0, 2\pi]$. Denote

$$\begin{aligned} A &= \begin{pmatrix} \sqrt{|\lambda_1|} I_n & \sqrt{|\lambda_2|} I_n & \cdots & \sqrt{|\lambda_m|} I_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \\ B &= \begin{pmatrix} \sqrt{|\lambda_1|} e^{i\theta_1} I_n & 0 & \cdots & 0 \\ \sqrt{|\lambda_2|} e^{i\theta_2} I_n & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{|\lambda_m|} e^{i\theta_m} I_n & 0 & \cdots & 0 \end{pmatrix}, \\ X &= \bigoplus_{i=1}^m A_i. \end{aligned} \quad (14)$$

Then,

$$AXB = (\lambda_1 A_1 + \lambda_2 A_2 + \dots + \lambda_m A_m) \oplus O_{nm-n}, \quad (15)$$

with contractive matrices A and B . By Theorem 1, this completes the proof. $\square$

Remark. This result admits the following important special cases:

- (1) Let $A_1, A_2, \dots, A_m \in M_n$, and let $\lambda_1, \lambda_2, \dots, \lambda_m$ be nonnegative real numbers with $\lambda_1 + \lambda_2 + \dots + \lambda_m = 1$. Then,

$$s_j(\lambda_1 A_1 + \lambda_2 A_2 + \dots + \lambda_m A_m) \leq s_j(A_1 \oplus A_2 \oplus \dots \oplus A_m), \quad j = 1, \dots, n. \quad (16)$$

- (2) Let $A, B \in M_n$. Then,

$$s_j\left(\frac{A + iB}{2}\right) \leq s_j(A \oplus B), \quad j = 1, \dots, n. \quad (17)$$

In particular, letting $B = iA^*$, we have

$$s_j(\operatorname{Im} A) \leq s_j(A \oplus A), \quad j = 1, \dots, n, \quad (18)$$

which can be regarded as a complement to Hirzallah and Kittaneh's inequality (see ([4], Corollary 2.3)), which says

$$s_j(\operatorname{Re} A) \leq s_j(A \oplus A), \quad j = 1, \dots, n, \quad (19)$$

where $\operatorname{Re} A$ and $\operatorname{Im} A$ denote the real part and the imaginary part of A , respectively.

Applying Theorem 2, we have the following result.

Corollary 2. Let $A \in M_n$. Then,

$$s_j(A) \leq s_j((A + A^*) \oplus (A - A^*)), \quad j = 1, \dots, n. \quad (20)$$

Proof. Considering the Cartesian decomposition of A , $A = \operatorname{Re} A + i \operatorname{Im} A$. By Theorem 2, we have for $1 \leq j \leq n$,

$$\begin{aligned} s_j(A) &= 2s_j\left(\frac{\operatorname{Re} A + i \operatorname{Im} A}{2}\right) \leq 2s_j(\operatorname{Re} A \oplus \operatorname{Im} A) \\ &= s_j((A + A^*) \oplus (A - A^*)). \end{aligned} \quad (21)$$

The last equality on the above is due to the fact that singular values are unitarily invariant: $s_j(UAV) = s_j(A)$, $j = 1, \dots, n$, for every A and all unitary U and V . $\square$

Remark. Next, we show that equality is possible in inequality (20) for some nonzero square matrix T . Consider the matrix $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then,

$$\begin{aligned} T + T^* &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ T - T^* &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned} \quad (22)$$

It is clear that $s_1(T) = s_1((T + T^*) \oplus (T - T^*)) = 1$.

The well-known arithmetic-geometric mean inequality for singular values due to Bhatia and Kittaneh [6] says that if $A, B \in M_n$, then

$$2s_j(AB^*) \leq s_j(AA^* + BB^*), \quad j = 1, \dots, n. \quad (23)$$

Zhan [5] asserts that for $A, B \in M_n$, if $A \geq O$ and $B \geq O$, then

$$s_j(A - B) \leq s_j(A \oplus B), \quad j = 1, \dots, n. \quad (24)$$

In [7], Tao has proved that if $A, B, C \in M_n$ such that $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq O$, then

$$s_j(B) \leq \frac{1}{2}s_j\left(\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}\right), \quad j = 1, \dots, n. \quad (25)$$

It has been pointed out [7] that the three inequalities (23)–(25) are equivalent.

Recently, Audeh and Kittaneh [8] proved that let $A, B, C \in M_n$. If $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq O$, then

$$s_j(B) \leq s_j(A \oplus C), \quad j = 1, \dots, n. \quad (26)$$

Next, we will interpolate Audeh and Kittaneh's inequality (26) by proving.

Theorem 3. Let $A, B, C \in M_n$ such that $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq O$. Then,

$$s_j(B) \leq \frac{1}{2}s_j\left(\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}\right) \leq s_j(A \oplus C), \quad j = 1, \dots, n. \quad (27)$$

Proof. Since $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq O$, we have

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} = (M \ N)^* (M \ N), \quad (28)$$

for some $M, N \in M_{2n,n}$. Then,

$$\begin{aligned} A &= M^* M, \\ C &= N^* N. \end{aligned} \quad (29)$$

Note that singular values and eigenvalues of positive semidefinite matrices are the same. By Theorem 1.27 in [3], for $1 \leq j \leq n$, then

$$\begin{aligned} s_j\left(\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}\right) &= s_j((M \ N)^* (M \ N)) \\ &= s_j((M \ N)(M \ N)^*). \end{aligned} \quad (30)$$

Applying Theorem 2 and Theorem 1.27 in [3], we obtain that for $1 \leq j \leq n$,

$$s_j(MM^* + NN^*) \leq 2s_j(MM^* \oplus NN^*) = 2s_j(M^* M \oplus N^* N), \quad (31)$$

i.e.,

$$\frac{1}{2}s_j\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \leq s_j(A \oplus C). \quad (32)$$

Combining inequality (25) by Tao and the above inequality gives (27).

Let $A = U \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)U^*$ be the spectral decomposition of $A \in H_n$ with U unitary and $\lambda_1 \geq \dots \geq \lambda_p \geq 0 > \lambda_{p+1} \geq \dots \geq \lambda_n$. Denote

$$\begin{aligned} A^+ &= U \operatorname{diag}(\lambda_1, \dots, \lambda_p, 0, \dots, 0)U^*, \\ A^- &= U \operatorname{diag}(0, \dots, 0, -\lambda_{p+1}, \dots, -\lambda_n)U^*. \end{aligned} \quad (33)$$

Then $A^+ \geq O$, $A^- \geq O$, and $A = A^+ - A^-$. This is called the *Jordan decomposition* of A . As is known, there are at least four different proofs of Zhan's result [5, 9]. Applying Theorem 1 and the Jordan decomposition, we give another direct proof of Zhan's inequality (8). $\square$

Proof. Denote $T = A - B \in H_n$. Considering the Jordan decomposition of T , $T = T^+ - T^-$. Recall from ([3], Theorem 3.6) that if $T = A - B$ with $A \geq O$, $B \geq O$, then

$$\begin{aligned} s_j(T^+) &\leq s_j(A), \\ s_j(T^-) &\leq s_j(B), \\ j &= 1, \dots, n. \end{aligned} \quad (34)$$

By Theorem 1, there exist two contractive matrices W_1 and W_2 such that

$$\begin{aligned} T^+ &= W_1^* A W_1, \\ T^- &= W_2^* A W_2. \end{aligned} \quad (35)$$

Denote $W = W_1 \oplus W_2$. Then, W is a contractive matrix and so $T^+ \oplus T^- = W^*(A \oplus B)W$. For $T \in H_n$,

$$s_j(T) = s_j(T \oplus O) = s_j(T^+ \oplus T^-), \quad j = 1, \dots, n, \quad (36)$$

where $O \in M_n$. By Theorem 1, this completes the proof. [10].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare there are no conflicts of interest.

Authors' Contributions

The authors read and approved the final manuscript.

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