Research Article

The Functional Orlicz Brunn-Minkowski Inequality for q -Capacity

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In this paper, we establish functional forms of the Orlicz Brunn-Minkowski inequality and the Orlicz-Minkowski inequality for the electrostatic q -capacity, which generalize previous results by Zou and Xiong. We also show that these two inequalities are equivalent.

1. Introduction

The classical Brunn-Minkowski inequality was inspired by questions around the isoperimetric problem. It is viewed as one of cornerstones of the Brunn-Minkowski theory, which is a beautiful and powerful tool to conquer all sorts of geometrical problems involving metric quantities such as volume, surface area, and mean width.

An excellent reference on this inequality is provided by Gardner [1].

In 2015, Colesanti, Nyström, Salani, Xiao, Yang, and Zhang (CNSXYZ) [2] introduced the electrostatic q -capacity. Let K be a compact set in the n -dimensional Euclidean space R^n . For $1 < q < n$, the electrostatic q -capacity, $C_q(K)$, of K is defined by

$$C_q(K) = \inf \left\{ \int_{R^n} |\nabla u|^q dx : u \in C_c^\infty(R^n) \text{ and } u \geq \chi_K \right\}, \quad (1)$$

where $C_c^\infty(R^n)$ denotes the set of functions from $C^\infty(R^n)$ with compact supports and χ_K is the characteristic function of K . If $q = 2$, then $C_2(K)$ is the classical electrostatic (or Newtonian) capacity of K . The Minkowski-type problems for the electrostatic q -capacity have attracted increasing attention [2–10]. The electrostatic q -capacity also has applications in analysis, mathematical physics, and partial differential equations (see [11–13]).

The electrostatic q -capacity can be extended on function spaces. Let $C(S^{n-1})$ denote the set of continuous functions defined on S^{n-1} , which is equipped with the metric induced by the maximal norm. Write $C_+(S^{n-1})$ for the set of strictly positive functions in $C(S^{n-1})$. For $1 < q < n$ and $f \in C_+(S^{n-1})$, define the electrostatic q -capacity $C_q(f)$ by

$$C_q(f) = C_q([f]), \quad (2)$$

where $[f]$ denotes the Aleksandrov body (also known as the Wulff shape) associated with f . For nonnegative $f \in C(S^{n-1})$, the Aleksandrov body $[f]$ is defined by

$$[f] = \bigcap_{\xi \in S^{n-1}} \{x \in R^n : x \cdot \xi \leq f(\xi)\}. \quad (3)$$

Obviously, $[f]$ is a compact convex set containing the origin and $h_{[f]} \leq f$, where $h_{[f]}$ denotes the support function of $[f]$. Moreover,

$$K = [h_K], \quad (4)$$

for every compact convex set K containing the origin. If $f \in C_+(S^{n-1})$, then $[f]$ is a convex body in R^n containing the origin in its interior. The Aleksandrov convergence lemma reads: if the

sequence $\{f_j\}_j \subset C_+(S^{n-1})$ converges uniformly to $f \in C_+(S^{n-1})$, then, $\lim_{j \rightarrow \infty} [f_j] = [f]$.

Suppose $a, b \geq 0$ (not both zero). If $f, g \in C_+(S^{n-1})$, then, the L_p Minkowski sum $a \cdot f +_p b \cdot g$ is defined by

$$a \cdot f +_p b \cdot g = (af^p + bg^p)^{1/p}, \quad (5)$$

where the L_p scalar multiplication $a \cdot f$ is defined by $a^{1/p}f$. By the definition of the Aleksandrov body (3), we have $[a \cdot h_{K+_\phi b \cdot L}] = a \cdot K +_\phi b \cdot L$ for convex bodies K and L containing the origin in their interiors. Here, $a \cdot K +_\phi b \cdot L$ denotes the L_p Minkowski sum of K and L , i.e.,

$$h_{a \cdot K +_\phi b \cdot L}^p = ah_K^p + bh_L^p, \quad (6)$$

for every $u \in S^{n-1}$, which was defined by Firey [14]. In the mid 1990s, it was shown in [15, 16] that when L_p Minkowski sum is combined with volume the result is an embryonic L_p -Brunn-Minkowski theory. Zou and Xiong ([7], Theorem 3.11) established the functional form of the L_p Brunn-Minkowski inequality for the electrostatic q -capacity. Suppose $1 < p < \infty$ and $1 < q < n$.

If $f, g \in C_+(S^{n-1})$, then

$$C_q(f +_p g)^{p/(n-q)} \geq C_q(f)^{p/(n-q)} + C_q(g)^{p/(n-q)}, \quad (7)$$

with equality if and only if $[f]$ and $[g]$ are dilates.

The Orlicz Brunn-Minkowski theory which was launched by Lutwak et al. in a series of papers [17–19] is an extension of the L_p Brunn-Minkowski theory. This theory has been considerably developed in the recent years. The Orlicz sum was introduced by Gardner et al. [20]. Let Φ be the class of convex, strictly increasing functions, $\phi : [0, \infty) \rightarrow [0, \infty)$ with $\phi(0) = 0$. Suppose $\phi \in \Phi$ and $a, b \geq 0$ (not both zero). If K and L are convex bodies that contain the origin in their interiors in R^n , then, the Orlicz sum $a \cdot K +_\phi b \cdot L$ is the convex body defined by

$$h_{a \cdot K +_\phi b \cdot L}(u) = \inf \left\{ \tau > 0 : a\phi\left(\frac{h_K(u)}{\tau}\right) + b\phi\left(\frac{h_L(u)}{\tau}\right) \leq \phi(1) \right\}, \quad (8)$$

for every $u \in S^{n-1}$. Gardner et al. ([20], Corollary 7.5) established the Orlicz Brunn-Minkowski inequality (see also ([21], Theorem 1). Same as the Orlicz sum of convex bodies, we extend the L_p Minkowski sum of functions to the Orlicz sum. For $\phi \in \Phi$, $f, g \in C_+(S^{n-1})$, and $a, b \geq 0$ (not both zero), the Orlicz sum $a \cdot f +_\phi b \cdot g$ is defined by

$$a \cdot f +_\phi b \cdot g = \inf \left\{ \tau > 0 : a\phi\left(\frac{f}{\tau}\right) + b\phi\left(\frac{g}{\tau}\right) \leq \phi(1) \right\}. \quad (9)$$

If we take $\phi(t) = t^p$ ($p \geq 1$) in (9), then it induces the L_p Minkowski sum (5). By the definition of the Aleksandrov body

(3), (8), (9), and (4), we have $[a \cdot h_{K+_\phi b \cdot L}] = a \cdot K +_\phi b \cdot L$ for convex bodies K and L containing the origin in their interiors.

The main aim of this paper is to establish the functional form of the Orlicz Brunn-Minkowski inequality for the electrostatic q -capacity.

Theorem 1. Suppose $\phi \in \Phi$ and $1 < q < n$. If $f, g \in C_+(S^{n-1})$, then,

$$\phi\left(\left(\frac{C_q(f)}{C_q(f+_\phi g)}\right)^{1/(n-q)}\right) + \phi\left(\left(\frac{C_q(g)}{C_q(f+_\phi g)}\right)^{1/(n-q)}\right) \leq \phi(1). \quad (10)$$

If ϕ is strictly convex, equality holds if and only if $[f]$ and $[g]$ are dilates.

2. Notation and Preliminary Results

For excellent references on convex bodies, we recommend the books by Gardner [22], Gruber [23], and Schneider [24].

We will work in R^n equipped with the standard Euclidean norm. Let $x \cdot y$ denote the standard inner product of $x, y \in R^n$. For $x \in R^n$, $|x| = \sqrt{x \cdot x}$ denotes the Euclidean norm of x . We write $B = \{x \in R^n : |x| \leq 1\}$ and S^{n-1} for the standard unit ball of R^n and its surface, respectively. Each compact convex set K is uniquely determined by its support function $h_K : R^n \rightarrow R$, which is defined by $h_K(x) = \max \{x \cdot y : y \in K\}$, for $x \in R^n$. Obviously, the support function is positively homogeneous of order 1.

The class of compact convex sets in R^n is often equipped with the Hausdorff metric δ_H , which is defined for compact convex sets K and L by

$$\delta_H(K, L) = \max \{|h_K(u) - h_L(u)| : u \in S^{n-1}\}. \quad (11)$$

Denote by K^n the set of convex bodies in R^n and by K_o^n the set of convex bodies which contain the origin in their interiors. For $s > 0$, the set $sK = \{sx : x \in K\}$ is called a dilate of convex body K . Convex bodies K and L are said to be homothetic, provided $K = sL + x$ for some $s > 0$ and $x \in R^n$. Let $K, L \in K^n$, the Minkowski sum of K and L is the convex body

$$K + L = \{x + y : x \in K, y \in L\}. \quad (12)$$

Some properties of the electrostatic q -capacitary measure are required [2, 3, 7, 8, 11]. The electrostatic q -capacitary measure, $\mu_q(E, \cdot)$, of a bounded open convex set E in R^n is the measure on the unit sphere S^{n-1} defined for $\omega \subset S^{n-1}$ and $1 < q < n$ by

$$\mu_q(E, \omega) = \int_{g^{-1}(\omega)} |\nabla U|^q dH^{n-1}, \quad (13)$$

where $g^{-1} : S^{n-1} \rightarrow \partial E$ (the set of boundary points of E) denotes the inverse Gauss map, H^{n-1} the $(n-1)$ -dimensional

Hausdorff measure, and U the q -equilibrium potential of E . If $K \in K^n$, then the electrostatic q -capacitary measure $\mu_q(K, \cdot)$ has the following properties. First, it is positively homogeneous of degree $(n - q - 1)$, i.e., $\mu_q(sK, \cdot) = s^{n-q-1}\mu_q(K, \cdot)$ for $s > 0$. Second, it is translation invariant, i.e., $\mu_q(K + x, \cdot) = \mu_q(K, \cdot)$ for $x \in \mathbb{R}^n$. Third, its centroid is at the origin, i.e., $\int_{S^{n-1}} u d\mu_q(K, u) = 0$. Moreover, it is absolutely continuous with respect to the surface area measure $S(K, \cdot)$. The weak convergence of the electrostatic q -capacitary measure is proved by CNSXYZ ([2], Lemma 4.1): if $\{K_j\} \subset K^n$ converges to $K \in K^n$, then $\mu_q(K_j, \cdot)$ converges weakly to $\mu_q(K, \cdot)$.

CNSXYZ [2] showed the Hadamard variational formula for the electrostatic q -capacity: for $K, L \in K^n$ and $1 < q < n$,

$$\left. \frac{dC_q(K + tL)}{dt} \right|_{t=0^+} = (q-1) \int_{S^{n-1}} h_L(u) d\mu_q(K, u). \quad (14)$$

And variational formula (14) leads to the following Poincare q -capacity formula:

$$C_q(K) = \frac{q-1}{n-q} \int_{S^{n-1}} h_K(u) d\mu_q(K, u). \quad (15)$$

The electrostatic q -capacity $C_q(K)$ has the following properties. First, it is increasing with respect to set inclusion; that is, if $K_1 \subseteq K_2$, then $C_q(K_1) \leq C_q(K_2)$. Second, it is positively homogeneous of degree $(n - q)$, i.e., $C_q(sK) = s^{n-q}C_q(K)$ for $s > 0$. Third, it is a rigid motion invariant, i.e.,

$C_q(\varphi K + x) = C_q(K)$ for $x \in \mathbb{R}^n$ and $\varphi \in O(n)$. If $q = 2$, then (15) induces the Poincare capacity formula

$$C_2(K) = \frac{1}{n-2} \int_{S^{n-1}} h_K(u) d\mu_2(K, u). \quad (16)$$

Let $C(S^{n-1})$ denote the set of continuous functions defined on S^{n-1} , which is equipped with the metric induced by the maximal norm. Write $C_+(S^{n-1})$ for the set of strictly positive functions in $C(S^{n-1})$. Let $K \in K_o^n$ and $g \in C(S^{n-1})$. There is a $t_0 > 0$ such that $h_K + tg \in C_+(S^{n-1})$ for $|t| < t_0$. The Aleksandrov body $[h_K + tg]$ is continuous in $t \in (-t_0, t_0)$. The Hadamard variational formula for the electrostatic q -capacity [2] states the following:

$$\left. \frac{dC_q(h_K + tg)}{dt} \right|_{t=0^+} = (q-1) \int_{S^{n-1}} g(u) d\mu_q(K, u). \quad (17)$$

For $f \in C_+(S^{n-1})$, define

$$C_q(f) = C_q([f]). \quad (18)$$

Obviously, $C_q(h_K) = C_q(K)$ for every $K \in K_o^n$. By the Aleksandrov convergence lemma and the continuity of C_q on K^n , the functional $C_q : C_+(S^{n-1}) \rightarrow (0, \infty)$ is continuous. For $K \in K_o^n$ and $g \in C(S^{n-1})$, the mixed elec-

trostatic q -capacity $C_q(K, g)$ is defined by

$$C_q(K, g) = \frac{1}{n-q} \left. \frac{dC_q(h_K + tg)}{dt} \right|_{t=0^+}. \quad (19)$$

Applying the Hadamard variational formula for the electrostatic q -capacity, the mixed electrostatic q -capacity $C_q(K, g)$ has the following integral representation:

$$C_q(K, g) = \frac{q-1}{n-q} \int_{S^{n-1}} g(u) d\mu_q(K, u). \quad (20)$$

Let $L \in K^n$. If $g = h_L$, then, $C_q(K, g)$ is the mixed electrostatic q -capacity $C_q(K, L)$, which has the following integral representation:

$$C_q(K, L) = \frac{q-1}{n-q} \int_{S^{n-1}} h_L(u) d\mu_q(K, u). \quad (21)$$

The Minkowski inequality for the electrostatic q -capacity ([2], Theorem 3.6) states the following: let $1 < q < n$.

If $K, L \in K^n$, then,

$$C_q(K, L) \geq C_q(K)^{(n-q-1)/(n-q)} C_q(L)^{1/(n-q)}, \quad (22)$$

with equality if and only if K and L are homothetic.

Let $1 \leq p < \infty$ and $1 < q < n$. For $K \in K_o^n$ and $g \in C(S^{n-1})$, the L_p Hadamard variational formula for the electrostatic q -capacity [7] states the following:

$$\left. \frac{dC_q(h_K + p^t \cdot g)}{dt} \right|_{t=0^+} = \frac{q-1}{p} \int_{S^{n-1}} g(u)^p h_K(u)^{1-p} d\mu_q(K, u). \quad (23)$$

The L_p mixed electrostatic q -capacity $C_{p,q}(K, g)$ is defined by

$$C_{p,q}(K, g) = \frac{1}{n-q} \left. \frac{dC_q(h_K + p^t \cdot g)}{dt} \right|_{t=0^+} = \frac{q-1}{n-q} \int_{S^{n-1}} g(u)^p h_K(u)^{1-p} d\mu_q(K, u). \quad (24)$$

Take $g = h_K$ in (24), and combine $C_q(K, g) = C_q(K)$ to obtain the Poincare q -capacity formula (15). Zou and Xiong ([7], Theorem 3.9) established the L_p Minkowski inequality for the L_p electrostatic q -capacity: let $1 < p < \infty$ and $1 < q < n$. If $K \in K_o^n$ and $g \in C(S^{n-1})$, then,

$$C_q(K, g) \geq C_q(K)^{(n-q-p)/(n-q)} C_q(L)^{p/(n-q)}, \quad (25)$$

with equality if and only if K and $[g]$ are dilates.

Based on the Orlicz sum (9), we define the Orlicz mixed electrostatic q -capacity as follows. For $K \in K_o^n$ and $g \in C(S^{n-1})$, the Orlicz mixed electrostatic q -capacity $C_{\phi,q}(K, g)$ is defined by

$$C_{\phi,q}(K, g) = \frac{1}{n-q} \frac{dC_q(h_K + \phi \cdot t \cdot g)}{dt} \Big|_{t=0^+}. \quad (26)$$

Indeed, the Orlicz mixed electrostatic q -capacity can be extended on function spaces. Let $\phi \in \Phi$ and $1 < q < n$. For $f \in C_+(S^{n-1})$ and $g \in C(S^{n-1})$, the Orlicz mixed electrostatic q -capacity $C_{\phi,q}([f], g)$ is defined by

$$C_{\phi,q}([f], g) = \frac{1}{n-q} \frac{dC_q(f + \phi \cdot t \cdot g)}{dt} \Big|_{t=0^+}. \quad (27)$$

If $f = h_K$ with $K \in K_o^n$, then, $C_{\phi,q}([h_K], g) = C_{\phi,q}(K, g)$. If $g = h_L$ with $L \in K_o^n$, then, $C_{\phi,q}([f], h_L)$ is the Orlicz mixed electrostatic q -capacity $C_{\phi,q}([f], L)$. In particular, $C_{\phi,q}([h_K], h_K) = \phi(1)C_q(K)$ for every $K \in K_o^n$.

3. Main Results

The following variational formula of electrostatic q -capacity plays a crucial role in our proof.

Lemma 2 ([2], Lemma 5.1). *Let $I \subset \mathbb{R}$ be an interval containing 0 in its interior, and let $h_t(u) = h(t, u): I \times S^{n-1} \rightarrow [0, \infty)$ be continuous such that the convergence in*

$$h'(0, u) = \lim_{t \rightarrow 0} \frac{h(t, u) - h(0, u)}{t} \quad (28)$$

is uniformly on S^{n-1} . Then,

$$\frac{dC_q([h_t])}{dt} \Big|_{t=0^+} = (q-1) \int_{S^{n-1}} h'(0, u) d\mu_q([h_0], u). \quad (29)$$

*Suppose $\phi \in \Phi$, $f, g \in C_q(S^{n-1})$, and $a, b \geq 0$ (not both zero). For every given $u \in S^{n-1}$, the function $t \mapsto a\phi(f(u)/t) + b\phi(g(u)/t)$ is strictly decreasing. By the definition of the Orlicz sum (9), we have $(a \cdot f + \phi \cdot b \cdot g)(u) = t$ and only if $a\phi(f(u)/t) + b\phi(g(u)/t) = \phi(1)$.
for every $u \in S^{n-1}$.*

The continuity properties of the Orlicz sum were established by Xi et al. [21].

Lemma 3 ([21], Lemma 3.1). *Suppose $\phi \in \Phi$ and $f \in C_+(S^{n-1})$, $g \in C(S^{n-1})$, and $a, b \geq 0$ (not both zero).*

- (i) *Let $\{f_i\}, \{g_i\} \subset C_+(S^{n-1})$ and $\{g_i\} \subset C(S^{n-1})$ such that $f_i \rightarrow f$ and $g_i \rightarrow g$, respectively. Then, $a \cdot f_i + \phi \cdot b \cdot g_i \rightarrow a \cdot f + \phi \cdot b \cdot g$*
- (ii) *Let $\{\phi_i\} \subset \Phi$ such that $\phi_i \rightarrow \phi$. Then, $a \cdot f + \phi_i \cdot b \cdot g \rightarrow a \cdot f + \phi \cdot b \cdot g$*
- (iii) *Let $a_i, b_i \geq 0$ (not both zero) such that $a_i \rightarrow a$ and $b_i \rightarrow b$. Then*

$$a_i \cdot f + \phi_i \cdot b \cdot g \rightarrow a \cdot f + \phi \cdot b \cdot g \quad (30)$$

Due to Lemma 2, the integral representation of the Orlicz mixed electrostatic q -capacity is given.

Lemma 4. *Suppose $\phi \in \Phi$ and $1 < q < n$. If $f \in C_+(S^{n-1})$ and $g \in C(S^{n-1})$, then*

$$C_{\phi,q}([f], g) = \frac{q-1}{n-q} \int_{S^{n-1}} \phi\left(\frac{g(u)}{f(u)}\right) f(u) d\mu_q([f], u). \quad (31)$$

Proof. Take an interval $I = [0, t_0]$ for $0 < t_0 < \infty$. Denote $h_t(u): I \times S^{n-1} \rightarrow (0, \infty)$ by

$$h_t(u) = h(t, u) = (f + \phi \cdot t \cdot g)(u). \quad (32)$$

Then, the definition of the Orlicz sum (9) and Lemma 3 imply that the function $h_t(u): I \times S^{n-1} \rightarrow (0, \infty)$ is continuous. By (9), we have

$$\phi\left(\frac{f(u)}{h_t(u)}\right) + t\phi\left(\frac{g(u)}{h_t(u)}\right) = \phi(1), \quad (33)$$

for every $u \in S^{n-1}$. Since $\phi \in \Phi$, we obtain

$$\frac{dh_t}{dt} = \frac{h_t^2 \phi(g/h_t)}{f\phi'(f/h_t) + t\phi'(f/h_t)}. \quad (34)$$

Note that $(f/h) \rightarrow 1^-$ as $t \rightarrow 0^+$ and the fact that $h_0 = f$. Thus,

$$\lim_{t \rightarrow 0^+} \frac{h_t - h_0}{t} = \frac{f\phi(g/f)}{\phi'_l(1)}, \quad (35)$$

uniformly on S^{n-1} , where $\phi'_l(1)$ denotes the left derivative of $\phi(t)$ at $t = 1$. Apply Lemma 2 and (35) to get

$$\frac{dC_q(f + \phi \cdot t \cdot g)}{dt} \Big|_{t=0^+} = \frac{q-1}{\phi'_l(1)} \int_{S^{n-1}} \phi\left(\frac{g(u)}{f(u)}\right) d\mu_q([f], u). \quad (36)$$

Thus, (27) and (36) yield the desired lemma.

Indeed, (36) can be considered as the Orlicz Hadamard variational formula for the electrostatic q -capacity. If we take $\phi(t) = t^p$ ($1 \leq p < \infty$) and $f = h_K$ with $K \in K_o^n$ in (36), then, we obtain the L_p Hadamard variational formula (23).

Note that $[h_K] = K$ for every $K \in K_o^n$. Take $f = h_K$ in Lemma 4 to get

Lemma 5. *Suppose $\phi \in \Phi$ and $1 < q < n$. If $K \in K_o^n$ and $g \in C(S^{n-1})$, then,*

$$C_{\phi,q}(K, g) = \frac{q-1}{n-q} \int_{S^{n-1}} \phi\left(\frac{g(u)}{h_K(u)}\right) h_K(u) d\mu_q(K, u). \quad (37)$$

A direct consequence of Lemma 4 and the homogeneity of the electrostatic q -capacitary measure can be obtained.

Corollary 6. Suppose $\phi \in \Phi$ and $1 < q < n$. If $f \in C_+(S^{n-1})$, then

$$C_{\phi,q}(c[f], f) = \phi\left(\frac{1}{c}\right) c^{n-q} C_q([f]) = \phi\left(\frac{1}{c}\right) c^{n-q} C_q(f), \quad (38)$$

for every $c > 0$.

Let $\phi \in \Phi$, $1 < q < n$, and $K, L \in K_o^n$. Note that $K +_\phi t \cdot L = [h_{K+\phi t \cdot h_L}]$, and apply (18) and (36) to obtain

$$\left. \frac{dC_q(K +_\phi t \cdot L)}{dt} \right|_{t=0^+} = \frac{q-1}{\phi'_1(1)} \int_{S^{n-1}} \phi\left(\frac{h_L(u)}{h_K(u)}\right) h_K(u) d\mu_q(K, u). \quad (39)$$

Based on (39), one can define the Orlicz mixed electrostatic q -capacity $C_{\phi,q}(K, L)$ of convex bodies K and L as follows:

$$C_{\phi,q}(K, L) = \frac{q-1}{n-q} \int_{S^{n-1}} \phi\left(\frac{h_L(u)}{h_K(u)}\right) h_K(u) d\mu_q(K, u), \quad (40)$$

which was first defined by Hong et al. ([10], Definition 3.1).

Lemma 7. Suppose $\phi \in \Phi$, $f \in C_+(S^{n-1})$, $g \in C(S^{n-1})$, and $1 < q < n$.

- (i) Let $\phi_1, \phi_2 \in \Phi$. If $\phi_1 \leq \phi_2$, then $C_{\phi_1,q}([f], g) \longrightarrow C_{\phi_2,q}([f], g)$
- (ii) Let $\{f_i\} \subset C_+(S^{n-1})$ and $\{g_i\} \subset C(S^{n-1})$ such that $f_i \longrightarrow f$ and $g_i \longrightarrow g$, respectively. Then, $C_{\phi,q}([f_i], g_i) \longrightarrow C_{\phi,q}([f], g)$
- (iii) Let $\{\phi_i\} \subset \Phi$ such that $\phi_i \longrightarrow \phi$. Then, $C_{\phi_i,q}([f], g) \longrightarrow C_{\phi,q}([f], g)$

Proof. It follows from (31) that (i) holds if $\phi_1 \leq \phi_2$.

Since $f > 0, f_i > 0, g \geq 0, g_i \geq 0$ and $f_i \longrightarrow f, g_i \longrightarrow g$ uniformly on S^{n-1} ; it follows that $g_i/f_i \longrightarrow g/f$ uniformly on S^{n-1} . Note that $\phi \in \Phi$, we have $\phi(g_i/f_i) f_i \longrightarrow \phi(g/f) f$ uniformly on S^{n-1} . The Aleksandrov convergence lemma implies that $[f_i] \longrightarrow [f]$ uniformly on S^{n-1} . Meanwhile, the convergence $[f_i] \longrightarrow [f]$ implies that $\mu_q([f_i], \cdot) \longrightarrow \mu_q([f], \cdot)$ weakly. Applying Lemma 4, one concludes that (ii) holds.

Clearly, there exists a compact interval $I \subset (0, \infty)$ such that $g/f \in I$ for all $u \in S^{n-1}$.

(iii) directly follows from Lemma 4 and the fact that the sequence $\{\phi_i(t)\}$ converges uniformly to $\phi(t)$ on I .

Next, we show that there is a natural Orlicz extension of the Minkowski inequality for the electrostatic q -capacity.

Theorem 8. Suppose $\phi \in \Phi$ and $1 < q < n$. If $f, g \in C_+(S^{n-1})$, then,

$$\frac{C_{\phi,q}([f], (g))}{C_q(f)} \geq \phi\left(\left(\frac{C_q(g)}{C_q(f)}\right)^{1/(n-q)}\right). \quad (41)$$

If ϕ is strictly convex, then equality holds if and only if $[f]$ and $[g]$ are dilates.

Proof. By the definition of the mixed electrostatic q -capacity (20) and the fact that $h_{[g]} \leq g$, we have

$$C_q([f], [g]) = C_q([f], h_{[g]}) \leq C_q([f], g). \quad (42)$$

for every $f, g \in C_+(S^{n-1})$. From (31), Jensen's inequality, (20), (42), (22), and (18), it follows that

$$\begin{aligned} \frac{C_{\phi,q}([f], g)}{C_q(f)} &= \frac{((q-1)/(n-q)) \int_{S^{n-1}} \phi(g(u)/f(u)) f(u) d\mu_q([f], u)}{C_q(f)} \\ &\geq \phi\left(\frac{C_q([f], g)}{C_q(f)}\right) \geq \phi\left(\frac{C_q([f], [g])}{C_q(f)}\right) \\ &\geq \phi\left(\frac{C_q(g)^{1/(n-q)}}{C_q(f)^{1/(n-q)}}\right) = \phi\left(\left(\frac{C_q(g)}{C_q(f)}\right)^{1/(n-q)}\right). \end{aligned} \quad (43)$$

It remains to prove the equality condition. Now, suppose ϕ is strictly convex. If equality in (41) holds, then, by the equality condition of Jensen's inequality, there exists an $s > 0$ such that $g = sf$ for almost every $u \in S^{n-1}$ with respect to the measure $(f(\cdot) d\mu_q([f], \cdot)) / (C_q(f))$. Then, we have

$$\begin{aligned} s &= \frac{(q-1)/(n-q) \int_{S^{n-1}} (g(u))/f(u) f(u) d\mu_q([f], u)}{C_q(f)} \\ &= \frac{C_q([f], g)}{C_q(f)} = \frac{C_q([f], [g])}{C_q(f)}, \end{aligned} \quad (44)$$

where the last step is from the equality condition of (42). The definition of Aleksandrov body implies that $h_{[g]} = sh_{[f]}$ for almost every $u \in S^{n-1}$ with respect to the measure $(f(\cdot) d\mu_q([f], \cdot)) / (C_q(f))$. Thus,

$$C_q([f], [g]) h_{[f]}(u) = C_q(f) h_{[g]}(u), \quad (45)$$

for almost every $u \in S^{n-1}$ with respect to the measure $(f(\cdot) d\mu_q([f], \cdot)) / (C_q(f))$. By the equality condition of the Minkowski inequality for the electrostatic q -capacity, there exists $x \in R^n$ such that $[g] = s[f] + x$.

Hence, for almost every $u \in S^{n-1}$ with respect to the measure $f(\cdot)d\mu_q([f], \cdot)$,

$$\begin{aligned} & \left(sC_q(f) + \frac{q-1}{n-q}x \cdot \int_{S^{n-1}} u d\mu_q([f], u) \right) h_{[f]}(u) \\ &= C_q(f) \left(sh_{[f]}(u) + x \cdot u \right). \end{aligned} \quad (46)$$

Since the centroid of $\mu_q([f], \cdot)$ is at the origin, we have that $x \cdot u = 0$ for almost every $u \in S^{n-1}$ with respect to the measure $(f(\cdot)d\mu_q([f], \cdot))/(C_q(f))$. Note that the electrostatic q -capacitary measure $\mu_q([f], \cdot)$ is not concentrated on any great subsphere of S^{n-1} . Hence, $x = 0$, which in turn implies that $[f]$ and $[g]$ are dilates.

Conversely, assume that $[f]$ and $[g]$ are dilates, say, $[f] = c[g]$ for some $c > 0$. From our assumption, Corollary 6, (18), and the fact that $C_q(c[g]) = c^{n-q}C_q([g])$, it follows that

$$\begin{aligned} \frac{C_{\phi,q}([f], g)}{C_q(f)} &= \frac{C_{\phi,q}(c[g], g)}{C_q(c[g])} = \frac{\phi(1/c)c^{n-q}C_q([g])}{c^{n-q}C_q([g])} \\ &= \phi\left(\frac{1}{c}\right) = \phi\left(\left(\frac{C_q(g)}{C_q(f)}\right)^{1/(n-q)}\right). \end{aligned} \quad (47)$$

This completes the proof.

By using the Orlicz-Minkowski inequality for the electrostatic q -capacity, we establish the following Orlicz Brunn-Minkowski inequality for the electrostatic q -capacity which is the general version of Theorem 1.

Theorem 9. Suppose $\phi \in \Phi$ and $1 < q < n$. If $f, g \in C_+(S^{n-1})$ and $a, b \geq 0$ (not both zero); then,

$$a\phi\left(\left(\frac{C_q(f)}{C_q(a \cdot f +_\phi b \cdot g)}\right)^{1/(n-q)}\right) + b\phi\left(\left(\frac{C_q(g)}{C_q(a \cdot f +_\phi b \cdot g)}\right)^{1/(n-q)}\right) \leq \phi(1). \quad (48)$$

If ϕ is strictly convex, then equality holds if and only if $[f]$ and $[g]$ are dilates.

Proof. By (31), (9), and the Orlicz-Minkowski inequality for the electrostatic q -capacity (41), we have

$$\begin{aligned} \phi(1) &= \frac{C_{\phi,q}([a \cdot f +_\phi b \cdot g], a \cdot f +_\phi b \cdot g)}{C_q(a \cdot f +_\phi b \cdot g)} \\ &= a \frac{C_{\phi,q}([a \cdot f +_\phi b \cdot g], f)}{C_q(a \cdot f +_\phi b \cdot g)} + b \frac{C_{\phi,q}([a \cdot f +_\phi b \cdot g], g)}{C_q(a \cdot f +_\phi b \cdot g)} \\ &\geq a\phi\left(\left(\frac{C_q(f)}{C_q(a \cdot f +_\phi b \cdot g)}\right)^{1/(n-q)}\right) \\ &\quad + b\phi\left(\left(\frac{C_q(g)}{C_q(a \cdot f +_\phi b \cdot g)}\right)^{1/(n-q)}\right). \end{aligned} \quad (49)$$

By the equality condition of the Orlicz-Minkowski inequality for the electrostatic q -capacity, we have that if ϕ is strictly convex, then equality in (48) holds if and only if $[f]$ and $[g]$ are dilates of $[a \cdot f +_\phi b \cdot g]$.

Remark 1. The case $\phi(t) = t^p$ ($1 \leq p < \infty$) of Theorem 9 was established by Zou and Xiong [7].

For $K, L \in K_o^n$, take $f = h_K$ and $g = h_L$ in Theorem 9 to obtain the following Orlicz-Brunn-Minkowski inequality for the electrostatic q -capacity, which was established by Hong et al. [10].

Corollary 10 ([10], Theorem 4.2). Suppose $\phi \in \Phi$ and $1 < q < n$. If $K, L \in K_o^n$, then

$$\begin{aligned} & a\phi\left(\left(\frac{C_q(K)}{C_q(a \cdot K +_\phi b \cdot L)}\right)^{1/(n-q)}\right) \\ & + b\phi\left(\left(\frac{C_q(L)}{C_q(a \cdot K +_\phi b \cdot L)}\right)^{1/(n-q)}\right) \leq \phi(1). \end{aligned} \quad (50)$$

If ϕ is strictly convex, then equality holds if and only if K and L are dilates.

Remark 2. The case $\phi(t) = t$ of Corollary 10 was obtained by Colesanti and Salani [25]. Borell [26] first established the Brunn-Minkowski inequality for the classical electrostatic capacity, while its equality condition was shown by Caffarelli et al. [4].

Theorem 11. Suppose $\phi \in \Phi$, $1 < q < n$, and $f, g \in C_+(S^{n-1})$. Then, the Orlicz-Brunn-Minkowski inequality for the electrostatic q -capacity implies the Orlicz-Minkowski inequality for the electrostatic q -capacity.

Proof. For $t \geq 0$ and $f, g \in C_+(S^{n-1})$, define the function $G(t)$ by

$$\begin{aligned} G(t) &= \phi(1) - \phi\left(\left(\frac{C_q(f)}{C_q(f +_\phi t \cdot g)}\right)^{1/(n-q)}\right) \\ &\quad - t\phi\left(\left(\frac{C_q(g)}{C_q(f +_\phi t \cdot g)}\right)^{1/(n-q)}\right). \end{aligned} \quad (51)$$

The Orlicz-Brunn-Minkowski inequality for the electrostatic q -capacity implies that $G(t)$ is nonnegative. Obviously, $G(0) = 0$. Thus,

$$\lim_{t \rightarrow 0^+} \frac{G(t) - G(0)}{t} \geq 0. \quad (52)$$

On the other hand, by (51) and the continuity of C_q , we have

$$\begin{aligned}
\lim_{t \rightarrow 0^+} \frac{G(t) - G(0)}{t} &= \lim_{t \rightarrow 0^+} \frac{\phi(1) - \phi\left(\left((C_q(f))/(C_q(f +_\phi t \cdot g))\right)^{1/(n-q)}\right) - t\phi\left(\left((C_q(f))/(C_q(f +_\phi t \cdot g))\right)^{1/(n-q)}\right)}{t} \\
&= \lim_{t \rightarrow 0^+} \frac{\phi(1) - \phi\left(\left((C_q(f))/(C_q(f +_\phi t \cdot g))\right)^{1/(n-q)}\right)}{t} - \phi\left(\left(\frac{C_q(g)}{C_q(f)}\right)^{1/(n-q)}\right) \\
&= \lim_{t \rightarrow 0^+} \frac{\phi(1) - \phi\left(\left((C_q(f))/(C_q(f +_\phi t \cdot g))\right)^{1/(n-q)}\right)}{1 - \left((C_q(f))/(C_q(f +_\phi t \cdot g))\right)^{1/(n-q)}} \cdot \frac{1 - \left((C_q(f))/(C_q(f +_\phi t \cdot g))\right)^{1/(n-q)}}{t} \\
&\quad - \phi\left(\left(\frac{C_q(g)}{C_q(f)}\right)^{1/(n-q)}\right).
\end{aligned} \tag{53}$$

Let $s = \left((C_q(f))/(C_q(f +_\phi t \cdot g))\right)^{1/(n-q)}$. Note that $s \rightarrow 1^-$ as $t \rightarrow 0^+$. Consequently,

$$\lim_{t \rightarrow 0^+} \frac{\phi(1) - \phi\left(\left((C_q(f))/(C_q(f +_\phi t \cdot g))\right)^{1/(n-q)}\right)}{1 - \left((C_q(f))/(C_q(f +_\phi t \cdot g))\right)^{1/(n-q)}} = \phi'_1(1). \tag{54}$$

The continuity of C_q and (27) imply

$$\begin{aligned}
\lim_{t \rightarrow 0^+} \frac{1 - \left((C_q(f))/(C_q(f +_\phi t \cdot g))\right)^{1/(n-q)}}{t} &= \lim_{t \rightarrow 0^+} \frac{C_q(f +_\phi t \cdot g) - (C_q(f))^{1/(n-q)}}{t} \\
&\quad \cdot \lim_{t \rightarrow 0^+} (C_q(f +_\phi t \cdot g))^{-1/(n-q)} \\
&= \frac{1}{n-q} (C_q(f))^{(1/(n-q)-1)} \cdot \lim_{t \rightarrow 0^+} \frac{(C_q(f +_\phi t \cdot g))^{1/(n-q)} - C_q(f)}{t} \\
&\quad \cdot (C_q(f))^{-1/(n-q)} = \frac{C_{\phi,q}([f], g)}{\phi'_1(1)C_q(f)}.
\end{aligned} \tag{55}$$

From (53), (54), (55), and (52), it follows that

$$\lim_{t \rightarrow 0^+} \frac{G(t) - G(0)}{t} = \left(\frac{C_{\phi,q}([f], g)}{C_q(f)}\right) - \phi\left(\left(\frac{C_q(g)}{C_q(f)}\right)^{1/(n-q)}\right) \geq 0, \tag{56}$$

which implies the Orlicz-Minkowski inequality for the electrostatic q -capacity (41).

Finally, we show an immediate application of the Orlicz-Minkowski inequality for the electrostatic q -capacity.

Lemma 12. Suppose $\phi \in \Phi$ and $1 < q < n$. If $f, g \in C_+(S^{n-1})$ and C is a subset of $C_+(S^{n-1})$ such that $f, g \in C$, then the following assertions hold:

- (i) $C_{\phi,q}([h], f) = C_{\phi,q}([h], g)$ for all $h \in C$; then $[f] = [g]$
- (ii) $(C_{\phi,q}([f], h))/(C_q(f)) = (C_{\phi,q}([g], h))/(C_q(f))$ for all $h \in C$; then $[f] = [g]$

Proof. We first show that (i) holds. Since $C_{\phi,q}([f], f) = \phi(1)C_q(f)$, it follows that $\phi(1) = (C_{\phi,q}([f], g))/(C_q(g))$ by the assumption. By the Orlicz-Minkowski inequality for the electrostatic q -capacity, we have $\phi(1) \geq \phi\left(\left((C_q(f))/(C_q(g))\right)^{1/(n-q)}\right)$. The monotonicity of ϕ and $1 < q < n$ imply that

$$\frac{C_q(f)}{C_q(g)} < 1, \tag{57}$$

with equality if and only if $[f]$ and $[g]$ are dilates. This inequality is reversed if interchanging f and g . So, $C_q(f) = C_q(g)$ and $[f]$ and $[g]$ are dilates. Assume that $s[f] = s[g]$ for some $s > 0$. The homogeneity of C_q implies $s = 1$. Thus, $[f] = [g]$.

Then, we can prove (ii) with the similar arguments in (i).

If the Orlicz mixed electrostatic q -capacity $C_{\phi,q}$ is restricted on convex bodies, then we obtain the following characterizations for identity of convex bodies, which were proved by Hong et al. [13].

Corollary 13 ([10], Theorem 3.3). Suppose $\phi \in \Phi$ and $1 < q < n$. If $K, L \in K^n_o$ and C is a subset of K^n_o such that $K, L \in C$, then the following assertions hold:

- (i) $C_{\phi,q}(Q, K) = C_{\phi,q}(Q, L)$ for all $Q \in C$; then $K = L$

- (ii) $(C_{\phi,q}(K, Q))/(C_q(K)) = (C_{\phi,q}(L, Q))/(C_q(L))$ for all $Q \in C$; then $K = L$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors contributed equally to this work. All authors have read and approved the final manuscript.

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