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Research Article

Doubly Degenerate Parabolic Equation with Time-Dependent Gradient Source and Initial Data Measures

Lihua Deng [1] **and Xianguang Shang** [2]

[1] *School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo 454000, China*
[2] *School of Mechanical and Power Engineering, Henan Polytechnic University, Jiaozuo 454000, China*

Correspondence should be addressed to Lihua Deng; denglihua@hpu.edu.cn

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This paper is devoted to the Cauchy problem for a class of doubly degenerate parabolic equation with time-dependent gradient source, where the initial data are Radon measures. Using the delicate a priori estimates, we first establish two local existence results. Furthermore, we show that the existence of solutions is optimal in the class considered here.

1. Introduction and Statement of the Main Results

We consider the nonnegative solutions of the following Cauchy problem:

$$u_t - \operatorname{div}\left(\left|Du^m\right|^{p-2} Du^m\right) = at^{-\sigma} u^q |Du|^l, \quad \text{in } S_T = R^N \times (0, T), \tag{1}$$

$$u(x, 0) = \mu, \quad \text{on } R^N, \tag{2}$$

where $m > 0$, $p > 1$, $m(p-1) > 1$, $a > 0$, $-\infty < \sigma < (p-l)/p$, $q \geq 0$, $0 < l < p$, $T > 0$, and $N \geq 1$ and μ is a nonnegative Radon measure. Equation (1) has been intensively investigated in the last decades because of both its mathematical interest and its potential for applications. In particular, it has been proposed as an appropriate model for surface growth by ballistic deposition and specifically for vapour deposition and the sputter deposition of thin films of aluminium and rare earth metals. One can refer to the bibliographies [1–7] and the references therein for more details on the physical situations.

For the homogeneous case $a = 0$, the existence of solutions and the initial trace problem were studied in [8–10] for the case $p = 2$. As $m = 1$, DiBenedetto and Herrero [11] established the existence of solutions under optimal assumptions on the initial data and initial trace for nonnegative solutions. When $m \neq 1$ and $p \neq 2$, one can refer to [12–14] for the existence of solutions, initial trace of nonnegative solutions, and the weak Harnack inequalities for weak supersolutions, respectively.

Concerning the case $\sigma = 0$, the existence of solutions and initial trace of nonnegative solutions to the problem (1) and (2) have been studied extensively (see, e.g., [15–27]). By a priori estimates and the compactness methods, Andreucci [18] proved the existence of solutions of (1) with $p = 2$ under optimal assumptions on the initial data. Later, Chen and Zhao [22] and Shang and Li [27] separately extended the results of [18] to the Cauchy problem (1) and (2) with $m = 1$, where the initial data are measured. Recently, Shang and Cheng [25] established the local existence of solutions to the Cauchy problem (1) and (2) with $m(p-1) > 1$ and initial data in $L^r_{\text{loc}}(R^N)$ with $r \geq 1$. However, the existence of solutions for initial data measures was left open. In the bounded domain, the critical extinction exponent, the existence, and uniqueness of weak solutions were studied [28, 29]. Weng [30] established the existence and stability of weak solutions to the double degenerate evolutionary $p(x)$-Laplacian equation.

As $\sigma > 0$, as well as we know there are few results in this direction. For the case $m = 1$, $p = 2$, and $l = 0$, the existence and nonexistence of solutions for (1) and (2) were obtained

by Meier [31]. Later, based on the a priori estimates and the improved De Giorgi iteration methods, Andreucci [17] established the local and global existence and nonexistence of solutions to (1) and (2) with $p = 2$ and $l = 0$ and initial data in $L^r_{\mathrm{loc}}(R^N)$ with $r \geq 1$. Recently, for the case $m = 1$ and $l = 0$, the local existence of solutions to the Cauchy problem (1) and (2) with initial data measures was studied by Shang [24].

Here we consider the existence of solutions to the Cauchy problem (1) and (2) in the spirit of [17, 18, 24, 25]. Due to the more complicated structure of the equation and lower regularity of the initial data, the existence issue considered here becomes more difficult. This makes it harder to get the uniform a priori L^∞-estimates and gradient estimates. Fortunately, using delicate estimates, we can overcome these difficulties and establish the local existence of solutions under optimal assumptions on the initial data.

As preparations, we first state several notations which will be used frequently later.

Definition 1. A nonnegative measurable function $u(x,t)$ defined in S_T is called a weak solution of (1) and (2), if

$$\begin{gathered} u \in C_{\mathrm{loc}}\left(0,T;L^1_{\mathrm{loc}}\left(R^N\right)\right) \cap L^\infty_{\mathrm{loc}}(S_T), \\ u^m \in L^p_{\mathrm{loc}}\left(0,T;W^{1,p}_{\mathrm{loc}}\left(R^N\right)\right), \\ t^{-\sigma}u^q|Du|^l \in L^1_{\mathrm{loc}}(S_T), \\ \int_0^t \int_{R^N} \left[-u\varphi_\tau + \left|Du^m\right|^{p-2} Du^m \cdot D\varphi\right] \mathrm{d}x\,\mathrm{d}\tau \\ = \int_0^t \int_{R^N} a\tau^{-\sigma}u^q|Du|^l \varphi\,\mathrm{d}x\,\mathrm{d}\tau, \end{gathered} \tag{3}$$

for all $\varphi \in C^1_0(S_T)$. Moreover,

$$\lim_{t\longrightarrow 0} \int_{R^N} u(x,t)\eta(x)\mathrm{d}x = \int_{R^N} \eta\,\mathrm{d}\mu, \quad \forall \eta \in C^1_0(S_T). \tag{4}$$

Weak subsolutions (resp. supersolutions) are defined in the same way except that the $=$ in (3) is replaced by $\leq$ (resp. $\geq$) and φ is taken to be nonnegative.

Set

$$\begin{gathered} [\mu] = \sup_{x\in R^N} \sup_{0<\rho<1} \rho^\theta \fint_{B_\rho(x)} \mathrm{d}\mu, \\ [u]_t = \sup_{0<\tau<t} \sup_{x\in R^N} \sup_{0<\rho<1} \rho^\theta \fint_{B_\rho(x)} u(y,\tau)\mathrm{d}y, \quad 0<t<T, \end{gathered} \tag{5}$$

where $0 \leq \theta \leq N$ and

$$\begin{gathered} \fint_E \mathrm{d}\mu = \frac{1}{|E|}\int_E \mathrm{d}\mu, \quad |E| \text{is the Lebesgue measure of } E, \\ B_\rho(x) = \{y \| y-x| < \rho\}. \end{gathered} \tag{6}$$

Moreover, we use $\gamma(a_1, a_2, \ldots, a_n)$ to denote positive constants depending only on specified quantities $a_1, a_2, \ldots, a_n$, which may vary from line to line.

We now state our main existence results as follows.

Theorem 1 (the case $q + l \geq m(p-1)$). *Let $[\mu]$ be finite and*

$$\sigma + \frac{\theta(p(q+l) - lm(p-1) - p + l)}{p(p + \theta(m(p-1)-1))} < \frac{p-l}{p}. \tag{7}$$

Then there exists a solution to (1) and (2) defined in $R^N \times (0,T_0)$, where $T_0 = T_0([\mu], N, m, p, a, q, l, \sigma, \theta)$, such that for all $0 < t < T_0$, we have

$$[u]_t \leq \gamma[\mu], \tag{8}$$

$$\|u(\cdot,t)\|_{\infty,R^N} \leq \gamma t^{-(\theta/p+\theta(m(p-1)-1))}[\mu]^{p/\kappa}, \tag{9}$$

$$\begin{aligned} &\int_0^t \int_{B_{\rho/2}(x_0)} \left|Du^m\right|^{p-1} \mathrm{d}x\,\mathrm{d}\tau \\ &\quad \leq \gamma\rho^{N-\theta} t^{1/p+\theta(m(p-1)-1)} [\mu]^{1+(m(p-1)-1/p+\theta(m(p-1)-1))}, \end{aligned} \tag{10}$$

$$\begin{aligned} &\int_0^t \int_{B_{\rho/2}(x_0)} \tau^{-\sigma}u^q|Du|^l \mathrm{d}x\,\mathrm{d}\tau \\ &\quad \leq \gamma\rho^{N-\theta} t^{(p-l/p)-\sigma-(\theta(p(q+l)-lm(p-1)-p+l)/p(p+\theta(m(p-1)-1)))} \\ &\qquad \cdot [\mu]^{1+(p(q+l)-lm(p-1)-p+l/p+\theta(m(p-1)-1))}, \end{aligned} \tag{11}$$

where $\gamma = \gamma(N, m, p, a, q, l, \sigma, \theta)$, $\kappa = N(m(p-1)-1) + p$, and $x_0 \in R^N$ is fixed.

Remark 1. Condition (7) is actually optimal in the class considered here in some sense (see Theorem 3 below). In fact, if $\sigma + ((\theta(p(q+l) - lm(p-1) - p + l))/(p(p + \theta(m(p-1) - 1)))) > ((p-l)/p)$, then $u_0(x) = |x|^{-\delta}\chi_{B_1(0)}(x)$ with $x \in R^N\backslash\{0\}$ and $0 < \delta < N$ can produce a counterexample such that no nonnegative solution to (1) and (2) may exist. This claim can be shown by following the method to prove Proposition 2.1 of [17] and Theorem 3 here, so we omit the details. Moreover, (7) extends the classical conditions in [17, 18] for $p = 2$ and [22, 24, 27] for $m = 1$ to the problem considered here.

Theorem 2 (the case $q + l < m(p-1)$). *Let $[\mu]$ be finite. Let $p(q+l) \geq lm(p-1)$. Then there exists a solution to (1) and (2) defined in $R^N \times (0,T_0')$, where $T_0' = T_0'([\mu], N, m, p, a, q, l, \sigma)$, such that for all $0 < t < T_0'$, we have*

$$[u]_t \leq \gamma([\mu] + 1), \tag{12}$$

$$\|u(\cdot,t)\|_{\infty,R^N} \leq \gamma t^{-(N/\kappa)}\left([\mu]^{p/\kappa} + 1\right), \tag{13}$$

$$\begin{aligned} &\int_0^t \int_{B_{\rho/2}(x_0)} \tau^{-\sigma}u^q|Du|^l \mathrm{d}x\,\mathrm{d}\tau \\ &\leq \gamma\left(\rho^{N-\theta}([\mu]+1) + \rho^{(Np-l\theta/p)}([\mu]+1)^{l/p}\right) \\ &\quad \cdot \left(t^{((p-l)/p)-\sigma-\left(l\theta r_0/p(p+\theta(m(p-1)-1))\right)}([\mu]+1)^{lr_0/\kappa} \right. \\ &\quad \left. + t^{((p-l)/p)-\sigma-\left(lr_0/p\right)}\right) \\ &\quad + \gamma t^{1-((p-l)/p)}\left(\rho^{(p(N-\theta)+l\theta)/p}([\mu]+1)^{(p-l)/p} + \rho^N\right), \end{aligned} \tag{14}$$

$$
\begin{aligned}
&\int_0^t \int_{B_{\rho/2}(x_0)} \left|Du^m\right|^{p-1} \mathrm{d}x\, \mathrm{d}\tau \\
&\le \gamma \rho^{N-\theta} ([\mu]+1) \\
&\quad \cdot \Big\{ t^{1/(p+\theta(m(p-1)-1))} ([\mu]+1)^{(m(p-1)-1)/\kappa} \\
&\quad + t^{((p-1)/p)\left(m-\left(\left((p+\theta m(p-1))r_1\right)/(p+\theta(m(p-1)-1))\right)\right)} \\
&\quad \cdot ([\mu]+1)^{((p-1)r_1)/\kappa} \\
&\quad + t^{(1/p)-\left(\left((p-1)r)_1/p\right)-\left(\left(\theta\left((p-1)\left(m-r_1\right)-1\right)\right)/(p(p+\theta(m(p-1)-1)))\right)} \\
&\quad \cdot ([\mu]+1)^{((p-1)(m-r_1)-1)/\kappa} + t^{(m(p-1))/p} \Big\} \\
&\quad + \gamma \rho^{(Np-\theta)/p} ([\mu]+1)^{1/p} \\
&\quad \cdot \Big(t^{1-((\sigma(p-1))/(p-l))-\left(\left(\theta\left((p-1)\left(m-r_1\right)-1\right)\right)/(p(p+\theta(m(p-1)-1)))\right)} \\
&\quad \cdot ([\mu]+1)^{((p-1)(m-r_1)-1)/\kappa} \\
&\quad + t^{\left(\left((p-1)\left(m-r_1+1\right)\right)/p\right)-((\sigma(p-1))/(p-l))} \Big),
\end{aligned}
\tag{15}
$$

where r_0 and r_1 depending on N, m, p, a, q, l, σ, and θ are positive constants such that the exponents in (14) and (15) are positive, respectively, $\gamma = \gamma(N, m, p, a, q, l, \sigma, \theta)$ and $x_0 \in R^N$ is fixed.

Remark 2. The dependence of T_0 and T_0' on the quantities specified in the statements of Theorems 1 and 2 can be made explicitly. One can refer to the proof of Theorems 1 and 2, respectively.

Remark 3. With minor revisions, one can prove that Theorems 1 and 2 also hold for $a < 0$. This paper is mainly devoted to the case of $a > 0$, so we do not give more details for the case $a < 0$. However, we would like to mention many important developments without completeness for theproblem (1) and (2) with $a < 0$; see [32–42] and the references therein, where the existence and nonexistence, quantitative properties, and asymptotic behavior of solutions were studied.

Lastly, we state a result in the direction of the optimality of the critical threshold for θ in (7).

Theorem 3. *Let $l \in [1, +\infty)$. Let u be a nonnegative solution to (1) in $R^N \times (0, T)$ such that $[u]_t < \infty$, $0 < t < T$, and*

$$\mu\left(B_\rho(x_0)\right) \ge \epsilon \rho^{N-\theta}, \quad 0 < \rho < \epsilon, \tag{16}$$

where $\epsilon > 0$ and $x_0 \in R^N$ are given. Then $\sigma + ((\theta(p(q+l) - lm(p-1) - p + l))/(p(p + \theta(m(p-1)-1)))) \le ((p-l)/p)$.

The rest of the paper will be divided into two sections: In Section 2, as preparations, we first establish some a priori estimates and then prove Theorems 1 and 2. In sequence, we finish the proof of Theorem 3 in Section 3.

2. Proofs of Theorems 1 and 2

In this section, we shall establish the L^∞-estimates and gradient estimates of solutions which are stated in Lemma 1 and Lemma 2 below, respectively, and thereby give proofs of Theorem 1.1 and Theorem 1.2 based on these key estimates.

For technical convenience, here for any $u \in L^\infty(R^N \times (\epsilon, T^*))$ with $\epsilon > 0$, we define

$$
\begin{aligned}
\langle u \rangle_t &= \sup_{0<\tau<t} \sup_{x \in R^N} \sup_{R(\tau)<\rho<1} \rho^\theta \fint_{B_\rho(x)} u(y, \tau) \mathrm{d}y, \\
R(t) &= \Gamma t^{1/(p+\theta(m(p-1)-1))},
\end{aligned}
\tag{17}
$$

for all $0 < t < T^*$, and

$$
\Gamma = \begin{cases} C[\mu]^{(m(p-1)-1)/(p+\theta(m(p-1)-1))}, & \text{if } q + l \ge m(p-1), \\ 1, & \text{if } q + l < m(p-1), \end{cases}
\tag{18}
$$

where $C > 0$ is to be chosen a priori dependent on $N, m, p, a, q, l, \sigma, \theta$. We also assume that T^* is chosen so that $R(T^*) \le 1$. Note that the last assumption is obvious for $q + l < m(p-1)$ and is meaningful for $q + l \ge m(p-1)$ because Γ is independent of T^*.

We now turn to the proof of Theorem 1 and Theorem 1.2. To achieve these goals, we first state and prove Lemma 1.

Lemma 1. *Let u be a nonnegative continuous weak subsolution of (1). Then the following two statements hold:*

(i) The case $q + l \ge m(p-1)$. Assume that a time $0 < T' < T^*$ is given such that

$$
\begin{aligned}
&\Gamma^{-p} t^{(\theta(m(p-1)-1))/(p+\theta(m(p-1)-1))} \|u(\cdot, t)\|_{\infty, R^N}^{m(p-1)-1} \\
&\quad + t^{1-(\sigma p/(p-l))} \|u(\cdot, t)\|_{\infty, R^N}^{((p(q+l)-lm(p-1))/(p-l))-1} \le 1, \quad \forall 0 < t < T'.
\end{aligned}
\tag{19}
$$

Then

$$
\|u(\cdot, t)\|_{\infty, R^N} \le \gamma t^{-(\theta/(p+\theta(m(p-1)-1)))} \Gamma^{(p(N-\theta))/\kappa} \langle u \rangle_t^{p/\kappa}, \quad \forall 0 < t < T',
\tag{20}
$$

where $\gamma = \gamma(N, m, p, a, q, l, \sigma, \theta)$ and $\kappa = N(m(p-1)-1) + p$.

(ii) The case $q + l < m(p-1)$. Assume also that a time $0 < T'' < T^*$ is given such that

$$
\begin{aligned}
&t^{(\theta(m(p-1)-1))/(p+\theta(m(p-1)-1))} \|u(\cdot, t)\|_{\infty, R^N}^{m(p-1)-1} \\
&\quad + t^{1-(\sigma p/(p-l))} \le 1, \quad \forall 0 < t < T''.
\end{aligned}
\tag{21}
$$

Then

$$\|u(\cdot,t)\|_{\infty,R^N} \le \gamma t^{-(\theta/(p+\theta(m(p-1)-1)))}\langle u\rangle_t^{p/\kappa} + t, \quad \forall 0<t<T'', \tag{22}$$

where $\gamma = \gamma(N,m,p,a,q,l,\sigma,\theta)$.

Proof. We divided the proof into two steps:

Step 1. Following the method to prove Lemma 1 in [25], if $q+l \ge m(p-1)$, we obtain

$$\|u(\cdot,t)\|_{\infty,B_{\rho/2}} \le \gamma t^{-((N+p)/\kappa)}\left(\int_0^t\int_{B_\rho} u\,\mathrm{d}x\,\mathrm{d}\tau\right)^{p/\kappa}, \tag{23}$$

for all $0<t<T'$. If $q+l<m(p-1)$, then we have

$$\|u(\cdot,t)\|_{\infty,B_{\rho/2}} \le \gamma t^{-((N+p)/\kappa)}\left(\int_0^t\int_{B_\rho} u\,\mathrm{d}x\,\mathrm{d}\tau\right)^{p/\kappa} + t, \tag{24}$$

for all $0<t<T''$.

Step 2. Set $\rho = R(t)$. Then (23) and (17) imply that

$$\begin{aligned}\|u(\cdot,t)\|_{\infty,B_{\rho/2}} &\le \gamma t^{-((N+p)/\kappa)}\left(\int_0^t\int_{B_{R(t)}} u\,\mathrm{d}x\,\mathrm{d}\tau\right)^{p/\kappa}\\ &\le \gamma t^{-(N/\kappa)}R^{(p(N-\theta))/\kappa}(t)\langle u\rangle_t^{p/\kappa}\\ &= \gamma t^{-(\theta/(p+\theta(m(p-1)-1)))}\Gamma^{(p(N-\theta))/\kappa}\langle u\rangle_t^{p/\kappa},\end{aligned} \tag{25}$$

for all $0<t<T'$. Therefore, (20) is obtained. Similarly, (24) and (17) yield (22) for all $0<t<T''$. □

Remark 4. It follows from the definitions of $\langle u\rangle_t$ and $[u]_t$ that $\langle u\rangle_t \le [u]_t$. Combining Lemma 1 and (17) and (18), we obtain

$$[u]_t \le \langle u\rangle_t + \gamma(C)[\mu]^{(N(m(p-1)-1))/\kappa}\langle u\rangle_t^{p/\kappa}, \quad \forall 0<t<T', \tag{26}$$

for the case $q+l \ge m(p-1)$, and

$$[u]_t \le \langle u\rangle_t + \langle u\rangle_t^{p/\kappa} + t^{1+(\theta/p+\theta(m(p-1)-1))}, \quad \forall 0<t<T'', \tag{27}$$

for the case $q+l<m(p-1)$, where T' and T'' are as in Lemma 1.

We now start to state and prove Lemma 2.

Lemma 2. *Let u be a nonnegative continuous weak subsolution of (1). Then for all $R(t) \le \rho \le 1$ and $B_\rho(x_0) \subset R^N$, the following two statements hold:*

(i) The case $q+l \ge m(p-1)$. Let (19) and (7) hold. Then for all $0<t<T'$, we have

$$\begin{aligned}&\int_0^t\int_{B_{\rho/2}(x_0)} \tau^{-\sigma}u^q|Du|^l\,\mathrm{d}x\,\mathrm{d}\tau\\ &\le \gamma G(t)t^{((p-l)/p)-\sigma-((\theta(p(q+l)-lm(p-1)-p+l))/(p(p+\theta(m(p-1)-1))))}\\ &\cdot\left(\Gamma^{N-\theta}\langle u\rangle_t\right)^{(p(q+l)-lm(p-1)-p+l)/\kappa},\end{aligned} \tag{28}$$

$$\begin{aligned}&\int_0^t\int_{B_{\rho/2}(x_0)} \left|Du^m\right|^{p-1}\mathrm{d}x\mathrm{d}\tau \le \gamma G(t)t^{1/(p+\theta(m(p-1)-1))}\\ &\cdot\left(\Gamma^{N-\theta}\langle u\rangle_t\right)^{(m(p-1)-1)/\kappa},\end{aligned} \tag{29}$$

where $G(t) = \sup_{0<\tau<t}\int_{B_\rho(x_0)} u(x,\tau)dx$, $\kappa = N(m(p-1)-1)+p$ *and* $\gamma = \gamma(N,m,p,a,q,l,\sigma,\theta)$.

(ii) The case $q+l<m(p-1)$. Let (21) hold. Then for all $0<t<T''$, we have

$$\begin{aligned}&\int_0^t\int_{B_{\rho/2}(x_0)} \tau^{-\sigma}u^q|Du|^l\mathrm{d}x\mathrm{d}\tau\\ &\le \gamma\left(G(t)+\rho^{(N(p-l))/p}G^{l/p}(t)\right)\\ &\cdot\left(t^{((p-l)/p)-\sigma-\left(l\theta r_0/(p(p+\theta(m(p-1)-1)))\right)}\langle u\rangle_t^{lr_0/\kappa}\right.\\ &\left.+\,t^{((p-l)/p)-\sigma-\left(lr_0/p\right)}\right)\\ &+\gamma t^{1-((p-l)/p)}\left(\rho^{Nl/p}G^{(p-l)/p}(t)+\rho^N\right),\end{aligned} \tag{30}$$

where $r_0 = r_0(N,m,p,a,q,l,\sigma,\theta) > 0$ is a constant such that the exponents in (30) are positive and $\gamma = \gamma(N,m,p,a,q,l,\sigma,\theta)$. Moreover,

$$\begin{aligned}&\int_0^t\int_{B_{\rho/2}(x_0)} \left|Du^m\right|^{p-1}\mathrm{d}x\,\mathrm{d}\tau\\ &\le \gamma G(t)\left\{t^{1/(p+\theta(m(p-1)-1))}\langle u\rangle_t^{(m(p-1)-1)/\kappa}\right.\\ &+\,t^{((p-1)/p)\left(m-\left(((p+\theta m(p-1))r_1)/(p+\theta(m(p-1)-1))\right)\right)}\langle u\rangle_t^{((p-1)r_1)/\kappa}\\ &+\,t^{(1/p)-\left((p-1)r_1/p\right)-\left(\left(\theta\left((p-1)\left(m-r_1\right)-1\right)\right)/(p(p+\theta(m(p-1)-1)))\right)}\\ &\left.\cdot\,\langle u\rangle_t^{\left((p-1)\left(m-r_1\right)-1\right)/\kappa}+t^{(m(p-1))/p}\right\}\\ &+\gamma\rho^{(N(p-1))/p}G^{1/p}(t)\\ &\cdot\left(t^{1-((\sigma(p-1))/(p-l))-\left(\left(\theta\left((p-1)\left(m-r_1\right)-1\right)\right)/(p(p+\theta(m(p-1)-1)))\right)}\right.\\ &\left.\cdot\,\langle u\rangle_t^{\left((p-1)\left(m-r_1\right)-1\right)/\kappa}+t^{\left((p-1)\left(m-r_1+1\right)\right)/p-(\sigma(p-1)/(p-l))}\right),\end{aligned} \tag{31}$$

where $r_1 = r_1(N,m,p,a,q,l,\sigma,\theta) > 0$ is a constant such that the exponents in (31) are positive and $\gamma = \gamma(N,m,p,a,q,l,\sigma,\theta)$.

Proof. In the following, we only prove (28) and (30), since (29) and (31) can be proved similarly and we omit the details.

We first prove (28). For notational convenience, we set $B_\rho = B_\rho(x_0)$. Take $\varphi = t^\beta u^r \zeta^p$ as a test function in (3), where ζ is a piecewise smooth cut-off function in B_ρ, such that $0 \le \zeta \le 1$ in B_ρ, $\zeta \equiv 1$ in $B_{\rho/2}$, and $|D\zeta| \le (2/\rho)$. Moreover, $\beta > 0$ and $r > 0$ are to be chosen. Then standard calculations imply

$$\begin{aligned} & rm^{p-1}\int_0^t\int_{B_\rho}\tau^\beta u^{(m-1)(p-1)+r-1}|Du|^p\zeta^p\,\mathrm{d}x\,\mathrm{d}\tau \\ & \le \frac{\beta}{r+1}\int_0^t\int_{B_\rho}\tau^{\beta-1}u^{r+1}\zeta^p\,\mathrm{d}x\,\mathrm{d}\tau + pm^{p-1}\int_0^t\int_{B_\rho}\tau^\beta u^{(m-1)(p-1)+r}|Du|^{p-1}|D\zeta|^{p-1}\,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t\int_{B_\rho}a\tau^{\beta-\sigma}u^{q+r}|Du|^l\zeta^p\,\mathrm{d}x\,\mathrm{d}\tau. \end{aligned} \tag{32}$$

Applying Young's inequality, together with (19) and (20), yield

$$\begin{aligned} & \int_0^t\int_{B_\rho}\tau^\beta u^{(m-1)(p-1)+r-1}|Du|^p\zeta^p\,\mathrm{d}x\,\mathrm{d}\tau \\ & \le \gamma\int_0^t\int_{B_\rho}\tau^{\beta-1}u^{r+1}\,\mathrm{d}x\,\mathrm{d}\tau + \gamma\int_0^t\int_{B_\rho}\tau^\beta u^{m(p-1)+r}|D\zeta|^p\,\mathrm{d}x\,\mathrm{d}\tau + \gamma\int_0^t\int_{B_\rho}\tau^{\beta-(\sigma p/(p-l))}u^{(p(q+l)-lm(p-1)/(p-l))+r}\,\mathrm{d}x\,\mathrm{d}\tau \\ & \le \gamma\int_0^t\int_{B_\rho}\tau^{\beta-1}\left(1+\rho^{-p}\tau u^{m(p-1)-1}+\tau^{1-(\sigma p/(p-l))}u^{(p(q+l)-lm(p-1)/(p-l))-1}\right)u^{r+1}\,\mathrm{d}x\,\mathrm{d}\tau \\ & \le \gamma G(t)\int_0^t\tau^{\beta-1}\|u(\cdot,\tau)\|_{\infty,B_\rho}^r\,\mathrm{d}\tau \\ & \le \gamma G(t)t^{\beta-(\theta r/(p+\theta(m(p-1)-1)))}\Gamma^{p(N-\theta)r/\kappa}\langle u\rangle_t^{pr/\kappa}, \end{aligned} \tag{33}$$

provided

$$\beta > \frac{\theta r}{p+\theta(m(p-1)-1)}. \tag{34}$$

Next, we estimate by Hölder's inequality

$$\begin{aligned} & \int_0^t\int_{B_\rho}\tau^{-\sigma}u^q|Du|^l\zeta^l\,\mathrm{d}x\,\mathrm{d}\tau \\ & \le \gamma\left(\int_0^t\int_{B_\rho}\tau^\beta u^{(m-1)(p-1)+r-1}|Du|^p\zeta^p\,\mathrm{d}x\,\mathrm{d}\tau\right)^{l/p}\left(\int_0^t\int_{B_\rho}\tau^{-((\sigma p+\beta l)/(p-l))}u^{(p(q+l)-lm(p-1)-rl)/(p-l)}\,\mathrm{d}x\,\mathrm{d}\tau\right)^{(p-l)/p}. \end{aligned} \tag{35}$$

Again using (20), we have

$$\begin{aligned} & \int_0^t\int_{B_\rho}\tau^{-((\sigma p+\beta l)/(p-l))}u^{(p(q+l)-lm(p-1)-rl)/(p-l)}\,\mathrm{d}x\,\mathrm{d}\tau \\ & \le G(t)\int_0^t\tau^{-(\sigma p+\beta l)/(p-l)}\|u(\cdot,\tau)\|_{\infty,B_\rho}^{(p(q+l)-lm(p-1))/(p-l)-1-(rl/(p-l))}\,\mathrm{d}\tau \\ & \le \gamma G(t)t^{1-(\sigma p+\beta l)/(p-l)-(\theta((p(q+l)-lm(p-1))/(p-l)-1-(rl/(p-l)))/p+\theta(m(p-1)-1))}\left(\Gamma^{p(N-\theta)/\kappa}\langle u\rangle_t^{p/\kappa}\right)^{((p(q+l)-lm(p-1))/(p-l))-1-(rl/(p-l))}, \end{aligned} \tag{36}$$

provided

$$\frac{p(q+l)-lm(p-1)}{p-l}-1\geq\frac{rl}{p-l}, \tag{37}$$

$$1-\frac{\sigma p+\beta l}{p-l}-\frac{\theta((p(q+l)-lm(p-1)/(p-l))-1-(rl/(p-l)))}{p+\theta(m(p-1)-1)}>0. \tag{38}$$

Inserting (33) and (36) into (35), we obtain (28). It is left to choose β and r such that (34), (37), and (38) hold. This is a trivial task if $\theta=0$. Assume that $\theta>0$, and

$$0<\beta<\frac{p-l}{l}. \tag{39}$$

Then (34) and (37) are implied by

$$r<\min\left\{\frac{p+\theta(m(p-1)-1)}{\theta}\beta,\frac{p-l}{l}\cdot\left(\frac{p(q+l)-lm(p-1)}{p-l}-1\right)\right\}\triangleq z_0. \tag{40}$$

Note that (38) is equivalent to

$$r>\left[\frac{p-l}{l}\left(\frac{\theta((p(q+l)-lm(p-1)/(p-l))-1)}{p+\theta(m(p-1)-1)}-1\right)+\frac{\sigma p}{l}+\beta\right]\frac{p+\theta(m(p-1)-1)}{\theta}\triangleq z_0'. \tag{41}$$

By virtue of (7) and (39), it is easy to check that $z_0'<z_0$. Finally, by noting that $z_0>0$, we can choose $r>0$ with $r\in(z_0',z_0)$ such that (34), (37), and (38) hold.

We now turn to prove (30). Comparing with the proof of (28), the main differences here are the estimates (33) and (36). Note that $u^{(p(q+l)-lm(p-1))/(p-l)+r}\leq u^{1+r}+1$, since $q+l<m(p-1)$ implies $(p(q+l)-lm(p-1))/(p-l)<1$, and we have

$$\begin{aligned}
&\int_0^t\int_{B_\rho}\tau^\beta u^{(m-1)(p-1)+r-1}|Du|^p\zeta^p\,dx\,d\tau\\
&\leq\gamma\int_0^t\int_{B_\rho}\tau^{\beta-1}u^{r+1}dx\,d\tau+\gamma\int_0^t\int_{B_\rho}\tau^\beta u^{m(p-1)+r}|D\zeta|^p dx\,d\tau+\gamma\int_0^t\int_{B_\rho}\tau^{\beta-(\sigma p/(p-l))}u^{(p(q+l)-lm(p-1))/(p-l)+r}dx\,d\tau\\
&\leq\gamma\int_0^t\int_{B_\rho}\tau^{\beta-1}\left(1+\rho^{-p}\tau u^{m(p-1)-1}+\tau^{1-(\sigma p/(p-l))}\right)u^{r+1}dx\,d\tau+\gamma\rho^N t^{\beta-(\sigma p/(p-l))+1}\\
&\leq\gamma G(t)\int_0^t\tau^{\beta-1}\|u(\cdot,\tau)\|^r_{\infty,B_\rho}d\tau+\gamma\rho^N t^{\beta-(\sigma p/(p-l))+1}\\
&\leq\gamma G(t)\left(t^{\beta-(\theta r/(p+\theta(m(p-1)-1)))}\langle u\rangle_t^{pr/\kappa}+t^{\beta+r}\right)+\gamma\rho^N t^{\beta-(\sigma p/(p-l))+1},
\end{aligned} \tag{42}$$

where we have used (21) and (22) in the third and last inequality, respectively, and we again used (34). Assume that

$$p(q+l)-lm(p-1)-rl>0. \tag{43}$$

Then we obtain

$$\begin{aligned}
&\int_0^t\int_{B_\rho}\tau^{-\sigma p+\beta l/(p-l)}u^{(p(q+l)-lm(p-1)-rl)/(p-l)}dx\,d\tau\\
&\leq\int_0^t\int_{B_\rho}\tau^{-(\sigma p+\beta l)/(p-l)}(u+1)dx\,d\tau\\
&\leq G(t)t^{1-((\sigma p+\beta l)/(p-l))}+\gamma\rho^N t^{1-((\sigma p+\beta l)/(p-l))},
\end{aligned} \tag{44}$$

where we have used the fact $u^{(p(q+l)-lm(p-1)-rl)/(p-l)}\leq u+1$, since $q+l<m(p-1)$ and (43) imply $(p(q+l)-lm(p-1)-rl)/(p-l)<1$. Inserting (42) and (44) into (35), we obtain (30). It is left to verify that there exist $\beta>0$ and $r>0$ such that (34) and (43) hold. This can be immediately proved by fixing β as (39) and choosing $r>0$ sufficiently small.

Based on Lemmas 1 and 2, we are ready to prove Theorem 1. □

Proof of Theorem 1. Consider the approximating problems:

$$\begin{cases}
u_{nt}-\operatorname{div}\left(|Du_n^m|^{p-2}Du_n^m\right)=a\min\left\{t^{-\sigma}u_n^q|Du_n|^l,n\right\}, & \text{in } B_n\times(0,\infty),\\
u_n(x,t)=0, & \text{in } \partial B_n\times(0,\infty),\\
u_n(x,0)=u_{0n}(x), & \text{on } B_n,
\end{cases} \tag{45}$$

where $B_n = \{x \in R^N \mid |x| < n\}$ and $u_{0n} \in C_0^\infty(R^N)$ is nonnegative and has compact support in B_n, which satisfy

$$\lim_{n\longrightarrow\infty} \int_{R^N} u_{0n}\eta(x)dx = \int_{R^N} \eta(x)d\mu, \quad \forall \eta \in C_0^\infty(R^N),$$
$$[u_{0n}] \le \gamma(N)[\mu]. \tag{46}$$

By the results of [4, 5, 43] and [44, 45], we can obtain the existence of solutions u_n for approximation problems (45). Moreover, these solutions are Hölder continuous. Then following the methods to prove Theorem 1 in [25], we can complete the proof of Theorem 1 whence we can show estimates (8)–(11) for u and u_0 replaced by u_n and u_{0n}, with constant γ independent of n. To prove these estimates, we will work with (45) and drop the subscript n.

Define

$$t_0 = \sup\left\{0 < T' \langle T^* \mid (2.3) \text{ holds}\right\}. \tag{47}$$

Choose $0 < t < t_0$ and let $B_\rho = B_\rho(x_0)$, where $R(t) \le \rho \le 1$ and $x_0 \in R^N$. Take ζ as the test function in (45), where ζ is as in the proof of Lemma 2. Direct calculation shows that

$$\int_{B_{\rho/2}} u(x,t)\mathrm{d}x \le \int_{B_\rho} \mathrm{d}\mu + \frac{2}{\rho}\int_0^t \int_{B_\rho} \left|Du^m\right|^{p-1}\mathrm{d}x\,\mathrm{d}\tau + \int_0^t \int_{B_\rho} a\tau^{-\sigma}u^q|Du|^l\mathrm{d}x\,\mathrm{d}\tau. \tag{48}$$

Multiplying (48) by $\rho^\theta|B_\rho|^{-1}$, together with (18), (20), (28), and (29), we obtain

$$\begin{aligned}\rho^\theta \fint_{B_{\rho/2}} u(x,t)\mathrm{d}x &\le \gamma_1(N)[\mu] + \gamma\langle u\rangle_t \\ &\cdot \left\{C^{-(p+\theta(m(p-1)-1))/\kappa}\left(\frac{\langle u\rangle_t}{[\mu]}\right)^{(m(p-1)-1)/\kappa}\right. \\ &+ \gamma t^{((p-l)/p)-\sigma-((\theta(p(q+l)-lm(p-1)-p+l))/(p(p+\theta(m(p-1)-1))))} \\ &\left.\cdot \left(\Gamma^{N-\theta}\langle u\rangle_t\right)^{(p(q+l)-lm(p-1)-p+l)/\kappa}\right\}.\end{aligned} \tag{49}$$

for all $0 < t < t_0$ and $R(t) \le \rho \le 1$. Note that $x_0 \in R^N$ is arbitrarily chosen, it is immediately seen that

$$\langle u\rangle_t \le \gamma_1(N)[\mu] + \gamma\langle u\rangle_t \left\{C^{-(p+\theta(m(p-1)-1))/\kappa}\left(\frac{\langle u\rangle_t}{[\mu]}\right)^{(m(p-1)-1)/\kappa} + M(t)\right\}, \tag{50}$$

where the meaning of $M(t)$ is obvious. Set

$$\begin{aligned} t_1 &= \sup\left\{0 < t < T^* \mid \langle u\rangle_t \le 4\gamma_1[\mu]\right\}, \\ t_2 &= \sup\left\{0 < t < T^* \mid M(t) < \delta\right\},\end{aligned} \tag{51}$$

where $\delta > 0$ (small) is to be chosen. Note that t_1 and t_2 are well defined because the stipulated assumptions make sure that $\langle u\rangle_t$ is continuous in $[0, T^*]$, and the exponent of t in (51) is positive. Let $t_3 = \min\{t_0, t_1, t_2\}$. Then for $0 < t < t_3$, we have

$$\begin{aligned}&\gamma C^{-(p+\theta(m(p-1)-1))/\kappa}\left(\frac{\langle u\rangle_t}{[\mu]}\right)^{(m(p-1)-1)/\kappa} \\ &\le \gamma C^{-(p+\theta(m(p-1)-1))/\kappa}(4\gamma_1)^{(m(p-1)-1)/\kappa} \le \frac{1}{4},\end{aligned} \tag{52}$$

provided C is suitably chosen. Then if we choose $\delta < (1/4\gamma)$, it follows from (50)

$$\langle u\rangle_t \le 2\gamma_1[\mu], \quad \forall 0 < t < t_3. \tag{53}$$

By (26) and (53), we get

$$[u]_t \le \gamma[\mu], \quad \forall 0 < t < t_3. \tag{54}$$

Therefore, (8)–(11) follow from (53), (20), (26), (28), and (29), provided that we can indeed find a quantitative estimates below $t_3 \ge T_0$. We may assume $t_3 < T^*$, since the estimate is otherwise trivial. First we note that (53) implies $t_3 < t_1$. Next, it is easy to rule out the case $t_3 = t_0$. In fact,

$$\begin{aligned}&\Gamma^{-p}t^{(\theta(m(p-1)-1))/(p+\theta(m(p-1)-1))}\|u(\cdot,t)\|_{\infty,R^N}^{m(p-1)-1} \\ &+ t^{1-(\sigma p/(p-l))}\|u(\cdot,t)\|_{\infty,R^N}^{((p(q+l)-lm(p-1))/(p-l))-1} \\ &\le C^{-(p+\theta(m(p-1)-1))/\kappa}\left(\frac{\langle u\rangle_t}{[\mu]}\right)^{(m(p-1)-1)/\kappa} \\ &\quad + t^{1-(\sigma p/(p-l))-((\theta(p(q+l)-lm(p-1)-p+l))/((p-l)(p+\theta(m(p-1)-1))))} \\ &\cdot \left(\Gamma^{N-\theta}\langle u\rangle_t\right)^{(p(p(q+l)-lm(p-1)-p+l))/(p-l)\kappa} \\ &\le \frac{1}{2},\end{aligned} \tag{55}$$

where we choose C large enough and δ sufficiently small. Finally, we are left with the task of estimating below $t_3 = t_2$. This can be accomplished at once, by replacing $\langle u\rangle_t$ with $2\gamma_1[\mu]$ in the definition of t_2 in (51), owing to (53).

Now we turn to prove Theorem 2, which is similar to the proof of Theorem 1. However, we give the details for the convenience of readers. □

Proof of Theorem 2. We again consider (45). If we show estimates (12)–(15) for u and u_0 replaced by u_n and u_{0n}, with constant γ independent of n, then Theorem 2 can be proved with similar reason as in Theorem 1. Set

$$t_0' = \sup\left\{0 < T'' \langle T^* \mid (22) \text{ holds}\right\}. \tag{56}$$

Again taking ζ as the test function in (45), where ζ is as in the proof of Lemma 2, we again obtain (48). Multiplying (48) by $\rho^\theta|B_\rho|^{-1}$, together with (22), (30), and (31), we obtain

$$
\begin{aligned}
\rho^{\theta}\fint_{B_{\rho/2}} u(x,t)\mathrm{d}x \le{}& \gamma(N)[\mu] \\
&+\gamma\left(\langle u\rangle_t+\rho^{(\theta(p-l))/p}\langle u\rangle_t^{l/p}\right)\left(t^{(p-l)/p-\sigma-\left(l\theta r_0/(p(p+\theta(m(p-1)-1)))\right)}\langle u\rangle_t^{lr_0/\kappa}+t^{((p-l)/p)-\sigma-\left(lr_0/p\right)}\right) \\
&+\gamma t^{1-((p-l)/p)}\left(\rho^{\theta l/p}\langle u\rangle_t^{(p-l)/p}+\rho^N\right) \\
&+\gamma\langle u\rangle_t\Big\{t^{1/(p+\theta(m(p-1)-1))}\langle u\rangle_t^{(m(p-1)-1)/\kappa}+t^{((p-1)/p)\left(m-(p+\theta m(p-1))r_1/(p+\theta(m(p-1)-1))\right)}\langle u\rangle_t^{((p-1)r_1)/\kappa} \\
&+t^{(1/p)-\left((p-1)r_1/p\right)-\left(\left(\theta\left((p-1)\left(m-r_1\right)-1\right)\right)/(p(p+\theta(m(p-1)-1)))\right)}\langle u\rangle_t^{\left((p-1)\left(m-r_1\right)-1\right)/\kappa}+t^{(m(p-1))/p}\Big\} \\
&+\gamma\rho^{(\theta(p-1))/p}\langle u\rangle_t^{1/p}\Big(t^{1-(\sigma(p-1)/(p-l))-\left(\left(\theta\left((p-1)\left(m-r_1\right)-1\right)\right)/(p(p+\theta(m(p-1)-1)))\right)}\langle u\rangle_t^{\left((p-1)\left(m-r_1\right)-1\right)/\kappa} \\
&+t^{\left((p-1)\left(m-r_1+1\right)\right)/p-(\sigma(p-1)/(p-l))}\Big),
\end{aligned}
\tag{57}
$$

for all $0<t<t_0'$ and $R(t)\le\rho\le 1$. Note that $x_0\in R^N$ is arbitrarily chosen, it is immediately seen that

$$
\langle u\rangle_t\le\gamma(N)[\mu]+\gamma N_1(t)\langle u\rangle_t+\gamma N_2(t),
\tag{58}
$$

where

$$
\begin{aligned}
N_1(t) ={}& t^{((p-l)/p)-\sigma-\left(l\theta r_0/(p(p+\theta(m(p-1)-1)))\right)}\langle u\rangle_t^{lr_0/\kappa}+t^{((p-l)/p)-\sigma-\left(lr_0/p\right)} \\
&+t^{1/(p+\theta(m(p-1)-1))}\langle u\rangle_t^{m(p-1)-1/\kappa}+t^{((p-1)/p)\left(m-\left(\left((p+\theta m(p-1))r_1\right)/(p+\theta(m(p-1)-1))\right)\right)}\langle u\rangle_t^{((p-1)r_1)/\kappa} \\
&+t^{(1/p)-\left(\left((p-1)r_1\right)/p\right)-\left(\left(\theta\left((p-1)\left(m-r_1\right)-1\right)\right)/(p(p+\theta(m(p-1)-1)))\right)}\langle u\rangle_t^{\left((p-1)\left(m-r_1\right)-1\right)/\kappa}+t^{m(p-1)/p}, \\
N_2(t) ={}& \rho^{(\theta(p-l))/p}\langle u\rangle_t^{l/p}\left(t^{((p-l)/p)-\sigma-\left(l\theta r_0/(p(p+\theta(m(p-1)-1)))\right)}\langle u\rangle_t^{lr_0/\kappa}+t^{((p-l)/p)-\sigma-\left(lr_0/p\right)}\right)+t^{1-((p-l)/p)}\left(\rho^{\theta l/p}\langle u\rangle_t^{(p-l)/p}+\rho^N\right) \\
&+\rho^{(\theta(p-1))/p}\langle u\rangle_t^{1/p}\Big(t^{1-(\sigma(p-1)/p-l)-\left(\left(\theta\left((p-1)\left(m-r_1\right)-1\right)\right)/(p(p+\theta(m(p-1)-1)))\right)}\langle u\rangle_t^{\left((p-1)\left(m-r_1\right)-1\right)/\kappa} \\
&+t^{\left((p-1)\left(m-r_1+1\right)\right)/p-(\sigma(p-1)/(p-l))}\Big).
\end{aligned}
\tag{59}
$$

Set

$$
t_1'=\sup\left\{0<t\langle T^*\middle|N_1(t)+N_2(t)<\epsilon\right\},
\tag{60}
$$

where $\epsilon>0$ (small) is to be chosen. Note that the exponents of t in (60) are positive. Then using (22), (58), and (60) and following the method to fix T_0 in Theorem 1, we can obtain a positive time T_0' such that

$$
\langle u\rangle_t\le\gamma([\mu]+1),\quad \forall 0<t<T_0'.
\tag{61}
$$

Therefore, (12)–(15) follow from (61), (22), (27), (30), and (31). □

3. Proof of Theorem 3

This section presents the proof of Theorem 3.

Proof of Theorem 3. Taking $\varphi=u^{\epsilon-1}\zeta^s$ as a test function in (2), where $0<\epsilon<1$, s is sufficiently large, and $\zeta(x)$ is as in the proof of Lemma 2, then standard calculations imply:

$$
\begin{aligned}
&\frac{1}{\epsilon}\int_{B_\rho}u^{\epsilon}(x,z)\zeta^s\mathrm{d}x-\frac{1}{\epsilon}\int_{B_\rho}u^{\epsilon}\left(x,\frac{t}{2}\right)\zeta^s\mathrm{d}x \\
&\quad+s\int_{t/2}^{z}\int_{B_\rho}u^{(m-1)(p-1)+\epsilon-1}|Du|^{p-2}Du\,D\zeta\zeta^{s-1}\mathrm{d}x\,\mathrm{d}\tau \\
&=(1-\epsilon)\int_{t/2}^{z}\int_{B_\rho}u^{(m-1)(p-1)+\epsilon-2}|Du|^{p}\zeta^s\mathrm{d}x\,\mathrm{d}\tau \\
&\quad+\int_{t/2}^{z}\int_{B_\rho}a\tau^{-\sigma}u^{q+\epsilon-1}|Du|^{l}\zeta^s\mathrm{d}x\,\mathrm{d}\tau,
\end{aligned}
\tag{62}
$$

where $0<(t/2)<z<t<T$. Applying Young's inequality, we obtain

$$
\begin{aligned}
&\int_{B_\rho}u^{\epsilon}(x,z)\zeta^s\mathrm{d}x-\int_{B_\rho}u^{\epsilon}\left(x,\frac{t}{2}\right)\zeta^s\mathrm{d}x \\
&\ge-\gamma\rho^{-p}\int_{t/2}^{z}\int_{B_\rho}u^{m(p-1)-1+\delta}\zeta^{s-p}\mathrm{d}x\,\mathrm{d}\tau \\
&\quad+a\int_{t/2}^{z}\int_{B_\rho}\tau^{-\sigma}u^{q+\epsilon-1}|Du|^{l}\zeta^s\mathrm{d}x\,\mathrm{d}\tau.
\end{aligned}
\tag{63}
$$

Note that

$$\int_{t/2}^{z}\int_{B_\rho}\tau^{-\sigma}u^{q+\epsilon-1}|Du|^l\zeta^s\mathrm{d}x\,\mathrm{d}\tau \geq \gamma\rho^{-l}\int_{t/2}^{z}\int_{B_\rho}\tau^{-\sigma}u^{q+l+\epsilon-1}\zeta^s\mathrm{d}x\,\mathrm{d}\tau, \tag{64}$$

where we have used Gagliardo–Nirenberg inequality in [18] and Lemma 5.1 in [46] with $\rho = \rho(t) = \gamma_0 t^{1/(p+\theta(m(p-1)-1))}$.

Inserting (64) into (63), we obtain

$$\begin{aligned}
&\int_{B_\rho}u^\epsilon(x,z)\zeta^s\mathrm{d}x - \int_{B_\rho}u^\epsilon\left(x,\frac{t}{2}\right)\zeta^s\mathrm{d}x\\
&\geq -\gamma\rho^{-p}\int_{t/2}^{z}\int_{B_\rho}u^{m(p-1)-1+\epsilon}\zeta^{s-p}\mathrm{d}x\,\mathrm{d}\tau + \gamma\rho^{-l}\int_{t/2}^{z}\int_{B_\rho}\tau^{-\sigma}u^{q+l+\epsilon-1}\zeta^s\mathrm{d}x\,\mathrm{d}\tau\\
&\geq -\gamma\rho^{-((p(q+l+\epsilon-1)-l(m(p-1)-1+\epsilon))/q+l-m(p-1))+N}\int_{t/2}^{z}\tau^{(\sigma(m(p-1)-1+\epsilon))/(q+l-m(p-1))}\mathrm{d}\tau + \gamma\rho^{-l}\int_{t/2}^{z}\int_{B_\rho}\tau^{-\sigma}u^{q+l+\epsilon-1}\zeta^s\mathrm{d}x\,\mathrm{d}\tau\\
&\geq -\gamma\rho^{-((p(q+l+\epsilon-1)-l(m(p-1)-1+\epsilon))/(q+l-m(p-1)))+N}\int_{t/2}^{z}\tau^{((\sigma(m(p-1)-1+\epsilon))/(q+l-m(p-1)))}\mathrm{d}\tau + \gamma\rho^{-l-((N(q+l-1))/\epsilon)}\\
&\quad\cdot\int_{t/2}^{z}\tau^{-\sigma}\left(\int_{B_\rho}u^\epsilon\zeta^s\mathrm{d}x\right)^{(q+l+\epsilon-1)/\epsilon}\mathrm{d}\tau,
\end{aligned} \tag{65}$$

for all $(t/2) < z < t$, where we have used Young's inequality and $q + l \geq m(p-1)$ in the second inequality and Hölder's inequality in the third inequality. Therefore, $\int_{B_\rho}u^\epsilon(x,z)\zeta^s\mathrm{d}x$ majorizes the solution $y(z)$ to

$$\begin{cases}
y' = -\gamma\rho^{-((p(q+l+\epsilon-1)-l(m(p-1)-1+\epsilon))/(q+l-m(p-1)))+N}z^{(\sigma(m(p-1)-1+\epsilon))/(q+l-m(p-1))} + \gamma\rho^{-l-(N(q+l-1)/\epsilon)}z^{-\sigma}y^{(q+l+\epsilon-1)/\epsilon}, & \text{in}\left(\frac{t}{2},t\right),\\
y\left(\frac{t}{2}\right) = \int_{B_\rho}u^\epsilon\left(x,\frac{t}{2}\right)\zeta^s\mathrm{d}x.
\end{cases} \tag{66}$$

Thus, the solution y of problem (66) is bounded over $(t/2, t)$. It follows from Lemma 4.1 in [17] that

$$\int_{B_\rho}u^\epsilon\left(x,\frac{t}{2}\right)\zeta^s\mathrm{d}x \leq \gamma\max\left\{\rho^{(\epsilon l/(q+l-1))+N}t^{-((\epsilon(1-\sigma))/(q+l-1))},\ \rho^{(\epsilon(l-p))/(q+l-m(p-1))+N}t^{\epsilon\sigma/(q+l-m(p-1))}\right\}. \tag{67}$$

By Lemma 5.1 of [46], we have

$$\int_{B_\rho}u^\epsilon\left(x,\frac{t}{2}\right)\zeta^s\mathrm{d}x \geq \gamma\rho^{-\epsilon\theta+N}, \tag{68}$$

for t small and $\rho = \rho(t)$ defined above.

Let $t \longrightarrow 0$, it is easily seen that (67) and (68) imply. $\sigma + ((\theta(p(q+l) - lm(p-1) - p + l))/(p(p+\theta(m(p-1) -1)))) \leq ((p-l)/p)$. □

4. Conclusion

The local existence of solutions of the doubly degenerate parabolic equation with time-dependent gradient source and initial data measures is studied in this paper. The equation is a class of non-Newtonian polytropic filtration equation and contains the heat equation, the porous medium equation, the evolutionary p-Laplacian equation, and so on. The main difficulties to establish the desired results here are the complicated structure of the equation and the lower regularity of the initial data. By making full use of the structure of the equation and the delicate a priori estimates, we obtain the local existence of weak solutions under the restriction condition $\sigma + ((\theta(p(q+l) - lm(p-1) - p + l))/(p(p+\theta(m(p-1)-1)))) < ((p-l)/p)$. Furthermore, if $\sigma + ((\theta(p(q+l) - lm(p-1) - p + l))/(p(p+\theta(m(p-1)-1)))) > ((p-l)/p)$, a counterexample indicates that the equation considered here has no nonnegative solution. We also remark that whether the existence of solutions holds for the limiting case $\sigma + ((\theta(p(q+l) - lm(p-1) - p + l))/(p(p+\theta(m(p-1)-1)))) = ((p-l)/p)$ is an interesting problem and left open, since the methods used here are invalid for this case.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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