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Research Article

Multiplicity of Quasilinear Schrödinger Equation

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In this paper, we study the quasilinear Schrödinger equation involving concave and convex nonlinearities. When the pair of parameters belongs to a certain subset of $\mathbb{R}^2$, we establish the existence of a nontrivial mountain pass-type solution and infinitely many negative energy solutions by using some new techniques and dual fountain theorem. Recent results from the literature are improved and extended.

1. Introduction and Main Results

In this paper, we are concerned with the following quasilinear Schrödinger equations:

$$-\Delta u + V(x)u - \Delta\left(|u|^{2\alpha}\right)|u|^{2\alpha-2}u = \lambda g(x,u) + \mu h(x,u),\ x \in \mathbb{R}^N, \tag{1}$$

where $N \geq 3$, $3/4 < \alpha \leq 1$, λ and μ are the parameters, $g, h \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, and the potential $V(x)$ satisfies the following assumption:

(V) $V \in C(\mathbb{R}^N, \mathbb{R})$, $0 < V_0 := \inf_{x\in\mathbb{R}^N} V(x)$, and for each $M > 0$, $\operatorname{meas}\{x \in \mathbb{R}^N : V(x) \leq M\} < +\infty$, where V_0 is a constant and meas denotes the Lebesgue measure in $\mathbb{R}^N$.

In the last two decades, Equation (1) has been studied extensively due to its strong physical background. In particular, under the suitable parameters, Equation (1) can describe the self-channeling of a high-power ultra-short laser in matter, for detail [1–3]. In addition, Equation (1) also appears the superfluid film in plasma physics [4] and some fluid mechanics [5].

For the case where $\alpha = 1$, $\lambda = 1$, and $\mu = 1$, solutions of (1) are standing wave solutions of the following quasilinear Schrödinger equation:

$$\begin{aligned} &i\Psi_t + \Delta\Psi - W(x)\Psi + k\Delta\left(h\left(|\Psi|^2\right)\right)h'\left(|\Psi|^2\right)\Psi \\ &\quad + g(x,\Psi) + h(x,\Psi) = 0, \quad x \in \mathbb{R}^N. \end{aligned} \tag{2}$$

Many researchers focus on the nonlinear quasilinear Schrödinger Equations (1) and (2). Adachi and Watanabe [6] discussed the uniqueness of the ground state solutions for the following quasilinear Schrödinger equations:

$$-\Delta u + \lambda u - k\Delta\left(|u|^{\alpha}\right)|u|^{\alpha-2}u = |u|^{p-1}u, \quad x \in \mathbb{R}^N, \tag{3}$$

via a dual approach. The Nehari method was adopted to establish the existence results of ground state solutions by Liu et al. in [7]. The Lagrange multiplier method was used in [8] to prove the existence of the ground state solutions. Recently, Liu et al. in [9] applied a perturbation method to get the existence of positive solutions and a sequence of high energy solutions. In [10], Zhang et al. established the existence of infinite solutions for a modified nonlinear Schrödinger equation by using a symmetric mountain pass theorem as well as dual approach. Recently, Zhang et al. [11] offered a new iterative technique to get the existence and nonexistence of blow-up radial solutions for a Schrödinger equation involving a nonlinear operator.

The common point of the above works is that the nonlinear terms are all superlinear; to our best knowledge, there are few results on the quasilinear Schrödinger equation involving concave and convex nonlinearities. Furthermore, as mentioned above, there are few results on the existence of negative energy solutions. The reason is that the combination of concave and convex will make it difficult to verify the mountain pass theorem and fountain theorem. Inspired by the above-mentioned works, this paper is aimed at considering quasilinear Schrödinger Equation (1) involving concave and convex nonlinearities and proving the existence of mountain pass-type of solution and a sequence of infinitely many solutions with negative energy.

Throughout this paper, we assume that g and h satisfy the following assumptions:

(g_1) There exist constants $1 < r_1 < r_2 < \cdots < r_m < 2$ and functions $h_i(x) \in L^{2/2-r_i}(\mathbb{R}^N, \mathbb{R}^+)$, $(i = 1, 2, \cdots, m)$, such that

$$|g(x,u)| \leq \sum_{i=1}^{m} h_i(x)|u|^{r_i-1}. \tag{4}$$

(g_2) There exist $\sigma \in (1,2)$, $c_1 > 0$ such that $G(x,u) \geq c_1|u|^\sigma$ for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$.

(g_3) $g(x,u)$ is odd in u.

(h_1) $3/4 < \alpha \leq 1$, $4\alpha < p < 2\alpha \cdot 2^*$, $|h(x,u)| \leq C(|u| + |u|^{p-1})$ holds for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$.

(h_2) $h \in C(\mathbb{R}^N, \mathbb{R})$ and $h(x,u) = o(|u|)$, $|u| \longrightarrow 0$ uniformly on $\mathbb{R}^N$.

(h_3) There exists $\theta > 4$ such that

$$0 < H(x,u) := \int_0^u h(x,s)ds \leq \frac{u}{\theta}h(x,u), \quad \text{for all } (x,u) \in \mathbb{R}^N \times \mathbb{R}. \tag{5}$$

(h_4) There exists $R > 0$ such that

$$\inf_{x\in\mathbb{R}^N, |u|>R} H(x,u) > 0. \tag{6}$$

(h_5) $h(x,u)$ is odd in u.

Theorem 1. *Assume (V), (g_1), and (h_1)–(h_4) hold. Then, there exist two positive constants $\bar{\mu}$, $\bar{\lambda}$ such that for $(\lambda, \mu) \in (0, +\infty) \times (0, \bar{\mu}] \cup [-\bar{\lambda}, \bar{\lambda}] \times (0, +\infty)$, (1) has at least a nontrivial mountain pass-type solution.*

Theorem 2. *Assume (V), (g_1)–(g_3) and (h_1)–(h_5) hold. Then, for $\lambda > 0$, $\mu > 0$, problem (1) has a sequence of solutions with negative energy.*

Remark 3. (V), (g_1), and (h_1)–(h_3) ensure the compactness of energy functional. Inspired by [12], we adopt a new approach to verify the mountain structure; on the other hand, the Tayler expansion technique plays an important role in verifying a dual fountain theorem. As mentioned above, our results may be regarded as a generalization and improvement of many existing results.

The main tools of this paper are variational methods combined with analysis techniques and suitable estimations; these methods and techniques are important for dealing with our equation and establishing the relative energy estimation. Thus, in order to make readers follow our work more easily, here, we briefly recall these helpful analysis techniques such as variational methods, iterative technique, fixed point theorems, and finite element methods. In [13–15], variational methods and some critical point theorems were employed to study the existence of solution for various elliptic equations. The iterative technique [16, 17] was also developed to establish the existence criterion of solutions as well as the corresponding convergence analysis for Hessian-type elliptic equations. In addition, fixed point theorems [18–21] and finite element methods [21, 22] also provide important theoretical and numerical tools for solving various nonlinear equations.

The rest of this paper is organized as follows: in Section 2, the variational framework and some lemmas are presented. Section 3 is devoted to the Proofs of Theorem 10 and Theorem 11.

2. Preliminaries and Functional Setting

In this paper, we make use of the following notations:

$$\begin{aligned} H^1(\mathbb{R}^N) &:= \{u \in L^2(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}, \\ \|u\|_{H^1}^2 &:= \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2), \\ D^{1,2}(\mathbb{R}^N) &:= \left\{u \in L^{2^*}(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\right\}, \\ \|u\|_{D^{1,2}}^2 &:= \int_{\mathbb{R}^N} |\nabla u|^2. \end{aligned} \tag{7}$$

Set $E := \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 < +\infty\}$; E is a Hilbert space with inner product $(u,v)_E = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv)$. By $|\cdot|_p$, we denote the usual L^p norm. B_R denotes the open ball centered at the origin and radius $R > 0$. Throughout this paper, C and C_i are used in various places to denote positive constants, which are not essential to the problem.

It follows from ([23], Lemma 2.1) that the embedding $E \hookrightarrow L^r(\mathbb{R}^N)(2 \leq r < 2^*)$ is compact, and there is $\eta_r > 0$ such that $|u|_r \leq \eta_r\|u\|$ for all $u \in E$. The energy functional $J : E \longrightarrow \mathbb{R}$ is given by

$$\begin{aligned} J(u) &= \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) + \frac{1}{4\alpha}\int_{\mathbb{R}^N} \left|\nabla\left(|u^{2\alpha}\right)\right|^2 \\ &\quad - \int_{\mathbb{R}^N} (\lambda G(x,u) + \mu H(x,u)) \\ &= \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) + \alpha\int_{\mathbb{R}^N} |u|^{2(2\alpha-1)}|\nabla u|^2 \\ &\quad - \int_{\mathbb{R}^N} (\lambda G(x,u) + \mu H(x,u)). \end{aligned} \tag{8}$$

According to [6], we can make the change of variable by $v = f^{-1}(u)$, where f is defined by

$$f' = \frac{1}{\sqrt{1 + 2\alpha|f(t)|^{2(2\alpha-1)}}}, \quad f(-t) = -f(t) \text{ on } t \in [0, +\infty). \tag{9}$$

After the change of variables, we obtain the following functional:

$$I(v) := J(f(v)) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla v|^2 + V(x)f^2(v)\right) - \int_{\mathbb{R}^N} \left(\lambda G(x, f(v)) + \mu H(x, f(v))\right), \tag{10}$$

which is well defined in E under the assumptions (V), (h_1), (h_2), and (g_1). Moreover, the critical points of the functional I correspond to the weak solutions of the following equation:

$$-\Delta v = \frac{1}{\sqrt{1 + 2\alpha |f(v)|^{2(2\alpha-1)}}}\left[g(x, f(v)) + h(x, f(v))V(x)f(v)\right] \text{ in } \mathbb{R}^N. \tag{11}$$

It is shown in [24] that if $v \in E$ is a critical point of the function I, then $u = f(v) \in E$ and u is a solution of (1). The function $f(t)$ enjoys some properties given in [25]. The Mountain Pass lemma in [26] allows us to find Cerami-type sequence.

Lemma 4. *Assume (V), (h_1), (h_2), and (g_1) hold. Then, $I \in C^1(E, \mathbb{R})$ and*

$$\left\langle I'(v), w\right\rangle = \int_{\mathbb{R}^N} \left(\nabla v \nabla w + V(x)f(v)f'(v)w\right) - \int_{\mathbb{R}^N} \left(\lambda g(x, f(v))f'(v)w + \mu h(x, f(v))f'(v)w\right), \tag{12}$$

for all $w \in E$ and $\psi' : E \longrightarrow E^*$ is compact, where $\psi(v) = \int_{\mathbb{R}^N} G(x, f(v))$.

Proof. The proof is standard; we omit it here.

Let X be a Banach space with the norm $\|\cdot\|$ and $X = \oplus_{i=1}^{\infty} X_i$ and $\dim X_i < +\infty$ for each $i \in \mathbb{N}$. Further, we set

$$Y_k = \oplus_{i=1}^{k} X_i, \qquad Z_k = \overline{\oplus_{i=k}^{\infty} X_i}. \tag{13}$$

Lemma 5 ([27]). *Let $\Phi \in C^1(X, \mathbb{R})$ satisfies $\Phi(-u) = \Phi(u)$. Assume that, for every $k > k_0$, there exists $\rho_k > \gamma_k > 0$ such that*

(A_1) $a_k := \inf_{u \in Z_k, \|u\| = \rho_k} \Phi(u) \geq 0$.

(A_2) $b_k := \max_{u \in Y_k, \|u\| = \gamma_k} \Phi(u) < 0$.

(A_3) $d_k := \inf_{u \in Z_k, \|u\| \leq \rho_k} \Phi(u) \longrightarrow 0$ as $k \longrightarrow +\infty$.

(A_4) Φ satisfies the $(C)_c$ condition for every $c \in [d_{k_0}, 0)$.

Then Φ has a sequence of negative critical values converging to 0.

Remark 6. If $1 < p < 2^*$, then we have

$$\beta_k := \sup_{u \in Z_k, \|u\| = 1} |u|_p \longrightarrow 0, \; k \longrightarrow \infty. \tag{14}$$

3. Proof of Main Results

Lemma 7. *If $\|v_n\|^2 \longrightarrow +\infty$, then $\|v_n\|_0^2 := \int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)f^2(v)) \longrightarrow +\infty$, as $n \longrightarrow \infty$.*

Proof. If not, there exists a positive constant $C > 0$ such that $\|v_n\|^2 \longrightarrow +\infty$ and $\|v_n\|_0^2 \leq C$, as $n \longrightarrow \infty$. Then, $(\|v_n\|_0^2)/(\|v_n\|^2) \longrightarrow 0$. Set $w_n = v_n/\|v_n\|$ and $h_n = (f^2(v_n))/(\|v_n\|^2)$, one has $\int_{\mathbb{R}^N} (|\nabla w_n|^2 + V(x)h_n(x)) \longrightarrow 0$. Hence,

$$\int_{\mathbb{R}^N} |\nabla w_n|^2 \longrightarrow 0, \int_{\mathbb{R}^N} V(x)h_n(x) \longrightarrow 0, \int_{\mathbb{R}^N} V(x)w_n^2(x) \longrightarrow 1. \tag{15}$$

It follows from [25]; for each $\varepsilon > 0$, there exists C_2 independent of n such that $\operatorname{meas}(\Omega_n) < \varepsilon$, where $\Omega_n := \{x \in \mathbb{R}^N : |v_n(x)| \geq C_2\}$. One has

$$\int_{\mathbb{R}^N \setminus \Omega_n} V(x)w_n^2 \leq C_3 \int_{\mathbb{R}^N \setminus \Omega_n} V(x)\frac{f^2(v_n)}{\|v_n\|^2} \leq C_3 \int_{\mathbb{R}^N} V(x)h_n(x) \longrightarrow 0. \tag{16}$$

On the other hand, there exists $\varepsilon > 0$ such that whenever $\Omega' \subset R^N$ and $\operatorname{meas}(\Omega') < \varepsilon$, $\int_{\Omega'} V(x)w_n^2 \leq 1/2$. Thus,

$$\int_{\mathbb{R}^N} V(x)w_n^2 = \int_{\mathbb{R}^N \setminus \Omega_n} V(x)w_n^2 + \int_{\Omega_n} V(x)w_n^2 \leq o(1) + \frac{1}{2}, \tag{17}$$

which contradicts with (15).

Lemma 8. *Suppose (V), (g_1), and (h_1)–(h_4) hold. Then, there exists two positive constants $\bar{\mu}, \bar{\lambda}$, such that (i) for $(\lambda, \mu) \in (0, +\infty) \times (0, \bar{\mu}] \cup [-\bar{\lambda}, \bar{\lambda}] \times (0, +\infty)$; there exist $\rho > 0$, $\eta > 0$ such that* $\inf\{I(v) : v \in E, \|v\| = \rho\} > \eta$; *(ii) there exists $e \in E$ with $\|e\| > \rho$ such that $I(e) < 0$.*

Proof.

(i) For $r > 0$, $v \in S_r = \{v \in E : \|v\| = r\}$, when r is large enough, by simple computation, one has

$$I(v) \geq \|v\|_0^2 \left(\frac{1}{4} - |\lambda| C_4 \|v\|^{r_m - 2} - \mu \frac{C_5}{p}\|v\|^{p-2}\right). \tag{18}$$

Let $m(s) = |\lambda| C_4 s^{r_m - 2} + \mu (C_5/p) s^{p-2}$, $m(s)$ is bounded below; thus, $m(s)$ admits a minimizer $s_0 = ((|\lambda| C_4 p(2 - r_m))/(\mu C_5 (p - 2)))^{1/(p - r_m)}$, then

$$\inf_{s\in(0,+\infty)} m(s) = m(s_0) = \mu^{(2-r_m)/(p-r_m)} \cdot |\lambda|^{(p-2)/(p-r_m)} \cdot \left[C_4 \left(\frac{C_4 p(2-r_m)}{C_5(p-2)} \right)^{(r_m-2)/(p-r_m)} + \frac{C_5}{p} \left(\frac{C_4 p(2-r_m)}{C_5(p-2)} \right)^{(p-2)/(p-r_m)} \right]. \quad (19)$$

Let $K := C_4(C_4 p(2-r_m)/C_5(p-2))^{(r_m-2)/(p-r_m)} + (C_5/p)(C_4 p(2-r_m)/C_5(p-2))^{(p-2)/(p-r_m)}$. For all $\mu > 0$ and

$$|\lambda| \le \left(\frac{1}{4\mu^{(2-r_m)/(p-r_m)} K} \right)^{(p-r_m)/(p-2)} := \bar{\lambda} \quad (20)$$

or for all $\lambda > 0$ and

$$\mu \le \left(\frac{1}{4\lambda^{(p-2)/(p-r_m)} K} \right)^{(p-r_m)/(2-r_m)} := \bar{\mu}, \quad (21)$$

one has $m(s_0) < 1/4$. For $\|v\| = s_0$, there exists $s' > 0$ such that $\|v\|_0 \in (s', s_0]$. Thus, there exists $\rho > 0$ such that $\inf_{v\in S_\rho} I(v) > 0 = I(0)$

(ii) The conclusion is easy to check, and we omit it here.

Lemma 9. *Suppose (V), (g_1), and (h_1)–(h_3) hold. Then I satisfies the $(C)_c$ condition*

Proof.

(1) Let $\{v_n\}$ be a $(C)_c$ sequence of I, that is, $I(v_n) \longrightarrow c$, $(1 + \|v_n\|)I'(v_n) \longrightarrow 0$. By Lemma 7, after some direct computation, one gets that the sequence $\{v_n\}$ is bounded

(2) Passing to a subsequence if necessary, assume that $v_n \rightharpoonup v$ in E. First, through a proof by contradiction with the help of $\alpha \in (3/4, 1]$, there exists $C_4 > 0$ such that

$$\int_{\mathbb{R}^N} \left[|\nabla(v_n - v)|^2 + V(x)\left(f(v_n)f'(v_n) - f(v)f'(v) \right)(v_n - v) \right] \ge C_4 \|v_n - v\|^2. \quad (22)$$

On the other hand, by Lemma 4, ψ' is compact, $\|\psi'(v_n) - \psi'(v)\| \longrightarrow 0$, and

$$\left\langle \psi'(v_n) - \psi'(v), v_n - v \right\rangle \longrightarrow 0 \, as \, n \longrightarrow \infty, \quad (23)$$

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \left(h(x, f(v_n))f'(v_n) - h(x, f(v))f'(v) \right)(v_n - v) = 0. \quad (24)$$

Therefore, by (22)–, one has

$$o(1) = \left\langle I'(v_n) - I'(v), v_n - v \right\rangle \ge C_6 \|v_n - v\|^2 + o(1), \quad (25)$$

which implies $\|v_n - v\| \longrightarrow 0$ as $n \longrightarrow \infty$

Proof of Theorem 10. As a consequence of Lemmas 8 and 9 and Mountain Pass theorem, we get the desired result.

Proof of Theorem 11. Since E is a reflexive and separable Banach space, there exist $\{e_j\} \subset E$ and $\{e_j^*\} \subset E^*$ such that $E = \text{span}\{e_j : j = 1, 2, \cdots\}$, $E^* = span\{e_j^* : j = 1, 2, \cdots\}$ in which

$$\left\langle e_i, e_j^* \right\rangle = \begin{cases} 1, & i = j, \\ 0, & i \ne j. \end{cases} \quad (26)$$

Set $E_j = \text{span}\{e_j\}$, $Y_k = \oplus_{i=1}^k E_i$, $Z_k = \oplus_{i=k}^\infty E_i$. We will use Dual Fountain theorem Lemma 5 to prove Theorem 2. Set

$$Q(v) := \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla v|^2 + V(x) f^2(v) \right), \quad (27)$$

Q is a C^2-functional on E. Now, by the Taylor formula and some simple computation, we get

$$\begin{aligned} Q(v) &= Q(0) + \left\langle Q'(0), v \right\rangle + \frac{1}{2} \left\langle Q''(0)v, v \right\rangle + o\left(\|v\|^2 \right) \\ &= \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla v|^2 + V(x) v^2 \right) dx + o\left(\|v\|^2 \right), \ as \ \|v\| \longrightarrow 0. \end{aligned} \quad (28)$$

On the other hand, on Z_k, for $\|v\|$ small enough, $\int_{\mathbb{R}^N} |G(x, f(v))| \le C_6 |v|_2^{r_1}$. Thus, by Remark 6 and simple computation, there exists $C > 0$ such that

$$I(v) \ge \|v\|^2 \left(\frac{C}{8} - \lambda C_6 \beta_k^{r_1} \|v\|^{r_1 - 2} \right), \quad (29)$$

which implies (A_1) holds. By (g_2), for $\|v\|$ which is small enough, we have

$$\begin{aligned} I(v) &= \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(x) f^2(v) - \lambda \int_{\mathbb{R}^N} G(x, f(v)) \\ &\quad - \mu \int_{\mathbb{R}^N} H(x, f(v)) \le \frac{1}{2} \|v\|^2 - C|v|_\sigma^\sigma + \mu C_2 |v|_2^2. \end{aligned} \quad (30)$$

Since Y_k is a finite dimensional space, (A_2) is satisfied for every $\gamma_k > 0$ which small enough, when $\lambda > 0$, $\mu > 0$. From (29), for $k > k_0$, $v \in Z_k$, $\|v\| \le \rho_k$,

$$I(v) \ge -\lambda C_6 \beta_k^{r_1} \|v\|^{r_1} \ge -\lambda C_6 \beta_k^{r_1} \rho_k^{r_1}, \quad (31)$$

since $\beta_k \longrightarrow 0$ and $\rho_k \longrightarrow 0$ as $k \longrightarrow \infty$; condition (A_3) is also checked. (A_4) follows from Lemma 9. Thus, by using Lemma 5, we get the desired results and complete the proof.

Data Availability

The authors declare that there is no data available.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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